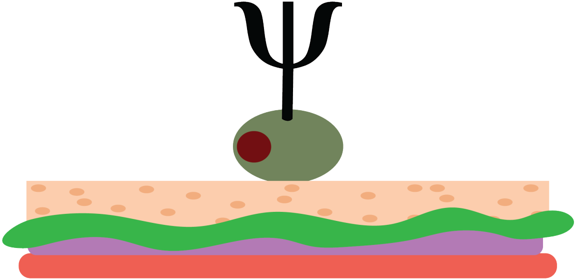
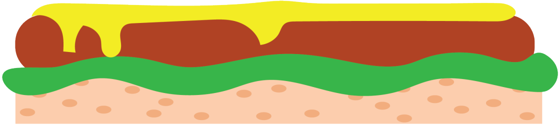


ABC's of M-estimation



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Thank you for attending

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github.com/pzivich/ABCs_of_M-estimation

Feedback on workshop: forms.gle/fAK6nQFj8zRknFQj7

Overview

Overview

Section 1: introduction

Section 2: applied examples

Destination

Three use-cases of M-estimators

- Estimate the variance of a marginal structural model
- Bridged treatment comparisons
- Sensitivity analyses

While other estimators can be used here, M-estimators have computational advantages

Variance with IPW

IPW for estimating marginal structural model (MSM)[1]

- Model propensity score
- Calculate weighted mean

Variance estimation is complicated

- Variance for MSM depends on variance of propensity scores
- Commonly use the 'GEE' trick
- Bootstrap is computationally intensive

M-estimation

- Not overly conservative
- Computationally simpler than bootstrap

[1] Robins et al. (2000) *Epidemiology*

Bridged Treatment Comparisons

Suppose we want to learn the causal effect of A vs C[2]

- Only have data that compares A vs B *and* B vs C

Bridged treatment comparisons link across data source

- Compare A to C through B
- Analytically account for differences between source
- Further correct for confounding or missing outcomes

M-estimation

- Consistently estimate the variance with models
- Holds for not identically distributed data

[2] Zivich et al. *arXiv:2206.04445*

Sensitivity Analysis

Missing data is a common problem

- Descriptive, predictive, causal

Missingness may depend on the missing variable

- Whether Y is observed depends on Y
- Missing not at random
- Sensitivity analyses to assess the impact

M-estimation

- Avoid having to use the bootstrap
- Evaluate a large number of scenarios

Overview

Section 1: introduction

Section 2: applied examples

Overview: Section 1

Review notation / definitions

M-estimator by-hand

M-estimator computationally

Some useful properties of M-estimators

Notation – Basics

O_i: observed data for unit i

- $O_i = (X_i, Y_i)$
- $O_i = (W_i, A_i, Y_i)$

$\sum_{i=1}^{n} i \; = \; 1 + 2 + ... + n$: cumulative sum

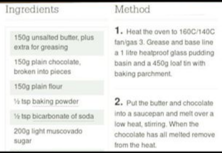
$\prod_{i=1}^{n} i \; = \; 1 \times 2 \times ... \times n$: cumulative product

expit$(a) = 1/(1 + \exp(-a))$

Notation – Basics

estimand θ
(parameter of interest)

estimator $\hat{\theta}$


Ingredients

- 150g unsalted butter, plus extra for greasing
- 150g plain chocolate, broken into pieces
- 150g plain flour
- ½ tsp baking powder
- ½ tsp bicarbonate of soda
- 200g light muscovado sugar

Method

1. Heat the oven to 160C/140C fan/gas 3. Grease and base line a 1 litre heatproof glass pudding basin and a 450g loaf tin with baking parchment.
2. Put the butter and chocolate into a saucepan and melt over a low heat, stirring. When the chocolate has all melted remove from the heat.

estimate 0.5

Notation – Matrix Algebra

Transpose

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Notation – Matrix Algebra

Dot product (multiplication)

$$\mathbf{B}\ \mathbf{C} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a\ w + b\ y & a\ x + b\ z \end{bmatrix}$$

- Elements in row of 1st must match elements in column of 2nd
  - $\mathbf{C}\ \mathbf{B}$ would not be defined
- Output has rows of 1st and columns of 2nd

Notation – Matrix Algebra

Inverse of matrix

$$\mathbf{D} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad \mathbf{D}^{-1} = \frac{1}{w \; z - x \; y} \begin{bmatrix} z & -y \\ -x & w \end{bmatrix}$$

- Only applies to matrices with same number of rows and columns

Derivatives – Basics

$$f'(x) = \frac{d}{dx} f(x)$$

Helpful to think of derivative as slope at a point

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives – Generalizations

If $\boldsymbol{x} = (x_1, x_2, ..., x_m)$, then the partial derivative is

$$\frac{\partial}{\partial x_1} f(\boldsymbol{x})$$

The gradient is

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \\ \frac{\partial}{\partial x_2} f(\boldsymbol{x}) \\ \vdots \\ \frac{\partial}{\partial x_m} f(\boldsymbol{x}) \end{bmatrix}$$

Derivatives – Generalizations

The Hessian is

$$\Delta H_f(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f(\boldsymbol{x}) & \dots & \frac{\partial^2}{\partial x_1 \partial x_m} f(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_m \partial x_1} f(\boldsymbol{x}) & \dots & \frac{\partial^2}{\partial x_m \partial x_m} f(\boldsymbol{x}) \end{bmatrix}$$

- Jacobian (transpose gradient, ∇^T) of the gradient

Notation – M-estimation

Estimating *function*

$$\psi(O_i; \theta)$$

Estimating *equation*

$$\sum_{i=1}^{n} \psi(O_i; \theta)$$

Definition: M-estimator

An M-estimator, $\hat{\theta}$, is the solution to

k-dimensional estimating function

k-dimensional parameter

$$\sum_{i=1}^{n} \boxed{\psi(O_i, \hat{\theta})} = 0$$

Observation i

root: where $f(x) = 0$

- Don't worry if any of the above isn't clear yet

M-estimator for the mean

Problem: Learn the Mean

Want to learn the population mean

- Estimand: $\mu = E[Y]$

Suppose we have the following observations to estimate μ

$$7, 1, 5, 3, 24$$

Usual method

Estimator for mean　　　　　　　Observed value for unit i

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

Total number of units

Applying to data in example (estimate)

$$\frac{7+1+5+3+24}{5} = \frac{40}{5} = 8$$

but let's use M-estimation instead

M-estimator steps

1. Determine estimating function
2. Find the roots of the estimating equations
3. Estimate variance via the sandwich

1. Determine Estimating Function

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} Y_i$$ definition

$$\hat{\mu} \; n = \sum_{i=1}^{n} Y_i$$ multiply by n

$$0 = \sum_{i=1}^{n} (Y_i) - \hat{\mu} \; n$$ subtract $\hat{\mu}n$

$$0 = \sum_{i=1}^{n} (Y_i) - \sum_{i=1}^{n} (\hat{\mu})$$

$$0 = \sum_{i=1}^{n} (Y_i - \hat{\mu})$$

1. Determine Estimating Function

This formula is our M-estimator for the mean

$$\sum_{i=1}^{n} \underbrace{\overbrace{\psi}^{\text{Estimating function}} (\underbrace{O_i}_{\text{Observation } i} , \overbrace{\hat{\theta}}^{\text{Parameter}})} = \sum_{i=1}^{n} (\underbrace{Y_i}_{\text{Observation } i} - \overbrace{\hat{\mu}}^{\text{Parameter}}) = 0$$

Alternative Derivation

Mean is value where

$$\min \sum_{i=1}^{n} \rho(O_i; \hat{\mu}) = \min \sum_{i=1}^{n} (Y_i - \hat{\mu})^2$$

At the minimum (or maximum), the slope is zero

$$\sum_{i=1}^{n} \rho'(O_i; \hat{\mu}) = 0$$

Therefore

$$\sum_{i=1}^{n} \rho'(O_i; \hat{\mu}) = \sum_{i=1}^{n} (Y_i - \hat{\mu}) = 0$$

2. Root-finding

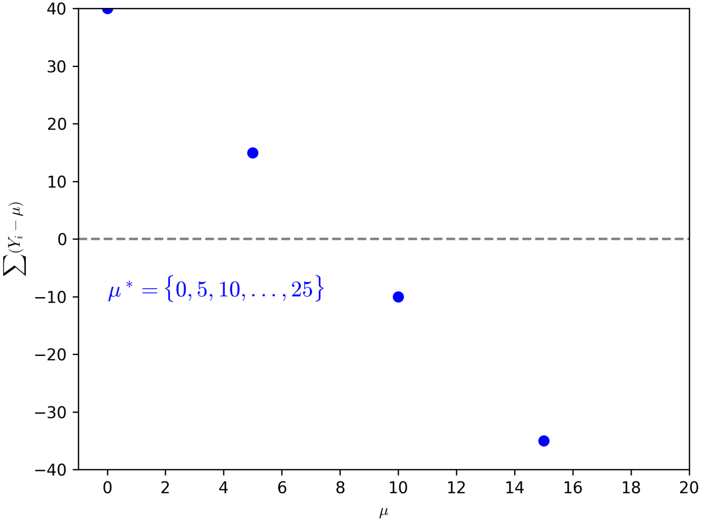
How can we find $\hat{\mu}$?

- Ignore the closed-form solution for the time

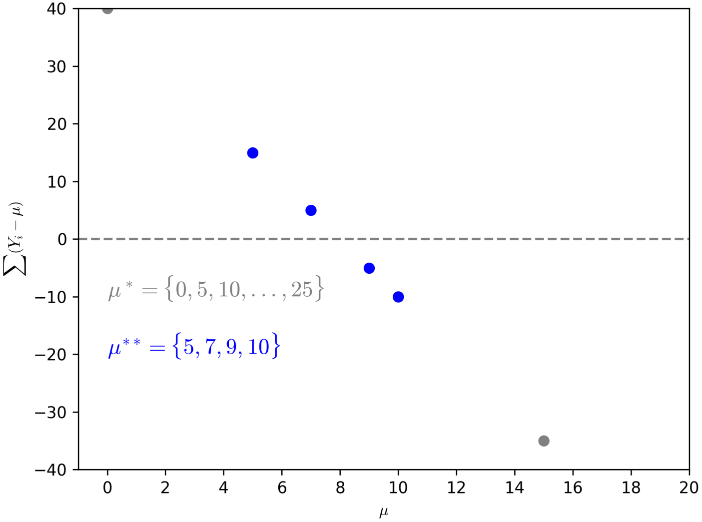
Broadly

- Take some guesses at $\hat{\mu}$, denoted as $\hat{\mu}^*$
- Compute $\sum_{i=1}^{n} \psi(O_i; \hat{\mu}^*)$
- Find the guesses that are close to zero
- Generate some new guesses, $\hat{\mu}^{**}$
- Repeat process until we find $\hat{\mu}$

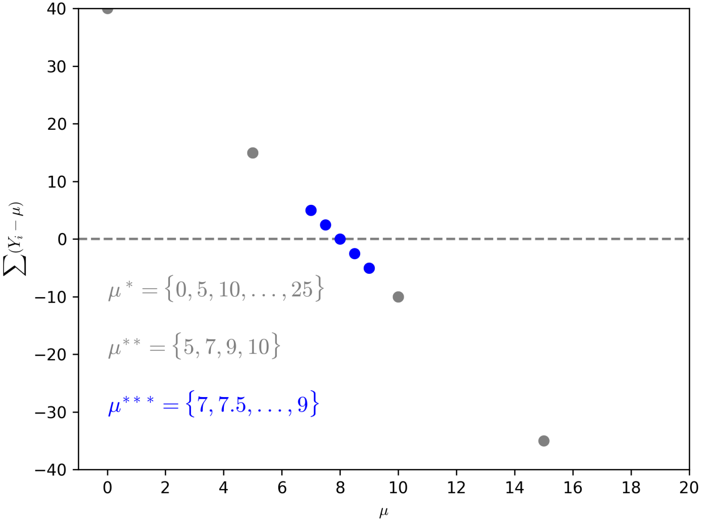
2. Root-finding



2. Root-finding



2. Root-finding



3. Variance

Closed-form estimator

$$\widehat{Var}(\hat{\mu}) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{\mu})^2$$

but let's use M-estimation instead

3. Sandwich Variance Estimator

Sandwich variance

Filling (meat) matrix

$$V(\hat{\theta}) = B(\hat{\theta})^{-1} \; F(\hat{\theta}) \left(B(\hat{\theta})^{-1} \right)^T$$

(inverse of) Bread matrix

3. Sandwich Variance Estimator

Bread matrix

$$\boxed{B(\hat{\theta})} = \frac{1}{n}\sum_{i=1}^{n}\left[-\boxed{\psi'(O_i, \hat{\theta})}\right]$$

Partial derivatives (Jacobian)

Filling matrix

$$\boxed{F(\hat{\theta})} = \frac{1}{n}\sum_{i=1}^{n}\left[\boxed{\psi(O_i, \hat{\theta})}\ \boxed{\psi(O_i, \hat{\theta})^T}\right]$$

Dot product of estimating functions

Baking the Bread: By-Hand

Need the derivative of $\psi(O_i; \mu)$

$$\begin{aligned}
\psi'(O_i; \hat{\mu}) =& \frac{d}{d\hat{\mu}} \psi(O_i; \hat{\mu}) && \text{definition of derivative} \\
=& \frac{d}{d\hat{\mu}} (Y_i - \hat{\mu}) && \text{definition of estimating function} \\
=& -1
\end{aligned}$$

Therefore

$$\underbrace{\frac{1}{n} \sum_{i=1}^{n} \left[-\psi'(O_i, \hat{\theta}) \right]}_{\text{Definition of Bread}} = \frac{1}{n} \sum_{i=1}^{n} \left[- \underbrace{-1}_{\text{From derivative above}} \right] = 1$$

Cooking the Filling: By-Hand

Definition of Filling

$$\boxed{\frac{1}{n}\sum_{i=1}^{n}\left[\psi(O_i,\hat{\theta})\psi(O_i,\hat{\theta})^T\right]} = \frac{1}{n}\sum_{i=1}^{n}\left[\boxed{(Y_i-\hat{\mu})(Y_i-\hat{\mu})}\right]$$

Plugging in estimating function

Therefore

$$\frac{1}{5}\sum_{i=1}^{5}\left[(Y_i-8)^2\right] = 68$$

Assembling the Sandwich: By-Hand

$$\underbrace{V(\hat{\mu})}_{\text{Sandwich variance}} = \underbrace{1^{-1}}_{\text{Bread}} \underbrace{68}_{\text{Filling}} \left(\underbrace{1^{-1}}_{\text{Bread}} \right)^T = 68$$

Confidence intervals

$$\hat{\mu} \pm z_\alpha \sqrt{\frac{V(\hat{\mu})}{n}} = 8 \pm 1.96 \sqrt{\frac{68}{5}} = (0.8, 15.2)$$

Computation for M-estimators

Computation for M-estimators

Solved for M-estimator of mean by-hand

- By-hand is not needed

Instead, consider how M-estimators can be implemented

- Root-finding
- Approximation of derivatives
- Matrix algebra

Follow along in `mean.R`, `mean.sas`, or `mean.py`

- Start of code inputs data and sets up estimating equations

Root-Finding – Algorithms

Performed a by-hand search for $\hat{\mu}$

- Similar to the *bisection method*

Variety of multidimensional root-finding algorithms exist

- Secant method (quasi-Newton)
- Levenberg-Marquardt
- Powell hybrid method

Root-Finding – Code

Under `Root-finding` see implementation

- SAS – `nlplm`
- R – `rootSolve::multiroot`
- Python – `scipy.optimize.root`

Derivatives – Back to the Definition

Derivative of function

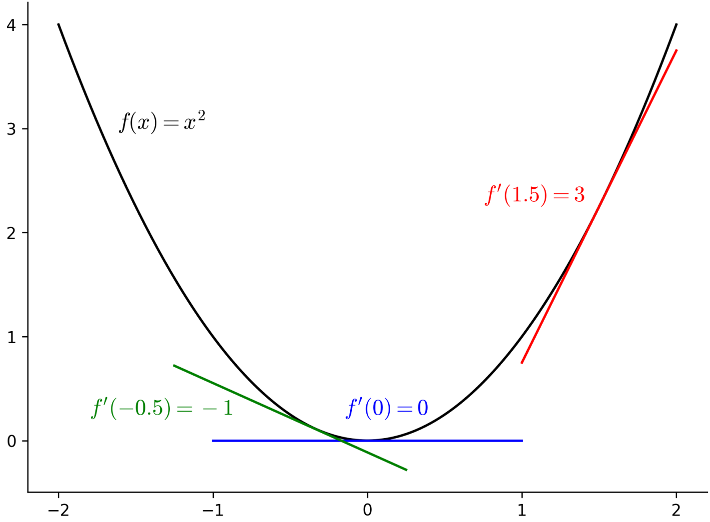
Change in output (rise)

$$\boxed{f'(x)} = \boxed{\lim_{h \to 0}} \frac{\boxed{f(x+h) - f(x)}}{\boxed{h}}$$

Behavior as h becomes small

Divided change in input (run)

Derivatives – Intuition



Derivatives – Numerical Approximation

Central Difference Method

$$\underbrace{\tilde{f}'(x)}_{\text{Approximation}} = \frac{f(\overbrace{x+a}^{\text{Slightly above } x}) - f(\overbrace{x-a}^{\text{Slightly below } x})}{2a}$$

Here a is a small value (e.g., 1×10^{-9})

Baking the Bread – Code

Under `Baking the bread` see implementation

- SAS – `nlpfdd`
- R – `numDeriv::jacobian`
- Python – `scipy.optimize.approx_fprime`

Cooking the Filling – Code

Under `Cooking the filling` see implementation

- Transpose
  - SAS – `'`
  - R – `base::t`
  - Python – `numpy.transpose`
- Dot product
  - SAS – `*`
  - R – `%*%`
  - Python – `numpy.dot`

Assembling the Sandwich – Code

Under `Assembling the sandwich` see implementation

- Inverse
  - SAS – `inv`
  - R – `base::solve`
  - Python – `numpy.linalg.inv`

Implications

To implement an M-estimator, we only need to provide

- Valid estimating functions
- Data

Everything else can be done by the computer

- Potential to simplify complex analyses

Extensions

But Why M-estimation?

So far, all we've done is calculate the mean in a complicated way

So why bother with M-estimation?

- Flexibility of the framework
    - Extensions of these basics
    - Simplified proofs for properties of estimators

How M-estimators are extended

As will be seen in applied examples

1. Stacking estimating functions
2. Automation of delta method

Stacking estimating functions

Often want to estimate more than 1 parameter

- Regression models
- Effect measure modification
- Inverse probability weighting requires estimating propensity scores

Stacking Estimating Functions

M-estimators extend by stacking estimating functions

$$\sum_{i=1}^{n} \begin{bmatrix} \psi_1(O_i;\theta) \\ \psi_2(O_i;\theta) \\ \psi_3(O_i;\theta) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \mathbf{0}$$

- Easy to stack estimating functions together
- Unlike maximizing a likelihood
  - Likelihood has a single value for individual contribution
  - Need each parameter to contribute correctly
  - More difficult to combine likelihood functions

Stacking Estimating Functions

Example

$$\sum_{i=1}^{n} \begin{bmatrix} \psi_1(O_i;\theta) \\ \psi_2(O_i;\theta) \end{bmatrix} = \begin{bmatrix} Y_i - \theta_1 \\ X_i - \theta_2 \end{bmatrix} = \mathbf{0}$$

- Stacking important when parameter depends on other parameters
- Concept explored further in next section

Delta Method

Theorem: smooth function of AN estimator is also AN

Application:

$$Var\left\{ \overbrace{\boxed{g(\alpha)}}^{\text{Transformation of } \alpha} \right\} \approx \underbrace{g'(\alpha)}_{\text{Derivative of transformation}} \overbrace{\Sigma_\alpha}^{\text{Covariance of } \alpha} \underbrace{g'(\alpha)}_{\text{Derivative of transformation}}$$

Delta Method

Many variance formulas you know are Delta method results

- $Var(RD)$, $Var(\log(RR))$, $Var(\log(OR))$
- Formulas follow from Delta method argument
- Don't need to manually solve due to known formulas
    - Not always the case

Delta Method with M-estimation

The estimating function for the transformed parameter, θ_t is

$$\psi_t(O_i; \theta, \theta_t) = g(\theta) - \theta_t$$

- Estimating function does not depend on data

Therefore, the stacked estimating equations are

$$\sum_{i=1}^{n} \begin{bmatrix} \psi^*(O_i; \theta) \\ \psi_t(O_i; \theta, \theta_t) \end{bmatrix} = 0$$

Delta Method with M-estimation

Following some derivatives and matrix algebra

$$V(\theta, \theta_t) = \begin{bmatrix} V^*(\theta) & g'(\theta)V^*(\theta) \\ V^*(\theta)g'(\theta)^T & g'(\theta)V^*(\theta)g'(\theta) \end{bmatrix}$$

where

$$V(\theta_t) = \underbrace{g'(\theta)}_{\text{Derivative of transformation}} \overbrace{V^*(\theta)}^{\text{Sandwich covariance for } \theta} \underbrace{g'(\theta)}_{\text{Derivative of transformation}}$$

- which is the same result from the delta method!

M-estimators automate the Delta method

Statistical Properties of M-estimators

Key Statistical Properties

Consistency (C)

- Estimator converges to the estimand as $n \to \infty$ in probability
- As your data increases, the estimate goes to the true value

Asymptotic Normality (AN)

- As $n \to \infty$ the distribution of $\hat{\theta}$ converges to a normal distribution

Desirable properties for estimators

- C: inconsistent estimators may not give 'right' answer
  - Even with massive amounts of data
- AN: Wald-type confidence intervals are justified

CAN Proofs with M-estimation

If you can prove that the estimating equations are unbiased[3]

- Following some regularity conditions
- Sufficient to prove M-estimator is CAN

[3] Section 7.8 (pg 327) of Boos & Stefanski's *Essential Statistical Inference*

Robust Variance

The sandwich variance is also known as the 'robust' variance

Maximum likelihood estimation

1. Inverse Hessian of the log-likelihood
   - Equivalent to the bread, $B(\theta)$
2. Residuals of the score function
   - Equivalent to the filling, $F(\theta)$

- These variance estimators asymptotically equivalent
  - When the model is correctly specified
  - $B(\theta) = F(\theta)$
  - Hessian is generally most efficient in finite samples

Robust Variance

When the model is not correctly specified

- $B(\theta) \neq F(\theta)$
- Example: log-Poisson model to estimate the risk ratio
  - Here, estimated variance is too large
- Here, sandwich variance works
  - By combining, sandwich is robust to assumptions
  - Variance estimator is consistent even if model is wrong

Caution:[4]

- Does not correct for bias in parameter estimates
- Okay to use for log-Poisson because unbiased for RR
- Otherwise inference is no longer for original θ

[4] See Freedman DA *Am Stat* 2006 for details

Applied Examples