THE BY-PRODUCTION MODEL

Let us assume a fixed level of inputs $\boldsymbol{x}_0 = (x_{10},...,x_{m0}) \in R_+^m$, which is divided into two groups: non-polluting inputs, $\boldsymbol{q} \in R_+^{m_1}$ and pollution-generating inputs, $\boldsymbol{p} \in R_+^{m_2}$, with $m_1 + m_2 = m$. Let also assume a fixed level of good outputs $\boldsymbol{y}_0 = (y_{10},...,y_{s0}) \in R_+^s$ and a fixed level of bad outputs $\boldsymbol{z}_0 = (z_{10},...,z_{s'0}) \in R_+^{s'}$. $\boldsymbol{z}_0 = (z_{10},...,z_{s'0}) \in R_+^{s'}$.

Ψ_1 is standard production technology with only good outputs, while Ψ_2 is production technology with bad outputs. The technology in general terms is defined as $\Psi = \Psi_1 \cap \Psi_2$, where

$$\Psi_1 = \left\{ \left(\boldsymbol{q},\boldsymbol{p},\boldsymbol{y},\boldsymbol{z}\right) \in R_+^{m_1+m_2+s+s'} : h\left(\boldsymbol{q},\boldsymbol{p},\boldsymbol{y}\right) \le 0 \right\}$$

$$\Psi_2 = \left\{ \left(\boldsymbol{q},\boldsymbol{p},\boldsymbol{y},\boldsymbol{z}\right) \in R_+^{m_1+m_2+s+s'} : \boldsymbol{z} \ge l\left(\boldsymbol{p}\right) \right\}$$

DEA FRAMEWORK

$$\begin{aligned}
\vec{B}_{\Psi_1}^{DEA}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0;\Psi_1\right) = \quad & Max \quad \beta^{\Psi_1} \\
s.t. \quad & \sum_{i=1}^{n} \lambda_i q_{ji} \le q_{j0}, && j = 1,...,m_1 \\
& \sum_{i=1}^{n} \lambda_i p_{ji} \le p_{j0}, && j = m_1+1,...,m \\
& \sum_{i=1}^{n} \lambda_i y_{ri} \ge y_{r0} + \beta^{\Psi_1} y_{r0}, && r = 1,...,s \\
& \beta^{\Psi_1} \ge 0, \\
& \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \ge 0, && i = 1,...,n
\end{aligned}$$

$$\begin{aligned}
\vec{B}_{\Psi_2}^{DEA}\left(\boldsymbol{p}_0,\boldsymbol{z}_0;\Psi_2\right) = \quad & Max \quad \beta^{\Psi_2} \\
s.t. \quad & \sum_{i=1}^{n} \mu_i p_{ji} \ge p_{j0}, && j = m_1+1,...,m \\
& \sum_{i=1}^{n} \mu_i z_{ki} \le z_{k0} - \beta^{\Psi_2} z_{k0}, && k = 1,...,s' \\
& \beta^{\Psi_2} \ge 0, \\
& \sum_{i=1}^{n} \mu_i = 1, \mu_i \ge 0, && i = 1,...,n
\end{aligned}$$

$$\vec{B}^{DEA}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0,\boldsymbol{z}_0\right) = \delta^{\Psi_1}\left(1 - \frac{1}{1+\beta^{\Psi_1 *}}\right) + \delta^{\Psi_2}\beta^{\Psi_2 *}$$

Note that $\beta^{\Psi_1 *} \in \left[0,+\infty\right)$ and $\beta^{\Psi_2 *} \in \left[0,1\right]$; and the parameters $\delta^{\Psi_1} \ge 0$ and $\delta^{\Psi_2} \ge 0$.

FDH FRAMEWORK

$$\vec{B}_{\Psi_1}^{FDH}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0;\Psi_1\right)= \quad Max \quad \beta^{\Psi_1}$$

$$\begin{aligned}
s.t. \quad & \sum_{i=1}^{n}\lambda_i q_{ji}\le q_{j0}, && j=1,\dots,m_1\\
& \sum_{i=1}^{n}\lambda_i p_{ji}\le p_{j0}, && j=m_1+1,\dots,m\\
& \sum_{i=1}^{n}\lambda_i y_{ri}\ge y_{r0}+\beta^{\Psi_1}y_{r0}, && r=1,\dots,s\\
& \beta^{\Psi_1}\ge 0, &&\\
& \sum_{i=1}^{n}\lambda_i=1,\lambda_i\in\{0,1\}, && i=1,\dots,n
\end{aligned}$$

$$\vec{B}_{\Psi_2}^{FDH}\left(\boldsymbol{p}_0,\boldsymbol{z}_0;\Psi_2\right)= \quad Max \quad \beta^{\Psi_2}$$

$$\begin{aligned}
s.t. \quad & \sum_{i=1}^{n}\mu_i p_{ji}\ge p_{j0}, && j=m_1+1,\dots,m\\
& \sum_{i=1}^{n}\mu_i z_{ki}\le z_{k0}-\beta^{\Psi_2}z_{k0}, && k=1,\dots,s'\\
& \beta^{\Psi_2}\ge 0, &&\\
& \sum_{i=1}^{n}\mu_i=1,\mu_i\in\{0,1\}, && i=1,\dots,n
\end{aligned}$$

$$\vec{B}^{FDH}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0,\boldsymbol{z}_0\right)=\delta^{\Psi_1}\left(1-\frac{1}{1+\beta^{\Psi_1*}}\right)+\delta^{\Psi_2}\beta^{\Psi_2*}$$

EAT FRAMEWORK

$$\vec{B}_{\Psi_1}^{EAT}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0;\Psi_1\right)= \quad Max \quad \beta^{\Psi_1}$$

$$\begin{aligned}
s.t. \quad & \sum_{t\in\tilde{T}_1^*(\aleph)}\lambda_t a^t_{jT_1^*(\aleph)}\le q_{j0}, && j=1,\dots,m_1\\
& \sum_{t\in\tilde{T}_1^*(\aleph)}\lambda_t a^t_{jT_1^*(\aleph)}\le p_{j0} && j=m_1+1,\dots,m\\
& \sum_{t\in\tilde{T}_1^*(\aleph)}\lambda_t d_{rT_1^*(\aleph)}\left(\boldsymbol{a}^t_{T_1^*(\aleph)}\right)\ge y_{r0}+\beta^{\Psi_1}y_{r0}, && r=1,\dots,s\\
& \beta^{\Psi_1}\ge 0, &&\\
& \sum_{t\in\tilde{T}_1^*(\aleph)}\lambda_t=1,\lambda_t\in\{0,1\}, && t\in\tilde{T}_1^*(\aleph)
\end{aligned}$$

$$\vec{B}_{\Psi_2}^{EAT}\left(\boldsymbol{p}_0,\boldsymbol{z}_0;\Psi_2\right)= \quad Max \quad \beta^{\Psi_2}$$

$$\begin{aligned}
s.t. \quad & \sum_{t\in\tilde{T}_2^*(\aleph)} \mu_t d_{jT_2^*(\aleph)}\left(\boldsymbol{a}_{T_2^*(\aleph)}^t\right) \geq p_{j0}, && j=m_1+1,\dots,m \\
& \sum_{t\in\tilde{T}_2^*(\aleph)} \mu_t a_{kT_2^*(\aleph)}^t \leq z_{k0} - \beta^{\Psi_2} z_{k0}, && k=1,\dots,s' \\
& \beta^{\Psi_2} \geq 0, && \\
& \sum_{t\in\tilde{T}_2^*(\aleph)} \mu_t = 1, \mu_t \in \{0,1\}, && t\in\tilde{T}_2^*(\aleph)
\end{aligned}$$

$$\vec{B}^{EAT}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0,\boldsymbol{z}_0\right)= \delta^{\Psi_1}\left(1-\frac{1}{1+\beta^{\Psi_1 *}}\right)+\delta^{\Psi_2}\beta^{\Psi_2 *}.$$

Note that $\left\{\left(\boldsymbol{a}_{T_1^*(\aleph)}^t, \boldsymbol{d}_{T_1^*(\aleph)}\left(\boldsymbol{a}_{T_1^*(\aleph)}^t\right)\right)\right\}_{t\in\tilde{T}_1^*(\aleph)}$ are created from the optimal tree structure $T_1^*(\aleph)$ of the EAT algorithm.

CEAT FRAMEWORK

$$\vec{B}_{\Psi_1}^{CEAT}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0;\Psi_1\right)= \quad Max \quad \beta^{\Psi_1}$$

$$\begin{aligned}
s.t. \quad & \sum_{t\in\tilde{T}_1^*(\aleph)} \lambda_t a_{jT_1^*(\aleph)}^t \leq q_{j0}, && j=1,\dots,m_1 \\
& \sum_{t\in\tilde{T}_1^*(\aleph)} \lambda_t a_{jT_1^*(\aleph)}^t \leq p_{j0} && j=m_1+1,\dots,m \\
& \sum_{t\in\tilde{T}_1^*(\aleph)} \lambda_t d_{rT_1^*(\aleph)}\left(\boldsymbol{a}_{T_1^*(\aleph)}^t\right) \geq y_{r0} + \beta^{\Psi_1} y_{r0}, && r=1,\dots,s \\
& \beta^{\Psi_1} \geq 0, && \\
& \sum_{t\in\tilde{T}_1^*(\aleph)} \lambda_t = 1, \lambda_t \geq 0, && t\in\tilde{T}_1^*(\aleph)
\end{aligned}$$

$$\vec{B}_{\Psi_2}^{CEAT}\left(\boldsymbol{p}_0,\boldsymbol{z}_0;\Psi_2\right)= \quad Max \quad \beta^{\Psi_2}$$

$$\begin{aligned}
s.t. \quad & \sum_{t\in\tilde{T}_2^*(\aleph)} \mu_t d_{jT_2^*(\aleph)}\left(\boldsymbol{a}_{T_2^*(\aleph)}^t\right) \geq p_{j0}, && j=m_1+1,\dots,m \\
& \sum_{t\in\tilde{T}_2^*(\aleph)} \mu_t a_{kT_2^*(\aleph)}^t \leq z_{k0} - \beta^{\Psi_2} z_{k0}, && k=1,\dots,s' \\
& \beta^{\Psi_2} \geq 0, && \\
& \sum_{t\in\tilde{T}_2^*(\aleph)} \mu_t = 1, \mu_t \geq 0, && t\in\tilde{T}_2^*(\aleph)
\end{aligned}$$

$$\vec{B}^{CEAT}\left(\boldsymbol{q}_0,\boldsymbol{p}_0,\boldsymbol{y}_0,\boldsymbol{z}_0\right)= \delta^{\Psi_1}\left(1-\frac{1}{1+\beta^{\Psi_1 *}}\right)+\delta^{\Psi_2}\beta^{\Psi_2 *}.$$