

# FYS4150 Project 1:

## 1-dimensional Poisson equation

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### Abstract

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## 1 Introduction

In this project we will solve the one-dimensional Poisson equation with Dirichlet boundary conditions by rewriting it as a set of linear equations and solving these numerically in a program written in C++. The equation to be solved is:

$$-u''(x) = f(x) \quad x \in (0, 1), \quad u(0) = u(1) = 0 \quad (1)$$

and we define the discretized approximation to  $u$  as  $v_i$  with grid points  $x_i = ih$  in the interval from  $x_0 = 0$  to  $x_{n+1} = 1$ . The step length or spacing is defined as  $h = 1/(n+1)$ . We then have the boundary conditions  $v_0 = v_{n+1} = 0$ . We can approximate the second derivative of  $u$  with:

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i \quad \text{for } i = 1, \dots, n, \quad (2)$$

where  $f_i = f(x_i)$ .

Eq. (2) can be written as a linear set of equations of the form:

$$\mathbf{A}\mathbf{v} = \tilde{\mathbf{b}} \quad (3)$$

where  $\tilde{b}_i = h^2 f_i$  and  $\mathbf{A}$  is an  $n \times n$  matrix of the form:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix} \quad (4)$$

This can be shown by writing it out. We then get

$$2v_1 - v_2 = \tilde{b}_1 \quad (5)$$

$$-v_1 + 2v_2 - v_3 = \tilde{b}_2 \quad (6)$$

$$-v_2 + 2v_3 - v_4 = \tilde{b}_3 \quad (7)$$

$$\dots \quad (8)$$

$$-v_{i-1} + 2v_n - v_{i+1} = \tilde{b}_i, \quad (9)$$

which is the same as equations (2).

We will assume that the source term is  $f(x) = 100e^{-10x}$ . Using the interval and boundary conditions as stated above, the above differential equation has a closed-form solution given by  $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$ . We will compare our numerical solution with this result.

## 2 Methods

### 2.1 Simple algorithm

In our case we are dealing with a simple tridiagonal matrix. We can therefore rewrite our matrix  $\mathbf{A}$  in terms of one-dimensional vectors  $a, b, c$  of length  $1:n$ . The linear equation then reads:

$$\mathbf{A} = \begin{pmatrix} b_1 & c_1 & 0 & \dots & \dots & \dots \\ a_2 & b_2 & c_2 & \dots & \dots & \dots \\ & a_3 & b_3 & c_3 & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \dots \\ \dots \\ \dots \\ \tilde{b}_n \end{pmatrix} \quad (10)$$

which can be written as:

$$a_i v_{i-1} + b_i v_i + c_i v_{i+1} = \tilde{b}_i \quad (11)$$

for  $i = 1, 2, \dots, n$ . We want to find a simple algorithm to solve this set of equations. To do this, we can rewrite Eq. 11 by a

method of elimination, which will lead to a system where there is only one unknown.

The first thing to do is to realize that Eq. 11 gives a system of three equations:

$$b_1 v_1 + c_1 v_2 = \tilde{b}_1, \quad i = 1 \quad (12)$$

$$a_i v_{i-1} + b_i v_i + c_i v_{i+1} = \tilde{b}_i, \quad i = 2, \dots, n-1 \quad (13)$$

$$a_n v_{n-1} + b_n v_n = \tilde{b}_n, \quad i = n \quad (14)$$

where the boundary conditions have been applied to simplify the equations. The idea now is to subtract one row with a scalar multiple of another to eliminate variables. First we want to eliminate  $v_1$ :

(something)

Then we can follow the same approach to eliminate  $v_2$ :

(something)

The algorithm for solving this equation can be stated as follows:

(algorithm)

## 2.2 LU decomposition

In addition to solving the linear second-order differential equation Eq. 1 using the simple algorithm described above, we want to solve the equation using LU decomposition, then compare the results. For this we will use the provided `lib.cpp` library.

(More coming here!!)

## 3 Results

### 3.1 Comparing the analytical and numerical solution

We can run our code for different values of  $n$  and plot the analytical and numerical solution to see how well they correspond. For certain values of  $n$  we get these plots:

### 3.2 Relative errors

The relative error in the data set is computed by calculating:

$$\epsilon_i = \log_{10} \left( \left| \frac{v_i - u_i}{u_i} \right| \right), \quad (15)$$

for  $i = 1, \dots, n$  for the function values  $u_i$  and  $v_i$ , where  $u_i$  represents the closed-form analytical solution, while  $v_i$  is the numerical solution given by the simple algorithm described above. The relative error will be a function of  $\log_{10}(h)$ . For each step length  $h$  we want to extract the max value of the relative error. Doing this while increasing  $n$  to  $n = 10000$  and  $n = 10^5$ , gives the following results:

$n$	Maximum error
10	1.18895
100	1.03632
$10^3$	1.00786
$10^4$	1.00076
$10^5$	XX

### 3.3 Number of floating point operations and time usage

Some of the methods described above may be more computationally heavy for a computer to do than others. One way to evaluate this is to calculate the precise number of floating point operations ("FLOPS") needed to solve the equations.

For our tridiagonal system,  $a \cdot O(n)$  (find  $a$ !!) FLOPS are required. Gaussian elimination requires  $\frac{2}{3}n^3 + O(n^2)$  FLOPS and LU decomposition requires  $O(n^3)$ .

The number of floating point operations and time usage in the different cases, are shown in the table below:

Method	$n$	FLOPS	Time (s)
Simple algorithm	10	XX	0
	100	XX	0
	1 000	XX	0
	10 000	XX	0
	100 000	XX	XX
Gaussian elimination	10	XX	-
	100	XX	-
	1 000	XX	-
	10 000	XX	-
	100 000	XX	-
LU decomposition	10	XX	0
	100	XX	0
	1 000	XX	3
	10 000	XX	XX
	100 000	XX	XX

## 4 Conclusions

...  
(Comment on the results in the previous section!!)

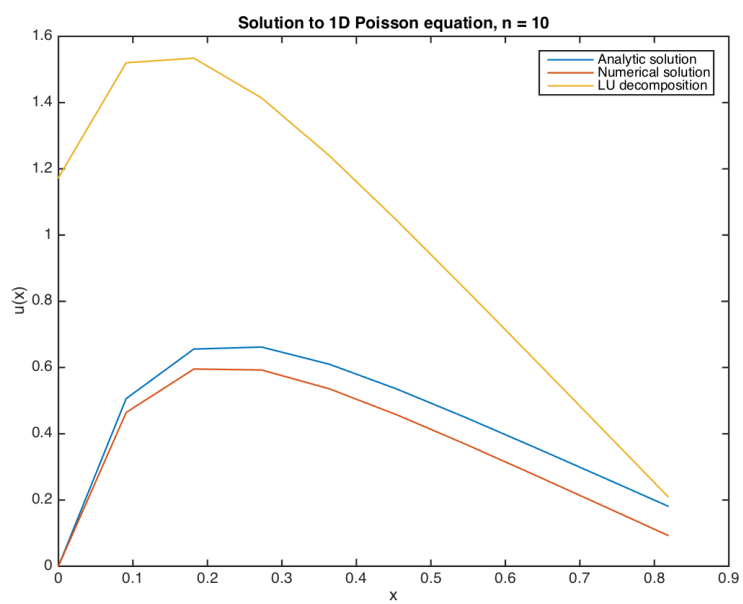
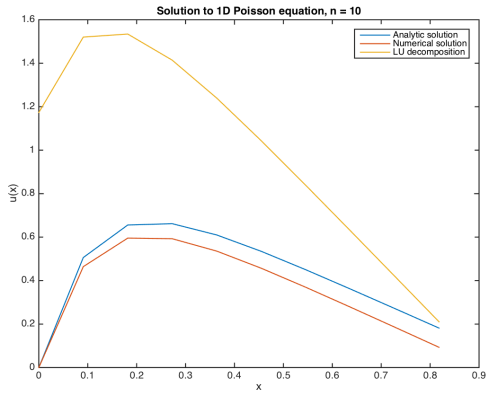
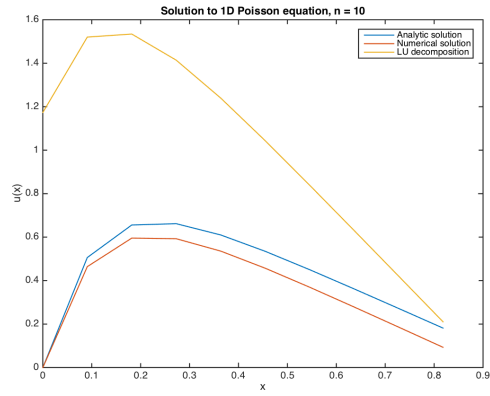


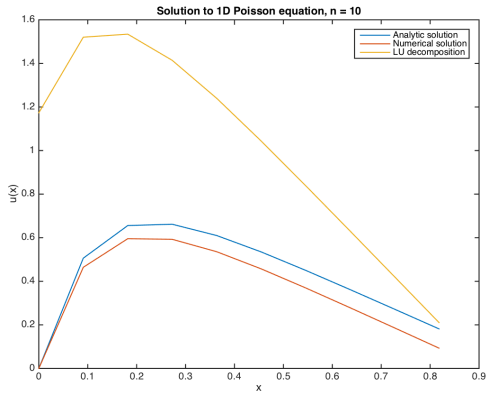
Figure 1: Awesome Image



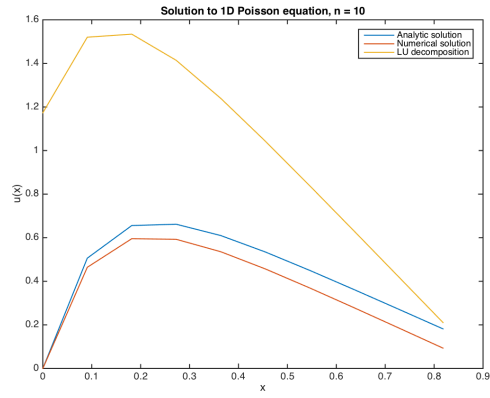
(a) first



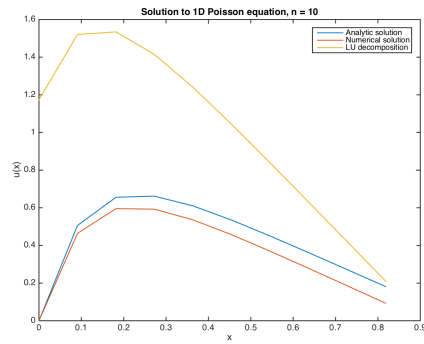
(b) second



(c) third



(d) fourth



(e) fifth

Figure 2: caption