

FYS4150 Project 1:

1-dimensional Poisson equation

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Abstract

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1 Introduction

In this project we will solve the one-dimensional Poisson equation with Dirichlet boundary conditions by rewriting it as a set of linear equations and solving these numerically in a program written in C++. The equation to be solved is:

$$-u''(x) = f(x) \quad x \in (0, 1), \quad u(0) = u(1) = 0 \quad (1)$$

and we define the discretized approximation to u as v_i with grid points $x_i = ih$ in the interval from $x_0 = 0$ to $x_{n+1} = 1$. The step length or spacing is defined as $h = 1/(n+1)$. We then have the boundary conditions $v_0 = v_{n+1} = 0$. We can approximate the second derivative of u with:

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i \quad \text{for } i = 1, \dots, n, \quad (2)$$

where $f_i = f(x_i)$. Eq. (2) can be written as a linear set of equations of the form:

$$\mathbf{A}\mathbf{v} = \tilde{\mathbf{b}} \quad (3)$$

where $\tilde{b}_i = h^2 f_i$ and \mathbf{A} is an $n \times n$ matrix of the form:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix} \quad (4)$$

This can be shown by writing it out. We then get

$$2v_1 - v_2 = \tilde{b}_1 \quad (5)$$

$$-v_1 + 2v_2 - v_3 = \tilde{b}_2 \quad (6)$$

$$-v_2 + 2v_3 - v_4 = \tilde{b}_3 \quad (7)$$

$$\dots \quad (8)$$

$$-v_{i-1} + 2v_i - v_{i+1} = \tilde{b}_i, \quad (9)$$

which is the same as Eq. (2).

We will assume that the source term is $f(x) = 100e^{-10x}$. Using the interval and boundary conditions as stated above, the above differential equation has a closed-form solution given by $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$. We will compare our numerical solution with this result.

2 Methods

2.1 Simple algorithm

In our case we are dealing with a simple tridiagonal matrix. We can therefore rewrite our matrix \mathbf{A} in terms of one-dimensional vectors a, b, c of length $1:n$. The linear equation then reads:

$$\mathbf{A} = \begin{pmatrix} b_1 & c_1 & 0 & \dots & \dots & \dots \\ a_2 & b_2 & c_2 & \dots & \dots & \dots \\ & a_3 & b_3 & c_3 & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ \dots \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \dots \\ \dots \\ \dots \\ \tilde{b}_n \end{pmatrix}. \quad (10)$$

which can be written as:

$$a_i v_{i-1} + b_i v_i + c_i v_{i+1} = \tilde{b}_i \quad (11)$$

for $i = 1, 2, \dots, n$. We want to find a simple algorithm to solve this set of equations. To do this, we can rewrite Eq. 11 by a method of elimination, which will lead to a system where there is only one unknown.

The first thing to do is to realize that Eq. 11 gives a system of three equations:

$$b_1 v_1 + c_1 v_2 = \tilde{b}_1, \quad i = 1 \quad (12)$$

$$a_i v_{i-1} + b_i v_i + c_i v_{i+1} = \tilde{b}_i, \quad i = 2, \dots, n-1 \quad (13)$$

$$a_n v_{n-1} + b_n v_n = \tilde{b}_n, \quad i = n \quad (14)$$

where the boundary conditions have been applied to simplify the equations. The idea now is to subtract one row with a scalar multiple of another to eliminate variables. First we want to eliminate v_1 :

(something)

Then we can follow the same approach to eliminate v_2 :

(something)

The algorithm for solving this equation can be stated as follows:

(algorithm)

2.2 LU decomposition

In addition to solving the linear second-order differential equation Eq. 1 using the simple algorithm described above, we want to solve the equation using LU decomposition, then compare the results. For this we will use the provided `lib.cpp` library.

(More coming here about what LU decomposition is!!)

3 Results

3.1 Comparing the analytical and numerical solution

We can run our code for different values of n and plot the analytical and numerical solution to see how well they correspond. For different values of n , see Fig. 1.

3.2 Relative errors

The relative error in the data set is computed by calculating:

$$\epsilon_i = \log_{10} \left(\left| \frac{v_i - u_i}{u_i} \right| \right), \quad (15)$$

for $i = 1, \dots, n$ for the function values u_i and v_i , where u_i represents the closed-form analytical solution, while v_i is the numerical solution given by the simple algorithm described above. The relative error will be a function of $\log_{10}(h)$. For each step length h we want to extract the max value of the relative error. Doing this while increasing n to $n = 10000$ and $n = 10^5$, gives the following results:

n	Maximum error
10	1.18895
100	1.03632
10^3	1.00786
10^4	1.00076
10^5	XX

Table 1: Maximum error as a function of step length.

3.3 Number of floating point operations and time usage

Some of the methods described above may be more computationally heavy for a computer to do than others. One way to evaluate this is to calculate the precise number of floating point operations ("FLOPS") needed to solve the equations.

Gaussian elimination requires $\frac{2}{3}n^3 + O(n^2)$ FLOPS and LU decomposition requires $O(n^3)$. By counting the floating point operations in our code, we find that for our tridiagonal system, $8n + 1$ FLOPS are required.

The number of floating point operations and time usage in the different cases, are shown in the table below.

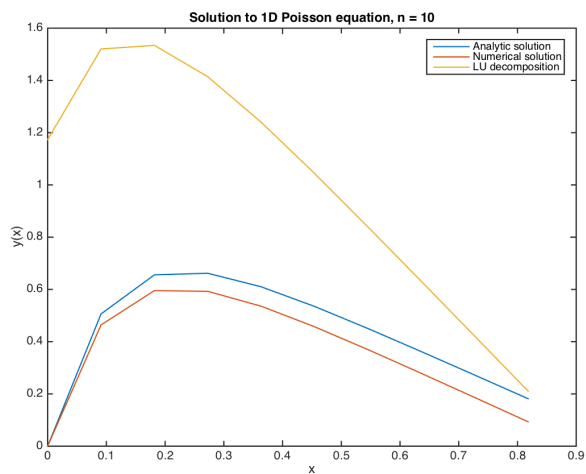
Method	n	FLOPS	Time (s)
Simple algorithm	10	XX	0
	100	XX	0
	1 000	XX	0
	10 000	XX	0
	100 000	XX	XX
Gaussian elimination	10	XX	-
	100	XX	-
	1 000	XX	-
	10 000	XX	-
	100 000	XX	-
LU decomposition	10	XX	0
	100	XX	0
	1 000	XX	3
	10 000	XX	XX
	100 000	XX	XX

Table 2: The number of floating point operations and time usage by method.

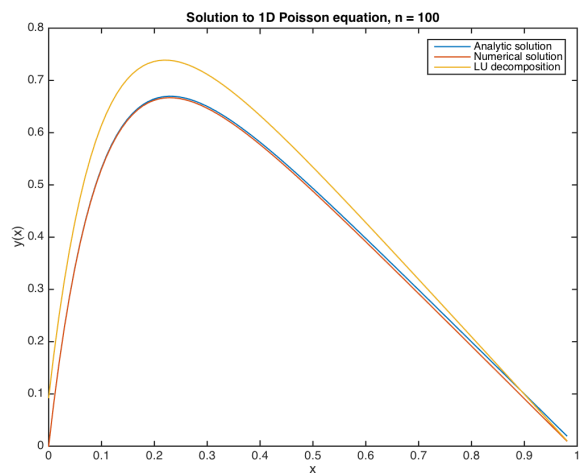
4 Conclusions

...

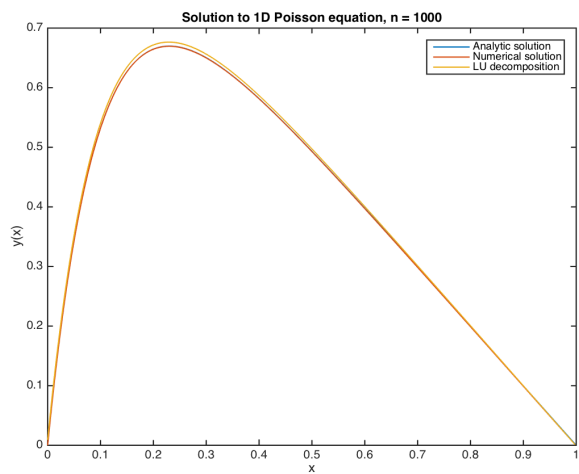
(Comment on the results in the previous section!!)



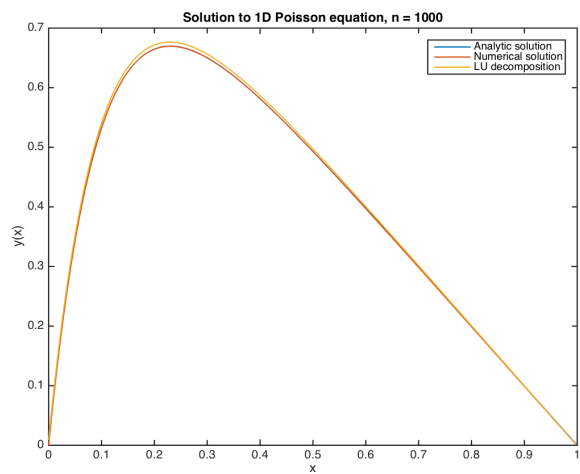
(a) $n = 10$



(b) $n = 100$



(c) $n = 1\,000$



(d) $n = 10\,000$ (still waiting on program)

Figure 1: Comparison of the three different solutions for different values of n .