COMPLEX NUMBERS:
ARITHMETIC OPERATIONS OF COMPLEX NUMBERS

★ Understanding and performing arithmetic operations of complex numbers

Cartesian complex number, z

$$\boxed{a+bi}$$

real part $\text{Re } z$ — imaginary part $\text{Im } z$

$i^2=-1$
$i=\sqrt{-1}$

Completely real
$z=x$

Completely imaginary
$z=yi$

$z_1=x+yi$
$z_2=a+bi$
if $z_1=z_2$
$\Rightarrow x=a$
$\Rightarrow y=b$

Complex Conjugate

$z^*=x-yi$

multiply Im z with -1

$$\sqrt{\frac{24i-8i}{i^7}}$$
$$=\sqrt{\frac{16i}{i^7}}=\frac{\sqrt{16}}{\sqrt{(-1)^3}}$$
$$=\frac{\sqrt{16}}{\sqrt{i^6}}=\sqrt{-16}$$
$$=\sqrt{16}(\sqrt{-1})$$
$$=\frac{\sqrt{16}}{\sqrt{(i^2)^3}}=4i$$

Roots!

$(x-\alpha)(x-\beta)=0$
$\alpha=a+bi$
$\beta=a-bi$

$$2x^2+3x+5=0$$
$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$\frac{-3+\sqrt{-31}}{4} \qquad \frac{-3-\sqrt{-31}}{4}$$
$$\frac{-3+\sqrt{31}(\sqrt{-1})}{4} \qquad \frac{-3-\sqrt{31}(\sqrt{-1})}{4}$$
$$x=\frac{-3+\sqrt{31}i}{4} \qquad x=\frac{-3-\sqrt{31}i}{4}$$

$$(2-3i)=\sqrt{a+bi}$$
$$(2-3i)^2=a+bi$$
$$4-12i+9i^2=a+bi$$
$$4-12i+9(-1)=a+bi$$
$$-5-12i=a+bi$$
$$a=-5,\ b=-12$$

$$\boxed{z_1=2-5i} \quad \boxed{z_2=3+4i}$$

ADDITION & SUBSTRACTION

$$z_1+z_2=5-i$$
$$z_2-z_1=1+9i$$

MULTIPLICATION

$$(2-5i)(3+4i)$$
$$6+8i-15i-20i^2$$
$$6-7i+20$$
$$26-7i$$

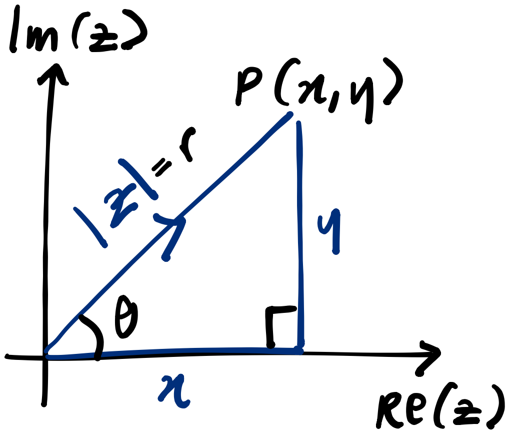
DIVISION

conjugate of denominator

$$\frac{z_2}{z_1}=\frac{3+4i}{2-5i}$$
$$=\frac{3+4i}{2-5i}\left(\frac{2+5i}{2+5i}\right)=\frac{(3+4i)(2+5i)}{(2-5i)(2+5i)}=\frac{6+23i-20}{4+25}$$
$$=\frac{6+23i+20i^2}{4-25i^2}=\frac{-14}{29}+\frac{23i}{29}$$

COMPLEX NUMBERS: ARGAND DIAGRAM

* Representing complex numbers geometrically by means of an Argand diagram



$-\pi < \theta < \pi$

rad

① Cartesian form

[Point $P(x, y)$]

$z = x + yi$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

② Modulus–argument form

$|z| = \sqrt{x^2+y^2}$

$\arg(z) = \theta \leftarrow \tan^{-1}\left(\frac{y}{x}\right)$

$z = r\cos\theta + i(r\sin\theta)$

$z = r(\cos\theta + i\sin\theta)$

③ Exponential (Euler) form

$r(\cos\theta + i\sin\theta) = e^{i}\theta$

$z = re^{i\theta}$

polar form

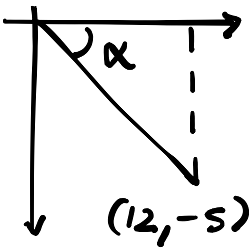
Multiplication

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Division

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

$\boxed{12 - 5i}$



$|z| = \sqrt{(-5)^2 + 12^2}$

$|z| = 13$

$\arg(z) = -\alpha$

$= -\tan^{-1}\left(\frac{5}{12}\right)$

$= -0.395\text{ rad}$

Modulus-argument form

$z = 13(\cos(-0.395) + i\sin(-0.395))$

Exponential form

$13e^{i(-0.395)}$

COMPLEX NUMBERS:
COMPLEX ROOTS OF EQUATION

$b^2 - 4ac$

$> 0 \longrightarrow \alpha$ and β

$= 0 \longrightarrow \alpha$ and α

$< 0 \longrightarrow z$ and z^*

$\uparrow \; x+iy \qquad \uparrow \; x-iy$

Square Roots

Find square root of $-2-(2\sqrt{3})i$

$x+yi = \pm\sqrt{-2-(2\sqrt{3})i}$

$(x+yi)^2 = -2-(2\sqrt{3})i$

$x^2+2xyi-y^2 = -2-(2\sqrt{3})i$

$x^2-y^2 = -2$

$2xy = 2\sqrt{3}$

$x = \pm\sqrt{-3}, \pm 1$

$x = \pm 1, \; y = \mp\sqrt{3}$

$\therefore 1-\sqrt{3}i$ and $-1+\sqrt{3}i$

Cube Roots of Unity

$\sqrt[3]{1}$

$= 1$

$= \dfrac{-1+i\sqrt{3}}{2}$

$= \dfrac{-1-i\sqrt{3}}{2}$

$z^3 - 1 = 0$

$z = \sqrt[3]{1} = 1$

Root: $z = 1$

Factor: $z - 1 = 0$

Quadratic^2

x_1, x_2 OR

$x+iy, x-iy$

Cubic^3

x_1, x_2, x_3 OR

x_1, x_2+iy, x_2-iy

Quartic^4

x_1, x_2, x_3, x_4 OR

x_1, x_2, x_3+iy, x_3-iy OR

$x_1+iy_1, x_1-iy_1, x_2+iy_2, x_2-iy_2$

* Solving equations with complex roots

①

$(1)(-1) = 1$

$(z-1)(z^2+bz+1)$

$z^3+(b-1)z^2+(1-b)z-1$

$b-1=0$

$b=1$

quadratic formula

$z = \dfrac{-1\pm\sqrt{3}i}{2}, 1$

②

$$
\begin{array}{r}
z^2 + z + 1 \\
z-1 \,\big)\, \overline{z^3 + 0z^2 + 0z - 1} \\
\underline{z^3 - z^2 \qquad\qquad} \\
z^2 + 0z \quad\; \\
\underline{z^2 - z \quad\;} \\
z - 1 \\
\underline{z - 1} \\
0
\end{array}
$$

* De Moivre Theorem

$z = r(\cos\theta + i\sin\theta)$

$z^n = [r(\cos\theta + i\sin\theta)]^n$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

z has n distinct roots:

$w_k = \sqrt[n]{r}\,(\cos\alpha + i\sin\alpha)$

$\alpha = \dfrac{\theta + 2k\pi}{n} \quad k = 0, 1, 2, \dots, n-1$

rad

SERIES: BINOMIAL EXPANSION

* Performing binomial expansion of $(a+b)^n$ where n is a positive integer

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$\dfrac{n!}{r!(n-r)!}$ OR ${}^nC_r$

$r = 0, 1, 2 \ldots n$

index of a decreases

index of b increases

$$\frac{n!}{2!(n-2)!} = \frac{n \times (n-1) \times (n-2) \times \cdots}{2 \times 1 \times (n-2) \times (n-3) \ldots} = \frac{n \times (n-1)}{2 \times 1}$$

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$$

SERIES: ARITHMETIC AND GEOMETRIC PROGRESSION

* Recognizing arithmetic progressions and geometric progressions

Arithmetic Progression

$2 \quad 5 \quad 8 \quad 11$ ($+3$)

$7 \quad 5 \quad 3 \quad 1$ (-2)

n^{th} term

$$\boxed{T_n = a + (n-1)d}$$

1st term — common difference

$$T_n = S_n - S_{n-1}$$

sum of n terms — last term

$$\boxed{S_n = \frac{n}{2}(a + l)}$$

$$\boxed{S_n = \frac{n}{2}(2a + (n-1)d)}$$

$$d = T_n - T_{n-1}$$

Geometric Progression

$2 \quad 4 \quad 8 \quad 16$ ($\times 2$)

$2 \quad -1 \quad \frac{1}{2} \quad -\frac{1}{4}$ ($\div(-2)$)

n^{th} term

$$\boxed{T_n = ar^{n-1}}$$

common ratio

$$r = \frac{T_n}{T_{n-1}}$$

sum of n terms

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

$-1 < r < 1$

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

$r > 1$ or $r \leq -1$

SERIES:
CONVERGENCE OF GEOMETRIC PROGRESSION

Infinite Geometric series

$n \to \infty$

$r^n \to 0$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$-1 < r < 1$

* Utilizing the condition for the convergence of a geometric progressions, and the formula for the sum to infinity of a convergent geometric progression

SERIES:
PARTIAL FRACTIONS IN BINOMIAL EXPANSIONS

If $n<0$, * Applying partial fractions in Binomial expansions

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

infinite

- n is a rational number
- $|x|<1$

If $n<0$,

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$$

infinite

$$= a^n\left[1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \dots\right]$$

- $\left|\frac{x}{a}\right|<1$

Partial Fractions

1 Distinct linear factors

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

2 Repeated linear factors

$$\frac{px+q}{(ax+b)} \equiv \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

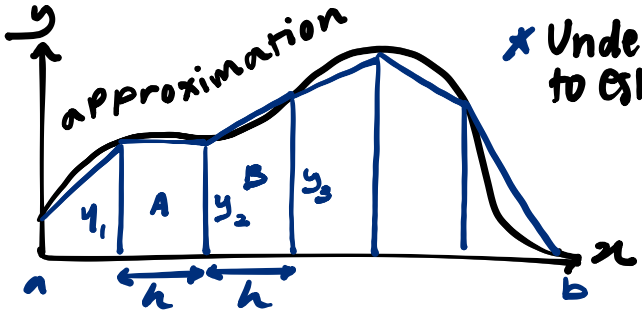
3 Quadratic factors

$$\frac{px+q}{(ax+b)(cx^2+d)} \equiv \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}$$

4 Deg(num) $\geq$ Deg(denom)

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

NUMERICAL SOLUTIONS:
TRAPEZIUM RULE IN NUMERICAL METHODS



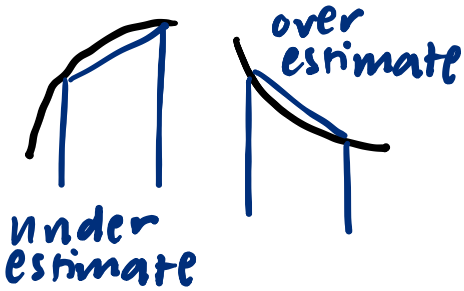
$$h = \frac{b-a}{n}$$

* Understanding and applying trapezium rule to estimate the value of a definite integral

$$A = \frac{1}{2}(y_1 + y_2)h$$
$$B = \frac{1}{2}(y_2 + y_3)h$$

Area $= \frac{1}{2}$ (width) (1st ordinate + 2(sum of middle ordinates) + last ordinate)

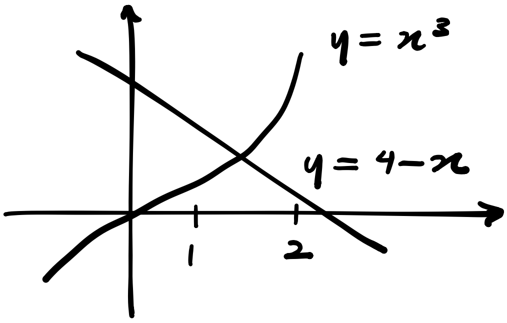
$$\boxed{\text{Area} = \frac{h}{2}(y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1}))}$$



NUMERICAL SOLUTIONS: APPROXIMATION ROOT OF EQUATION

Graphical Method

* Locating approximately a root of an equation, by means of graphical considerations and/ or searching for a sign change



$\therefore 1 < \alpha < 2$

Change of Sign Approach

Show that $f(x) = x^5 + x - 1 = 0$ has a root between 0 and 1

$f(0) = 0^5 + 0 - 1 = -1 < 0$

$f(1) = 1^5 + 1 - 1 = 1 > 0$

$\therefore$ Change of sign

Root: $0 < \alpha < 1$

NUMERICAL SOLUTIONS: IMPROVING SOLUTION

* Improving solutions using a sequence of approximations which converges to a root of an equation may be introduced

iterative process to find values closer to the root (trial & error)

Iterative Formula

$$\boxed{x_{n+1} = F(x_n)}$$

- stop until x_n and x_{n+1} are the same correct to 1 s.f.
- $\alpha = x_{n+1}$