QUADRATICS

$y = ax^2 + bx + c$ (2nd degree)

- $a \neq 0$ (general form)
- has max/min value
- symmetrical graph

* Completing the square for a quadratic polynomial ax^2+bx+c
* Discriminant and its application
* Solutions to quadratic equations and inequalities in one unknown
* Solutions to simultaneous equations involving one linear and one quadratic equation
* Recognition and solution of equations that are quadratic in some function of x
* Relationship between the graph of quadratic function and its algebraic equation
* Relationship between points of intersection of graphs and the solutions of equations

Roots

· values of x

Factor vs Root

$(x-2)$ $\quad$ $x=2$

$$x^2-4x+4=0$$
$$(x-2)(x-2)=0$$
$$x-2=0$$
$$x=2$$

lol!

$$x=\alpha,\ x=\beta$$
$$(x-\alpha)(x-\beta)=0$$
$$x^2-\underbrace{(\alpha+\beta)}_{\text{sum of roots}}x+\underbrace{\alpha\beta}_{\text{product of roots}}=0$$

Square Root Property

$[x^2=a,\ x=\pm\sqrt{a}]$

$$3(x+5)^2-75=0$$
$$3(x+5)^2=75$$
$$(x+5)^2=25$$
$$x+5=\pm\sqrt{25}$$
$$x+5=5 \text{ and } x+5=-5$$
$$x=0 \text{ and } x=-10$$

Completing the Square

$$p(x-q)^2+r$$
$$p(x^2-2qx+q^2)+r$$
$$px^2-2pqx+pq^2+r$$
$$ax^2+bx+c$$

a $\quad$ b $\quad$ c

$$3x^2+2x-4=0$$
$$3x^2+2x=4$$
$$3\left(x^2+\frac{2}{3}x\right)=4$$
$$3\left(x^2+\frac{2}{3}x+\left(\frac{2}{6}\right)^2\right)=4+3\left(\frac{2}{6}\right)^2$$
$$3\left(x+\frac{1}{3}\right)^2=\frac{13}{3}$$
$$3\left(x+\frac{1}{3}\right)^2-\frac{13}{3}=0$$

$x=-\frac{1}{3}$ $\quad$ $y=-\frac{13}{3}$

vertex

Quadratic Formula

from CTS $\rightarrow$ $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Solve Simultaneous Equations

two pairs of solutions!

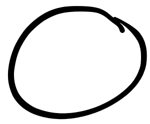
$y=x^2-4 \longrightarrow$ ①

$y=2x-1 \longrightarrow$ ②

$$x^2-4=2x-1$$

General Form of Quadratic in Two Unknowns!

$ax^2 + bxy + cy^2 + dx + ey + f = 0$,
$a, b, c \neq 0$



shape of graph
"conic"

More Complex Quadratics

$4x^4 - 37x^2 + 9 = 0$
let $y = x^2$
$4y^2 - 37y + 9 = 0$
$(y-9)(4y-1) = 0$
$y = 9,\ y = \frac{1}{4}$
$x^2 = y$
$x^2 = 9$ and $x^2 = \frac{1}{4}$
$x = \pm\sqrt{9}$ $\quad x = \pm\sqrt{\frac{1}{4}}$
$x = \pm 3$ $\quad x = \pm\frac{1}{2}$

$x - 4\sqrt{x} - 12 = 0$
let $y = \sqrt{x}$
$y^2 - 4y - 12 = 0$
$(y-6)(y+2) = 0$
$y = 6,\ y = -2$
$y = \sqrt{x}$
$\sqrt{x} = 6$ and $\sqrt{x} = -2$
$x = 36$ $\quad$ ↑ no solution

Max and Min Values

Sketching!
- general shape
- axis intercepts
- coordinate of vertex

Discriminant

$\boxed{b^2 - 4ac}$

> 0, Two distinct real roots
Intersect at two points

$= 0$, Two equal real roots
Tangent

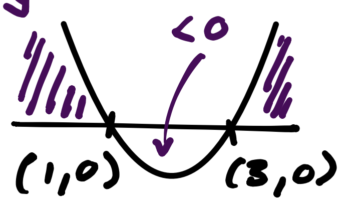
< 0, No real roots
Does not intersect

Quadratic Inequalities

$x^2 - 4x + 3 > 0$

① Graphical method

Let $y = x^2 - 4x + 3$
$y = (x-1)(x-3)$
$0 = (x-1)(x-3)$
$x = 1,\ x = 3$ ← critical values



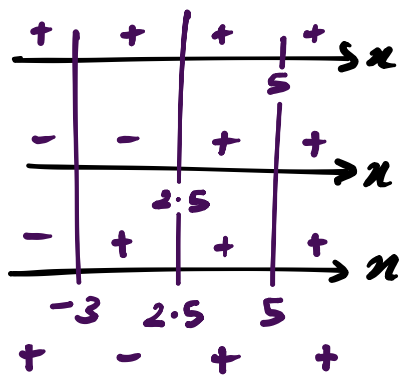
$x < 1,\ x > 3$

② Sign Diagram Method

$$\frac{5}{2x^2 + x - 15} < 0$$

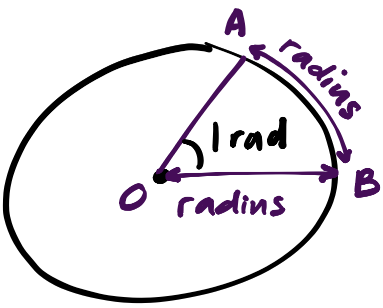
$$\frac{5}{(2x-5)(x+3)} < 0$$

Factors	Critical
5	0
$2x-5$	$x = 2.5$
$x+3$	$x = -3$



$-3 < x < 2.5$

- Radians
- Length of Arc
- Area of Sector



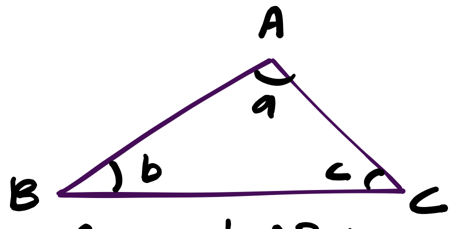
Circumference

$= 2\pi r$ ← irrational number!

2π radians $= 360°$

π radians $= 180°$

$\pi = 180°$

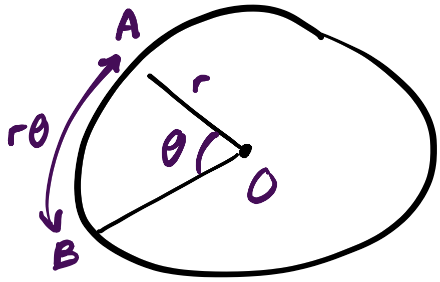


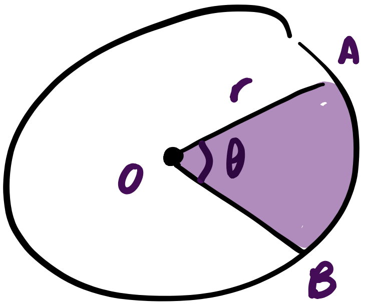
Area $= \frac{1}{2} AB \sin c$

Length of Arc

Arc AB $= \boxed{r\theta}$

in radians





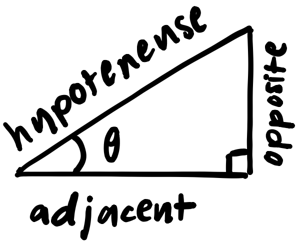
Area of Sector

$$\frac{OAB}{\text{Circle}} = \frac{\angle AOB}{\text{Complete angle}}$$

$$\frac{OAB}{\pi r^2} = \frac{\theta}{2\pi}$$

$$OAB = \boxed{\frac{1}{2} r^2 \theta}$$

TRIGONOMETRY I

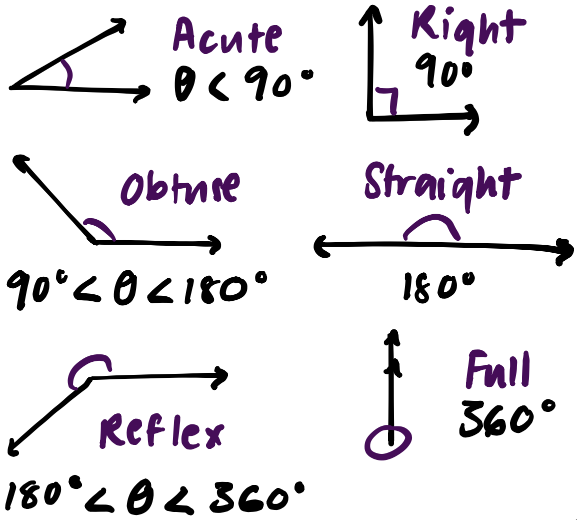


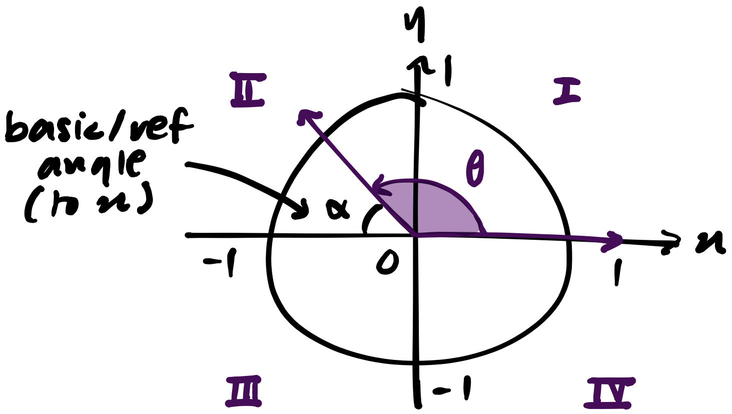
$\sin\theta = \frac{o}{h}$

$\cos\theta = \frac{a}{h}$

$\tan\theta = \frac{o}{a}$

* Angles between 0°-90°
* General definition of angle
* Trigonometric ratios of general angles
* Graphs of trigonometric functions
* Transformation of trigonometric functions



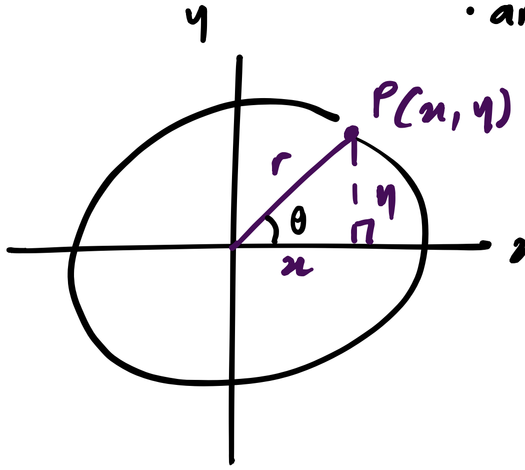


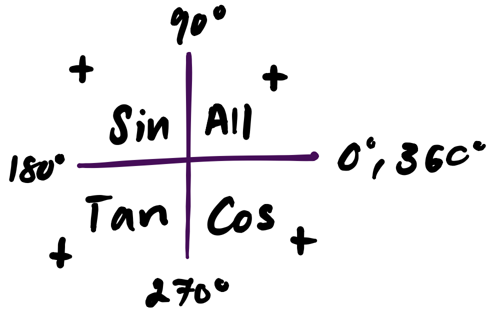
* from +x axis
* anticlockwise is +ve

$\sin\theta = \frac{x}{r}$

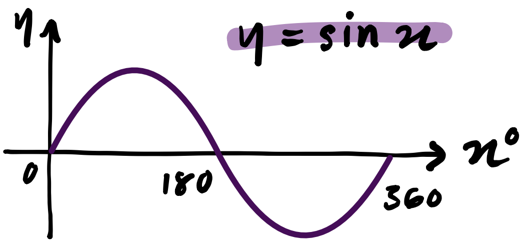
$\cos\theta = \frac{y}{r}$

$\tan\theta = \frac{y}{x}$

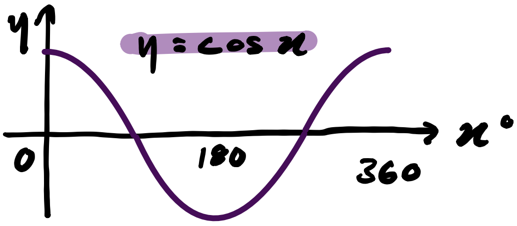




Graphs



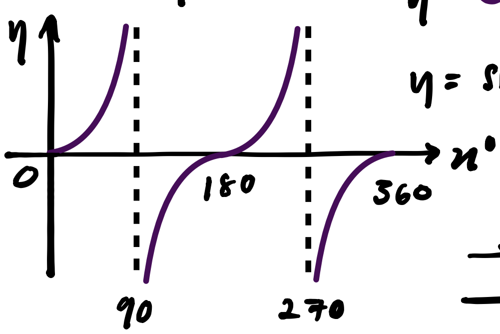
* displacement from x-axis



* displacement from y-axis

Period (one cycle)
2π or 360°

Amplitude
(max displacement)



$y = A\sin x$ — stretch y by factor A

$y = \sin Bx$ — compress x by factor B
└ period $= \frac{2\pi}{B}$

$y = C + \sin x$ — vertical shift +C

$y = \sin(x + D)$ — horizontal shift −D

→ No amplitude
→ Period $= \pi = 180°$

MODULUS / Absolute value

(magnitude of a number)

$|3| = 3$ OR $|-3| = 3$

always +ve

$|ax+b| = k$

1. $ax+b = k$
2. $ax+b = -k$

validate values after solving

- Modulus function
- Graphs of $y = |f(x)|$
- Modulus inequalities
- Division of polynomials
- Factor theorem
- Remainder theorem

$$|a|^2 = a^2$$

$$a^2 - b^2 = |a|^2 - |b|^2$$
$$= (|a|+|b|)(|a|-|b|)$$

$|a| = |b| \longrightarrow a^2 = b^2$

$|a| > |b| \longrightarrow a^2 > b^2$

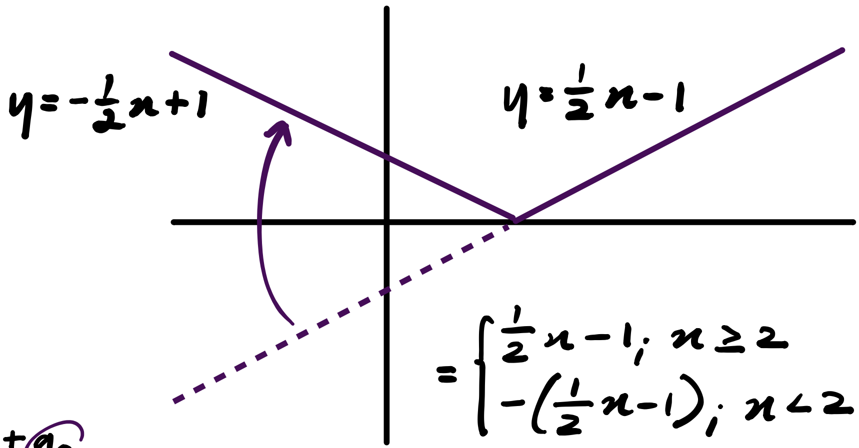
$$|a| \le |b|$$
$$-b \le a \le b$$
$$|a| \ge |b|$$
$$a \le -b,\ a \ge b$$

$$|ax+b| = |cx+d|$$

- $ax+b = +(cx+d)$
- $ax+b = -(cx+d)$

$$y = |f(x)|$$

$$y = \left|\tfrac{1}{2}x - 1\right|$$



$$= \begin{cases} \frac{1}{2}x - 1; & x \ge 2 \\ -\left(\frac{1}{2}x - 1\right); & x < 2 \end{cases}$$

POLYNOMIALS

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

leading coefficient — descending variable index — constant

Long division

$$\left[\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}\right]$$

$$\begin{array}{r} 2x^2 - 6x + 17 \\ x+3 \overline{\smash{)}\,2x^3 + 0x^2 - x + 52} \\ -(2x^3 + 6x^2) \\ \hline -6x^2 - x \\ -(-6x^2 - 18x) \\ \hline 17x + 52 \\ -(17x + 51) \\ \hline 1 \end{array}$$

remainder

$$= 2x^2 - 6x + 17 + \frac{1}{x+3}$$

$q(x) = 2x^2 - 6x + 17$

$r(x) = 1$

Steps

1. $\dfrac{\text{Leading term}}{1^{\text{st}} \text{ term divisor}} = A$
2. Divisor $\times A = B$
3. Substract B

If remainder $= 0$, quotient and divisor are factors of dividend

Synthetic division

only if divisor $= x - c$, $c \ne 0$

$$\frac{5x^4 - 4x^2 + x + 6}{x - 3}$$

3	5	0	−4	1	6
	×↗	15	45	123	372
	5	15	41	124	378

$r(x)$

$(qx) = 5x^3 + 15x^2 + 41x + 124$

The Factor Theorem

$P(x) = (x-c)Q(x)$

poly linear poly

if $x = c$, $(x-c)$

$P(c) = 0 \Rightarrow$ is a factor

Remainder Theorem

$P(c) = a$, $x \neq 0$

remainder

Rational Roots Test

$ax^3 + bx^2 + cx + d$

Possible rational roots $= \pm \dfrac{\text{factors of } d}{\text{factors of } a}$

Proper fraction

$$\frac{22}{35} = \frac{1}{5} + \frac{3}{7}$$

Improper fraction

$$\frac{57}{20} = 2 + \frac{1}{4} + \frac{3}{5}$$

$\frac{P(x)}{Q(x)}$ degree of $P(x) \geq Q(x)$

↳ Improper fraction

$$\frac{x^3 - 3x^2 + 7}{x-2} = x^2 - x - 2 + \frac{3}{x-2}$$

- Form selection for rational function decomposition
- Decomposition of rational functions
  - Linear factors
  - Repeated linear factors
  - Irreducible quadratic factors

Common denominator →

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{2x^2 - x - 1}$$

← Partial fraction (decomposition)

1

$$\boxed{\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}}$$

$$\frac{4+x}{(1+x)(2-x)} = \frac{A}{(1+x)} + \frac{B}{(2-x)}$$

$$= \frac{A(2-x) + B(1+x)}{(1+x)(2-x)}$$

If $x = 2$,
$6 = 3B$
$B = 2$

If $x = -1$,
$3 = 3A$
$A = 1$

2

$$\boxed{\frac{px+q}{(ax+b)^2} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}}$$

$$\frac{3x^2 - 4x + 2}{(x+1)(x-5)^2} = \frac{A}{(x+1)} + \frac{B}{(x-5)} + \frac{C}{(x-5)^2}$$

3

$$\boxed{\frac{px+q}{(ax+b)(cx^2+d)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}}$$

$$\frac{2x+3}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

4

$$\boxed{\frac{px^2+qx+r}{(ax+b)(cx+d)} = N + \frac{A}{(ax+b)} + \frac{B}{(cx+d)}}$$

$$\frac{3x^2 - 3x - 2}{(x-1)(x-2)} = \frac{3x^2 - 3x - 2}{x^2 - 3x + 2}$$

$$= 3 + \frac{6x-8}{(x-1)(x-2)}$$

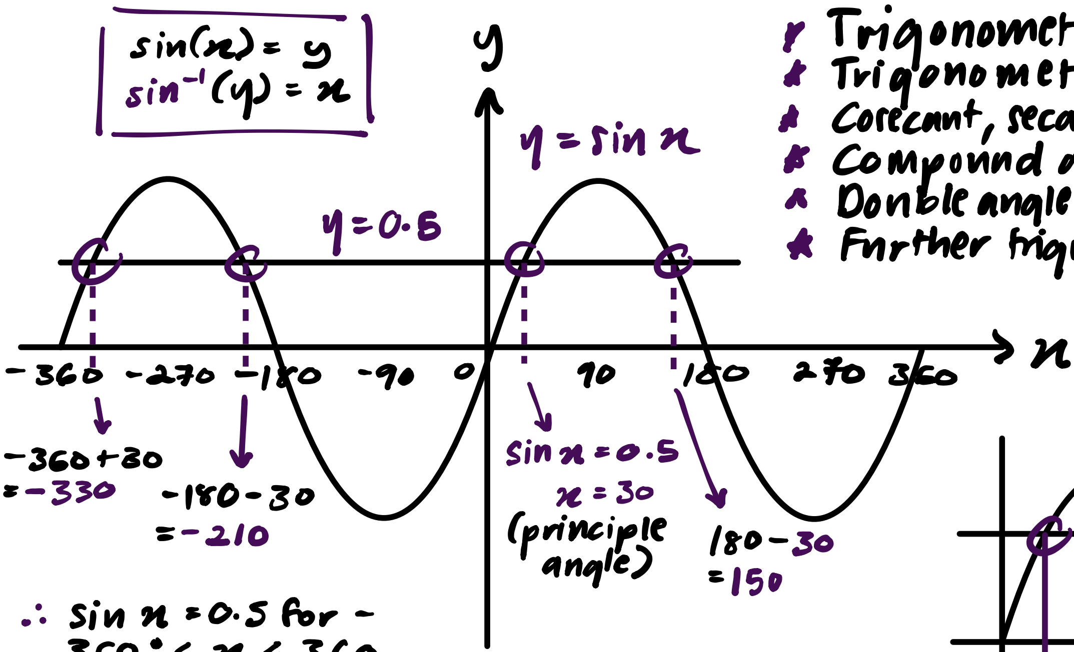
long division

remainder

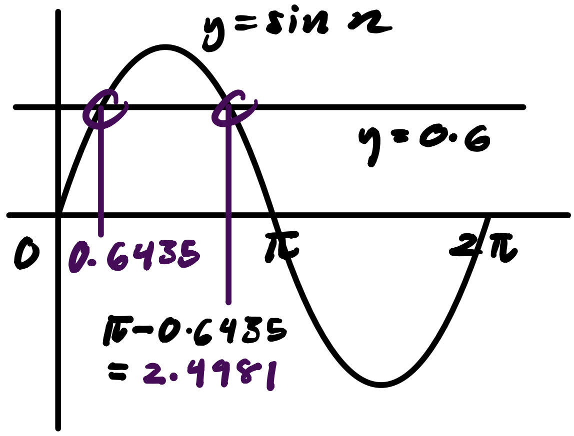
$$= 3 + \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$\sin(x) = y$
$\sin^{-1}(y) = x$

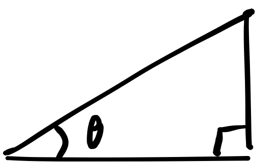
- Trigonometric equations
- Trigonometric identities
- Cosecant, secant and cotangent ratios
- Compound angle formulae
- Double angle formulae
- Further trigonometric identities



$\therefore \sin x = 0.5$ for $-360° \leq x \leq 360$,
$x = -330°, -210°, 30°, 150°$



IDENTITIES!



$$\boxed{\tan\theta \equiv \frac{\sin\theta}{\cos\theta}} \quad \cos\theta \neq 0$$

$$\boxed{\cos^2\theta + \sin^2\theta \equiv 1}$$

$3\cos^2 x - \sin x \cos x = 0$
$\cos x(3\cos x - \sin x) = 0$
$\cos x = 0 \quad , \quad 3\cos x = \sin x$
$\tan x = 3$

$\cos x = 0 \Rightarrow x = 90°, 270°$
$\tan x = 3 \Rightarrow x = 71.56°, 251.56°$
$x = 71.56°, 90°, 251.6°, 270°$

Prove! $\left(\tan x + \frac{1}{\cos x}\right)^2 \equiv \frac{1+\sin x}{1-\sin x}$

$\left(\tan x + \frac{1}{\cos x}\right)^2$

$\left(\frac{\sin x}{\cos x} + \frac{1}{\cos x}\right)^2$

$\left(\frac{\sin x + 1}{\cos x}\right)^2$

$\frac{(\sin x + 1)(\sin x + 1)}{\cos^2 x}$

$\frac{(\sin x + 1)(\sin x + 1)}{(1+\sin x)(1-\sin x)}$

$\frac{\sin x + 1}{1 - \sin x}$

$\sin\left(2A + \frac{\pi}{6}\right) = 0.6$
$0 \leq A \leq \pi$

let $x = 2A + \frac{\pi}{6}$,

$\sin x = 0.6$
$x = 0.6435$ rad
$x = 2.4981$ rad

When $x = 0.6435$,
$2A + \frac{\pi}{6} = 0.6435$
$A = 0.0600$

When $x = 2.4981$,
$2A + \frac{\pi}{6} = 2.4981$
$A = 0.9872$

- verifying range of x!

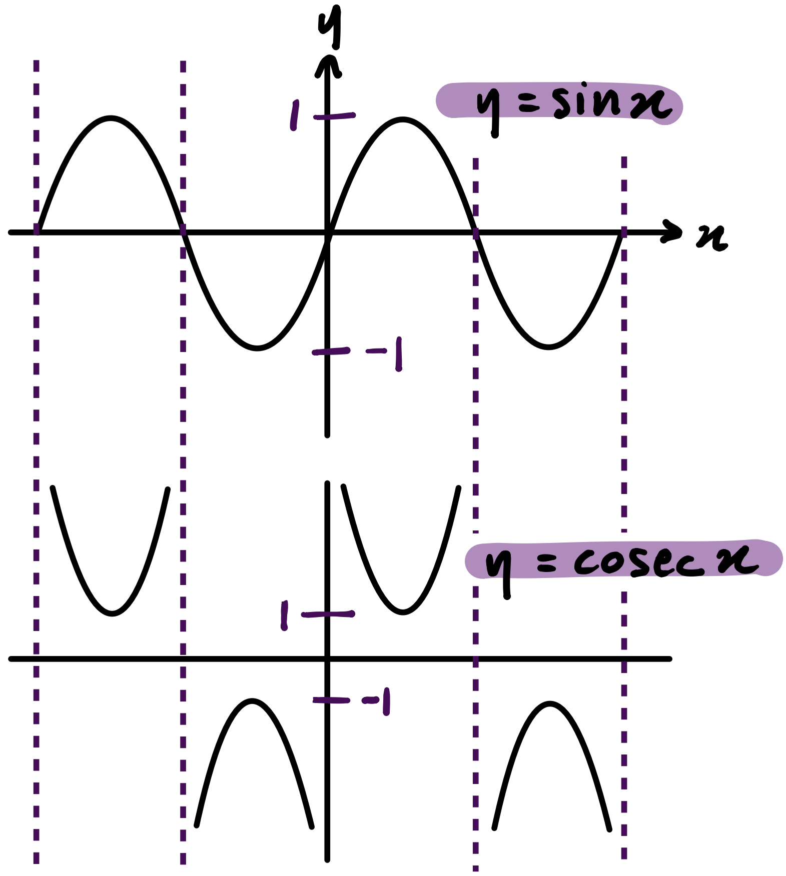
$0 \leq A \leq \pi$
$\frac{\pi}{6} \leq 2A + \frac{\pi}{6} \leq 2\pi + \frac{\pi}{6}$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

$$\boxed{\begin{aligned} 1 + \tan^2\theta &\equiv \sec^2\theta \\ \cot^2\theta + 1 &\equiv \operatorname{cosec}^2\theta \end{aligned}}$$



Compound Angle Formula

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Double Angle Formula

$\sin(2A) = 2\sin A\cos A$

$\cos(2A) = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$

$$\boxed{a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)}$$

$a\sin\theta + b\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$a = R\cos\alpha \longrightarrow ①$

$b = R\sin\alpha \longrightarrow ②$

$② \div ①: \frac{b}{a} = \tan\alpha$

$①^2 + ②^2: a^2 + b^2 = R^2$

$R = \pm\sqrt{a^2+b^2}$

$R\sin(\theta+\alpha)$

amplitude

phase shift