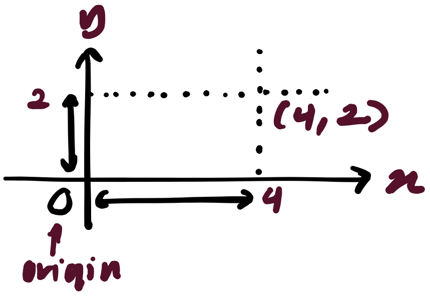
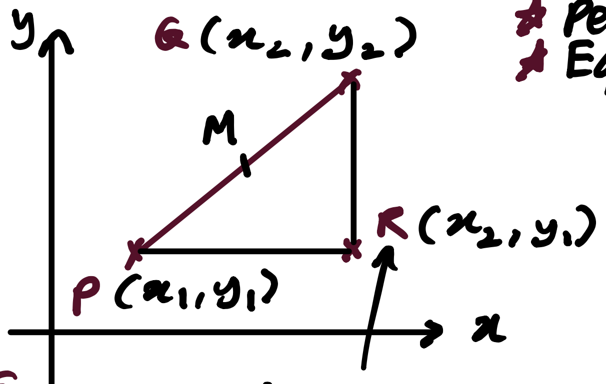
COORDINATE GEOMETRY



- Midpoint of a line segment
- Distance between two points
- Gradient of a line segment
- Properties of parallel and perpendicular lines
- Equation of a straight line
- Cartesian equation of a straight line
- Angle between two straight lines
- Perpendicular distance of a point from line
- Equation of perpendicular bisector



labelling
clockwise / anticlockwise

① Length

Pythagoras's Theorem

$$PQ^2 = PR^2 + QR^2$$

$$PQ = \sqrt{PR^2 + QR^2}$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

② Area

shoelace formula

$$PQR = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & \cdots & x_n & x_1 \\ y_1 & y_2 & \cdots & y_n & y_1 \end{vmatrix}$$

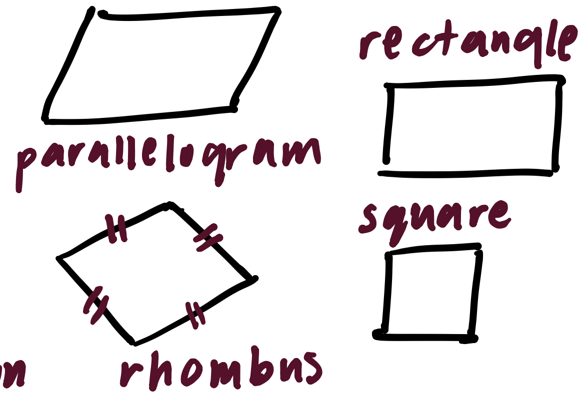
③ Midpoint

$PM = MQ$

$$x = \frac{x_1 + x_2}{2}$$

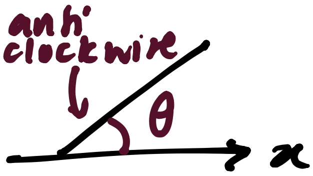
$$y = \frac{y_1 + y_2}{2}$$

* Midpoint of diagonals bisect e/o!



④ Gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



$m = \tan\theta$ ← angle of inclination positive from x-axis

$\tan(\pi - \theta) = -\tan\theta$ ← for -ve gradient

Parallel

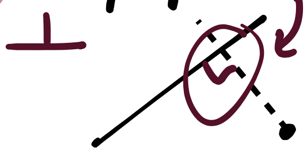
$$m_1 = m_2$$

$$ax + by + c = 0$$
$$ax + by + d = 0$$

Perpendicular

$$m_1 m_2 = -1$$

foot of perpendicular



⑤ Equation of Straight Line

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

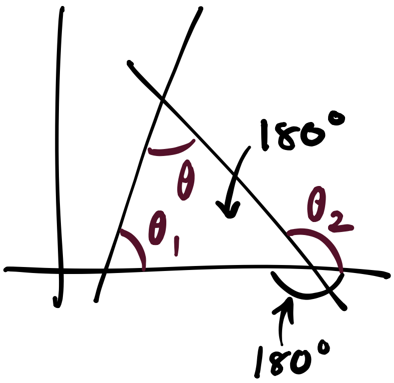
$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1)$$

$$y = mx + c$$

c ← y-intercept

⑥ Angle between Two Lines



$$\theta + \theta_1 = \theta_2$$

$$\theta = \theta_2 - \theta_1$$

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}$$

$$\tan\theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$$

modulus!
→ acute angle

$m_1 = m_1$

num = 0

· $\tan\theta = 0$

$m_1 m_2 = -1$

· $\tan\theta$ is undefined

denom = 0

⑦ Perpendicular Bisector

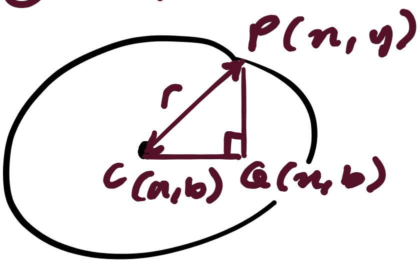
Find midpoint + gradient!

⑧ Intersection of Straight Line and Curve

Solve simultaneous equation

* Discriminant (ref Part 1)

⑨ Circle



$$CQ^2 + PQ^2 = r^2$$

$$(x-a)^2 + (y-b)^2 = r^2$$

completed square form

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

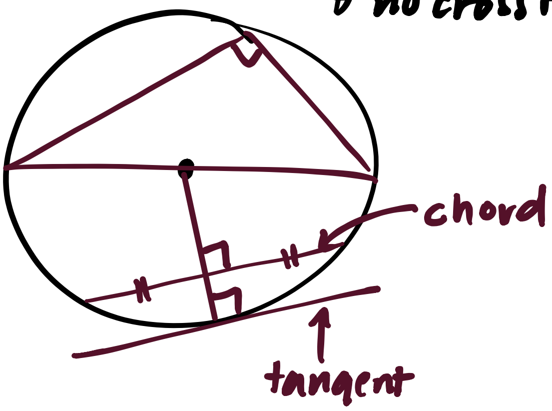
constant

$$x^2 + y^2 + 2gx + 2fx + c = 0$$

general form

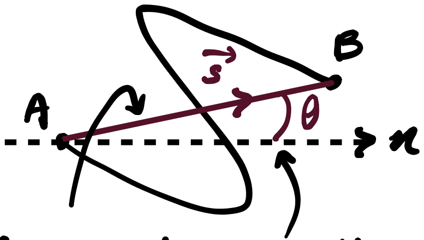
x^2 and y^2 have equal coefficient & no cross term (xy)

$$c = (-g, -f)$$
$$r = \sqrt{g^2 + f^2 - c}$$



Vector notation

$\vec{s}_{AB}$ or $\vec{AB}$



Magnitude

$|\vec{s}|$, s, $|\vec{AB}|$, AB

Direction

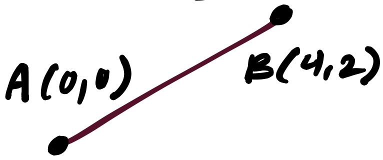
1. anticlockwise +ve
2. clockwise −ve

- ★ Displacement vectors in 2D and 3D
- ★ Parallel vectors
- ★ Addition and subtraction of vectors
- ★ Negative vectors, magnitudes, and unit vectors
- ★ Position vs free vectors
- ★ Linear combinations of vectors in 2D/3D
- ★ Scalar product (dot product)
- ★ Parallel and perpendicular vectors via the scalar product
- ★ Properties and applications of the scalar prod
- ★ Component form of the scalar product

Quadrant	Direction
I	θ
II	$180-\alpha$ ← ref angle
III	$180+\alpha$
IV	$360-\alpha$

Column vector

$$\vec{AB} = \begin{pmatrix}4\\2\end{pmatrix}$$



$$AB = \sqrt{4^2+2^2}$$

$$\theta = \tan^{-1}\left(\frac{2}{4}\right)$$

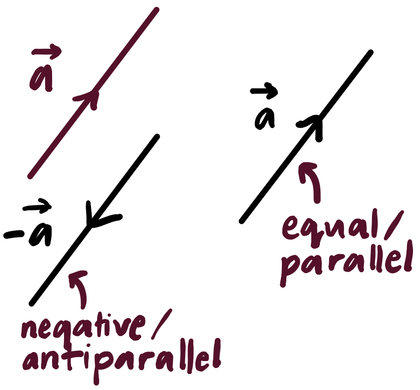
Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Standards

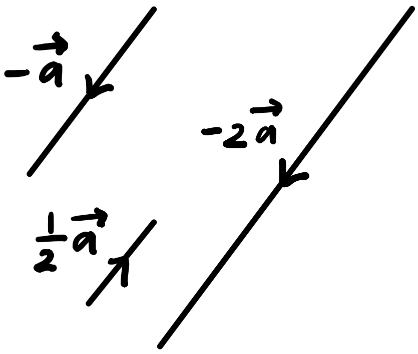
$\hat{i} = \begin{pmatrix}1\\0\end{pmatrix}$ $\hat{j} = \begin{pmatrix}0\\1\end{pmatrix}$ $\hat{k}$ ← z-axis

$$\vec{A} = \begin{pmatrix}a\\b\end{pmatrix} = a\hat{i} + b\hat{j}$$



[Multiplication]

by a SCALAR



by a VECTOR

$$\boxed{\begin{aligned}\vec{AB} &= \vec{AX} + \vec{XB}\\ \vec{AB} &= -\vec{BA}\end{aligned}}$$

$\vec{AB} = 3\vec{CD}$ — parallel

$\vec{AB} = 3\vec{AC}$ — collinear

[Addition or Substraction]

$$\vec{c} + \vec{d}$$

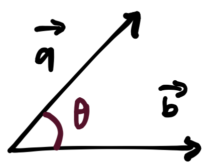
$$\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}5\\-7\end{pmatrix}$$

$$\begin{pmatrix}3+5\\4+(-7)\end{pmatrix}$$

$$\begin{pmatrix}8\\-3\end{pmatrix}$$

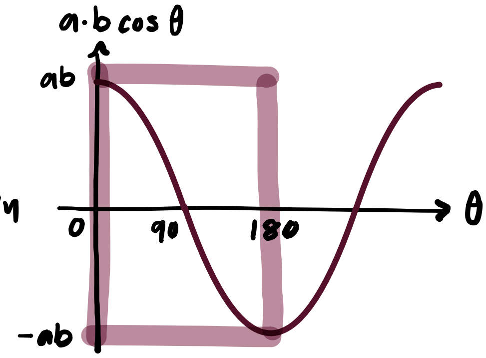
Scalar Product

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

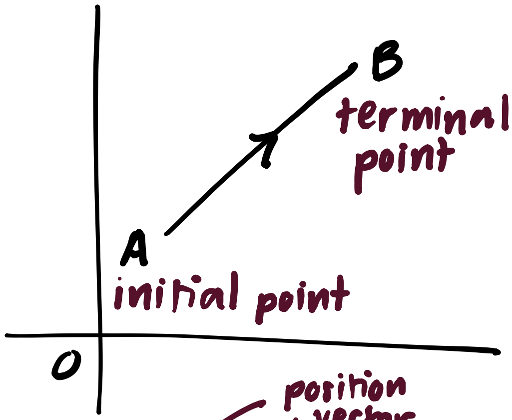


$0^\circ \leq \theta \leq 180^\circ$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$



Position Vector



position vector

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$$

free vector

MATRIX ALGEBRA 3

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \rightarrow m \text{ rows}$$

$\downarrow$ n columns

Order $m \times n$

- Types of matrices
- Conditions for matrix equality
- Matrix addition and subsraction
- Scalar multiplication of matrices
- Matrix multiplication
- Matrix transposition

① Row Matrix $\begin{bmatrix} 3 & 0 & -1 \end{bmatrix}$

② Column Matrix $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

③ Square Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

④ Zero Matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

⑤ Diagonal Matrix

leading diagonal $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

⑥ Identity Matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

± integer

⑦ Triangular Matrix

- Upper $\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -7 \\ 0 & 0 & 1 \end{bmatrix}$
- Lower $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & -7 & 1 \end{bmatrix}$

⑧ Symmetric Matrix $\begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & -7 \\ 1 & -7 & 1 \end{bmatrix}$

$A = B$

if $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Addition & Subsraction

(element by element)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 9 & 6 \end{bmatrix}$$

1. $A + B = B + A$
2. $(A+B)+C = A+(B+C)$

Scalar Multiplication

$\alpha A = A\alpha = [\alpha a_{ij}]$

1. $(\alpha\beta)A = \alpha(\beta A)$
2. $(\alpha+\beta)A = \alpha A + \beta A$
3. $\alpha(A+B) = \alpha A + \alpha B$

Matrix Multiplication

$AB = (a_{ij})_{m\times n}(b_{ij})_{n\times p} = (c_{ij})_{m\times p}$

1. $AB \neq BA$
2. $AB = 0$, not necessarily $A = 0$ or $B = 0$
3. $AB = AC$ not necessarily $B = C$

$AB - AC = 0$, $A(B-C) = 0$ ✓

$B - C \neq 0$

$B \neq C$

$AB - CA = 0$, $A(B-C) = 0$ ✗

4. $\lambda(AB) = (\lambda A)B = A(\lambda B)$
5. $(AB)C = A(BC)$
6. $(B+C)A = BA + CA$

Transposition

(interchanging rows and columns)

$A^T = (a_{ij})^T = (a_{ji})$

$$\begin{bmatrix} 1 & 2 \\ 7 & 9 \\ 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 8 \\ 2 & 9 & 6 \end{bmatrix}$$

1. A is $m\times n$, A^T is $n \times m$
2. A is symmetric, $A = A^T$
3. $(A^T)^T = A$
4. $(A+B)^T = A^T + B^T$
5. $(\lambda A)^T = \lambda A^T$
6. $(AB)^T = B^T A^T$

DETERMINANT AND INVERSE MATRIX

- Determinant
- Properties of determinant
- Inverse matrix
- Properties of inverse matrix

(only for square matrices)

det (A) or |A|

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

not modulus! (value can be −ve)

2×2

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ checkboard rule!

$|B| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$\begin{matrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{matrix}$ $\begin{matrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{matrix}$ $\begin{matrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{matrix}$

Properties!

① if A has rows/columns of zeros

$\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 5 \\ 2 & 0 & 9 \\ 3 & 0 & 2 \end{vmatrix} = 0$

② if A has two proportional rows/columns

$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ -4 & 2 & 8 \end{vmatrix} = \begin{vmatrix} -2 & 1 & 3 \\ 3 & 5 & 15 \\ 1 & 6 & 18 \end{vmatrix} = 0$

③ interchanging row and column changes sign of determinant

$\begin{vmatrix} 1 & 6 & 2 \\ 3 & 5 & -7 \\ -2 & 1 & 4 \end{vmatrix} = 65, \quad \begin{vmatrix} 1 & 6 & 2 \\ -2 & 1 & 4 \\ 3 & 5 & -7 \end{vmatrix} = -65$

④ $|A^T| = |A|$

$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 5 & 6 \\ 4 & -7 & 2 \end{vmatrix} = -65$

⑤ multiply row/column by a constant

$\begin{vmatrix} -2 & 6 & 1 \\ 1 & 10 & 6 \\ 4 & -14 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & 3 & 1 \\ 1 & 5 & 6 \\ 4 & -7 & 2 \end{vmatrix}$

$\begin{vmatrix} -4 & 2 & 8 \\ 3 & 5 & -7 \\ 3 & 18 & 6 \end{vmatrix} = (2)(3)\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$

$\Rightarrow |\lambda A| = \lambda^n |A|$

scalar multiplication of matrix

n × n matrix A

⑥ $|A| = \begin{vmatrix} 3 & 2 & 4 \\ 2 & 2 & -3 \\ 6 & 0 & 7 \end{vmatrix}$

$|B| = \begin{vmatrix} 3 & 2 & 4 \\ 2 & 3 & 2 \\ 6 & 0 & 7 \end{vmatrix}$

$|C| = \begin{vmatrix} 3 & 2 & 4 \\ 4 & 5 & -1 \\ 6 & 0 & 7 \end{vmatrix}$

$\Rightarrow |C| = |A| + |B|$

for both row or column

⑦ $|AB| = |A||B|, \quad |A^n| = |A|^n$

⑧ $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 13 & 8 \\ 0 & -13 & -13 \\ 1 & 6 & 2 \end{vmatrix}$

$r_1 \rightarrow 2r_3 + r_1$

$r_2 \rightarrow -3r_3 + r_2$

If A is a square matrix,

Inverse of $A = A^{-1}$ ← $\det(A) \neq 0$ (invertible, non-singular)

$\times \frac{1}{A}$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Properties!

1. $(A^{-1})^{-1} = A$
2. $(\lambda A)^{-1} = \lambda^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $(AB)^{-1} = B^{-1}A^{-1}$
5. $|A^{-1}| = |A|^{-1}$
6. $(A^n)^{-1} = (A^{-1})^n$

1. $|A| = 14$
2. $\mathrm{cof}(A)$

$$\begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -3 & 5 \end{vmatrix} & + \\ -\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & + & - \\ + & - & + \end{bmatrix}$$

3. $A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$

3×3

$$\boxed{A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ -3 & 4 & 5 \end{bmatrix}$$

minor

cofactor

$$C_{ij} = (-1)^{i+j}M_{ij}$$

1 $M_{32} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

2 $C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 7$

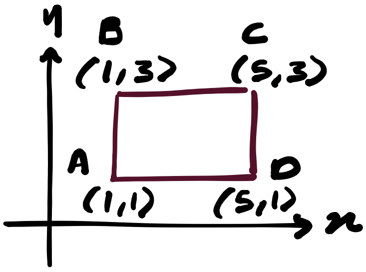
cofactor matrix

3 $\mathrm{cof}(A) = \begin{bmatrix} 4 & -7 & 8 \\ 2 & 14 & 10 \\ -2 & 7 & -4 \end{bmatrix}$

adjoint matrix

4 $\mathrm{adj}(A) = \begin{bmatrix} 4 & 2 & -2 \\ -7 & 14 & 7 \\ 8 & 10 & -4 \end{bmatrix}$

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T$$



$$\begin{array}{c c} \begin{matrix} A & B & C & D \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix} & \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix} \end{array}$$

* Reflection
* Scaling
* Rotation

Stretching / Enlargement

$AA^{-1} = I$

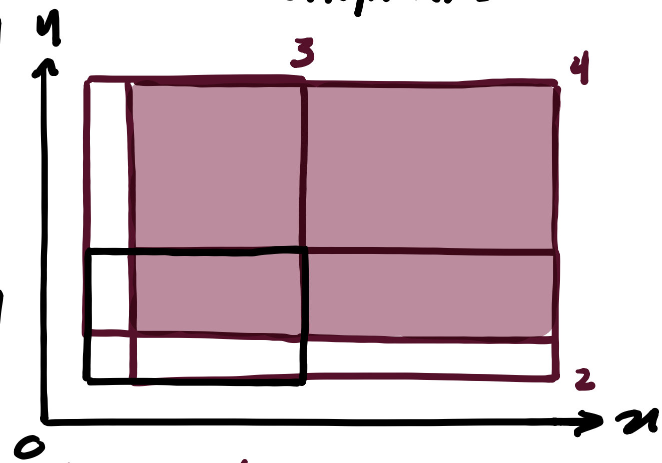
applying inverse returns a transformation to original state

① $I\begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

② $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 10 & 10 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

③ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 & 5 \\ 2 & 6 & 6 & 2 \end{bmatrix}$

④ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 10 & 10 \\ 2 & 6 & 6 & 2 \end{bmatrix}$



x stretch : $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

enlargement: $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

y stretch : $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

scale factor

$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \underset{\text{2nd}}{\begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}}\underset{\text{1st}}{\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}}$

Invariant lines : $x=0$, $y=0$

Invariant point: 0

unchanged by transformation

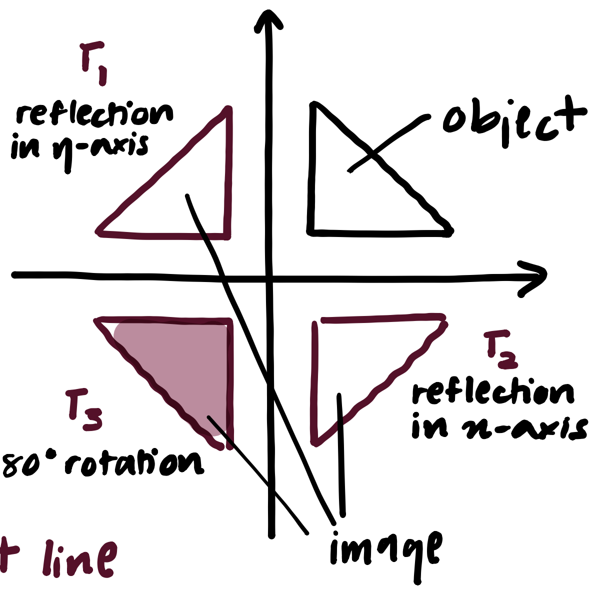
determinant = area change

Reflection

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -4 \\ 1 & 4 & 1 \end{bmatrix} \; T_1$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ -1 & -4 & -1 \end{bmatrix} \; T_2$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -4 \\ -1 & -4 & -1 \end{bmatrix} \; T_3$



$T_3 = T_1T_2 = T_2T_1$

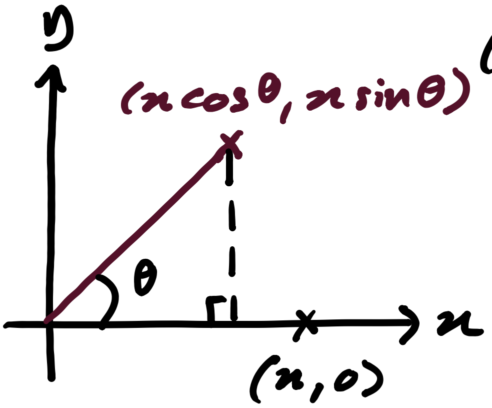
clockwise ↓ ←

anticlockwise ← ↓

Invariant line

↳ any line passing through the origin

Rotation



$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(only for anticlockwise)

if clockwise, $\theta < 0$

$$\boxed{\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta \end{aligned}}$$

Invariant Line

$y = mx$, $c = 0$ (line passes through origin)

$$\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} t \\ mt \end{bmatrix} = \begin{bmatrix} T \\ mT \end{bmatrix}$$

$t \leftarrow x$, $mt \leftarrow y$

$4t + 3mt = T \longrightarrow (1)$

$-3t - 2mt = mT \longrightarrow (2)$

From $\frac{(1)}{(2)}$:

$m = -1$

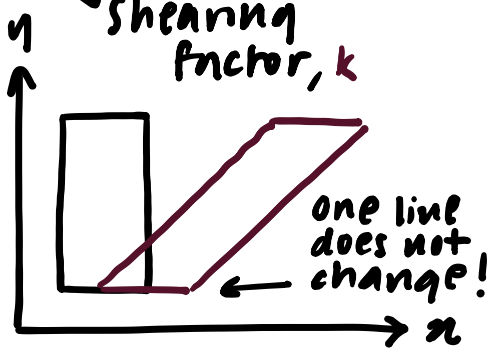
$\therefore y = -x$

Shearing

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 8 & 6 \\ 1 & 1 & 5 & 5 \end{bmatrix} \quad \begin{aligned} x &\rightarrow x + y \\ y &\rightarrow y \end{aligned}$$

Shearing factor, k



Further from $y = 0$ ⇒ Larger displacement

$\hookrightarrow k y_0$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 7 & 4 \end{bmatrix} \quad \begin{aligned} x &\rightarrow x \\ y &\rightarrow \underbrace{2x}_{k x_0} + y \end{aligned}$$

SYSTEM OF LINEAR EQUATIONS AND MATRICES

- Elementary row operation
- Inverse matrix by row operation
- Cramer's rule

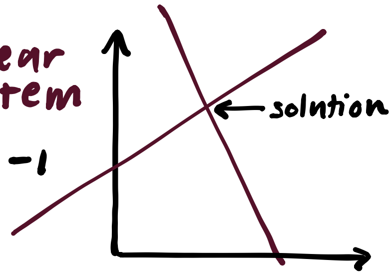
Linear equation: $a_1 x_1 + a_2 x_2 = b$ ← index: 1

a: coefficient
x: variable

- Two variables
  - Line
- Three variables
  - Plane

$$\left.\begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned}\right\} \text{Linear system}$$

Solution: $x_1 = 1,\ x_2 = 2,\ x_3 = -1$



$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & \vdots & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_m & \cdots & & a_{mn} & b_m \end{bmatrix}$$

Elementary Row Operations

- → multiply a row with a constant $\neq 0$ ①
- → interchange two rows ②
- → add multiple of one row to another ③

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

↓ ① ② ③

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$$

Reduced Row-Echelon Form (Gauss–Jordan)

- if a row is not all zeros, the first non-zero element has to be one (1)
- all-zero rows are grouped at the bottom
- leading one on lower rows are further right
- all elements in a column with a leading one = 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Gaussian Elimination)

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$ Row-Echelon Form

↳ does not satisfy fourth condition

Unique Solution

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Infinite Solutions

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Solutions

$x_1 = 7 - 2r - 3s,\ x_2 = r,$
$x_3 = 1,\ x_4 = s,\ x_5 = 2$

No Solution

$$\begin{bmatrix} 1 & -3 & 2 & 3 \\ 0 & 1 & -2/3 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 + 0 + 0 = 1$$

HOMOGENOUS LINEAR SYSTEM

$$ax + by + cz = 0$$
$$dx + ey + fz = 0$$

Trivial solution

$$x + y + z = 0$$

Nontrivial solution

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[A \mid I] \xrightarrow[\text{Operations}]{\text{Row}} [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$Ax = b \longrightarrow x = A^{-1}b$$

Cramer's Rule

$$\boxed{x_n = \frac{\det(A_n)}{\det(A)}}$$

$$x_1 + x_2 + 2x_3 = 9$$
$$2x_1 + 4x_2 - 3x_3 = 1$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

$$\underset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}}$$

$$A_1 \begin{bmatrix} 9 & 1 & 2 \\ 1 & 4 & -3 \\ 0 & 6 & -5 \end{bmatrix} \quad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$A_2 \begin{bmatrix} 1 & 9 & 2 \\ 2 & 1 & -3 \\ 3 & 0 & -5 \end{bmatrix} \quad x_2 = \frac{\det(A_2)}{\det(A)}$$

$$A_3 \begin{bmatrix} 1 & 1 & 9 \\ 2 & 4 & 1 \\ 3 & 6 & 0 \end{bmatrix} \quad x_3 = \frac{\det(A_3)}{\det(A)}$$

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

* Base 10 and base logarithms
* Laws of logarithms
* Solving logarithmic equations
* Solving exponential equations
* Solving exponential inequalities
* Natural logarithms
* Transforming relationships to linear form

$a = b^{c}$ — base b; index/exponent c

$y = 2^{x}$ — exponential function

$$\boxed{y = a^{x} \longleftrightarrow x = \log_a y}$$

logarithmic function

inverse

① $\log_a a = 1$
② $\log_a a^{x} = x$
③ $\log_a 1 = 0$

$\log_a x$
- $a > 0,\ a \neq 1$
- $x > 0$

Exponential Inequalities

$\log_a x$, $a > 1$ – inequality symbol remains
$a < 1$ – symbol is reversed

Natural Logarithms

$e \approx 2.718\ldots\ldots$

$y = e^{x}$

$\ln x = \log_e x$

Laws of Logarithms

Multiplication

$$\log_a xy = \log_a x + \log_a y$$

Division

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

Power

$$\log_a x^{m} = m \log_a x$$

$$\log_a \left(\frac{1}{x}\right) = -\log_a x$$