FUNCTIONS

$x \rightarrow f(x)$
$a \rightarrow f(a)$
} relation

domain (input)

range (output / codomain)

unique! output } function

* Domain, range, and one-to-one functions
* Composite functions
* Inverse of one-to-one functions
* Graphs of simple functions and their inverse, including piecewise functions
* Transformations of functions and graphs

Types of relations

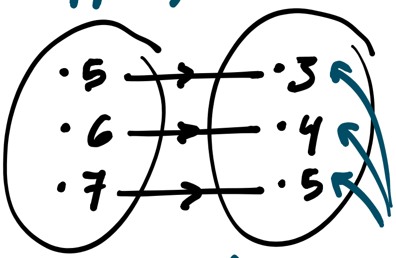
1. one to one
2. many to one
3. one to many
4. many to many

= unique output
= functions!

X a function

Representing functions

Mapping ①



elements

Domain = {5, 6, 7}
Range = {3, 4, 5}

sets!

Set Notation ②

$f(x) = \{(5,3), (6,4), (7,5)\}$

x y

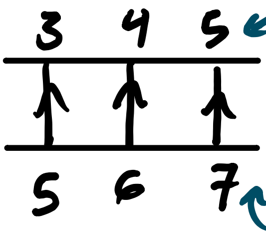
$P \xrightarrow{f} Q$

$f: P \rightarrow Q$

$a \in P$, $b \in Q$

$\therefore f(a) = b$

Arrow Diagram ③

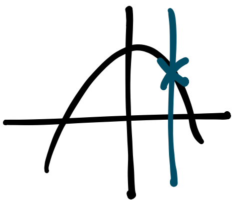


y

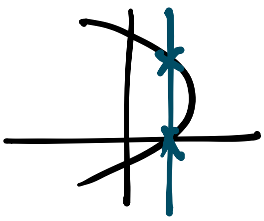
f

x

Vertical line test

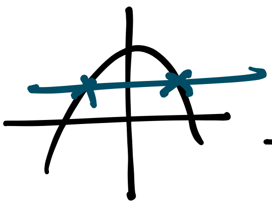


intersect at one point
✓ function

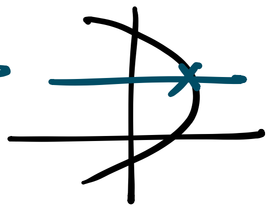


intersect at > one point
X function

Horizontal line test



many to one



one to one

Types of Functions

1. Linear function

$f(x) = mx + c$

constant

gradient

2. Quadratic function

$f(x) = ax^2 + bx + c$

$\neq 0$ coefficient constant

3. Root function

$f(x) = x^{\frac{1}{n}}$ or $f(x) = \sqrt[n]{x}$

>1

Finding DOMAIN

$f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$

$f(x) = \frac{1}{x^n}$, $n = 2, 4, 6, \ldots$

≥ 0 ✓ $\sqrt{4}$ ✗ $\sqrt{-4}$

4. Reciprocal function

$f(x) = \frac{1}{x}$ or $f(x) = x^{-1}$

$\neq 0$

5. Rational function

$f(x) = \frac{p(x)}{q(x)}$ ← numerator, ← denominator

$\neq 0$

Finding RANGE

① $f(x) = k$, assign value

② $\min \leq f(x) \leq \max$, range

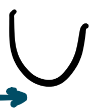
③ $f(x) = \frac{p(x)}{q(x)} = \frac{ax+b}{cx+d}$

④ Completing the square

$f(x) = ax^2 + bx + c$

$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \left(\frac{b}{2a}\right)^2$

if $a > 0$,



minimum value

Range $= [\min, \infty)$

includes

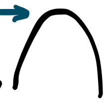
does not include

$$\boxed{\begin{aligned} y &= c - \frac{b^2}{4a} \\ x &= -\frac{b}{2a} \end{aligned}}$$

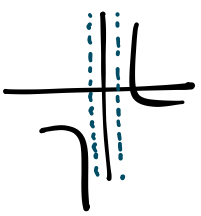
Range $= (-\infty, \max]$

maximum value

if $a < 0$,

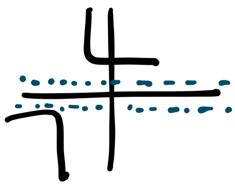


Vertical asymptote



$x = -\frac{d}{c}$

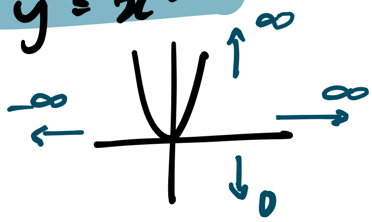
Horizontal asymptote



$y = \frac{a}{c}$, if degree of $p(x) = q(x)$

$y = 0$, if degree of $p(x) < q(x)$

$y = x^2$

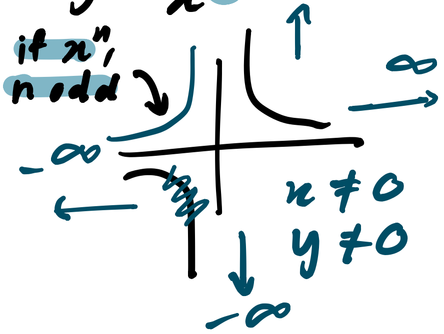


$x = (-\infty, \infty)$

$y = [0, \infty)$

$y = \frac{1}{x}$

$x \neq 0$



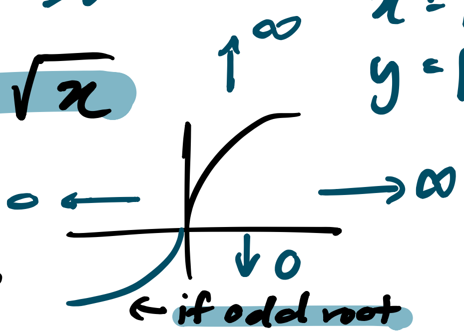
$x = (-\infty, 0) \cup (0, \infty)$

$y = (-\infty, 0) \cup (0, \infty)$

$x \neq 0$

$y \neq 0$

$y = \sqrt{x}$



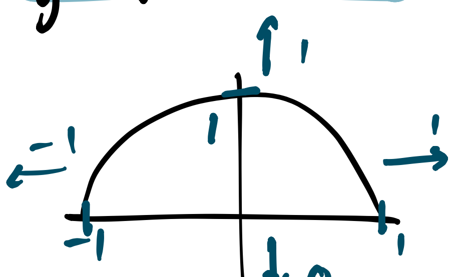
$x = [0, \infty)$

$y = [0, \infty)$

$x \geq 0$

$y \geq 0$

$y = \sqrt{1 - x^2}$



$x = [-1, 1]$, $y = [0, 1]$

Completing the Square!

$3x^2 + 2x - 4$

$3\left(x^2 + \frac{2}{3}x\right) - 4$

$3\left(x^2 + \frac{2}{3}x + \left(\frac{2}{6}\right)^2\right) - 4 - 3\left(\frac{2}{6}\right)^2$

$3\left(x + \frac{1}{3}\right)^2 - 4 - \frac{1}{3}$

$3\left(x + \frac{1}{3}\right)^2 - \frac{13}{3}$

$x + \frac{1}{3} = 0$ $\quad y = -\frac{13}{3}$

$x = -\frac{1}{3}$

Graph Sketching

1. Find vertex
2. Find y-intercept
3. Find x-intercept
4. Find range

Rational Function!

$\frac{9x-9}{2x+7}$

Finding domain:

$2x + 7 \neq 0$

$x \neq -\frac{7}{2}$

$x = \left(-\infty, -\frac{7}{2}\right) \cup \left(-\frac{7}{2}, \infty\right)$

Finding range:

$\frac{9x-9}{2x+7} = k$

$9x - 9 = 2kx + 7k$

$9x - 2kx = 7k + 9$

$x(9 - 2k) = 7k + 9$

$x = \frac{7k+9}{9-2k}$

$9 - 2k \neq 0$

$9 - 2k < 0$ $\quad 9 - 2k > 0$

$-2k < -9$ $\quad -2k > 9$

$k < \frac{9}{2}$ $\quad k > \frac{9}{2}$

$k = y = \left(-\infty, \frac{9}{2}\right) \cup \left(\frac{9}{2}, \infty\right)$

COMPOSITE FUNCTIONS

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[rr, bend right, "fg"'] & g(x) \arrow[r, bend left, "f"] & f(g(x)) \end{tikzcd} \leftarrow fg(x)$$

Domain of Composite Functions — Find the domain for the composite and nested function!!

① $f(x) = \sqrt{1-x^2}$ $\qquad g(x) = \frac{1}{x} \rightarrow \boxed{x \neq 0}$

$fg(x) = \sqrt{1-\left(\frac{1}{x}\right)^2}$

$fg(x) = \sqrt{1-\frac{1}{x^2}}$

$fg(x) = \sqrt{\frac{x^2-1}{x^2}}$

$x^2 \rightarrow x^2 \neq 0$

$\boxed{x \neq 0}$

$\frac{x^2-1}{x^2} \geq 0$

$x^2 - 1 \geq 0$

$x^2 \geq 1$

$\pm x \geq \sqrt{1}$

$\boxed{x \geq 1}$

$\boxed{x \leq -1}$

$D_f = (-\infty, -1] \cup [1, \infty)$

② $f(x) = \sqrt{1-x^2}$ $\qquad g(x) = \frac{1}{x}$

$gf(x) = \frac{1}{\sqrt{1-x^2}}$

$1 - x^2 \geq 0$

$-x^2 \geq -1$

$x^2 \leq 1$

$\pm x \leq \sqrt{1}$

$\boxed{x \leq 1}$ $\quad \boxed{x \geq -1}$

$\sqrt{1-x^2} \neq 0$

$1 - x^2 \neq 0$

$x^2 \neq 1$

$\pm x \neq \sqrt{1}$

$\boxed{x \neq 1}$ $\quad \boxed{x \neq -1}$

$D_f = (-1, 1)$

INVERSE FUNCTION

$$\begin{tikzcd} x \arrow[r, bend left, "f(x)"] & y \arrow[l, bend left, "f^{-1}(x)"] \end{tikzcd}$$

only for one-one function

Domain of $f^{-1}(x)$ $\rightarrow$ Range of $f(x)$

Range of $f^{-1}(x)$ $\rightarrow$ Domain of $f(x)$

$f^{-1} \cdot f(x) = f \cdot f^{-1}(x) = x$

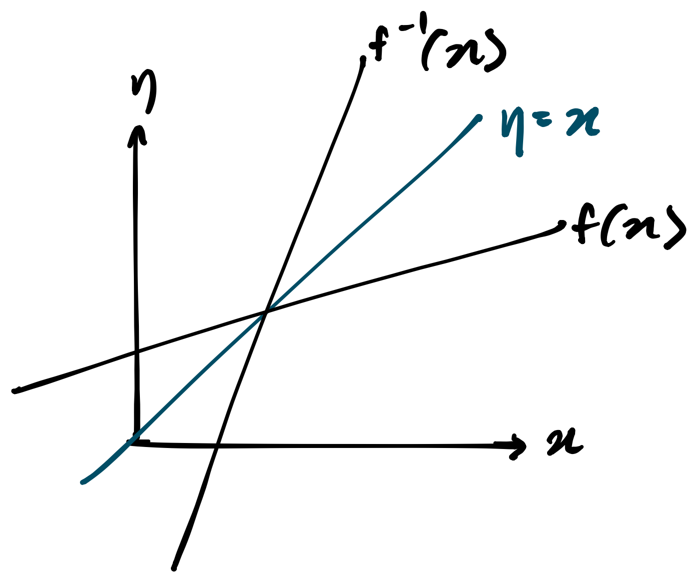
identity function

Show that $g(x) = \frac{x-1}{2}$ is the inverse of $f(x) = 2x+1$

$$gf(x) = g(2x+1)$$
$$= \frac{(2x+1)-1}{2}$$
$$= \frac{2x}{2}$$
$$gf(x) = x$$

$$fg(x) = f\left(\frac{x-1}{2}\right)$$
$$= 2\left(\frac{x-1}{2}\right)+1$$
$$= x-1+1$$
$$fg(x) = x$$

$$f(x) = \sqrt{x+2} - 7$$
$$y = \sqrt{x+2} - 7$$
$$x = \sqrt{y+2} - 7$$
$$x+7 = \sqrt{y+2}$$
$$(x+7)^2 = y+2$$
$$y = (x+7)^2 - 2$$
$$f^{-1}(x) = (x+7)^2 - 2$$



GRAPH TRANSFORMATION

① Translation

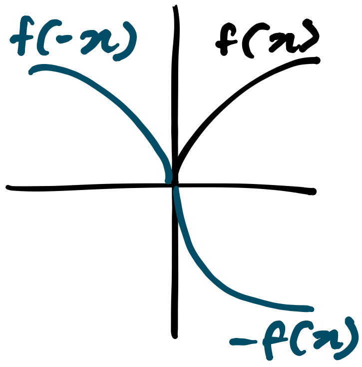
$y = f(x) + a$
$a > 0, \uparrow a$
$a < 0, \downarrow a$
} vertical shift

$y = f(x + a)$
$a > 0, \overset{\leftarrow}{a}$
$a < 0, \vec{a}$
} horizontal shift

② Reflection

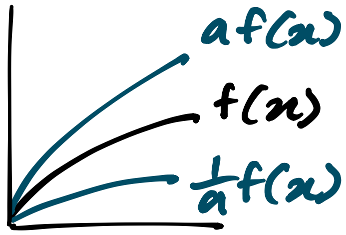
$y = -f(x)$

$y = f(-x)$

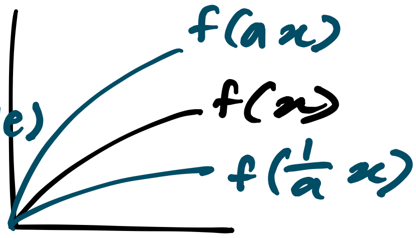


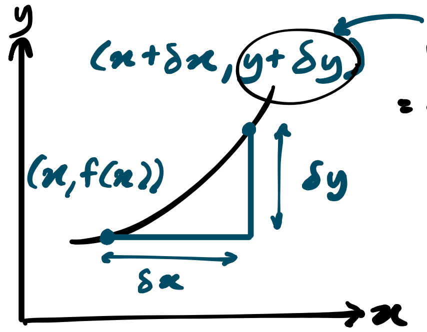
③ Scaling

$y = a f(x)$
$y = \frac{1}{a} f(x)$
} vertical scaling (same roots)



$y = f(ax)$
$y = f(\frac{1}{a} x)$
} horizontal scaling ($\text{vertex}_x \times \text{scale}$)





$$y+\delta y = f(x+\delta x)$$

$$m = \frac{\delta y}{\delta x}$$

- Gradient of a curve at a point as the limit of gradients and chords
- Derivative of a rational function using the first principle
- The chain rule
- Applications of differentiation to gradients, tangents and normals
- Second derivatives

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{x+\delta x - x} = \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\frac{dy}{dx} = f'(x) = \boxed{\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}} \Rightarrow \text{First Principle}$$

[FIRST DERIVATIVE]

as δx approaches 0, gradient at (x, y) can be obtained

Power / Scalar Multiple Rule

$$f(x) = kx^n$$

$$\boxed{f'(x) = nkx^{n-1}}$$

Chain Rule

$$y = (2x-1)^9$$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$y = u^9$

$\frac{dy}{du} = 9u^8$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

$= 9u^8 \times 2$

$= 18u^8$

$\frac{dy}{dx} = 18(2x-1)^8$

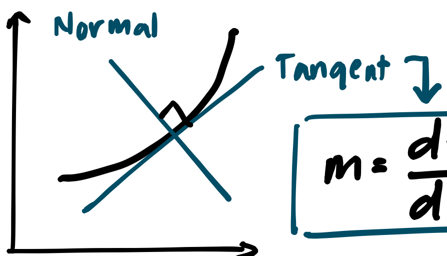
Addition/Subtraction Rule

$$f(x) \pm g(x)$$

$$\frac{d}{dx}[f(x) \pm g(x)]$$

$$= \boxed{\frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)}$$

Tangents & Normals



$$\boxed{m = \frac{dy}{dx}}$$

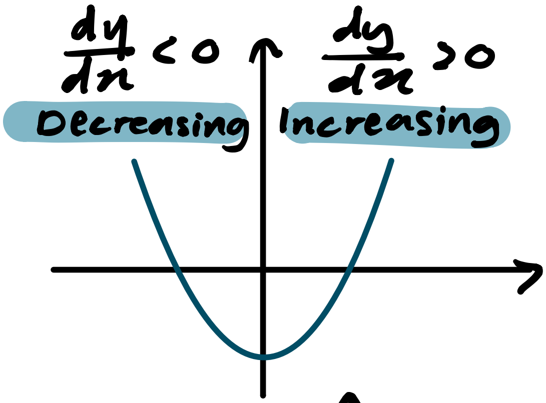
$$m_\perp = -\frac{1}{m}$$

$E_T: y_2 - y_1 = m(x_2 - x_1)$

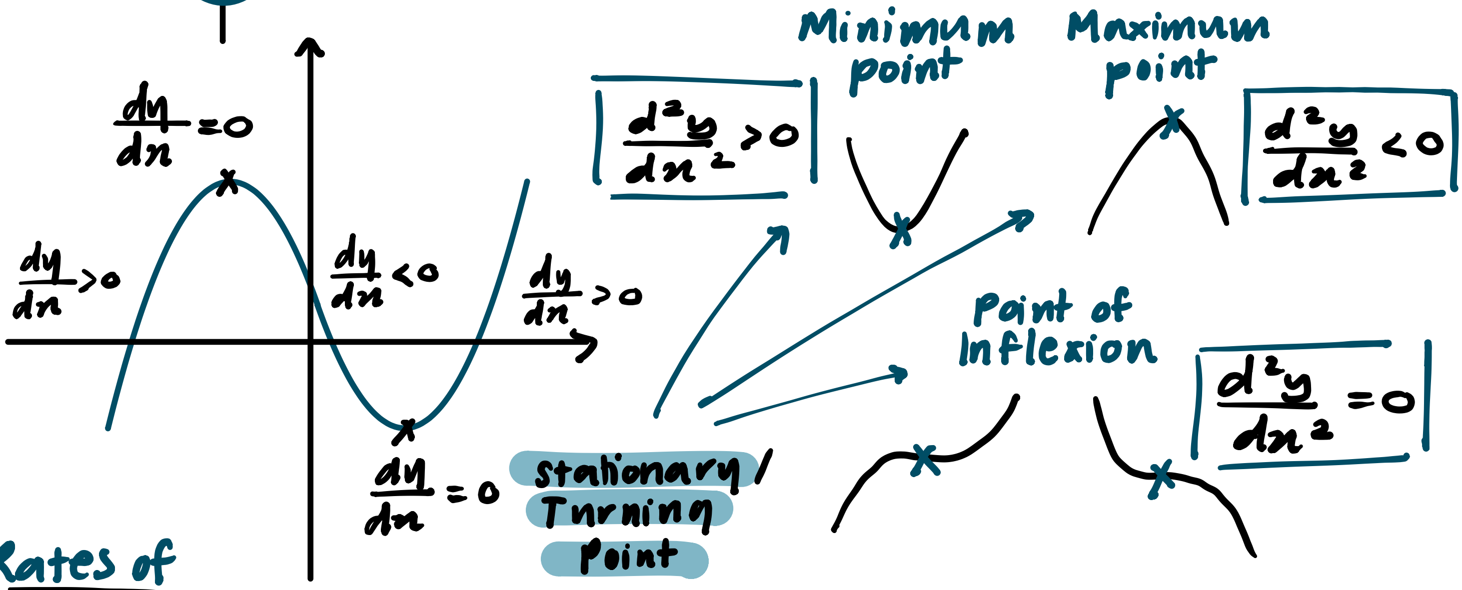
$E_N: y_2 - y_1 = -\frac{1}{m}(x_2 - x_1)$

[SECOND DERIVATIVE] $f''(x) \mid \frac{d^2y}{dx^2}$

$$\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right)}$$



* Increasing and decreasing functions
* Stationary points (critical points) and their nature
* Rates of change
* Applications of connected rates of change
* Practical maximum and minimum problems



1. $\frac{dy}{dx}$
2. let $\frac{dy}{dx} = 0$, $x = ?$
3. sub x into $f(x)$
4. $\frac{d^2y}{dx^2}$

Rates of Change

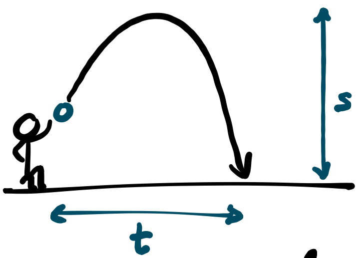
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \approx \frac{dy}{dx}$$

average rate of change

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

instantaneous rate of change

TWO VARIABLES



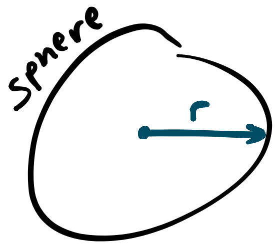
velocity $= \frac{ds}{dt}$

acceleration $= \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Connected Rates of Change

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$ ← THREE VARIABLES

$$\frac{dy}{dt} = \frac{1}{\frac{dt}{dy}}$$ } Reciprocal form



$A = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

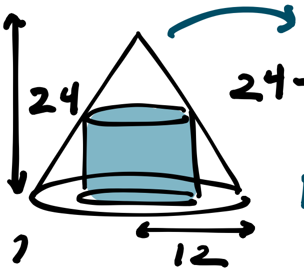
$\frac{dr}{dt} = \frac{1}{5\pi}$

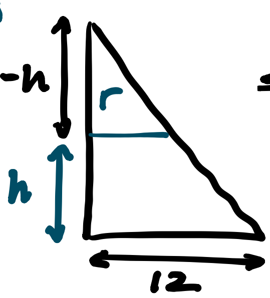
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

Practical Max / Min

* largest cylinder in the cone?





$\frac{24}{12} = \frac{24-h}{r}$

$h = 24 - 2r$

$V = \pi r^2 h = 24\pi r^2 - 2\pi r^3$

$\frac{dV}{dr} = 0, r = ?$

$0 < r < 12$

INTEGRATION I

L anti-derivative

- Integration as the reverse process of differentiation
- Evaluation of the constant of integration
- Integration of $(ax+b)^n$ (for any rational n except -1) with constant multiples, sums, and differences
- Definite integrals
- Improper integrals

$$y = x^n \qquad \frac{dy}{dx} = nx^{n-1}$$

$m = n-1$

$$\int nx^{n-1}\,dx = \frac{1}{m+1}x^{m+1} + c$$

INDEFINITE INTEGRAL

arbitrary constant

from coordinate

1. Power Rule

2. Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx$$

3. Sums & Differences Rule

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

4. $(ax+b)^n$

$$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$$

DEFINITE INTEGRAL

(area & volume)

upper limit

$$\int_a^b f(x)\,dx = \big[f(x)\big]_a^b = F(b) - F(a)$$

lower limit

1. Limit reversal

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2. Additivity

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

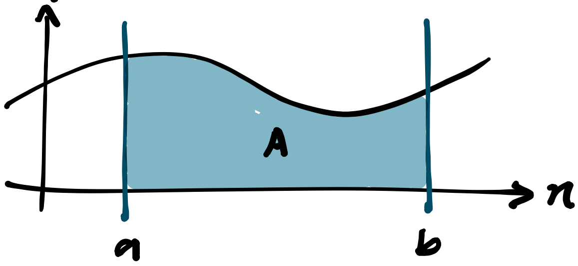
IMPROPER INTEGRAL

$$\int_a^{\infty} f(x)\,dx \qquad \int_{-\infty}^{a} f(x)\,dx$$

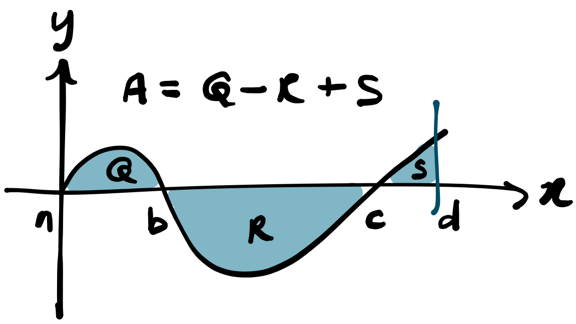
$$\int_0^2 \frac{5}{x^2}\,dx$$

Area Under Curve

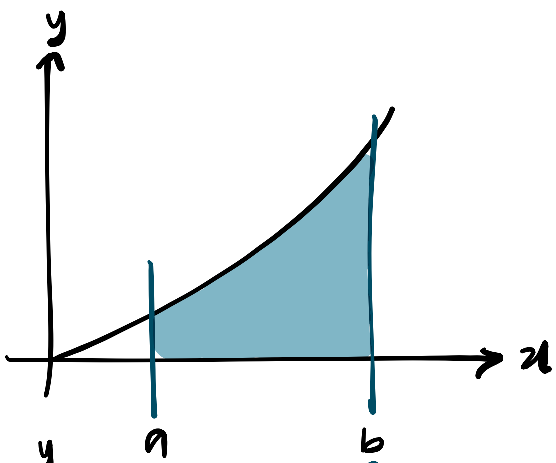
$$A = \int_a^b y\,dx$$

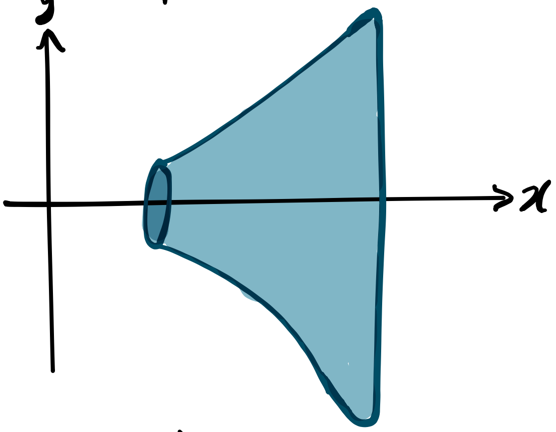


$$A = Q - R + S$$



Volume of Revolution

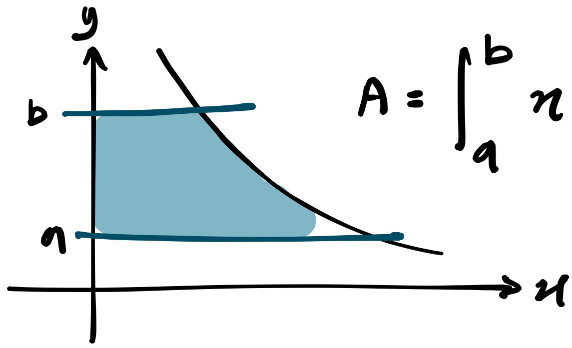




$$V = \int_a^b \pi y^2\,dx$$

- Area under curves
- Area of a region bounded by a curve and lines parallel to the axes, or between a curve and a line or between two curves
- Volume of revolution about an axis

Bounded Area



$$A = \int_a^b x\,dy$$

$$A = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$