# DIFFERENTIATION RULES

## * Power Rule

$y = 2x^1 + 3^0$

$\frac{dy}{dx} = (1)(2)x^{1-1} + (0)(3)x^{0-1}$

$\frac{dy}{dx} = 2$

## * Chain Rule

$y = (2x+3)^5$

$u = 2x+3$

$\frac{du}{dx} = 2, \quad \frac{dy}{du} = 5u^4$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 5(2x+3)^4(2)$

$\frac{dy}{dx} = 10(2x+3)^4$

## * Differentiating a function using product rule and quotient rule

### ① Product Rule

$$\boxed{\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u}$$

$u = (x+1)^2 \qquad v = (-3x+2)^3$

$\frac{d}{dx}(uv) = \frac{d}{dx}(x+1)^2 \times (-3x+2)^3$
$\quad + (x+1)^2 \times \frac{d}{dx}(-3x+2)^3$

$\frac{d}{dx}(uv) = 2(x+1)(-3x+2)^3$
$\quad -9(x+1)^2(-3x+2)^2$

$\Rightarrow \frac{d}{dx}(uvw) = \frac{du}{dx}vw + \frac{dv}{dx}uw$
$\quad + \frac{dw}{dx}uv$

$uv^{-1}$

### ② Quotient Rule

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

$u = (x+1)^2 \qquad v = (-3x+2)^3$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(-3x+2)^3 \times \frac{d}{dx}(x+1)^2 - (x+1)^2 \times \frac{d}{dx}(-3x+2)^3}{((-3x+2)^3)^2}$$

$$= \frac{2(x+1)(-3x+2)^3 + 9(x+1)^2(-3x+2)^2}{(-3x+2)^6}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(x+1)(6x+11)}{(-3x+2)^4}$$

* Find the first derivative of exponential, logarithmic and trigonometric functions

Algebraic Function vs Transcendental Function

Algebraic Function: finite number of polynomials and operations
- $\frac{(x^2-1)^{\frac{3}{2}}}{(2x-x^3)^2}$

Transcendental Function
- $1+x+x^2+x^3+\dots$
- $x^{\pi}$
- trigonometric
- exponential
- logarithmic

## 1 Trigonometric Function

$$\frac{d}{dx}\sin(x)=\cos(x)$$

$$\frac{d}{dx}\cos(x)=-\sin(x)$$

$f(x)=2\sin(x)\cos(x)$

$f'(x)=2\frac{d}{dx}(\sin(x)\cos(x))$ — product rule

$=2\left(\frac{d}{dx}\sin(x)\cdot\cos(x)+\frac{d}{dx}\cos(x)\cdot\sin(x)\right)$

$=2(\cos(x)\cdot\cos(x)+(-\sin(x))\cdot\sin(x))$

$f'(x)=2(\cos^2(x)-\sin^2(x))$

$f'(x)=2\cos(2x)$

$f(x)=\sin(2x)$

$f'(x)=\cos(2x)\cdot 2$ — chain rule

$f'(x)=2\cos 2x$

$f(x)=\tan^3(x)=(\tan(x))^3$

$f'(x)=3(\tan(x))^2\sec^2(x)$

$f'(x)=3\tan^2(x)\sec^2(x)$

$f(x)=\tan(x)=\frac{\sin(x)}{\cos(x)}$

$f'(x)=\frac{\cos(x)\cdot\cos(x)-\sin(x)\cdot(-\sin(x))}{\cos^2(x)}$

$=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$

$=\frac{1}{\cos^2(x)}$

$f'(x)=\sec^2(x)$

## 2 Exponential Function

$\frac{d}{d(\;)}e^{(\;)}=e^{(\;)}$

natural: $\frac{d}{dx}e^x=e^x$

$\frac{d}{dx}e^{3x^3}=\frac{d}{d(3x^3)}e^{3x^3}\cdot\frac{d(3x^3)}{dx}$

$=e^{3x^3}\cdot 9x^2$

general: $\frac{d}{dx}a^x=a^x\ln(a)$

$\frac{d}{d(\;)}a^{(\;)}=a^{(\;)}\ln(a)$

$\frac{d}{dx}3^{x^2}=\frac{d}{dx^2}3^{x^2}\cdot\frac{dx^2}{dx}$

$=3^{x^2}\ln(3)\cdot 2x$

## 3 Logarithmic Function

natural: $\frac{d}{dx}\ln(x)=\frac{1}{x}$

$\frac{d}{d(\;)}\ln(\;)=\frac{1}{(\;)}$

$\frac{d}{dx}\ln(2x+1)=\frac{1}{2x+1}\cdot 2$

# IMPLICIT DIFFERENTIATION AND PARAMETRIC DIFFERENTIATION

* Find the first derivative of an implicit function using implicit differentiation, and find the first derivative of a parameterized function using parametric differentiation and applying the computation of these derivatives in finding the equation of tangents and normals

y is an explicit function of x

$$y = \underbrace{x^2 + x}_{\text{only one expression}}$$

y is an implicit function of x

$$y^2 = x \quad \begin{cases} y = +\sqrt{x} & (1) \\ y = -\sqrt{x} & (2) \end{cases}$$

**IMPLICIT DIFFERENTIATION**

$$\frac{d}{dx} y^2 = \frac{d}{dx} x = 1$$

chain rule

$$2(y)' \cdot \frac{d}{dx} y$$

$$= 2y\frac{dy}{dx} \longrightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$\frac{d}{dx}(xy)$$

$$= \frac{dx}{dx}y + x\frac{dy}{dx}$$

$$= y + x\frac{dy}{dx}$$

$\longrightarrow$ if $y = 3$: $y\frac{dx}{dx}$ (y is a constant)

$\longrightarrow$ if $y = x^2$: $3x^2$ (y is a function)

$$\frac{d}{dx}(x^2y^3)$$

$$= 2xy^3 + x^2(3y^2)\frac{dy}{dx}$$

$$\frac{d}{dy}(x^2y^3)$$

$$= 2x\left(\frac{dx}{dy}\right)y^3 + x^2 3y^2$$

**PARAMETRIC DIFFERENTIATION**

$x = \cos(t)$, $y = \sin(t)$

$0 \le t \le 2\pi$

Gradient of tangent, $m_T = \frac{dy}{dx}$

Gradient of normal, $m_N = -\frac{1}{\frac{dy}{dx}}$

$$\frac{dx}{dt} = -\sin(t), \quad \frac{dy}{dt} = \cos(t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}}$$

$$= \cos(t) \times \frac{1}{-\sin(t)}$$

$$\frac{dy}{dx} = -\cot(t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

# INTEGRATION OF EXPONENTIAL AND LOGARITHMIC FUNCTION

* Find the integral of exponential function and the integral of the reciprocal of a linear polynomial

## 1 Exponential Function

$$\boxed{\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c}$$

straight line $x'$

$\frac{d}{dx}(ax+b)$

arbitrary constant

$$\int a e^{bx+c}\,dx = a\int e^{bx+c}\,dx$$

$$= a\,\frac{e^{bx+c}}{b} + d$$

## 2 Logarithmic Function

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \frac{d}{dx}\ln(-x) = \frac{1}{x}$$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$$

modulus

$$\int \frac{a}{bx+c}\,dx = a\int \frac{1}{bx+c}\,dx$$

$$= \frac{a\ln|bx+c|}{b} + d$$

## Trig. Identities:

✗ Find the integral of trigonometric functions

$$\sin(A)\cdot\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\cos(A)\cdot\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\sin(A)\cdot\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

$$\int \sin(7x)\cos(4x)\,dx$$
$$= \int \frac{1}{2}(\sin(7x-4x) + \sin(7x+4x)\,dx$$
$$= \frac{1}{2}\int \sin(3x) + \sin(11x)$$
$$= \frac{1}{2}\left(-\frac{\cos(3x)}{3} - \frac{\cos(11x)}{11}\right) + c$$
$$= -\frac{1}{2}\left(\frac{\cos(3x)}{3} + \frac{\cos(11x)}{11}\right) + c$$

$\int (1-5x)^{\frac{1}{2}} dx$

★ Find the integral of a composite function using substitution

By formula,

$$= \frac{(1-5x)^{\frac{1}{2}}}{(\frac{1}{2}+1)(-5)} + c = -\frac{2}{15}(1-5x)^{\frac{3}{2}} + c$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\frac{1}{ax+b} dx = \frac{\ln|ax+b|}{a} + c$$

By substitution,

$u = 1-5x$

$x = \frac{1-u}{5}$

$\frac{du}{dx} = -5$

$\frac{du}{-5} = dx$

$$\int u^{\frac{1}{2}} dx$$

$$= \int u^{\frac{1}{2}} \frac{du}{-5}$$

$$= -\frac{1}{5}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$= -\frac{2}{15}(u^{\frac{3}{2}}) + c$$

$$= -\frac{2}{15}(1-5x)^{\frac{3}{2}} + c$$

$$\boxed{\int \frac{1}{1+e^{-x}} dx}$$

1

$u = 1+e^{-x}$

$du = -e^{-x} dx$

$dx = \frac{du}{-e^{-x}}$

$dx = \frac{du}{u-1}$

$$\int \frac{1}{u}\left(\frac{du}{u-1}\right)$$

$$= \int \frac{du}{u(u-1)}$$

= cont. w partial fraction

2

$$\frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

$u = e^x+1$

$du = e^x dx$

$$\int \frac{e^x}{e^x+1} dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + c$$

$$= \ln|e^x+1| + c$$

$$f(f^{-1}(x)) = x$$

$$\sin\left(\frac{\pi}{2}\right) = 1, \quad \sin^{-1}(1) = \frac{\pi}{2}$$

* $0 \leq \theta \leq 2\pi$

* Find the derivative of the arctan function and the integral of a function using trigonometric function

## Differentiation

$$[y = \sin^{-1}(x)]$$

$$\sin(y) = \sin(\sin^{-1}(x))$$

$$\sin(y) = x$$

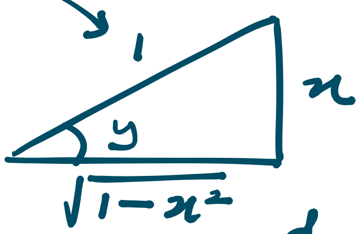
$$\frac{d}{dx}\sin(y) = \frac{d}{dx}x$$

$$\cos(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

(where $-1 < x < 1$)




$$[y = \cos^{-1}(x)]$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$[y = \tan^{-1}(x)]$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

Extended:

$$\frac{d}{dx}\sin^{-1}(\ ) = \frac{1}{\sqrt{1-(\ )^2}} \times \frac{d}{dx}(\ )$$

$$\frac{d}{dx}\cos^{-1}(\ ) = -\frac{1}{\sqrt{1-(\ )^2}} \times \frac{d}{dx}(\ )$$

$$\frac{d}{dx}\tan^{-1}(\ ) = \frac{1}{1+(\ )^2} \times \frac{d}{dx}(\ )$$

## Integration

I: $\displaystyle\int \frac{1}{\sqrt{A^2 - X^2}}\,dX$ $\quad X = A\sin\theta$

or

$\displaystyle\int \sqrt{A^2 - X^2}\,dX$ $\quad X = A\cos\theta$

II: $\displaystyle\int \frac{1}{A^2 + X^2}\,dX$ $\quad X = A\tan\theta$

III: $\displaystyle\int \frac{1}{X\sqrt{X^2 - A^2}}\,dX$ $\quad X = A\sec\theta$

$$\left[\int \frac{1}{\sqrt{3-2x-x^2}}\,dx\right]$$

CTS:

$$3 - 2x - x^2 = 2^2 - (x+1)^2$$

$X = A\sin\theta$

$$x + 1 = 2\sin\theta$$

$$dx = 2\cos\theta\,d\theta$$

$$\int \frac{1}{\sqrt{2^2 - 2^2\sin^2\theta}}\,2\cos\theta\,d\theta$$

$$= \int \frac{2\cos\theta}{\sqrt{2^2\cos^2\theta}}\,d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{x+1}{2}\right) + c$$

[ $\int \frac{1}{s^2+10s+30}\, ds$

CTS:

$$= \int \frac{1}{(s+5)^2+5}\, ds$$

$$= \int \frac{1}{(s+5)^2+(\sqrt{5})^2}\, ds$$

$X = A\tan\theta$

$$s+5 = \sqrt{5}\tan\theta$$

$$ds = \sqrt{5}\sec^2\theta\, d\theta$$

$$\int \frac{1}{5\tan^2\theta+5}\sqrt{5}\sec^2\theta\, d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5(\tan^2\theta+1)}\, d\theta$$

$$= \int \frac{\sqrt{5}\sec^2\theta}{5\sec^2\theta}\, d\theta$$

$$= \int \frac{1}{\sqrt{5}}\, d\theta$$

$$= \frac{1}{\sqrt{5}}\theta + c$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{s+5}{\sqrt{5}}\right) + c$$

# INTEGRATION BY PARTIAL FRACTIONS

Rational:

$$\frac{g(x)}{f(x)}$$ ← polynomial

* Find the integral of a rational function using partial fractions

$$\frac{2x+1}{4} + \frac{-\frac{21}{4}x + \frac{41}{4}}{2x^2+3x-1}$$

Long division

$$\int \frac{x^3+2x^2-5x+10}{2x^2+3x-1}\,dx$$

degree (num) ≥ degree (denom)

Partial fraction

$$\int \frac{2x^2-5x+10}{(x+1)(x^2+2x)}\,dx$$

degree (num) < degree (denom)

## ① Distinct linear factors

$$\frac{3x+5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$3x+5 = A(x+3) + B(x-2)$

when $x = 2$,

$3(2)+5 = A(2+3) + B(2-2)$

$5A = 11$

$A = \frac{11}{5}$

when $x = -3$,

$3(-3)+5 = A(-3+3) + B(-3-2)$

$-5B = -4$

$B = \frac{4}{5}$

$$\frac{3x+5}{(x-2)(x+3)} = \frac{11}{5(x-2)} + \frac{4}{5(x+3)}$$

## ② Repeated linear factors

$$\frac{10x^2+7x+3}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

## ③ Irreducible quadratic factors

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

# INTEGRATION BY PARTS

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

* Apply the integration by parts method to find the integral of a composite function

$$\int \frac{d}{\cancel{dx}}(uv)\,\cancel{dx} = \int u\frac{dv}{\cancel{dx}}\,\cancel{dx} + \int v\frac{du}{\cancel{dx}}\,\cancel{dx}$$

$$\int \cancel{d}(uv) = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

LIATE

$u$ = log func / inverse trig
= algebraic func
= exponential / sin / cos func

$$\left[\int x\,e^{3x}\,dx\right]$$

$u = x$

$dv = e^{3x}\,dx$

$du = 1\,dx$

$v = \frac{e^{3x}}{3}$

$$\begin{array}{cccc}
+ & \xrightarrow{\times} & x & e^{3x} \\
- & \xrightarrow{\times} & 1 & \frac{e^{3x}}{3} \\
+ & & \underline{\underline{0}} \xrightarrow{+} & \frac{e^{3x}}{9}
\end{array}$$

$$\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + \int (0)\left(\frac{e^{3x}}{9}\right) + c$$

$$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c$$

$$uv - \int v\,du = x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx$$

$$\int xe^{3x}\,dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c$$

#

$$F = ma$$
$$F(t, v) = m\left(\frac{dv}{dt}\right)$$
$$F\left(t, u, \frac{du}{dt}\right) = m\left(\frac{d^2u}{dt^2}\right)$$

* Model a physical problem in terms of a differential equation and solve a separable differential equation using method of separation

## Separable Differential Equation

$$xy\frac{dy}{dx} = 3xy^2 + 4y^2$$
$$\underbrace{\frac{y}{y^2}\,dy}_{y} = \underbrace{\frac{3x+4}{x}\,dx}_{x}$$
$$\int \frac{1}{y}\,dy = \int \left(3 + \frac{4}{x}\right) dx$$
$$\ln|y| = 3x + 4\ln|x| + c$$

## Initial Value Problem (IVP)

$$xy\frac{dy}{dx} = 3xy^2 + 4y^2; \quad y(1) = 1$$
$$\ln|y| = 3x + 4\ln|x| + c$$
$$c = \ln|y| - 3x - 4\ln|x|$$
$$c = -3$$
$$\ln|y| = 3x + 4\ln|x| - 3$$