A possible model for bifurcation in 2D fracture networks

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1 Geometrical construction

We consider the case of three fractures γ_i, $i = 0, 1, 2$, described as $2D$ curves that intersect. We make the following geometrical assumptions

1. The fractures intersect in a region that may be approximated by a triangle I;

2. In the vicinity of the intersection fractures may be assumed to be straight and with constant thickness. More precisely, there exist three points $\boldsymbol{c}_i$, with $i = 0, 1, 2$, on each γ_i, respectively, such that the mid-line of fracture γ_i (which we indicate with the same symbol) may be described in the vicinity of the intersection by equation

$$\gamma_i = \gamma_i(s_i) = \boldsymbol{c}_i + s_i\boldsymbol{t}_i \quad s_i \in (-S_1, S_i). \tag{1}$$

   Here, $\boldsymbol{t}_i$ is the tangent vector to γ_i, s_i is a parameter and $S_i > 0$ provides the bound so that the given expression describes the fracture up to the intersection I. In this portion of the fracture the width d_i is constant. The situation is described in figure 1.

3. A simplified setting is when we assume that the mid lines intersect at a single point $\boldsymbol{c}$. In that case we may take $\boldsymbol{c}_0 = \boldsymbol{c}_1 = \boldsymbol{c}_2 = \boldsymbol{c}$. We will see that this stronger assumptions leads to some simplifications. See figure 2.

4. In the case of tee junction, we use the construction of figure 3. (WE NEED TO BE MORE PRECISE HERE).

1.1 Construction of the intersecting triangle

We denote with $\boldsymbol{z}_i$ the normal to γ_i such that the 2D cross product $\boldsymbol{t}_i \times \boldsymbol{z}_i$ is positive. We denote with γ_i^+ and γ_i^- the borders of fracture γ_i laying respectively opposite and in the direction of $\boldsymbol{z}_i$, as illustrated in the cited figures.
We have that

$$\gamma_i^{\pm} = \boldsymbol{c}_i + s_i\boldsymbol{t}_i \pm \frac{d_i}{2}\boldsymbol{z}_i, \quad i = 0, 1, 2, \quad s_i \in (-S_1, S_i) \tag{2}$$

We denote with $\boldsymbol{P}_{ji}$ the intersection of γ_j^- and γ_i^+, where $i = 0, 1, 2$ and $j = (i + 1) \mod 3$. By simple calculations we have

$$\boldsymbol{P}_{ji} = \gamma_j^-(s_{ji}), \tag{3}$$

where

$$s_{ji} = \left(\left(\boldsymbol{c}_i - \boldsymbol{c}_j + \frac{d_j}{2}\boldsymbol{z}_j\right) \cdot \boldsymbol{z}_i + \frac{d_i}{2}\right) / \boldsymbol{t}_j \cdot \boldsymbol{z}_i. \tag{4}$$

If we adopt assumption 3 we have

$$s_{ji} = \left(\frac{d_j}{2}\boldsymbol{z}_j \cdot \boldsymbol{z}_i + \frac{d_i}{2}\right)/\boldsymbol{t}_j \cdot \boldsymbol{z}_i. \tag{5}$$

1.1.1 The case of the tee junction

In the case of the tee junction, like that of figure 3 there are two indexes i, j, with $j = i + 1 \mod 3$ such that $\boldsymbol{t}_j \cdot \boldsymbol{z}_i = 0$[1].
In this case we set

$$\boldsymbol{P}_{ji} = \frac{\boldsymbol{c}_j - (d_j/2)\boldsymbol{z}_j + \boldsymbol{c}_i + (d_i/2)\boldsymbol{z}_i}{2} \tag{6}$$

1.2 Interface conditions

To identify the interface conditions we consider the intersection triangle I we employ a mimetic hybridized finite difference scheme [3, 1]. We assume that triangle I is non degenerate and we refer to figure 4 for notation. In particular we indicate with $\boldsymbol{n}_i$ the outward normal to $i-th$ triangle edge e_i, and we set[2]

$$\boldsymbol{\nu}_i = |e_i|\boldsymbol{n}_i, \tag{7}$$

as the area (length) scaled normal. We indicate with $\boldsymbol{E}_i$ the mid point of edge e_i and we set

$$\boldsymbol{g}_i = \boldsymbol{E}_i - \boldsymbol{c}_I, \tag{8}$$

where here $\boldsymbol{c}_I$ is the center of mass of triangle I. We use notation inspired from [3] since it is of immediate implementation. In particular with u_i we indicate the mass flux across edge e_i. We not that a positive u_i denotes a flux that is exiting the triangle across the corresponding edge. We use the mass flux as variable and not the average velocity as done elsewhere because it is the quantity more readily interfaced with the reduced model for the fracture network presented in [2]. Of course the expressions that follow can be recast using average velocity by a simple rescaling.

1.2.1 Preliminaries

We introduce the following matrices

$$\mathrm{N} = \begin{bmatrix} \boldsymbol{\nu}_0^T \\ \boldsymbol{\nu}_1^T \\ \boldsymbol{\nu}_2^T \end{bmatrix} \quad \text{and } \mathrm{C} = \begin{bmatrix} \boldsymbol{g}_0^T \\ \boldsymbol{g}_1^T \\ \boldsymbol{g}_2^T, \end{bmatrix} \tag{9}$$

which are of dimension 3×2.

Proposition 1. *The nullspace of both* C^T *and* N^T *is spanned by the constant vector* $\mathbf{1}_3 = [1, 1, 1]^T$.

[1] In practice we identify the tee junction if $|\boldsymbol{t}_j \cdot \boldsymbol{z}_i| \leq \tau$, τ being a suitable tolerance.
[2] Note that the triangle nodes are numbered clockwise.

Proof. The fact that

$$\mathrm{N}^{\mathrm{T}}\mathbf{1}_3 = \sum_{\mathrm{i}=0}^{3} \boldsymbol{\nu}_{\mathrm{i}} = \mathbf{0}$$

and

$$\mathrm{C}^{\mathrm{T}}\mathbf{1}_3 = \sum_{\mathrm{i}=0}^{3} \boldsymbol{g}_{\mathrm{i}} = \mathbf{0}$$

is an immediate consequence of the fact that a triangle is a polygon and of the definition of the matrices. We need to prove that $\mathbf{1}_3$ generates the whole nullspace. The proof is analogous for the two matrices, so we consider just N.
Let assume that there is a non null vector $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \alpha_2]^T \neq \beta\mathbf{1}_3$ for any $\beta \in \mathbb{R}$ such that

$$\mathrm{N}^{\mathrm{T}}\boldsymbol{\alpha}_3 = \sum_{\mathrm{i}=0}^{3} \alpha_{\mathrm{i}}\boldsymbol{\nu}_{\mathrm{i}} = \mathbf{0}.$$

Without loss of generality we assume $\alpha_0 \neq 0$. It means that there exists two real numbers $a_1 = \alpha_1/\alpha_0$ and $a_2 = \alpha_2/\alpha_0$ such that

$$\boldsymbol{\nu}_0 = -a_1\boldsymbol{\nu}_1 - a_2\boldsymbol{\nu}_2.$$

Yet, we have already demonstrated that

$$\boldsymbol{\nu}_0 = -\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2.$$

Then, $(1 - a_1)\boldsymbol{\nu}_1 = -(1 - a_2)\boldsymbol{\nu}_2$. Since all $\boldsymbol{\nu}_i$ are different from zero and $\boldsymbol{\nu}_1 \neq \boldsymbol{\nu}_2$ otherwise the triangle is degenerate, we may conclude that the previous expression implies $a_1 = a_2 = 1$, and thus $\alpha_0 = \alpha_1 = \alpha_2$, which is excluded by hypothesis. $\square$

As a consequence, $\mathrm{Q}_{\mathrm{C}}^{\perp} = \mathrm{Q}_{\mathrm{N}}^{\perp} = \frac{1}{\sqrt{3}}\mathbf{1}$ is the 3×1 matrix whose column is an orthogonal basis of $\ker(\mathrm{C}^{\mathrm{T}})$ and $\ker(\mathrm{N}^{\mathrm{T}})$, respectively.
Let now Q_{C} be the matrix whose columns form an orthogonal basis of the image of C. Since $\dim(\mathrm{range}(\mathrm{C})) = 3 - \dim(\ker(\mathrm{C}^{\mathrm{T}})) = 2$ the matrix is 3×2.

Proposition 2. *The 3×3 matrix*

$$\mathrm{P}_{\mathrm{C}}^{\perp} = \mathrm{Id}_3 - \mathrm{Q}_{\mathrm{C}}\mathrm{Q}_{\mathrm{C}}^{\mathrm{T}} \tag{10}$$

represents the orthogonal projector on the null space of C^{T}. As a consequence, for all $\boldsymbol{v} = [v_0, v_1, v_2]^T$

$$\mathrm{P}_{\mathrm{C}}^{\perp}\boldsymbol{v} = \overline{\mathrm{v}}\mathbf{1}_3, \quad \textit{with } \overline{\mathrm{v}} = \frac{1}{3}\sum_{\mathrm{i}=1}^{3} \mathrm{v}_{\mathrm{i}}. \tag{11}$$

Here, Id_3 *is the 3×3 identity matrix.*

Proof. Indeed, the nullspace of $\mathrm{Q}_{\mathrm{C}}^{\mathrm{T}}$ coincides with that of C^{T} by construction. Thus if $\boldsymbol{w} = \alpha\mathbf{1}_3$ for an arbitrary $\alpha \in \mathbb{R}$ we have, for any $\boldsymbol{v} \in \mathbb{R}^3$

$$\boldsymbol{v}^T\mathrm{P}_{\mathrm{C}}^{\perp}\boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{w}.$$

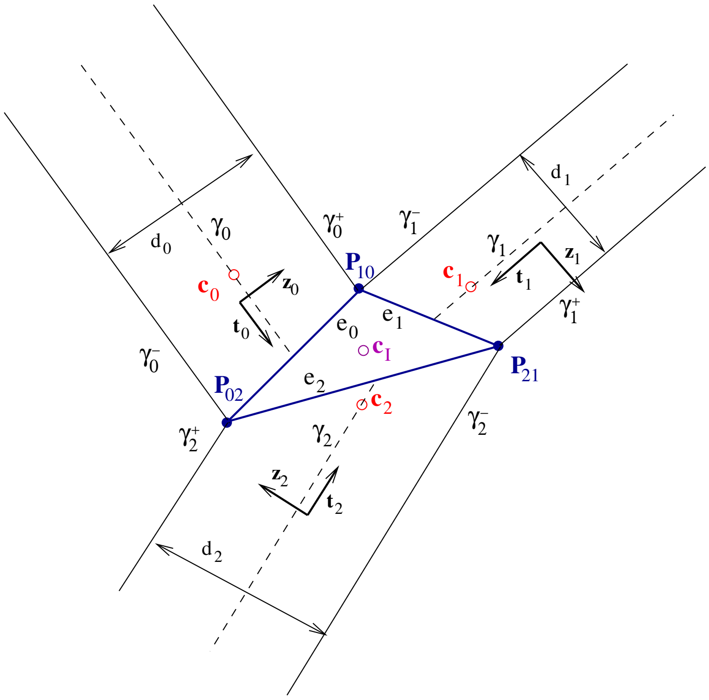

Figure 1:

which proves that $\mathrm{P}_{\mathrm{C}}^{\perp}$ is an orthogonal projector.
Since $\mathrm{P}_{\mathrm{C}}^{\perp}$ is an orthogonal projection on a space whose orthogonal basis is formed just by the vector $\boldsymbol{k} = 3^{-1/2}\mathbf{1}_3$ we have that

$$\mathrm{P}_{\mathrm{C}}^{\perp}\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{k} = \overline{\mathrm{v}}\mathbf{1}_3 \tag{12}$$

□

Proposition 3. *Let* S *be a* 3×3 *symmetric positive definite matrix, with elements* S_{sl}. *We have*

$$\mathrm{P}_{\mathrm{C}}^{\perp}\mathrm{S}\mathrm{P}_{\mathrm{C}}^{\perp}\boldsymbol{v} = \hat{\mathrm{s}}\mathrm{Q}_{\mathrm{C}}^{\perp}(\mathrm{Q}_{\mathrm{C}}^{\perp})^{\mathrm{T}}\boldsymbol{v} = \hat{\mathrm{s}}\overline{\mathrm{v}}\mathbf{1}_3, \tag{13}$$

with

$$\hat{s} = \frac{1}{3}\sum_{ls=0}^{3} S_{ls} \tag{14}$$

Proof. It is a simple application of the previous result and of the definition of $\mathrm{Q}_{\mathrm{C}}^{\perp}$. □

Therefore,

$$\mathrm{rank}(\mathrm{P}_{\mathrm{C}}^{\perp}\mathrm{S}\mathrm{P}_{\mathrm{C}}^{\perp}) = 1.$$

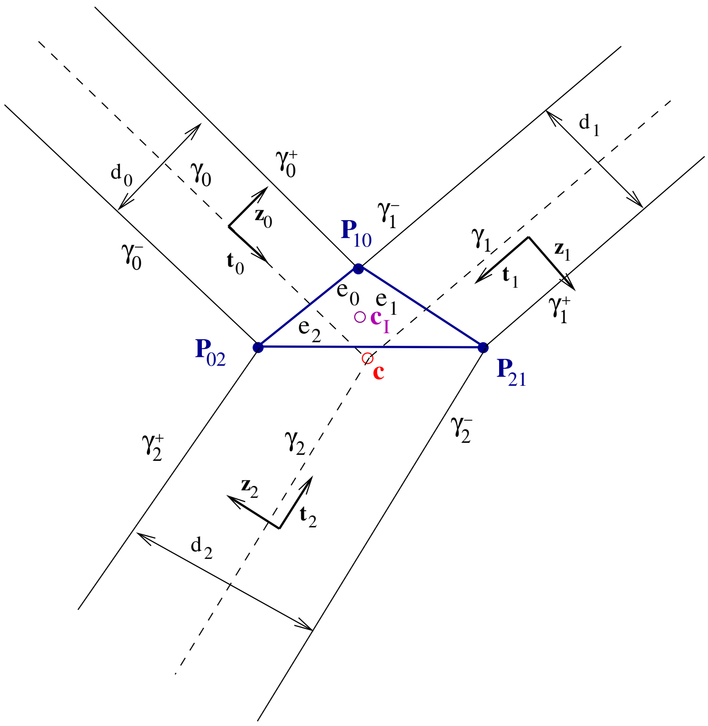

Figure 2:

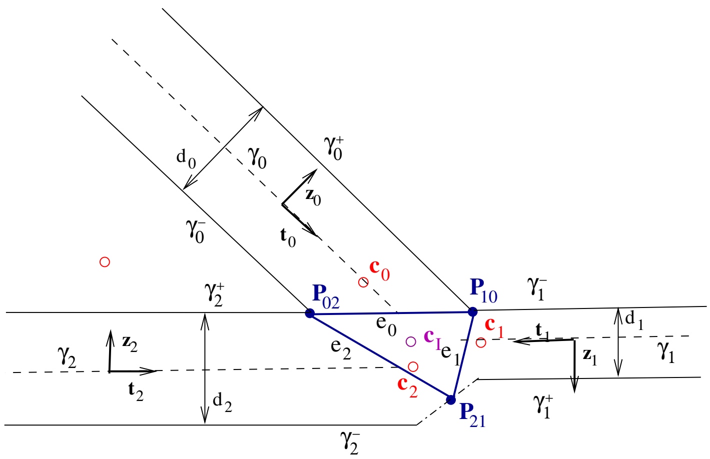

Figure 3:

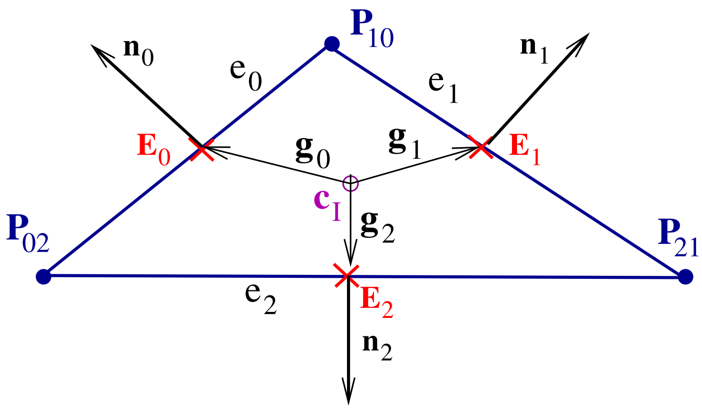

Figure 4:

1.2.2 The derivation of the interface conditions

Let $\mathrm{K_I}$ be the symmetric 2×2 matrix representing the permeability at the intersection, p_I the pressure at the intersection, located at $\boldsymbol{c}_i$ and π_i, $i \in \{0,1,2\}$ the Lagrange multipliers that represent the values of pressure at $\boldsymbol{E}_i$.
Let $\boldsymbol{u} = [u_0, u_1, u_2]^T$ be the vector of fluxes, and $\Pi = [\pi_0, \pi_1, \pi_2]^T$ that of the edge pressures. The application of a mimetic finite difference scheme on the triangle I for the Darcy's constitutive relation, ignoring gravity effects, leads to a relation of the type

$$\boldsymbol{u} = \mathrm{T}(\mathrm{p_I}\mathbf{1}_3 - \Pi), \tag{15}$$

where T is the *transmissibility matrix* that takes the general form

$$\mathrm{T} = \frac{1}{|\mathrm{I}|}\mathrm{NK_IN^T} + \mathrm{P_C^\perp S P_C^\perp}, \tag{16}$$

or, equivalently,

$$\mathrm{T} = \frac{1}{|\mathrm{I}|}\mathrm{NK_IN^T} + \mathrm{sQ_C^\perp(Q_C^\perp)^T} = \frac{1}{|\mathrm{I}|}\mathrm{NK_IN^T} + \frac{\mathrm{s}}{3}\mathcal{I}, \tag{17}$$

where $\mathcal{I} = \mathbf{1}_3\mathbf{1}_3^T$ is the matrix with all elements equal to 1. Here, S is a suitable symmetric positive definite matrix, and $s > 0$ a constant.
Typical solutions are

$$\mathrm{S} = t\,\mathrm{diag}(\mathrm{NK_IN^T}) \tag{18}$$

being $t > 0$ a parameter (a possible value is $t = 6$), or

$$\mathrm{S} = t\,\mathrm{trace}(\mathrm{K_I})\mathrm{Id}_3 \tag{19}$$

Proposition 4. *Matrix* T *is symmetric positive definite.*

Proof. Indeed, by using (12), (13), and exploiting the symmetry of $\mathrm{P_C^\perp}$, for any $\boldsymbol{v} \neq \mathbf{0}$

$$\boldsymbol{v}^T\mathrm{T}\boldsymbol{v} = \frac{1}{|\mathrm{I}|}\boldsymbol{v}^\mathrm{T}\mathrm{NK_IN^T}\boldsymbol{v} + 3\overline{\mathrm{v}}^2\hat{\mathrm{s}}.$$

Now, $\hat{s} > 0$ since S is s.p.d. and $\boldsymbol{v}^T\mathrm{NK_IN^T}\boldsymbol{v} \geq 0$ since $\mathrm{K_I}$ is s.p.d.. This concludes the proof. □

We now use the continuity equation $\nabla \cdot u = q$ in I, which in the mimetic difference framework is approximated as

$$\sum_{i=0}^{2} u_i = \mathbf{1}_3^T \boldsymbol{u} = Q, \quad \text{with } Q = \int_I q \, d\Omega \tag{20}$$

Using (15) we have

$$Q = \mathbf{1}_3^T \boldsymbol{u} = \mathbf{1}_3^T \mathrm{T}(\mathrm{p_I}\mathbf{1}_3 - \Pi), \tag{21}$$

We now use (16) and (13) to obtain the following

Proposition 5. *We have that*

$$p_I = \frac{Q}{3\hat{s}} + \overline{\pi}, \tag{22}$$

where $\overline{\pi} = \frac{1}{3}\sum_{i=0}^{2} \pi_i$. *In particular, in the absence of source terms* $p_I = \overline{\pi}$.

Proof. We start by noting, thank to symmetry, that

$$\mathbf{1}_3^T \mathrm{P}_\mathrm{C}^\perp = \mathbf{1}_3^\mathrm{T} \quad \text{and } \mathrm{P}_\mathrm{C}^\perp \mathbf{1}_3 = \mathbf{1}_3.$$

and we recall that

$$\mathrm{N}\mathbf{1}_3 = \mathbf{0}.$$

Thus, by using (13) and the fact that $\overline{\mathbf{1}_3} = 1$, we get

$$p_I \mathbf{1}_3^T \mathrm{T} \mathbf{1}_3 = \mathrm{p_I}\hat{\mathrm{s}} \mathbf{1}_3^\mathrm{T} \mathbf{1}_3 = 3\mathrm{p_I}\hat{\mathrm{s}}.$$

While,

$$\mathbf{1}_3^T \mathrm{T}\Pi = \mathbf{1}_3^\mathrm{T} \mathrm{T}\Pi = \hat{\mathrm{s}}\overline{\pi}\mathbf{1}_3^\mathrm{T}\mathbf{1}_3 = 3\hat{\mathrm{s}}\overline{\pi}.$$

Substituting into (21) we have

$$3\hat{s}(p_I - \overline{\pi}) = Q,$$

which allows us to conclude the proof. $\square$

Consequently, we have the following system

$$\begin{cases} \boldsymbol{u} - p_I \mathrm{T}\mathbf{1}_3 - \mathrm{T}\Pi = \mathbf{0} \\ \sum_{i=0}^{2} u_i = Q \\ p_I - \overline{\pi} = \dfrac{Q}{3\hat{s}}. \end{cases} \tag{23}$$

When coupling with the reduced model of fractures we will identify u_i and π_i as the fluxes (BEWARE OF THE SIGN CONVENTION) and pressures at the end of the fracture located at the bifurcation. While, p_I is the pressure at the bifurcation, which can be computed (if needed) as post-processing stage since system (23) is obviously equivalent to

$$\begin{cases} \boldsymbol{u} - \mathrm{T}\left(\overline{\pi}\mathbf{1}_3 - \Pi\right) = \dfrac{\mathrm{Q}}{3\hat{\mathrm{s}}}\mathrm{T}\mathbf{1}_3 \\ \sum_{i=0}^{2} u_i = Q. \end{cases} \tag{24}$$

We finally recall that $\mathrm{T}\mathbf{1}_3 = \hat{\mathrm{s}}\mathbf{1}_3$.

References

[1] Franco Brezzi, Konstantin Lipnikov, and Valeria Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *Mathematical Models and Methods in Applied Sciences*, 15(10):1533–1551, 2005.

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[3] Knut-Andreas Lie, Stein Krogstad, Ingeborg Skjelkvåle Ligaarden, Jostein Roald Natvig, Halvor Møll Nilsen, and Bård Skaflestad. Open-source matlab implementation of consistent discretisations on complex grids. *Computational Geosciences*, 16(2):297–322, 2012.