1. Generalities

Fix a field K.

Definition 1. A valuation $v\colon K \to \mathbb{Z}$ on K is discrete if it is non-archimedean and surjective.

We have the

Proposition 1.0.1 ([Ser68, Chap I,§1, Proposition 1])**.** *If K is complete with respect to a discrete valuation v, then its unit ball K° is a discrete valuation ring.*

Lemma 1.0.2 ([Ser68, Chap II,§2, last part of Proposition 3])**.** *If K is complete with respect to a discrete valuation v and if L/K is a finite extension, then L has a discrete valuation $w\colon L \twoheadrightarrow \mathbb{Z}$ inducing v and L is complete with respect to w.*

Moreover,

Proposition 1.0.3 ([Ser68, Chap II,§2, Proposition 3])**.** *If K is complete with respect to a discrete valuation v and if L/K is a finite extension, then the integral closure of K° inside L coincides L° and so, in particular, it is a discrete valuation ring by 1.0.1. Moreover, L° is a finite, free K°-module of rank $n = [L : K]$.*

2. Local Fields

Definition 2. A *(nonarchimedean) local field* is a field complete with respect to a discrete valuation and with finite residue field.

Definition 3. A *mixed characteristic local field* is a finite field extension of the field $\mathbb{Q}_p$ of p-adic numbers, for some prime p.

Definition 4. An *equal characteristic local field* is a finite field extension of the field $\mathbb{F}_p(\!(X)\!)$, for some prime p.

Lemma 2.0.1. *A mixed characteristic local field is a local field.*

Lemma 2.0.2. *An equal characteristic local field is a local field.*

2.1. Ramification Index.

(1) Define a local ramification index to get rid of the variables $\mathfrak{p}$ and $\mathfrak{P}$ in the mathlib definition
(2) Obtain from there the formula $n = ef$
(3) Prove that $\mathbb{Q}_p$ and $\mathbb{F}_p(\!(X)\!)$ are unramified
(4) If possible, go through [Ser68, Chap. I, §6] (before completion, the results there hold for DVR's).

3. Global to Local

Starting with a Dedekind domain R and a non-zero maximal ideal $\mathfrak{p}$, let $K = \mathrm{Frac}(R)$. Then we prove the

Proposition 3.0.1 ([Ser68, Chap. II, §2, Théorème 1])**.** *The completion $K_\mathfrak{p}$ has a discrete valuation extending the valuation $v_\mathfrak{p}$ and such that ${K_\mathfrak{p}}^\circ = R_\mathfrak{p}$. In particular, this localization is a DVR by Proposition 1.0.1.*

Proposition 3.0.2. *Let F be a number field and let v be a finite place of residue characteristic p. Then F_v is a mixed characteristic local field with residue characteristic p.*

Proposition 3.0.3. *Let F be a function field and let v be any place. Then F_v is an equal characteristic local field with residue characteristic p.*

References

[Ser67] Jean-Pierre Serre. Local class field theory. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 128–161. Thompson, Washington, D.C., 1967.

[Ser68] Jean-Pierre Serre. *Corps locaux*. Hermann, Paris, 1968. Deuxième édition.