Classification

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Outline

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- To fit the model, we use a method called **maximum likelihood**

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- The coefficients β_0 and β_1 are estimated based on the available *training data* using the **maximum likelihood** method.

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- The null hypothesis implies that $p(X) = \frac{e^{\beta_0}}{1+e^{\beta_0}}$.
- To make predictions, we simply put the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ into the equation $\hat{p} = \frac{e^{\hat{\beta}_0+\hat{\beta}_1 X}}{1+e^{\hat{\beta}_0+\hat{\beta}_1 X}}$.

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for $k = 1, \cdots, K-1$, and

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- Thus, rather than estimating coefficients for $K - 1$ classes, we actually estimate coefficients for all K classes.

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- To apply the Bayes classifier we still have to estimate the parameters π_k, μ_k and σ^2.

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- The Bayes decision boundaries are the set of values x for which $\delta_k(x) = \delta_l(x)$ for $k \neq l$.
- Once again, we need to estimate the unknown parameters $\mu_1, \cdots, \mu_K$, $\pi_1, \cdots, \pi$, and Σ.
- The formulas are similar to those used in the one-dimensional case.

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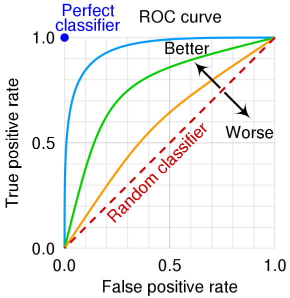

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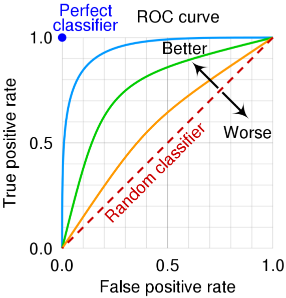

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- So the QDA classifier involves plugging estimates for μ_k, Σ_k and π_k into (13), and then assigning an observation $X = x$ to the class for which this quantity is **largest**.

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  for $k = 1, \cdots, K$.
- To estimate the one-dimensional density function f_{kj} using training data $x_{1j}, \cdots, x_{nj}$, we have a few options.

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     $\rightarrow$ We assume that within each class, the jth predictor is drawn from a (univariate) normal distribution.
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Naive Bayes

Estimating the density function

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   - Simply count the proportion of training estimator observations for the jth predictor corresponding to each class.

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- The bias-variance trade-off, etc.

Thank you!

Any question?