Support Vector Machines

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Outline

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One can determine on which side of the hyperplane a point lies by simply calculating the sign of the left hand side of (1).

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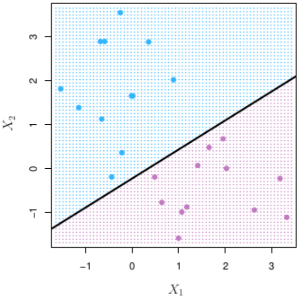

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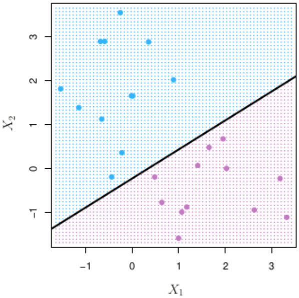

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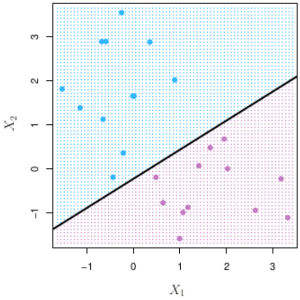

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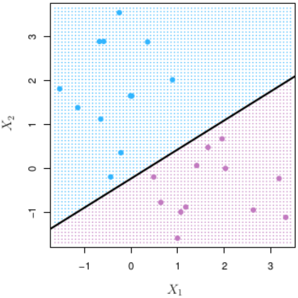

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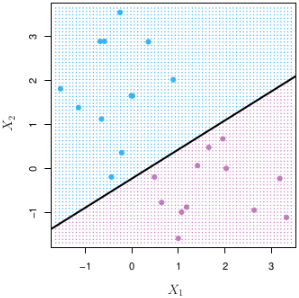

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- Then we classify a test observation based on which side of the maximal margin hyperplane it lies.

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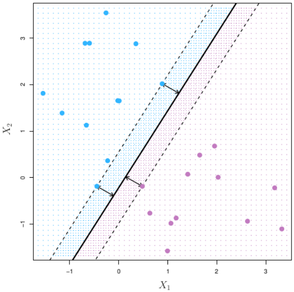

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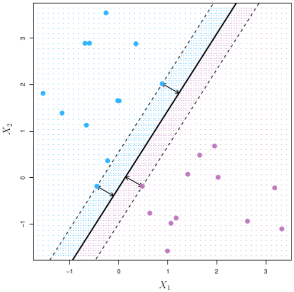

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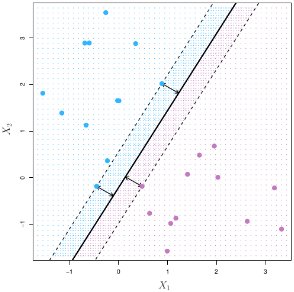

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- They "support" the MMH: if they were moved slightly then the maximal margin hyperplane would move as well.

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- So, eq. (9) ensures that each observation is at least a distance M from the hyperplane.

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- The generalization of the maximal margin classifier to the non-separable case is known as the **support vector classifier**.

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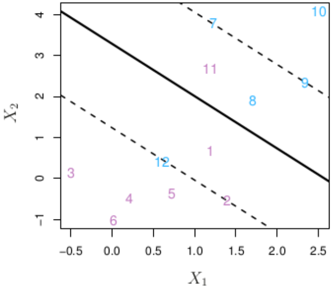

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$$\text{s.t. } y_i \left(\beta_0 \sum_{j=1}^{p} \beta_{j1} x_{ij} + \sum_{j=1}^{p} \beta_{j2} x_{ij}^2 \right) \geq M(1 - \epsilon_i) \tag{11}$$

$$\epsilon_i \leq C, \epsilon_i \geq 0, \sum_{j=1}^{p} \sum_{k=1}^{2} \beta_{jk}^2 = 1. \tag{12}$$

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- M is the width of the margin.

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- C is a tuning parameter: bounds the sum of the ϵ_i's and it's chosen by Cross-validation.

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- Then, eq. (14) is known as a **linear kernel** because the support vector classifier is linear in the features.
- But one could instead choose another form for the kernels

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- Only nearby training observations have an effect on the class label of a test observation.

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  4. The previous amount is the level of level of confidence that x^* belongs to the kth class rather than $(k-1)$.

Thank you!

Any question?