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Maria Kesa
(YAFMM 093159)

Analysis of the super-exponential model of human population growth

Essay

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1. Description of the problem and the model

About a million years ago, the population of early human species consisted of about 55,000 individuals—dangerously close to extinction (Huff et al, 2010). As time progressed humans have spread all over the planet and even explored outer space. There are approximately 6.8 billion of us now and we are growing at a rate of 134 million per year. How the human population will grow in the future is a key question that will determine its impact on the environment, resources available per person, the size of cities, social organization and many other things. It is not clear how many people the environment can sustain and whether this limit can be extended with the advances of science and technology. Predicting what the world population is likely to be in the future is also of great practical interest to governments who have to plan infrastructure and resource allocation to schools and pensions based on projections of population growth. In this essay I will describe and critically analyze the super-exponential model of population growth. In order to give a thorough understanding of the problem I will describe how this model is related to the Malthusian and the logistic models of growth.

Human population growth was first described mathematically by Thomas Malthus in his essay „An Essay on the Principle of Population“ published in 1798. Malthus' model assumes that the size of a population increases by a fixed proportion of its initial size (base) independently of the size of the population. Mathematically this results in unbounded exponential growth. The differential equation corresponding to Malthus' model is:

$$\frac{dP(t)}{dt} = rP(t),$$

where $P(t)$ is the population at time *t* and *r* is the Malthusian parameter that determines the population growth rate (the population will grow exponentially if $r \geq 0$; if $r = 0$ the population will remain consant and if $r < 0$ the population will decrease exponentially towards zero*)*. Malthus realized that if the population was growing faster than the resources available to sustain it, then a proportion of people would be left in scarcity. Many would die of hunger, but with new births a competition for limited resources would soon begin again. This cycle of misery was called the Malthusian trap and it was used to explain the famines taking place in England at that time.

Malthus himself did not put his idea of a limiting factor into mathematical form. This was done in 1838 by Pierre Verhulst who created the equation of logistic growth by introducing the concept of a carrying capacity (the same model was created independently by Raymond Pearl). The carrying capacity in this model is a fixed quantity that represents the population size that the environment can sustain (Cohen, 1995). The population growth rate in this model decreases as population size grows and goes to zero when it reaches the carrying capacity. The carrying capacity creates a horizontal asymptote for population size. The Malthusian trap represents the population size oscillating up and down around an average value which is the carrying capacity. The model is given by the modified differential equation:

$$\frac{dP(t)}{dt} = rP(t)\frac{[K(t) - P(t)]}{K(t)}$$

where *K(t)* is the carrying capacity.

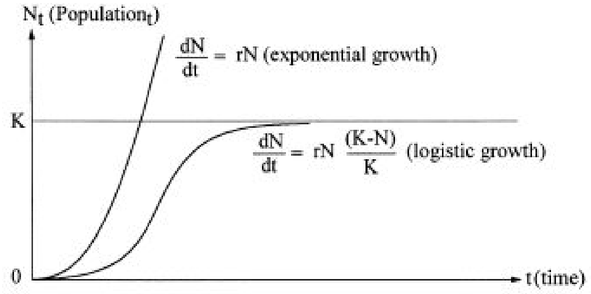

Figure 1. Illustration of the exponential and logistic growth models (From Seidl and Tisdell, 1999).

Malthus was a religious man (he was a reverend) and he believed that the suffering that people experienced from the scarcity of resources had a higher reason—that humanity would progress to overcome the scarcity of resources. In Malthus' words: "Had population and food increased in the same ratio, it is probable that man might never have emerged from the savage state." The idea that the carrying capacity is not a fixed quantity, but instead is increased by the progress of technology is the basis of the super-exponential model formulated mathematically by von Foerster in 1960 (this form of growth is also called hyperbolic growth). It has the following form:

$$\frac{dP(t)}{dt} = rP(t)[K(t) - P(t)]$$

This model assumes a positive feedback relationship between population size and technological progress. As population increases more people work on research and development, more tax money can be collected by governments to allocate to science, etc. Therefore additional carrying capacity is created which allows for even more people to be sustained in the next generation, which in turn allows for more additional carrying capacity to be created. Foerster's model was called the doomsday model, because the super-exponential model has a finite time singularity, which means that it „explodes", reaches infinity in finite time. Technological progress is what makes the model super-exponential and therefore leads to the singularity. Johansen and Sornette (2001) phrased it very well: „Any new innovation is deeply embedded in the very existence of a singularity, it *feeds* it."

The singularity is an important feature of the super-exponential (hyperbolic) growth model. This singularity is „spontaneous" („movable"), because it is determined by the initial condition (constant of integration). The singularity is investigated by Johansen and Sornette, 2001. They used a variety of data-fitting techniques and showed that this singularity should happen in 2056+-10. Von Foerster originally predicted that human population would grow to infinity on 13th November, 2026. However these studies were parameter fitting exercises, that don't validate the model itself. The difficulties associated with the interpretation of the singularity will be discussed in section 5 of this essay.

Exponential, logistic and super-exponential growth models are highly non-linear. They have an important similarity in that they are convex functions (exponential and super-exponential functions are fully convex, the first half of the logistic growth is convex—the part of the curve that is before the inflection point, where the second derivative turns negative). However they behave very differently as the input (time) gets large. The logistic function has a horizontal asymptote, therefore the population has a finite limit as time goes to infinity; exponential growth goes to infinity as time goes to infinity (but is finite in finite time) and super-exponential model goes to infinity in finite time.

These three growth models—exponential, logistic and super-exponential, express different relationships between the increase of population and increase of carrying capacity. In these models the rate of change of the carrying capacity is directly proportional to the rate of change of the population size:

$$\frac{dK(t)}{dt} = c\frac{dP(t)}{dt}.$$

If $c=1$ then each person creates additional carrying capacity to sustain himself, therefore the carrying capacity does not become a limiting factor and the population grows exponentially. If $c<1$, the population growth rate smoothly approaches zero and the population grows logistically. If $c>1$, then each additional person increases the carrying capacity to maintain themselves and even creates surplus carrying capacity. Then the term $K(t)-P(t)$ in the model increases with time and this results in growth that is faster than exponential thereby giving the super-exponential model. In summary, the super-exponential model can be collapsed to the other two models based on the value of *c*.

2. Analysis of assumptions

Here I will discuss four assumptions that I have identified as central to the super-exponential model.

2.1. An increase in carrying capacity necessarily increases population

It is a well known fact that fertility in developed countries is falling and in some places is below replacement levels. Close to 95% of population growth currently comes from developing nations in Africa and Asia. This suggests that there is a link between economic development and fertility. One hypothesis that can be advanced is that there is a proportional relationship between fertility and mortality levels. In areas where economic development is poor and mortality is high parents hedge against probable deaths of their children by having lots of them. In addition in areas relying on agriculture it is beneficial to have many children as they can be exploited as workforce. This suggests that there is a potential for demographic transition in countries that are currently poor if they became economically better off. If fertility levels all over the world would fall then the population growth rate would also slow and the super-exponential model would no longer describe reality.

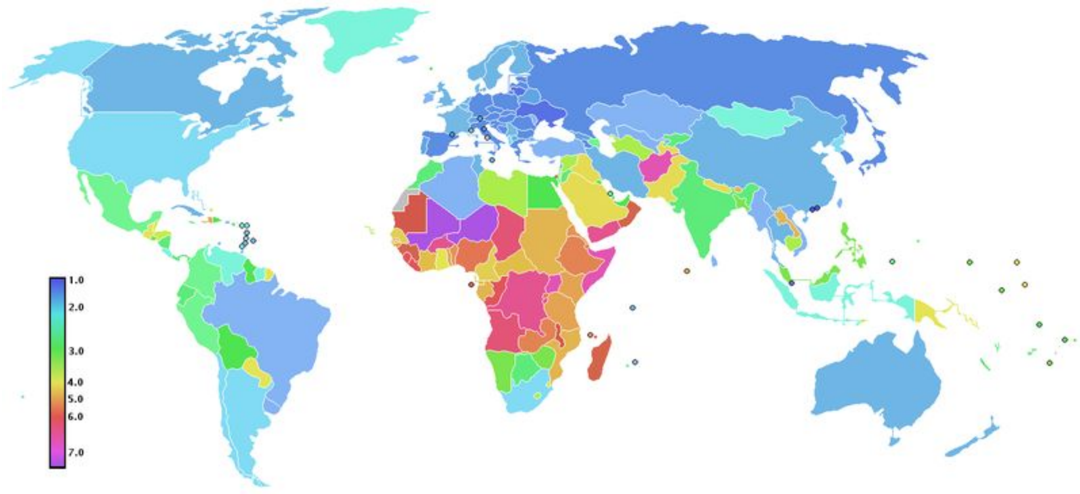

Figure 2. World fertility levels (from Wikipedia).

2.2. Technological progress is unbounded

The assumption of the super-exponential model is that we will always be able to overcome limiting factors presented to us by the environment through progress of our technology. This is a rather bold assumption considering that the current environmental crises and the depletion of many important natural resources such as oil, fish, etc. Furthermore we do not know the limits of our knowledge and of our ingenuity. We cannot predict the nature of future science and technology—we do not know if we will be able to overcome resource shortage crises and extend our carrying capacity infinitely, because have not yet invented and realized the things that would allow us to do so. If technological progress is bounded then the super-exponential growth cannot last forever and the logistic model is a proper descriptor of reality.

2.3. Technological growth always fuels carrying capacity

Certain technological innovations do not lead to a sustainable increase in the carrying capacity. Short term boosts in carrying capacity can undermine the environment which results in long-term negative impact on the carrying capacity. For example, advances in naval technology allowed for fish to be harvested at a drastic rate, therefore increasing the carrying capacity, but the consequences of over-fishing are that fishing stocks have declined and some fish populations will never recover. Also historically populations have grown according to different growth laws during different times. Malthus' exponential model is an accurate description of Europe from about 1200s to 1700s according to the data available. After the industrial revolution the growth rate became super-exponential. This means that there is potential for non-smoothness in the rate of population growth and the behavior of the growth rate might be cyclical, or divided into phases. This type of phenomenon if witnessed in the data could be incorporated into the model by making it multi-phase.

The present super-exponential model is a one-phase model with a finite end point. To augment the model one could incorporate multiple phases—these phases would correspond to different growth regimes (exponential, super-exponential, logistic) and therefore incorporate different relationships between the rate of change of population growth and rate of change of the carrying capacity. The multi-phase model would have different derivatives in time. The

transition between phases could be either made periodic (then the model would be periodic in the rate of change) or stochastic.
This would describe situations where the society would grow very fast due to innovative boosts in technology and this growth would eventually slow and stabilize when the negative impacts of this technology would emerge.

2.4. The rate of carrying capacity improvement is always greater than the population growth rate.

This expresses the relationship between the rate of change of the carrying capacity and population growth

$$\frac{dK(t)}{dt} = c\frac{dP(t)}{dt},$$

where c is always greater than one.
It is questionable if c is a constant. The assumption that c is greater than one for some population sizes and growth rates is plausible, because with the growth of the population there is a greater amount of people working to improve technology, more resources allocated to research and development, a larger market for technology etc. However as population becomes very large and therefore the amount of people added in a new generation becomes very large, disrupting factors arise. These disrupting factors are called diseconomies of scale. For example, as population grows communication between research organizations and scientific groups and individual scientists may become a problem. Miscoordination between a very large number of research efforts will likely result in a duplication of work. More and more resources will have to be put into communication and management, which means that more and more people will be working solely to maintain coordination between different research and engineering efforts instead of actually creating carrying capacity. As the system grows larger it will become more and more inert and resistant to change—it will become more and more expensive to upgrade to new innovations as the size of the system to be renewed grows. All of the different factors of diseconomies of scale combine and this will mean that eventually a massive influx of people in the new generation is not going to lead to a massive boost in carrying capacity that is needed to sustain it. Therefore the increase in carrying capacity will be slower than the rate of population growth and it is questionable if the singularity will be reached or the growth will become logistic.

Mathematically diseconomies of scale could be defined to be a function whose rate of change would increase with population size and counteracts technological progress--this function could be subtracted from the parameter c. For example, coordination and communication problems are not an issue with small and medium population size, but as population nears infinity the rate of increase in communication problems also nears infinity.

I can think of two arguments in defense of the present super-exponential formulation of the model. Firstly communication and coordination problems can be remedied with advances of information technology. For example the invention of the telephone and the internet have transformed the efficiency with which new knowledge is created. However it is questionable whether any communication technology can remedy the duplication of work and miscoordination as population approaches infinity.

Secondly it can be argued that even though the rate of technological improvement might become slower than population growth, more people would be working to adopt technology in society to increase the carrying capacity. Therefore the needed increase in carrying capacity would come through the exploitation of existing inventions. However it is questionable if the increase in carrying capacity gained through adoption of existing technology could last indefinetly and could increase the carrying capacity enough to produce a singularity.

A more fundamental point that I want to make is that, the idea that the carrying capacity is not a fixed quantity and increases with time can be incorporated into the population growth model without resulting in super-exponential growth. For example if the carrying capacity grew exponentially or linearly, then population would have and exponential or linear asymptote. It is possible that these types of growth would better for modeling human population growth.Also these forms of growth do not contain a singularity and therefore are very different qualitatively (the difficulty in interpreting the singularity will be discussed in section 5 of this essay).

3. Validation of the model using past data

Population data time series has to be fitted to the super-exponential model by determining parameters. This is best done on the log scale because when fitting an exponential curve deviations at different points in time can be compared evenly. The resulting curve has to be checked—the fitted equation has to reproduce the data used for fitting. This can be done via some goodness-of-fit test (for example chi-square, Kolmogorov-Smirnov test, lack-of-fit sum of squares test, etc.) to ensure that the fitted model reasonably matches the shape of past data. Goodness-of-fit tests can be performed on other models and the results can be compared with the super-exponential model (for example exponential vs super exponential). This way it can be ascertained which models are in best correspondence with empirical data.

Then the model can be back-tested. This means fitting the model to a past data set, say data up until 1990, then using that fitted model to make predictions up until the present and then comparing the back fitted model's predictions with the actual outcomes. This gives a very good understanding of the model's predictive power. If the model wouldn't have predicted the past period's population growth, what basis do we have to expect it to predict future population growth?

The parameters of the model can be tested for stability via sensitivity analysis. Sensitivity analysis is a technique for systematically changing the parameters in the model and determining the effects of such changes. This clarifies which parameters the model is most sensitive to and thus allows to counteract possible problems by making these parameters more accurate. The model can also be tested for sensitivity to data sets. This means fitting the model to different ranges of historical data, and examining how much the output varies. This both looks at smoothness of the underlying parameters, how much the model depends on very old data. If the model behaves in an unexpected or illogical way under some parameters or inputs, this can indicate mathematical problems in the model and may lead to reformulation of the model.

The first assumption that can be tested is the assumption of smoothness or non-periodicity (this was assumption number 4, described in the previous section). If empirical population data exhibits cyclicality with the population growth rate changing over time, then the super-exponential model is not the best descriptor of population growth rate (then the model can be augmented as suggested in the previous section). In fact it is a well known trend that currently world population growth is beginning to slow. Johansen and Sornette (2001) fitted population, economic and financial data to the super-exponential model dismiss the deceleration in population growth as a „finite size effect“ that will not prevent the singularity from occuring and that this deceleration can even be interpreted as the transition of humanity to a new state. Their paper uses different quantitative techniques to show that humanity is nearing singularity, which should occur in 2052+-10 years. However I believe that this just one possible interpretation of the empirical data and that there is no conclusive reason to believe that the deceleration in population growth is signifying an approach to a singularity rather than showing a cultural trend that is the result of changing attitudes toward fertility, the emancipation of women, widespread use of contraceptives etc. (if it is the case that the world population is actually transitioning to a different growth regime, then the singularity will certainly not take place). The paper by Johansen and Sornette was a parameter fitting exercise and that *assumed a super-exponential model* and therefore doesn’t validate the model itself. In fact it is not clear if the world population will continue to grow in a super-exponential fashion, what is the nature of the singularity and wheter it will ever be realized for the human population. I will further deal with the interpretation of the singularity in section 5 of this essay.

Another assumption that could be tested is that the Malthusian parameter r, that determines the growth rate, remains constant. This assumption could be tested by fitting the data and determining the value of r and then fitting different time periods and checking if the value of r remains the same.

4. Predicting the future

Even if the super-exponential model fits past data well, it is still questionable how well it will predict the future. Predicting the future with this model assumes the continuation of past trends, which will not necessarily be the case. Understanding of population growth seems to require qualitative understanding of the underlying processes, such as the economic development of different parts of the world, cultural transition, the nature of new scientific and technological advances and their transformative effect on the society, the developments in international politics, the emergence of global pandemics, natural disasters due to global warming, different measures that governments might take to limit or increase fertility (a good example of this is China’s one-child policy) etc. We can draw many examples of such changes in growth rate from history—for example, the demographic transition in Europe during industrialization, when birth rates drastically fell.

Such qualitative changes in the population growth regime cannot be predicted quantitatively. I think that in order to accurately predict the future some form of human expert oppinion should be incorporated. A demography or a technology expert could identify major trend changing events, that would likely change the way that the population grows (such as a new medical discovery, a law passed in order to limit fertility, etc) and the population model could be altered to account for these events. This would make prediction more robust towards unpredictable shocks. The problem with this solution though is that it relies on subjective opinion and lacks mathematical precision.

5. How to interpret the singularity?

The singularity is a material feature of the super-exponential model— it represents the world population growing to infinity in finite time. Singularities most commonly figure in physics of phase transition, for example they describe transformations from ice to water. They can model rupture or failure. Spontaneous singularity is also exhibited by the equations of General Relativity and predict the emergence of black holes (Johansen and Sornette, 2001). Singularity is a mathematical idealization and therefore the question arises—how can this singularity be interpreted, what is it's real world counterpart in the context of population growth?

After the occurence of the singularity human kind changes „state"—the population can no longer be considered as the same population. It is an abrupt and complete qualitative transition after which the system can no longer be observed. This transition has been interpreted as a massive catastrophe that wipes out humanity (Avila and Rekker, 2010), for instance a nuclear war. Johansen and Sornette (2010) describe several scenarios one of which is the slow depletion of Earth's resources leading to the extinction of humanity, another a transition to an ecologically friendly lifestyle (my question for this interpretation is how is such a transition really a singularity—wouldn't the human population still be an observable and quantifiable system when the nature of changes is such?) and the achievement of space travel. I think if humanity faced a fundamental limitation in resources then the catastrophe resulting from that would better be characterized by approaching the limit of the carrying capacity (Malthusian trap). Futurist Ray Kurzweil describes a technological singularity that signifies the creation of machines superior in intelligence to humans (Kurzweil, 2005). Other possible scenarios include somehow uploading all human minds to the internet and engineering humans genetically to live forever (they would become a different species). This is a fundamentally different type of singularity as all people could transition to a different state (become a different species or merge with machines or have a nuclear war or whatever) with finite population size—therefore it isn't associated with the singularity in population size.

My opinion is that we cannot describe the nature of this singularity through mathematics or any other scientific way. For me the singularity for the human population is a theoretical vacuum that could only be the subject of speculation—I could well theorize that at the singularity all people would turn into pink elephants and noone would be able to rigorously disprove me. Singularity is a feature of the model that noone knows the meaning of and I don't think it has any link to the validity of the model before the singular behavior. My suggestion would be to use the super-exponential model where it fits data well or seems like a reasonable prediction (because no matter how steep you make the exponential model it would never grow as fast as the super-exponential model) and treat the singularity as a black box.

6. Summary

In this essay I described the super-exponential model of population growth. This model assumes a positive feedback relationship between human population growth and the creation of carrying capacity through improvement in technology. I identified the assumptions material to this model and analyzed their validity. I proposed several improvements to the model, including the incorporation of different growth regimes into the model (making population switch between exponential, super-exponential and logistic growth) and the incorporation of linear or exponential asymptotes different from the horizontal and vertical asymptotes exhibited by the logistic and exponential model. I described ways in which the model could be validated, the most important of which is back-testing and goodness of fit tests. In the final section I describe the problems associated with interpreting the singularity in the super-exponential model.

Citations

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