Appendix A

Proofs

A.1 Questions

A.1.1. FACT. [Rigidity and Answerhood] Let $M = \langle D, W \rangle$ be a standard model.

(i) $Pt \vartriangleright_M ?xPx \iff t$ is rigid in M

(ii) $!Pt \,_c\vartriangleright_M ?xPx \iff t$ is rigid in M

proof:

$\Leftarrow$ I give first an intuitive idea of why this direction holds. Suppose t is rigid in M. It then follows:

(i) the proposition associated with Pt in M, namely $\{w \mid [t]_{M,w} \in w(P)\}$ corresponds to the union of a non-empty subset of the blocks in the partition determined by $?xPx$, namely $\{w \mid d \in w(P)\}$ where $d = [t]_{M,w}$ for all $w \in W$. Thus Pt constitutes a partial answer to $?xPx$.

(ii) the proposition associated with $!Pt$ in M, $\{w \in W \mid w(P) = \{[t]_{M,w}\}\}$ corresponds to exactly one block in the partition determined by $?xPx$, namely $\{w \in W \mid w(P) = \{d\}\}$ where $d = [t]_{M,w}$ for all $w \in W$. Thus $!Pt$ constitutes a complete answer to $?xPx$.

I give now a more detailed proof of (i) $\Leftarrow$.
Suppose t is rigid in M and let $X = \{\alpha \in [\![?xPx]\!]_M \mid \alpha \subseteq [\![Pt]\!]_M\}$. Clearly:

(a) $X \neq \emptyset$, since, e.g., $\alpha = \{w \mid w(P) = \{[t]_{M,w}\}\} \in X$.

(b) $X \subset [\![?xPx]\!]_M$, since, $X \subseteq [\![?xPx]\!]_M$ by construction and, e.g., $\{w \mid w(P) = \emptyset\} \notin X$ and $\{w \mid w(P) = \emptyset\} \in [\![?xPx]\!]_M$.

(c) $[\![Pt]\!]_M = \cup X$. Indeed,

    (i) $\cup X \subseteq [\![Pt]\!]_M$ holds trivially by definition of X;

    (ii) $[\![Pt]\!]_M \subseteq \cup X$ holds because suppose $w \in [\![Pt]\!]_M$. Since $[\![?xPx]\!]_M$ is a partition of W, there must be a unique $\alpha \in [\![?xPx]\!]_M$ such that $w \in \alpha$. For any $w' \in \alpha$, $w(P) = w'(P)$, since $\alpha \in [\![?xPx]\!]_M$, and, since t is rigid, $w' \in [\![Pt]\!]_M$ as well. Since this holds for any $w' \in \alpha$, then $\alpha \in X$ and, therefore, $w \in \cup X$.

From (a), (b) and (c) we have that

$$\exists X \subset [\![?xPx]\!]_M : [\![Pt]\!]_M = \cup X \neq \emptyset$$

and therefore $Pt \rhd_M ?xPx$.

$\Rightarrow$ Suppose now t is not rigid in M. Let α be the set $\{w \in W \mid w(P) = \{d\}\}$ for some $d \in D$. α is obviously an element of $[\![?xPx]\!]_M$. Consider now the two worlds w and w' such that $w(P) = \{d\}$ and $w(t) = d$, and $w'(P) = \{d\}$ and $w'(t) = d'$ where $d \neq d'$. Since M is standard, and t is not rigid, w and w' will be elements of W and therefore, given the denotation they assign to P, they are elements of α. Obviously $w \in [\![Pt]\!]_M$ and $w \in [\![!Pt]\!]_M$, but $w' \notin [\![Pt]\!]_M$ and $w' \notin [\![!Pt]\!]_M$. This means that the following holds:

$$\exists \alpha \in [\![?xPx]\!]_M : (\alpha \bigcap [\![(!)Pt]\!]_M) \neq \emptyset \ \& \ \alpha \nsubseteq [\![(!)Pt]\!]_M$$

which implies that:

(i) $[\![Pt]\!]_M$ can not be equivalent to the union of a set of blocks in $[\![?xPx]\!]_M$, i.e. for no $X \subseteq [\![?xPx]\!]_M$ the following holds: $[\![Pt]\!]_M = \cup\{X\}$. Thus Pt does not partially answer $?xPx$, $Pt \ntriangleright_M ?xPx$.

(ii) $[\![Pt]\!]_M$ is not an element of $[\![?xPx]\!]_M$, thus it does not completely answer the question, $!Pt \ _c\ntriangleright_M ?xPx$. $\qquad\qquad\square$

A.1.2. Fact. [Rigidity and Triviality]

$$t \text{ is rigid in } M \ \Leftrightarrow \ ?x\ t = x \text{ is trivial in } M$$

proof: This fact holds trivially, $?x\ t = x$ places those worlds in different blocks in which t denotes different individuals; however, if t is rigid and, only in this case, there are no such pairs of worlds, thus the question groups all worlds together.

A.1.3. FACT. [Cardinality] Let M be a model, $\wp$ be a conceptual perspective, g an assignment function and $\alpha \in [\![?\vec{x}\phi]\!]^{\wp}_{M}$ be a block in the partition determined by $?\vec{x}\phi$ in M under $\wp$. Then

$$\forall w, w' \in \alpha : |\lambda\vec{d}\,[\![\phi]\!]^{\wp}_{M,w,g[\vec{x}/\vec{d}]}| = |\lambda\vec{d}\,[\![\phi]\!]^{\wp}_{M,w',g[\vec{x}/\vec{d}]}|$$

proof: Let $f^{\vec{CC}}_{w,w'}$ be a relation between sequences $\vec{d}$ of n individuals figuring in w to sequences $\vec{d}'$ of n individuals figuring in w' such that

$$f^{\vec{CC}}_{w,w'}(\vec{d}, \vec{d}') \;\Leftrightarrow\; \exists \vec{c} \in \prod_{i \in n}(CC_i) : \vec{c}(w) = \vec{d}\,\&\, \vec{c}(w') = \vec{d}'$$

We prove that $f^{\vec{CC}}_{w,w'}$ is a total function and is bijective.

(i) Suppose $f^{\vec{CC}}_{w,w'}(\vec{d}, \vec{d}')$ and $f^{\vec{CC}}_{w,w'}(\vec{d}'', \vec{d}''')$. We show that

$$\vec{d} = \vec{d}'' \;\Leftrightarrow\; \vec{d}' = \vec{d}'''$$

By definition of $f^{\vec{CC}}_{w,w'}$ we have that

$$\exists \vec{c}, \vec{c'} \in \prod_{i \in n}(CC_i) : \vec{c}(w) = \vec{d}\,\&\, \vec{c}(w') = \vec{d}'\,\&\, \vec{c'}(w) = \vec{d}''\,\&\, \vec{c'}(w') = \vec{d}'''$$

$\Rightarrow$ Suppose $d = d''$. By the uniqueness condition on conceptual covers since $\vec{c}(w) = \vec{c'}(w)$ it follows that $\vec{c} = \vec{c'}$ which implies that $\vec{c}(w') = \vec{d}'$ is the same sequence as $\vec{c'}(w') = \vec{d}'''$.

$\Leftarrow$ As above.

We have proved that $f^{\vec{CC}}_{w,w'}$ is a function ($\Rightarrow$) and is injective ($\Leftarrow$).

(ii) By the existence condition on conceptual covers, $\forall \vec{d} \in D^n : \forall w \in W : \exists \vec{c} \in \prod_{i \in n}(CC_i) : \vec{c}(w) = \vec{d}$. Since the concepts in $\vec{c}$ are total functions, it holds that $\forall w' \in W : \exists \vec{d}' \in D^n : \vec{c}(w') = \vec{d}'$. Thus the following holds for all w and w':

$$\forall \vec{d} \in D^n : \exists \vec{c} \in \prod_{i \in n}(CC_i) : \exists \vec{d}' \in D^n : \vec{c}(w) = \vec{d}\,\&\, \vec{c}(w') = \vec{d}'$$

Therefore by definition of $f^{\vec{CC}}_{w,w'}$ it holds that

(a) $\forall \vec{d} \in D^n : \exists \vec{d}' \in D^n : f^{\vec{CC}}_{w,w'}(\vec{d}, \vec{d}')$;

(b) $\forall \vec{d}' \in D^n : \exists \vec{d} \in D^n : f^{\vec{CC}}_{w,w'}(\vec{d}, \vec{d}')$.

We have proved that $f_{w,w'}^{\vec{CC}}$ is total ($dom(f_{w,w'}^{\vec{CC}}) = D^n$ from (a)), and surjective ($range(f_{w,w'}^{\vec{CC}}) = D^n$ from (b)).

I will write $f_{w,w'}^{\vec{CC}}(\vec{d}) = \vec{d'}$ for $f_{w,w'}^{\vec{CC}}(\vec{d}, \vec{d'})$. Now, since $f_{w,w'}^{\vec{CC}}$ is a bijection, in order to prove proposition A.1.3, it is enough to prove the following lemma:

A.1.4. LEMMA. Let $M = \langle D, W \rangle$ be a model, $\wp$ be a conceptual perspective, g an assignment function and $\alpha \in [\![?\vec{x}\phi]\!]_M^\wp$ be a block in the partition determined by $?\vec{x}\phi$ in M under $\wp$. Then

$$\forall w, w' \in \alpha : \forall \vec{d} \in D^n : [\![\phi]\!]_{M,w,g[\vec{x}/\vec{d}]}^\wp = 1 \quad \Leftrightarrow \quad [\![\phi]\!]_{M,w',g[\vec{x}/f_{w,w'}^{\vec{CC}}(\vec{d})]}^\wp = 1$$

proof: By definition of $f_{w,w'}^{\wp(\vec{x})}$, the following holds:

$$\forall \vec{d} \in D^n : \exists \vec{c} \in \prod_{i \in n}(\wp(\vec{x_i})) : \ \vec{c}(w) = \vec{d} \ \& \ \vec{c}(w') = f_{w,w'}^{\wp(\vec{x})}(\vec{d})$$

and this sequence $\vec{c}$ is clearly unique. Therefore $[\![\phi]\!]_{M,w,g[\vec{x}/\vec{d}]}^\wp = 1$ holds iff (i) holds:

(i) $[\![\phi]\!]_{M,w,g[\vec{x}/\vec{c}(w)]}^\wp = 1$

Since w and w' belong to the same block α in the partition determined by $?\vec{x}\phi$, by the semantics of interrogatives, (i) is the case iff the following is the case:

(ii) $[\![\phi]\!]_{M,w',g[\vec{x}/\vec{c}(w')]}^\wp = 1$

and (ii) clearly holds iff $[\![\phi]\!]_{M,w',g[\vec{x}/f_{w,w'}^{\vec{CC}}(\vec{d})]}^\wp = 1$. □

A.1.5. FACT. [K-Answerhood and Rigidity] Let $M = \langle D, W \rangle$ be a standard model.

$$Pt \ _K\rhd_M ?xPx \quad \Leftrightarrow \quad t \text{ is rigid in } M$$

proof: The proof follows trivially from the Karttunen semantics of constituent questions which, as is easily seen, assigns the following set of propositions as denotation to our question:

$$_K[\![?xPx]\!]_{M,w,g} = \{\{v \mid d \in v(P)\} \mid d \in D \ \& \ d \in w(P)\}$$

$\Leftarrow$ Suppose t is rigid in M. This means that $[\![Pt]\!]_M = \{v \mid d \in v(P)\}$ for $d = [t]_M$. By the Karttunen analysis of constituent questions, this means $[\![Pt]\!]_M \in {}_K[\![?xPx]\!]_{M,w,g}$ for all $w \in W$ and g such that $[\![Pt]\!]_{M,w,g} = 1$. Since M is standard (and Pt is consistent) there is such a world w (and assignment g). Thus $\exists w \in W$,

$\exists g$: $[\![Pt]\!]_M \in {}_K[\![?xPx]\!]_{M,w,g}$, which means by the definition of singular positive answers $Pt \ {}_K\!\rhd_M ?xPx$.

$\Rightarrow$ Suppose t is not rigid in M. Then since M is standard, there is no $d \in D$ such that $[\![Pt]\!]_M = \{v \mid d \in v(P)\}$. So for no world w (and no assignment g), $[\![Pt]\!]_M$ is in the K-denotation of $?xPx$ in w (with respect to g), so the sentence does not constitute a singular positive answer to the question. □

A.2 Belief

Comparison with MPL

A.2.1. PROPOSITION. Let ϕ be a wff in $\mathcal{L}_{CC}$.

$$\models_{CCC} \phi \quad \text{iff} \quad \models_{MPL} \phi$$

proof: To prove this proposition I first show that given a classical CC-model M, we can define an equivalent ordinary modal predicate logic model M', that is, an MPL-model that satisfies the same wffs as M. Let M be $\langle W, R, D, I, \{CC\} \rangle$. We define an equivalent model $M' = \langle W', R', D', I' \rangle$ as follows. $W' = W$, $R' = R$, $D' = CC$. For I' we proceed as follows.

(i) $\forall \langle c_1, ..., c_n \rangle \in CC^n$, $w \in W$, $P \in \mathcal{P}$:

$$\langle c_1, ..., c_n \rangle \in I'(P)(w) \quad \text{iff} \quad \langle c_1(w), ..., c_n(w) \rangle \in I(P)(w)$$

(ii) $\forall c \in CC$, $w \in W$, $a \in \mathcal{C}$:

$$I'(a)(w) = c \quad \text{iff} \quad I(a)(w) = c(w)$$

Clause (ii) is well-defined because the uniqueness condition on covers guarantees that there is a unique $c \in CC$ such that $I(a)(w) = c(w)$.

In our construction we take the elements of the conceptual cover in the old model to be the individuals in the new model, and we stipulate that they do, in all w, what their instantiations in w do in the old model. Clause (i) says that a sequence of individuals is in the denotation of a relation P in w in the new model iff the sequence of their instantiations in w is in P in w in the old model. In order for clause (ii) to be well-defined, it is essential that CC is a conceptual cover, rather than an arbitrary set of concepts. In M', an individual constant a will denote in w the unique c in CC such that $I(a)(w) = c(w)$. That there is such a unique c is guaranteed by the uniqueness condition on conceptual covers. We

have to prove that this construction works. I will use g, g' for assignments within M and h, h' for assignments within M'. Note that for all assignments g within M: $g(n) = CC$ for all CC-indices n, since CC is the unique cover available in M. I will say that g corresponds with h iff $g = h \cup \{\langle n, CC \rangle \mid n \in N\}$. This means that the two assignments assign the same values to all individual variables x_n for all n, and g assigns the cover CC to all CC-indices n.

A.2.2. THEOREM. Let g and h be any corresponding assignments. Let w be any world in W and ϕ any wff in $\mathcal{L}_{CC}$. Then

$$M, w, g \models_{CC} \phi \text{ iff } M', w, h \models_{MPL} \phi$$

proof: The proof is by induction on the construction of ϕ. We start by showing that the following holds for all terms t:

(A) $[t]_{M,w,g} = [t]_{M',w,h}(w)$

Suppose t is a variable. Then $[t]_{M,w,g} = g(t)(w)$. By definition of corresponding assignments, $g(t)(w) = h(t)(w)$. Since $[t]_{M',w,h} = h(t)$, this means that $[t]_{M,w,g} = [t]_{M',w,h}(w)$. Suppose now t is a constant. Then $[t]_{M,w,g} = I(t)(w)$. By the existence and uniqueness conditions on conceptual covers, there is a unique $c \in CC$, such that $I(t)(w) = c(w)$. By clause (ii) of the definition of I': $I'(t)(w) = c$. Since $[t]_{M',w,h} = I'(t)(w)$, this means $[t]_{M,w,g} = c(w) = [t]_{M',w,h}(w)$. We can now prove the lemma for atomic formulae.

Suppose ϕ is $Pt_1, ..., t_n$. Now $M, w, g \models_{CC} Pt_1, ..., t_n$ holds iff (a) holds:

(a) $\langle [t_1]_{M,w,g}, ..., [t_n]_{M,w,g} \rangle \in I(P)(w)$

By (A), (a) holds iff (b) holds:

(b) $\langle [t_1]_{M',w,h}(w), ..., [t_n]_{M',w,h}(w) \rangle \in I(P)(w)$

which, by definition of I', is the case iff (c) holds:

(c) $\langle [t_1]_{M',w,h}, ..., [t_n]_{M',w,h} \rangle \in I'(P)(w)$

which means that $M', w, h \models_{MPL} Pt_1, ..., t_n$.

Suppose now ϕ is $t_1 = t_2$. $M, w, g \models_{CC} t_1 = t_2$ holds iff (d) holds:

(d) $[t_1]_{M,w,g} = [t_2]_{M,w,g}$

By (A) above, (d) holds iff (e) holds:

(e) $[t_1]_{M',w,h}(w) = [t_2]_{M',w,h}(w)$

which, by the uniqueness condition on conceptual covers, is the case iff (f) holds:

(f) $[t_1]_{M',w,h} = [t_2]_{M',w,h}$

which means that $M', w, h \models_{MPL} t_1 = t_2$.

The lemma is proved for atomic cases. The induction for $\neg$, $\exists$, $\wedge$ and $\Box$ is immediate. $\qquad\square$

Now it is clear that if a classical CC-model M and an ordinary MPL-model M' correspond in the way described then the theorem entails that any wff in $\mathcal{L}_{CC}$ is CC-valid in M iff it is MPL-valid in M'. Thus, given a classical CC-model, we can define an equivalent MPL-model, but also given an MPL-model, we can define an equivalent classical CC-model $\langle W, R, D, \{CC\}, I\rangle$ by taking CC to be the rigid cover. This suffices to prove proposition A.2.1. $\qquad\square$

Axiomatization

Recall our definition of a CC-theorem. CC consists of the following axiom schemata:

Basic propositional modal system

PC All propositional tautologies.

K $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

Quantifiers Recall that $\phi[t]$ and $\phi[t']$ differ only in that the former contains the term t in one or more places where the latter contains t'.

EGa $\phi[t] \rightarrow \exists x_n \phi[x_n]$ (if ϕ is atomic)

EGn $\phi[y_n] \rightarrow \exists x_n \phi[x_n]$

BFn $\forall x_n \Box\phi \rightarrow \Box\forall x_n \phi$

Identity

ID $t = t$

SIa $t = t' \rightarrow (\phi[t] \rightarrow \phi[t'])$ (if ϕ is atomic)

SIn $x_n = y_n \rightarrow (\phi[x_n] \rightarrow \phi[y_n])$

LNIn $x_n \neq y_n \rightarrow \Box\, x_n \neq y_n$

Let AX_{CC} be the set of axioms of CC. The set of CC-theorems T_{CC} is the smallest set such that:

AX $AX_{CC} \subseteq T_{CC}$

MP If ϕ and $\phi \to \psi \in T_{CC}$, then $\psi \in T_{CC}$

∃I If $\phi \to \psi \in T_{CC}$ and x_n not free in ψ, then $(\exists x_n \phi) \to \psi \in T_{CC}$

N If $\phi \in T_{CC}$, then $\Box \phi \in T_{CC}$

I use the standard notation and write $\vdash_{CC} \phi$ for $\phi \in T_{CC}$.

I list now a number of theorems and rules that will be used later on, I omit the derivations which are standard.

First of all, by **∃I** and **EGn**, we can derive all standard predicate logic theorems and rules governing the behaviour of quantifiers which do not involve any shift of index, among others the rule of introduction of the universal quantifier **∀I**, and the principle of renaming of bound variables **PRn**, and the two principles **PL1n** and **PL2n**.

∀I $\vdash_{CC} \phi \to \psi \;\; \Rightarrow \;\; \vdash_{CC} \phi \to \forall x_n \psi$ (provided x_n is not free in ϕ)

PRn $\vdash_{CC} \exists x_n \phi \leftrightarrow \exists y_n \phi[x_n/y_n]$

PL1n $\vdash_{CC} \forall x_n(\phi \to \psi) \to (\phi \to \forall x_n \psi)$ (provided x_n not free in ϕ)

PL2n $\vdash_{CC} \exists z_n(\exists x_n \phi \to \phi[x_n/z_n])$ (provided z_n not free in $\exists x_n \phi$)

Furthermore, we can prove the following two identity theorems. The derivation of **ID'** uses **ID** and **SIa**, whereas the derivation of **LIn** uses **ID**, **N** and **SIn**.

LIn $\vdash_{CC} x_n = y_n \to \Box \, x_n = y_n$

ID' $\vdash_{CC} t = t' \to t' = t$

Recall our definition of CC-validity.

A.2.3. DEFINITION. [CC-Validity] Let ϕ be in $\mathcal{L}_{CC}$. Then

$$\models_{CC} \phi \;\; \text{iff} \; M \models_{CC} \phi \;\; \text{for all CC-models } M$$

Soundness

A.2.4. THEOREM. [Soundness] If $\vdash_{CC} \phi$, then $\models_{CC} \phi$.

proof: The proof that **MP**, $\exists$**I** and **N** preserve validity is standard. The validity of **PC** and **K** is obvious. The validity of **EGa** and **SIa** follows from proposition 2.4.9. **LNIn** is valid by proposition 2.4.8. The validity of **SIn** may be established by induction on the construction of ϕ. We show that **EGn** and **BFn** are valid.

EGn Suppose $M, w \models_g \phi[y_n]$. This implies $M, w \models_{g[x_n/g(y_n)]} \phi[x_n]$. Since $g(y_n) \in g(n)$, $M, w \models_g \exists x_n \phi[x_n]$.

BFn Suppose $M, w \models_g \forall x_n \Box \phi$. Let h be any x_n-alternative of g, $g[x_n]h$, i.e. let h be such that: $\forall v \in (N \cup \mathcal{V}_N)$: $v \neq x_n \Rightarrow g(v) = h(v)$ and $h(x_n) \in h(n)$. Let wRw'. Then $M, w \models_h \Box \phi$, and hence $M, w' \models_h \phi$. Since this holds for every x_n-alternative h of g, we have $M, w' \models_g \forall x_n \phi$; and since this holds for all w' such that wRw', we finally have $M, w \models_g \Box \forall x_n \phi$. $\qquad\Box$

Completeness

We show that for any ϕ which is not a theorem in CC we can define a CC-model in which ϕ is not valid. The technique we will use in the construction of these models varies only slightly from the standard technique used for modal predicate logic with identity (see in particular Hughes and Cresswell (1996)). The worlds of these models will be maximal CC-consistent sets of wffs which have the *witness property*, that is, for all indices n, for all wffs of the form $\exists x_n \phi$, there is a n-indexed variable y_n such that $\exists x_n \phi \rightarrow \phi[x_n/y_n] \in w$. In order to obtain this we will follow the common practice and consider an expanded language $\mathcal{L}^+$, which is $\mathcal{L}_{CC}$ with the addition of a denumerable set of fresh variables.

We assume the standard results about maximal consistent sets of wffs with respect to a system S.

A.2.5. THEOREM. [Lindenbaum's Theorem] Any S-consistent set of wffs Δ can be enlarged to a maximal S-consistent set of wffs Γ.

A.2.6. THEOREM. Suppose Γ is a maximal consistent set of wffs with respect to S. Then

1. for each wff ϕ, exactly one member of $\{\phi, \neg\phi\}$ belongs to Γ.

2. for each pair of wffs ϕ and ψ, $\phi \wedge \psi \in \Gamma$ iff $\phi \in \Gamma$ and $\psi \in \Gamma$.

3. if $\vdash_S \phi$, then $\phi \in \Gamma$.

4. if $\phi \in \Gamma$ and $\vdash_S \phi \rightarrow \psi$, then $\psi \in \Gamma$.

The basic theorem about the witness property is the following:

A.2.7. THEOREM. Let Λ be a consistent set of wffs of $\mathcal{L}_{CC}$. Let $\mathcal{V}^+$ be a denumerable set of new variable symbols and let $\mathcal{L}^+$ be the simple expansion of $\mathcal{L}_{CC}$ formed by adding $\mathcal{V}^+$. Then there is a consistent set Δ of wffs of $\mathcal{L}^+$ with the witness property such that $\Lambda \subseteq \Delta$.

proof: I follow the standard proof (see Hughes and Cresswell (1996), pp. 258–259), the only difference is that we have not just one sort of variables, but many. We assume that all wffs of the form $\exists x_n \phi$ for any wff ϕ of $\mathcal{L}^+$ and any index n, and any variable x_n for any index n are enumerated so that we can speak of the first, the second and so on. We define a sequence of sets Δ_0, Δ_1, ... etc. as follows:

$$\Delta_0 = \Lambda$$

$$\Delta_{m+1} = \Delta_m \cup \{\exists x_n \phi \rightarrow \phi[x_n/y_n]\}$$

where $\exists x_n \phi$ is the $m + 1$th wff in the enumeration of wff of that form and y_n is the first variable indexed with n not in Δ_m or in ϕ. Since Δ_0 is in $\mathcal{L}$ and Δ_m has been formed from it by the addition of only m wffs there will be infinitely many n-indexed variables from $\mathcal{V}^+{}_n$ to provide such a y_n.

Δ_0 is assumed to be consistent so we shall show that Δ_{m+1} is if Δ_m is. Suppose not. Then there will be $\psi_1 \wedge ... \wedge \psi_m$ in Δ_m such that:

(i) $\vdash_{CC} (\psi_1 \wedge ... \wedge \psi_m) \rightarrow \exists x_n \phi$

(ii) $\vdash_{CC} (\psi_1 \wedge ... \wedge \psi_m) \rightarrow \neg\phi[x_n/y_n]$

Since y_n does not occur in Δ_m, it is not free in $(\psi_1 \wedge ... \wedge \psi_m)$ and so from (ii) by $\forall\mathbf{I}$ we have:

(iii) $\vdash_{CC} (\psi_1 \wedge ... \wedge \psi_m) \rightarrow \forall y_n \neg\phi[x_n/y_n]$

which can be rewritten as:

(iv) $\vdash_{CC} (\psi_1 \wedge ... \wedge \psi_m) \rightarrow \neg\exists y_n \phi[x_n/y_n]$

Now since y_n did not occur in ϕ, $\exists y_n \phi[x_n/y_n]$ is a bound alphabetic variant of $\exists x_n \phi$, and so by **PRn**:

(v) $\vdash_{CC} (\psi_1 \wedge ... \wedge \psi_m) \rightarrow \neg\exists x_n \phi$

But (i) and (v) give

(vi) $\vdash_{CC} \neg(\psi_1 \wedge ... \wedge \psi_m)$

which contradicts the consistency of Δ_m. Let Δ be the union of all the Δ_ms. It is easy to see that Δ is consistent and has the witness property. $\qquad\square$

Once a set Δ has the witness property each extension of Δ in the same language also has the witness property. Lindenbaum's theorem guarantees that if Δ is consistent there is a maximal consistent set Γ such that $\Delta \subseteq \Gamma$, and so since Δ has the witness property, Γ does too. The standard result we can prove about maximal consistent sets with the witness property in modal logic is the following:

A.2.8. Theorem. If Γ is a maximal consistent set of wffs in some language (say $\mathcal{L}^+$) of modal predicate logic, and Γ has the witness property, and α is a wff such that $\Box\alpha \notin \Gamma$, then there is a consistent set Δ of wff of $\mathcal{L}^+$ with the witness property such that $\{\psi \mid \Box\psi \in \Gamma\} \bigcup \{\neg\alpha\} \subseteq \Delta$.

proof: Again we can use the standard construction (see Hughes and Cresswell (1996), pp. 259–261). Again we assume that all wffs of the form $\exists x_n \phi$ for any wff ϕ of $\mathcal{L}^+$ and any index n, and any variable x_n for any index n are enumerated so that we can speak of the first, the second and so on. We define a sequence of wffs γ_0, γ_1, γ_2, ... etc. as follows:

γ_0 is $\neg\alpha$

γ_{m+1} is $\gamma_m \wedge (\exists x_n \phi \to \phi[x_n/y_n])$

where $\exists x_n \phi$ is the $m+1$th wff in the enumeration of that form and y_n is the first variable indexed by n such that:

(*) $\{\psi \mid \Box\psi \in \Gamma\} \cup \{\gamma_m \wedge (\exists x_n \phi \to \phi[x_n/y_n])\}$ is consistent.

In order for this construction to succeed we have to be sure that there always will be a n-indexed variable y_n satisfying (*).

Since γ_0 is $\neg\alpha$, $\{\psi \mid \Box\psi \in \Gamma\} \cup \{\gamma_0\}$ is consistent from a standard result of propositional modal logic. We show that provided $\{\psi \mid \Box\psi \in \Gamma\} \cup \{\gamma_m\}$ is consistent, there always will be a n-indexed variable y_n satisfying (*).

Suppose there were not. Then for every variable y_n in $\mathcal{V}_n^+$, there will exist some $\{\psi_1, ..., \psi_k\} \subseteq \{\psi \mid \Box\psi \in \Gamma\}$ such that

$\vdash_{CC} (\psi_1 \wedge ... \wedge \psi_k) \to (\gamma_m \to \neg(\exists x_n \phi \to \phi[x_n/y_n]))$

so, by propositional modal logic (**N**, **K** and $\Box$-distribution):

$\vdash_{CC} (\Box\psi_1 \wedge ... \wedge \Box\psi_k) \to \Box(\gamma_m \to \neg(\exists x_n \phi \to \phi[x_n/y_n]))$

But Γ is maximal consistent and $\Box\psi_1, ..., \Box\psi_k \in \Gamma$, and so

(i) $\Box(\gamma_m \to \neg(\exists x_n \phi \to \phi[x_n/y_n])) \in \Gamma$

Let z_n be some n-indexed variable not occurring in ϕ or in γ_m. Since Γ has the witness property, then we have for some n-witness y_n:

$$\exists z_n(\neg\Box(\gamma_m \to \neg(\exists x_n\phi \to \phi[x_n/z_n]))) \to \neg\Box(\gamma_m \to \neg(\exists x_n\phi \to \phi[x_n/y_n])) \in \Gamma$$

Since (i) holds for all $y_n \in \mathcal{V}_n^+$, we then have that:

$$\neg\exists z_n\neg\Box(\gamma_m \to \neg(\exists x_n\phi \to \phi[x_n/z_n])) \in \Gamma$$

But Γ is CC-maximal consistent and hence by **BFn** we have:

$$\Box\forall z_n(\gamma_m \to \neg(\exists x_n\phi \to \phi[x_n/z_n])) \in \Gamma$$

Since z_n does not occur γ_m, then by **PL1n** we have:

$$\Box(\gamma_m \to \neg\exists z_n(\exists x_n\phi \to \phi[x_n/z_n])) \in \Gamma$$

But since z_n does not occur in ϕ by **PL2n**, we have

$$\vdash_{CC} \exists z_n(\exists x_n\phi \to \phi[x_n/z_n])$$

but then $\Box\neg\gamma_m \in \Gamma$ and so $\neg\gamma_m \in \{\psi \mid \Box\psi \in \Gamma\}$ which would make $\{\psi \mid \Box\psi \in \Gamma\} \cup \{\gamma_m\}$ inconsistent against our assumption.

Let Δ be the union of $\{\psi \mid \Box\psi \in \Gamma\}$ and all the γ_ms. Since each $\{\psi \mid \Box\psi \in \Gamma\} \cup \{\gamma_m\}$ is consistent, and since $\vdash_{CC} \gamma_m \to \gamma_k$ for $m \geq k$, so is their union Δ. Any maximal consistent extension of Δ has all the required properties and so the theorem is proved. $\Box$

I will now show that for each CC-consistent set Δ of wffs of $\mathcal{L}$, we can construct a model M_Δ containing a world in which all the wffs in Δ are true. These models M_Δ are based on *cohesive* sub-frames of the frame $F = \langle W_F, R_F \rangle$ where:

(a) W_F is the set of CC-maximal consistent sets of wffs of $\mathcal{L}^+$ which have the witness property.

(b) wR_Fw' iff for every wff $\Box\phi$ of $\mathcal{L}^+$, if $\Box\phi \in w$, then $\phi \in w'$.

Cohesive frames are frame in which each two worlds are linked by means of some forwards or backwards R-chain. The reason why we need to consider cohesive models is that in a cohesive model for CC for each index n any world verifies exactly the same identity formulas between variables in $\mathcal{V}_n$.

Let Δ be a CC-consistent set of wffs of $\mathcal{L}_{CC}$. We show how to construct $M_\Delta = \langle W, R, D, I, C \rangle$ in which there is a world w^* such that $\Delta \subseteq w^*$. The construction of W, R, D, I are standard. C will be some extra work.

W Given some world $w^* \in W_F$ such that $\Delta \subseteq w^*$, we let W be the set of all and only those worlds in W_F which are reachable from w^* by a chain of forwards R_F-steps. These worlds are maximal consistent sets of wffs of $\mathcal{L}^+$, which satisfy the witness property.

R R is R_F restricted to W.

D Let $\sim$ be the following relation over the set $\mathcal{V}_0^+$ of variables of $\mathcal{L}^+$ with index 0.

$$v_0 \sim x_0 \text{ iff } v_0 = x_0 \in w$$

Since W is cohesive (every w, w' in W are linked by some R-chain) and every w contains **LIn** and **LNIn**, it makes no difference which w is selected for this purpose. We can prove the following lemma:

A.2.9. LEMMA. $\forall w, w' \in W : \forall n \in N : \forall x, y \in \mathcal{V}^+ : \ x_n = y_n \in w$ iff $x_n = y_n \in w'$

proof: Consider any two worlds w, w' such that wRw' or $w'Rw$. Suppose $x_n = y_n \in w$. If wRw', then by **LIn**, $\Box x_n = y_n \in w$ and so $x_n = y_n \in w'$. If $w'Rw$, then if $x_n = y_n \notin w'$, then $x_n \neq y_n \in w'$ and so by **LNIn** $\Box x_n \neq y_n \in w'$ and so $x_n = y_n \notin w$, contradicting the assumption. Now since our M_Δ is cohesive, then any two worlds w and w' in W are linked by a chain of backwards or forwards R-steps, and so if $x_n = y_n \in w$ then $x_n = y_n \in w'$, and if $x_n = y_n \in w'$ then $x_n = y_n \in w$. $\qquad\square$

It is easy to see that $\sim$ is an equivalence relation (we use **ID** and **SIa** and the maximal consistency of each w). Now for each $x_0 \in \mathcal{V}_0^+$, let

$$[x_0] = \{y_0 \in \mathcal{V}_0^+ \mid x_0 \sim y_0\}$$

be the equivalence class of x_0. We take the domain D to be the set of all these equivalence classes $[x_0]$, for $x_0 \in \mathcal{V}_0^+$ and so define $D = \{[x_0] \mid x_0 \in \mathcal{V}_0^+\}$.

I We now define the interpretation function I for the predicate and individual constant symbols of the language.

(i) For each n-placed relation symbol P in $\mathcal{L}_{CC}$, for each $w \in W$ we define the interpretation $I(P)(w)$ of the symbol P in w as follows: $\forall \langle [x_{0_1}], ..., [x_{0_n}] \rangle \in D^n$,

$$\langle [x_{0_1}], ..., [x_{0_n}] \rangle \in I(P)(w) \text{ iff } P x_{0_1}, ..., x_{0_n} \in w$$

The definition is independent of the representatives of the equivalence classes $[x_{0_1}], ..., [x_{0_n}]$, because by **SIa** (or **SIn**) we have

$$\vdash_{CC} P x_{0_1}, ..., x_{0_n} \wedge x_{0_1} = y_{0_1} \wedge ... \wedge x_{0_n} = y_{0_n} \rightarrow P y_{0_1}, ..., y_{0_n}$$

(ii) Let a be a constant symbol of $\mathcal{L}$ and $w \in W$. From **ID** and **EGa**, we have

$$\vdash_{CC} \exists x_0(a = x_0)$$

So $\exists x_0(a = x_0) \in w$, and because w has the witness property, there is a 0-indexed variable y_0 such that

$$a = y_0 \in w$$

y_0 may not be unique, but its equivalence class $[y_0]$ is unique because, using **SIa** we have:

$$\vdash_{CC} a = y_0 \wedge a = z_0 \rightarrow y_0 = z_0$$

The interpretation $I(a)(w)$ in w is this (uniquely determined) element $[y_0]$ of D.

$$I(a)(w) = [y_0] \quad \text{iff} \quad a = y_0 \in w$$

C Let c_{v_n} be the function from W to D such that for all $w \in W$:

$$c_{v_n}(w) = [y_0] \quad \text{iff} \quad v_n = y_0 \in w$$

The proof that for all w there is such a unique element $[y_0] \in D$ is parallel to the one in the second clause of the definition of I. By **ID** and **EGa**, we have $\vdash_{CC} \exists x_0(v_n = x_0)$ and so $\exists x_0(v_n = x_0) \in w$ for all w, and, therefore, by the witness property of w, $v_n = y_0 \in w$ for some 0-witness y_0. y_0 may not be unique, but its equivalence class $[y_0]$ is, because, by using **SIa**, we have: $\models_{CC} v_n = y_0 \wedge v_n = z_0 \rightarrow z_0 = y_0$.

We let now $CC_n = \{c_{v_n} \mid v_n \in \mathcal{V}_n^+\}$ and we define $C = \{CC_n \mid n \in N\}$.

We have to show that these sets CC_n are conceptual covers.

(i) Existence Condition: $\forall w \in W : \forall [x_0] \in D : \exists c_{v_n} \in CC_n : c_{v_n}(w) = [x_0]$.
   proof: Take any w and $[x_0]$. By **ID** and **EGa**, we have $\exists y_n(y_n = x_0) \in w$ and because w has the witness property we know that $v_n = x_0 \in w$ for some n-witness v_n. Consider now c_{v_n} which is in CC_n. By definition $c_{v_n}(w) = [x_0]$.

(ii) Uniqueness Condition: $\forall w \in W : \forall c_{v_n} c_{z_n} \in CC_n : c_{v_n}(w) = c_{z_n}(w) \Rightarrow c_{v_n} = c_{z_n}$
   proof: Note firstly that for all $w' \in W$ the following holds:

(A) $c_{x_n}(w') = c_{y_n}(w') \Leftrightarrow x_n = y_n \in w'$

($\Rightarrow$) by definition of c_{x_n} and c_{y_n}, and **SIa**; ($\Leftarrow$) suppose $x_n = y_n \in w'$ by **ID**, **EGa**, **SIa**, and witness property of w', we have for some z_0 and v_0, $x_n = z_0 \in w'$, $y_n = v_0 \in w'$ and $z_0 = v_0 \in w'$, which by definition of c_{x_n} and c_{y_n} means $c_{x_n}(w') = c_{y_n}(w')$.

Suppose now $c_{v_n}(w) = c_{z_n}(w)$ for some $c_{v_n}, c_{z_n} \in CC_n$, and w. By (A) this implies $v_n = z_n \in w$, which, by lemma A.2.9, implies that for any $w' \in W$, $v_n = z_n \in w'$ and so, again by (A), $c_{v_n}(w') = c_{z_n}(w')$. Since this holds for all $w' \in W$, we have $c_{v_n} = c_{z_n}$.

We define the *canonical assignment* g as follows: $\forall n \in N, \forall x_n \in \mathcal{V}_n^+$: $g(n) = CC_n$ and $g(x_n) = c_{x_n}$. We can now prove the following theorem:

A.2.10. THEOREM. For any $w \in W$, and any wff $\phi \in \mathcal{L}^+$,

$$M_\Delta, w \models_g \phi \quad \text{iff} \quad \phi \in w$$

proof: the proof is by induction on the construction of ϕ. I start by showing that (B) holds for all t in $\mathcal{L}^+$:

(B) $[t]_{M_\Delta, w, g} = [x_0]$ iff $t = x_0 \in w$

Suppose t is an indexed variable v_n in $\mathcal{V}_N^+$. Then $[t]_{M_\Delta, w, g} = g(v_n)(w)$. By definition of canonical assignment $g(v_n)(w) = c_{v_n}(w)$. By definition of c_{v_n}, $c_{v_n}(w) = [x_0]$ iff $v_n = x_0 \in w$.

Suppose t is a constant symbol a in $\mathcal{L}$. Then $[t]_{M_\Delta, w, g} = I(a)(w)$. By clause (ii) in the definition of I, $I(a)(w) = [x_0]$ iff $a = x_0 \in w$. We can now prove the theorem for atomic formulae.

(a) Consider $Rt_1, ..., t_n$. Let $w \in W$. By (B), the denotation of the terms $t_1, ..., t_n$ in w will be some $[x_{0_1}], ..., [x_{0_n}]$ where $t_1 = x_{0_1}, ..., t_n = x_{0_n} \in w$. Then

$$M_\Delta, w \models_g Rt_1, ..., t_n \Leftrightarrow \langle [x_{0_1}], ..., [x_{0_n}] \rangle \in I(R)(w) \Leftrightarrow Rx_{0_1}, ..., x_{0_n} \in w$$

But $t_1 = x_{0_1}, ..., t_n = x_{0_n} \in w$, thus by various application of **SIa** we have that

$$Rx_{0_1}, ..., x_{0_n} \leftrightarrow Rt_1, ..., t_n \in w$$

and so $Rx_{0_1}, ..., x_{0_n} \in w \Leftrightarrow Rt_1, ..., t_n \in w$.

(b) $M_\Delta, w \models_g t_1 = t_2$ iff $[t_1]_{M_\Delta, w, g} = [t_1]_{M_\Delta, w, g}$. By (B) above this is the case iff $[x_{0_1}] = [x_{0_2}]$ for some $[x_{0_1}]$ and $[x_{0_2}]$ such that $t_1 = x_{0_1} \in w$ and $t_2 = x_{0_2} \in w$. Obviously $x_{0_1} = x_{0_2} \in w$, and therefore by various applications of **SIa** we have that $t_1 = t_2 \in w$.

(c) $M_\Delta, w \models_g \neg\phi$ iff $M_\Delta, w \not\models_g \phi$ iff $\phi \notin w$ iff $\neg\phi \in w$.

(d) $M_\Delta, w \models_g \phi \wedge \psi$ iff $M_\Delta, w \models_g \phi$ and $M_\Delta, w \models_g \psi$ iff $\phi \in w$ and $\psi \in w$ iff $\phi \wedge \psi \in w$.

(e) Suppose $\exists x_n \phi \in w$. By the witness property, for some y_n, we have $\phi[x_n/y_n] \in w$. But then by induction hypothesis $M_\Delta, w \models_g \phi[x_n/y_n]$ which, by standard principle of replacement, implies $M_\Delta, w \models_{g[x_n/g(y_n)]} \phi$. Since $g(y_n) = c_{y_n}$ is an element of $g(n)$ this implies $M_\Delta, w \models_g \exists x_n \phi$.

Suppose $M_\Delta, w \models_g \exists x_n \phi$. Then $M_\Delta, w \models_{g[x_n/c_{v_n}]} \phi$ for some $c_{v_n} \in g(n)$. Since by definition of canonical assignment $g(v_n) = c_{v_n}$, by standard principle of replacement, we have $M_\Delta, w \models_g \phi[x_n/v_n]$. But then, by induction hypothesis, $\phi[x_n/v_n] \in w$ and by **EGn** $\exists x_n \phi \in w$.

(f) Suppose $\Box \phi \in w$ and wRw'. Then $\phi \in w'$ and so $M_\Delta, w' \models_g \phi$. Since this holds for all w' such that wRw', we have $M_\Delta, w \models_g \Box \phi$.

Suppose $\Box \phi \notin w$. Then $\neg \Box \phi \in w$. But then by theorem A.2.8 (in combination with A.2.5) we know that there is some $w' \in W_F$ such that $wR_F w'$ and $\phi \notin w'$. w' is clearly in W as well since it is accessible from w. Thus by induction hypothesis $M_\Delta, w' \not\models_g \phi$. Since wRw', we can conclude $M_\Delta, w' \not\models_g \Box \phi$.   □

A.2.11. THEOREM. [Completeness] If $\models_{CC} \phi$, then $\vdash_{CC} \phi$

proof: Suppose $\not\vdash_{CC} \phi$. Then $\neg \phi$ is CC-consistent. We then know that $\neg \phi$ is an element of some world of the model $M_{\{\neg\phi\}}$ generated by $\{\neg\phi\}$ and therefore by theorem A.2.10 true in some world in that model. This means that $M_{\neg\phi} \not\models_{CC} \phi$. Since $M_{\neg\phi}$ is a CC-model, $\not\models_{CC} \phi$.   □

We have shown that the system CC is sound and complete with respect to the class of all CC-models. By standard techniques we can show that CC+**D**+**4**+**E** is sound and complete with respect to all serial, transitive and euclidean CC-models.

A.3   Dynamics

A.3.1. PROPOSITION. Let ϕ be a novel sentence. Then

$$\approx_{old} \phi \; \Leftrightarrow \; \approx_{new} \phi$$

proof: One direction of the proof hinges on the fact that (α) given a new state s, a new assignment g and a novel sentence ϕ, we can construct an old state σ and perspective $\wp$, such that if $s \approx_g \phi$ then $\sigma \approx_\wp \phi$. For the other direction, we show that (β) given an old state σ connected under a perspective $\wp$ we can find a new

state s and a new assignment g such that for all ϕ, if $\sigma \approx_\wp \phi$, then $s \approx_g \phi$. The two constructions are straightforward.

(α) Let s be a new state, g be a new assignment, and $X \subseteq \mathcal{V}_N$ a set of indexed variables. Let $a_{(g,w,X)}$ be an old assignment such that $dom(a_{(g,w,X)}) = X$ and $\forall v \in X : a_{(g,w,X)}(v) = g(v)(w)$. Then $\sigma_{(s,g,X)}$ is the following set: $\{\langle w, a_{(g,w,X)} \rangle \mid w \in s\}$; and $\wp_{(g)}$ is such that for all indices $n \in N$: $\wp_{(g)}(n) = g(n)$.

(β) Suppose now σ is an old state CC-accessible under a perspective $\wp$. Then let $s_\sigma = \{w \in W \mid \exists a : \langle w, a \rangle \in \sigma\}$. Suppose v is an indexed variable defined in σ. I will write $\sigma(v)$ to denote the function $f : s_\sigma \to D$ such that $\forall w \in s_\sigma: f(w) = a(v)$ where a is the unique assignment such that $\langle w, a \rangle \in \sigma$. The uniqueness of such an a is guaranteed by the definiteness of σ, since σ is an accessible state. Now let $G_{(\sigma,\wp)}$ be the following set $\{g \in (C^N \cup (D^W)^{\mathcal{V}_N}) \mid \forall n \in N : g(n) = \wp(n) \ \& \ \forall v \in dom(\sigma) : \forall w \in s_\sigma : g(v)(w) = \sigma(v)(w)\}$. The $\wp$-uniformity of σ guarantees that for all n and x_n, $g(x_n) \in g(n)$. Thus any g in $G_{(\sigma,\wp)}$ is a new assignment.

We can prove the following theorem: (I write $QV(\phi)$ to denote the set of quantified variables in ϕ, $AQV(\phi)$ to denote the set of dynamically active quantified variables in ϕ and $FV(\phi)$ to denote the set of variables in ϕ which are neither syntactically nor dynamically bound in ϕ)

A.3.2. THEOREM. Let ϕ be novel.

(α) Let $X \subseteq \mathcal{V}_N$ be such that $FV(\phi) \subseteq X$ and $QV(\phi) \cap X = \emptyset$. Then

$$s[\phi]^g_h = t \quad \Rightarrow \quad \sigma_{(s,g,X)}[\phi]^{\wp_{(g)}} \sigma_{(t,h,(X \cup AQV(\phi)))}$$

(β) Let σ be CC-accessible under $\wp$ and $g \in G_{(\sigma,\wp)}$. Then

$$\sigma[\phi]^\wp \tau \quad \Rightarrow \quad \exists h \in G_{(\tau,\wp)} : s_\sigma[\phi]^g_h = s_\tau$$

proof: the proof is by mutual induction on the complexity of ϕ. I will just give the atomic cases **(a)** $Rt_1, ..., t_n$, and **(b)** $\exists x_n$ and the induction step for **(c)** dynamic conjunction. Notice that the existential quantifier can be treated as an atomic update also in the old formulation of the semantics:

$$\sigma[\exists x_n]^\wp \tau \quad \text{iff} \quad \sigma[x_n/c]\tau \text{ for some } c \in \wp(n)$$

In definition 3.4.2, I chose the classical format for defining existential quantification in uniformity with the definitions in 3.2.8 of the other styles of dynamic quantification, in which the existential was not treated as an atomic action, since MS quantification could not have been formulated in such a fashion.

(a)　ϕ is $Rt_1, ... t_n$.

(α) Suppose $s[Rt_1, ... t_n]_h^g = t$. This is the case iff the following holds:

$$g = h \ \& \ t = \{w \in s \mid \langle [t_1]_{w,g}, ..., [t_n]_{w,g}\rangle \in w(R)\}$$

By construction of $\sigma_{(t,h,X)}$, this means:

(i)　$\sigma_{(t,h,X)} = \{\langle w, a_{(g,w,X)}\rangle \mid w \in s \ \& \ \langle [t_1]_{w,g}, ..., [t_n]_{w,g}\rangle \in w(R)\}$

Notice that for all terms t defined in $\langle w, a_{(g,w,X)}\rangle$ the following holds:

(ii)　$[t]_{w,g} = \langle w, a_{(g,w,X)}\rangle(t)$

Indeed, if t is a constant: $[t]_{w,g} = w(t) = \langle w, a_{(g,w,X))}\rangle(t)$. If t is a variable, and if t is in X, then $[t]_{w,g} = g(t)(w) = a_{(g,w,X)}(t) = \langle w, a_{(g,w,X))}\rangle(t)$.
Since, by assumption, $FV(Rt_1, ..., t_n) \subseteq X$, from (i) and (ii), we then have:

$$\sigma_{(t,h,X)} = \{\langle w, a_{(g,w,X)}\rangle \mid w \in s \ \& \ \langle\langle w, a_{(g,w,X)}\rangle(t_1), ..., \langle w, a_{(g,w,X)}\rangle(t_n)\rangle \in w(R)\}$$

By construction of $\sigma_{(s,g,X)}$, this implies:

$$\sigma_{(t,h,X)} = \{i \in \sigma_{(s,g,X)} \mid \langle i(t_1), ..., i(t_n)\rangle \in i(R)\}$$

which means for all $\wp$:

$$\sigma_{(s,g,X)}[Rt_1, ..., t_n]^\wp \sigma_{(t,h,X)}$$

So in particular for $\wp_{(g)}$. Since $(X \cup AQV(Rt_1, ... t_n)) = X$, we then have:

$$\sigma_{(s,g,X)}[Rt_1, ..., t_n]^{\wp_{(g)}} \sigma_{(t,h,(X \cup AQV(Rt_1,...,t_n)))}$$

(β) Suppose $\sigma[Rt_1, ..., t_n]^\wp \tau$. Then $\tau = \{i \in \sigma \mid \langle i(t_1), ..., i(t_n)\rangle \in i(R)\}$. By definition of s_τ, this means:

$$s_\tau = \{w \in W \mid \exists a : \langle w, a\rangle \in \sigma \ \& \ \langle\langle w, a\rangle(t_1), ..., \langle w, a\rangle(t_n)\rangle \in w(R)\}$$

By definition of s_σ and, since $g \in G_{(\sigma,\wp)}$, we have $\langle w, a\rangle = [t]_{w,g}$. So we have:

$$s_\tau = \{w \in s_\sigma \mid \langle [t_1]_{w,g}, ..., [t_n]_{w,g}\rangle \in w(R)\}$$

which means:

$$s_\sigma[Rt_1, ..., t_n]_g^g = s_\tau$$

Since $\tau \subseteq \sigma$, then $G_{(\sigma,\wp)} \subseteq G_{(\tau,\wp)}$. Thus $\exists h \in G_{(\tau,\wp)} : s_\sigma[Rt_1, ..., t_n]_h^g = s_\tau$.

(b) ϕ is $\exists x_n$.

(α) Suppose $s[\exists x_n]_h^g = t$. Then $g[x_n]h$ and $t = s$. By definition of $g[x_n]h$ and construction of $\sigma_{(t,h,(X\cup\{x_n\}))}$, we then have (note that by assumption X does not contain x_n):

$$\sigma_{(t,h,(X\cup\{x_n\}))} = \{\langle w,a\rangle \mid w \in s \ \& \ dom(a) = X \cup \{x_n\} \ \& \ a(x_n) = h(x_n)(w) \ \& \ \forall v \in X : a(v) = g(v)(w)\}$$

But this means:

$$\sigma_{(t,h,(X\cup\{x_n\}))} = \{\langle w, a_{(g,w,X)}\rangle[x_n/c(w)] \mid w \in s \ \& \ c = h(x_n)\}$$

which implies by construction of $\sigma_{(s,g,X)}$ and since $h(n) = g(n)$ for all n:

$$\sigma_{(t,h,(X\cup\{x_n\}))} = \{i[x_n/c(w_i)] \mid i \in \sigma_{(s,g,X)} \ \& \ c \in g(n)\}$$

which implies by definition of c-extension and of $\wp_{(g)}$:

$$\sigma_{(s,g,X)}[x_n/c]\sigma_{(t,h,(X\cup\{x_n\}))} \ \text{for some} \ c \in \wp_{(g)}$$

which means $\sigma_{(s,g,X)}[\exists x_n]^{\wp(g)}\sigma_{(t,h,(X\cup AQV(\exists x_n)))}$.

(β) Suppose $\sigma[\exists x_n]^\wp \tau$. Then $\sigma[x_n/c]\tau$ for some $c \in \wp(n)$, which means the following:

(i) $\tau = \{i[x_n/c(w_i)] \mid i \in \sigma \ \& \ c \in \wp(n)\}$

Since no world occurring in σ is eliminated in τ, a consequence of (i) is (ii):

(ii) $s_\sigma = s_\tau$

Consider now $g \in G_{(\wp,\sigma)}$. It is easy to see that:

(iii) any assignment h such that $g[x_n]h$ and $\forall w \in s_\tau : \ h(x_n)(w) = \tau(x_n)(w)$ is in $G_{(\wp,\tau)}$.

Clause (iii) follows from the fact that given (i), $G_{(\wp,\tau)}$ is the following set:

$$G_{(\wp,\tau)} = \{h \in (C^N \cup (D^W)^{V_N}) \mid \forall n \in N : h(n) = \wp(n) \ \& \ \forall w \in s_\tau : \ \forall v \in dom(\tau) : ((v \neq x_n \ \Rightarrow \ h(v)(w) = \sigma(v)(w)) \ \& \ (v = x_n \ \Rightarrow \ h(v)(w) = \tau(v)(w)))\}$$

from (ii) and (iii), it follows that $s_\sigma = s_\tau$ and $\exists h \in G_{(\tau,\wp)} : g[x_n]h$, which means $\exists h \in G_{(\tau,\wp)} : s_\sigma[\exists x_n]_h^g = s_\tau$.

(c) ϕ is $\psi_1 \wedge \psi_2$.

(α) Suppose now $s[\psi_1 \wedge \psi_2]_h^g = t$. This means $\exists k : (s[\psi_1]_k^g)[\psi_2]_h^k = t$. By induction hypothesis we then have:

$$\exists k : \sigma_{(s,g,Y)}[\psi_1]^{\wp(g)} \sigma_{((s[\psi_1]_k^g),k,(Y \cup AQV(\psi_1)))} \ \& \ \sigma_{((s[\psi_1]_k^g),k,Z)}[\psi_2]^{\wp(k)} \sigma_{(t,h,(Z \cup AQV(\psi_2)))}$$

where

(i) $FV(\psi_1) \subseteq Y$ & $QV(\psi_1) \cap Y = \emptyset$ and

(ii) $FV(\psi_2) \subseteq Z$ & $QV(\psi_2) \cap Z = \emptyset$

Consider any $X \subseteq \mathcal{V}_N$ such that $FV(\psi_1 \wedge \psi_2) \subseteq X$ and $QV(\psi_1 \wedge \psi_2) \cap X = \emptyset$. It is easy to see that X satisfies (i) and $(X \cup AQV(\psi_1))$ satisfies (ii) (recall that $\psi_1 \wedge \psi_2$ is novel). But then we have for all such X:

$$\exists k : \sigma_{(s,g,X)}[\psi_1]^{\wp(g)} \sigma_{((s[\psi_1]_k^g),k,(X \cup AQV(\psi_1)))} \ \& \ \sigma_{((s[\psi_1]_k^g),k,(X \cup AQV(\psi_1)))}[\psi_2]^{\wp(k)} \sigma_{(t,h,((X \cup AQV(\psi_1)) \cup AQV(\psi_2)))}$$

Since $\wp_{(g)} = \wp_{(k)}$ this implies:

$$\exists \tau : \sigma_{(s,g,X)}[\psi_1]^{\wp(g)} \tau \ \& \ \tau[\psi_2]^{\wp(g)} \sigma_{(t,h,(X \cup AQV(\psi_1 \wedge \psi_2)))}$$

which means $\sigma_{(s,g,X)}[\psi_1 \wedge \psi_2]^{\wp(g)} \sigma_{(t,h,(X \cup AQV(\psi_1 \wedge \psi_2)))}$.

(β) Suppose $\sigma[\psi_1 \wedge \psi_2]^\wp \tau$. Then $\exists \chi$: $\sigma[\psi_1]^\wp \chi[\psi_2]^\wp \tau$. By induction hypothesis, we have the following two facts:

(i) $\exists k \in G_{(\chi,\wp)} : s_\sigma[\psi_1]_k^g = s_\chi$

and

(ii) $\forall k' \in G_{(\chi,\wp)} : \exists h \in G_{(\tau,\wp)} : s_\chi[\psi_2]_h^{k'} = s_\tau$

which imply:

(iii) $\exists h \in G_{(\tau,\wp)} : \exists k : (s_\sigma[\psi_1]_k^g)[\psi_2]_h^k = s_\tau$

which means $\exists h \in G_{(\tau,\wp)} : s_\sigma[\psi_1 \wedge \psi_2]_h^g = s_\tau$. The other induction steps are left to the reader. Notice that case of negation requires the mutual induction. $\square$

A.3.3. COROLLARY. Let ϕ be novel and $X \subseteq \mathcal{V}_N$ be such that $FV(\phi) \subseteq X$ and $QV(\phi) \cap X = \emptyset$. Then

$$s \mathrel{\approx\!\!\!\approx}_g \phi \;\;\Rightarrow\;\; \sigma_{(s,g,X)} \mathrel{\approx\!\!\!\approx}_{\wp(g)} \phi$$

proof: Suppose $s \mathrel{\approx\!\!\!\approx}_g \phi$. Then $\exists h : s[\phi]^g_h = s$. Since ϕ is novel and X is specified as above, by theorem A.3.2, clause (α), this means $\exists h : \sigma_{(s,g,X)}[\phi]^{\wp(g)} \sigma_{(s,h,(X \cup AQV(\phi)))}$. Since for each possibility $\langle w, a_{(g,w,X)} \rangle \in \sigma_{(s,g,X)}$, it holds that

(i) $\langle w, a_{(h,w,(X \cup AQV(\phi)))} \rangle \in \sigma_{(s,h,(X \cup AQV(\phi)))}$ (by construction of the two states)

(ii) $a_{(g,w,X)} \subseteq a_{(h,w,(X \cup AQV(\phi)))}$ (since g and h may differ only in the values they assign to the variables in $AQV(\phi)$)

we then have that $\sigma_{(s,g,X)} \prec \sigma_{(s,h,(X \cup AQV(\phi)))}$. It follows that $\sigma(s,g,\phi) \mathrel{\approx\!\!\!\approx}_{\wp(g)} \phi$. $\square$

A.3.4. COROLLARY. Let σ be CC-accessible under $\wp$ and $g \in G_{(\wp,\sigma)}$. Then

$$\sigma \mathrel{\approx\!\!\!\approx}_{\wp} \phi \;\;\Rightarrow\;\; s_\sigma \mathrel{\approx\!\!\!\approx}_g \phi$$

proof: Suppose $\sigma \mathrel{\approx\!\!\!\approx}_{\wp} \phi$. Then $\exists \tau$: $\sigma[\phi]_\wp \tau$ & $\sigma \prec \tau$. Since σ is CC-accessible and $g \in G_{(\wp,\sigma)}$, this implies, by theorem A.3.2, clause (β), that $\exists h$: $s_\sigma[\phi]^g_h = s_\tau$. Since $\sigma \prec \tau$, we have $s_\sigma = s_\tau$. It follows that $s_\sigma \mathrel{\approx\!\!\!\approx}_g \phi$. $\square$

The proof of proposition A.3.1 is now a trivial exercise. Suppose $\mathrel{\approx\!\!\!\approx}_{new} \phi$ and ϕ is novel. Then $\exists M, g$, $\exists s \in S_M$: $s \neq \emptyset$ & $s \mathrel{\approx\!\!\!\approx}_g \phi$. Since ϕ is novel, by corollary A.3.3, it follows that for all $X \subseteq \mathcal{V}_N$ such that $FV(\phi) \subseteq X$ and $QV(\phi) \cap X = \emptyset$: $\sigma_{(s,g,X)} \mathrel{\approx\!\!\!\approx}_{\wp(g)} \phi$. Furthermore, since $s \neq \emptyset$, then such a $\sigma_{(s,g,X)} \neq \emptyset$, which means $\mathrel{\approx\!\!\!\approx}_{old} \phi$. Suppose now $\mathrel{\approx\!\!\!\approx}_{old} \phi$. Then $\exists M$, $\exists \wp, \exists \sigma \in \Sigma_M : \mathrm{Acc}_\wp(\sigma)$ & $\sigma \neq \emptyset$ & $\sigma \mathrel{\approx\!\!\!\approx}_{\wp} \phi$. By corollary A.3.4, it follows that $\forall g \in G_{(\wp,\sigma)}$: $s_\sigma \mathrel{\approx\!\!\!\approx}_g \phi$. Since $G_{(\wp,\sigma)} \neq \emptyset$, and, since $\sigma \neq \emptyset$, also $s_\sigma \neq \emptyset$, we then have $\mathrel{\approx\!\!\!\approx}_{new} \phi$. $\square$

A.4 Formal and Pragmatic Aspects of Conceptual Covers

A.4.1. PROPOSITION. Let CC be a cover over (W, D). The method of cross-identification R_{CC} determined by CC is proper.

proof: Given a conceptual cover CC over $\langle W, D \rangle$ the corresponding R_{CC} is defined as follows:

$$\langle w, d \rangle R_{CC} \langle w', d' \rangle \text{ iff } \exists c \in CC : c(w) = d \ \& \ c(w') = d'$$

We have to prove that R_{CC} satisfies conditions (i) and (ii) on proper methods of cross-identification:

(i) R_{CC} is an equivalence relation:

    (a) Reflexivity: $\forall w, d : \exists c \in CC : c(w) = d$ (by existence) $\Rightarrow \forall w, d : \langle w, d \rangle R_{CC} \langle w, d \rangle$;

    (b) Symmetry: $\langle w, d \rangle R_{CC} \langle w', d' \rangle \Rightarrow$ (by construction) $\exists c \in CC : c(w) = d \ \& \ c(w') = d' \Rightarrow \langle w', d' \rangle R_{CC} \langle w, d \rangle$;

    (c) Transitivity: $\langle w, d \rangle R_{CC} \langle w', d' \rangle$ and $\langle w', d' \rangle R_{CC} \langle w'', d'' \rangle \Rightarrow \exists c : c(w) = d$ and $c(w') = d'$ and $\exists c' : c'(w') = d'$ and $c'(w'') = d'' \Rightarrow$ (by uniqueness) $c = c' \Rightarrow \exists c : c(w) = d$ and $c(w'') = d'' \Rightarrow \langle w, d \rangle R_{CC} \langle w'', d'' \rangle$.

(ii) Each individual has one and only counterpart in each world:

    (a) $\forall w, w', d : \exists d' : \langle w, d \rangle R \langle w', d' \rangle$:
    By existence $\forall w, d : \exists c \in CC : c(w) = d$ and since c is a total function $\exists d' : c(w') = d' \Rightarrow \forall w, d : \forall w' : \exists d' : \langle w, d \rangle R_{CC} \langle w', d' \rangle$;

    (b) $\forall w, w', d, d', d'' : \langle w, d \rangle R \langle w', d' \rangle \ \& \ \langle w, d \rangle R \langle w', d'' \rangle \Rightarrow d' = d''$:
    $\langle w, d \rangle R_{CC} \langle w', d' \rangle \ \& \ \langle w, d \rangle R_{CC} \langle w', d'' \rangle \Rightarrow$ (by construction) $\exists c : c(w) = d \ \& \ c(w') = d'$, and $\exists c' : c'(w) = d \ \& \ c'(w') = d'' \Rightarrow$ (by uniqueness) $c = c' \Rightarrow d' = d''$. $\square$

A.4.2. PROPOSITION. The set of classes of pairs CP_R induced by a proper cross-identification method R is a conceptual cover.

proof: The set of classes of pairs induced by R is the following set:

$$CP_R = \{ [w, d]_R \mid w \in W \ \& \ d \in D \}$$

where $[w, d]_R = \{ \langle w', d' \rangle \mid \langle w, d \rangle R \langle w', d' \rangle \}$.
The result in A.4.2 follows from the following two lemmas:

A.4.3. LEMMA. Let R be a proper method of cross-identification over (W, D). Then $\forall \alpha \in CP_R, \forall w \in W, \exists! d \in D : \langle w, d \rangle \in \alpha$.

proof: For any α, by construction, $\alpha = [w, d]_R$ for some w, d, and by condition (iia), there is a $\langle w, d \rangle$ in α for all w. Suppose now $\langle w, d \rangle \in \alpha$ and $\langle w, d' \rangle \in \alpha$. This means by construction of CP_R that for some w'', d'', $\langle w'', d'' \rangle R \langle w, d \rangle$ and $\langle w'', d'' \rangle R \langle w, d' \rangle$. Since R is an equivalence relation: $\langle w, d \rangle R \langle w, d' \rangle$, which implies by condition (iib) that $d = d'$. $\square$

A.4.4. Lemma. Let R be a proper method of cross-identification over (W, D). Then $\forall w \in W$, $\forall d \in D$, $\exists! \alpha \in CP_R$: $\alpha(w) = d$.

proof: Since R is an equivalence relation, CP_R is a partition of the set of all world-individual pairs. This means that $\forall w, d : \exists! \alpha \in CP_R$: $\langle w, d \rangle \in \alpha$. $\square$