Random Variables. (RV)

Informal definition: RV denotes a value that depends on the result of some random experiment.

Formal definition: Ω – sample space (space of all poss. outcomes) of some random exp.

$$X: \Omega \to \mathbb{R}$$
$$\omega \qquad X(\omega)$$

$\boxed{Ex}$ We toss a fair coin, $\Omega = \{H, T\}$ $\quad P(H) = \frac{1}{2} = P(T)$

X – number of heads

outcomes	H	T
x	1	0
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

probability distribution of X

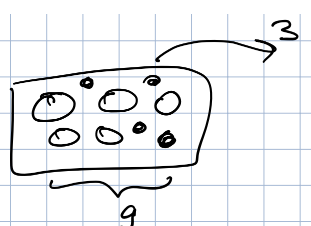
$\boxed{Ex}$ We toss a fair coin 3 times. X – number of heads

HHH THH
HTH TTH
HHT THT
HTT TTT

X	0	1	2	3
Prob.	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Problem 1. Assume that we have a box with 5 white balls and 4 black balls. We take 3 random balls from the box. Let X be the number of white balls taken. Write probability distribution of X.



probability dist.

X	0	1	2	3
Prob.	$\frac{4}{84}$	$\frac{30}{84}$	$\frac{40}{84}$	$\frac{10}{84}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$P(X=0) = \frac{\binom{4}{3}}{\binom{9}{3}} = \frac{\frac{4!}{3!\,1!}}{\frac{9!}{3!\,\cancel{6!}}} = \frac{4}{\frac{7\cdot \cancel{8}\cdot \cancel{9}}{\cancel{2}\cdot \cancel{3}}} = \frac{4}{84}$$

$$P(X=1) = \frac{\binom{5}{1}\cdot\binom{4}{2}}{\binom{9}{3}} = \frac{30}{84}$$

$$P(X=2) = \frac{\binom{5}{2}\cdot\binom{4}{1}}{\binom{9}{3}} = \frac{40}{84}$$

$$P(X=3) = \frac{\binom{5}{3}\binom{4}{0}}{\binom{9}{3}} = \frac{10}{84}$$

probability mass function $\quad pmf(x) = P(X=x)$

Binomial distribution $\quad \boxed{X \sim Bin(n, p)}$

$\boxed{Ex}$ We toss a coin n times. $P(H) = p \quad P(T) = 1-p.$

X - number of heads

$$\boxed{P(X=k) = \binom{n}{k}\cdot p^k \cdot (1-p)^{n-k}}$$

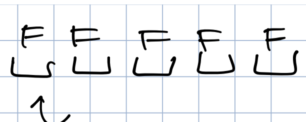
$\underbrace{HHH\ldots H}_{k}\;\underbrace{T\ldots T}_{n-k}$

$TH\ldots HT\ldots T$

usually we can interpret X as a number of "successful" outcomes in a series of n exp.

Problem 2. During the transmission of a message each symbol is distorted with probability 0.1. A message consisting of 5 symbols is send.

1. What is the probability that no symbols will be distorted?
2. What is the probability that there will be at least two distorted symbols?
3. What is the probability that there will be more non-distorted symbols than distorted ones?



X - number of dist. symb

$X \sim Bin(5, 0{,}1)$

① $P(\text{,,FFFFF''}) = (0{,}9)^5$

$p(\text{,,success''}) = P(\text{symbol is dist.}) = 0{,}1$

$P(\text{,,failure''}) = P(\text{symbol is not dist}) = 1 - 0{,}1 = \underline{0{,}9}$

② $P(\text{number of dis. symb} \geq 2) = 1 - P(\text{number of dist. symb.} < 2) = 1 - \underbrace{(0{,}9)^5}_{\text{number of dist} = 0} - \underbrace{\binom{5}{1} 0{,}1 (0{,}9)^4}_{\text{numb of dist} = 1}$

③ $P(\#\text{non-dist.} > \#\text{dist.}) = P(\#\text{dist} \in \{0,1,2\}) = P(X \in \{0,1,2\}) =$

$= (0{,}9)^5 + \binom{5}{1}(0{,}9)^4(0{,}1)^1 + \binom{5}{2}(0{,}1)^2(0{,}9)^3$

Problem 3. In the evening restaurant accepts only guests who made a reservation beforehand. The owner knows that 10% of people who made a reservation, ultimately don't come. The restaurant has 28 tables and the owner received 30 reservations. Calculate the probability that there will be a problem - the number of clients that made a reservation and came will be greater than the number of tables.

X - number of people who came.

$P(\text{,,success''}) = P(\text{a person came}) = 0{,}9 \qquad P(\text{,,failure''}) = 0{,}1$

$X \sim Bin(30, 0{,}9)$

$P(X > 28) = P(X \in \{29, 30\}) = P(X = 29) + P(X = 30) = \binom{30}{29}(0{,}9)^{29}(0{,}1)^{30-29} + \binom{30}{30}(0{,}9)^{30}(0{,}1)^0$

$$\boxed{P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}}$$

$= \binom{30}{29}(0{,}9)^{29}(0{,}1) + (0{,}9)^{30}$

Expectation

$$E(X) = \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{n} x_i P(X=x_i) =$$

x	x_1	x_2	...	x_n
prob	p_1	p_2	...	p_n

$$= x_1 p_1 + x_2 p_2 + \ldots + x_n p_n$$

Problem 4. Complete the table if we know that $\mathbb{E}X = 0$

X	-1	0	a	4
P	0,6	p	0,1	0,1

2

0,2

$$0{,}6 + p + 0{,}1 + 0{,}1 = 1.$$

$$p = 1 - 0{,}8 = 0{,}2.$$

$$EX = (-1)\cdot 0{,}6 + 0\cdot 0{,}2 + a\cdot 0{,}1 + 4\cdot 0{,}1 = 0$$

$$0{,}1 a = 0{,}6 - 0{,}4 = 0{,}2$$

$$\boxed{a = 2}$$

Problem 5. We toss 2 dice independently. Let X be the sum of points on them. Find probability distribution of X and $\mathbb{E}X$.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

36 outcomes

X	2	3	4	5	6	7	8	9	10	11	12
prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\mathbb{E}X = \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = \boxed{7}$$

Variance $\quad Var(X) = \mathbb{E}((X-EX)^2)$

$\sigma(X) = \sqrt{Var(X)}$ standard deviation.

Ex

X	−1	0	1	2
P	0,1	0,2	0,3	0,4

$$EX = (-1)\cdot 0{,}1 + 0\cdot 0{,}2 + 1\cdot 0{,}3 + 2\cdot \underbrace{0{,}4}_{=0{,}8} = \boxed{1}$$

$X-\underset{=1}{EX}$	−2	−1	0	1
prob	0,1	0,2	0,3	0,4

$(X-EX)^2 = (X-1)^2$	0	1	4
Prob	0,3	$\underbrace{0{,}2+0{,}4}_{=0{,}6}$	0,1

$$Var(X) = \mathbb{E}((X-EX)^2) = 0\cdot 0{,}3 + 1\cdot 0{,}6 + 4\cdot 0{,}1 = \boxed{1}$$

Problem 6. Complete the table if $\mathbb{E}X = 0$, $Var(X) = 5.4$.

X	−2	−1	0	a	4
P	0,4	0,2	p_1	0,1	p_2

(1) $\quad 0{,}4 + 0{,}2 + p_1 + 0{,}1 + p_2 = 1$

$$p_1 + p_2 = 0{,}3$$

(2) $EX = (-2)\cdot 0{,}4 + (-1)\cdot 0{,}2 + 0\cdot p_1 + a\cdot 0{,}1 + 4p_2 = 0$

$$-0{,}8 - 0{,}2 + \underline{a}\cdot 0{,}1 + 4p_2 = 0$$

$$0{,}1\cdot a = 1 - 4p_2 \qquad 4p_2 = 1 - 0{,}1a$$

(3) $Var(X) = E\big((\underbrace{X - EX}_{=0})^2\big) = E(X^2) =$

X^2	0	1	4	16	a^2
p	p_1	0,2	0,4	p_2	0,1

$$= 0{,}2 + 4\cdot 0{,}4 + \underbrace{16p_2}_{=4\cdot 4p_2} + a^2\cdot 0{,}1 = 5{,}4$$

$$a^2 - 4a + 4 = 0 \qquad \boxed{a = 2}$$

From (2) $\quad 4p_2 = 1 - 0{,}2 = 0{,}8 \quad \boxed{p_2 = 0{,}2}$

From (1) $\quad \boxed{p_1} = 0{,}3 - 0{,}2 = \boxed{0{,}1}$

(didn't solve at the seminar, try to solve yourself!)

Problem 7. In a lottery you choose a three-digit number from 000 to 999. If you guess one digit (for example, the winning number is 366 and your number is 436, then you guessed only the right digit) – you get 5 dollars. If you guessed two digits you get 50 dollars, if you guessed all three digits you get 500 dollars. Let X be a random variable - your payoff in the lottery. Construct a pmf for X and write it in tabular form. Calculate an expected payoff and answer the question: how much should a lottery ticket cost?

000 999 [each digit can be 0,1,2,...9]

Probability to guess each digit is 0,1

[this is „success"]

Probability of failure $= 1 - 0{,}1 = \boxed{0{,}9}$

Y – number of correct digits, $Y \sim Bin(3; 0{,}1)$

Y	0	1	2	3
P	$(0{,}9)^3$ $0{,}729$	$\binom{3}{1}(0{,}9)^2(0{,}1)$ $= 0{,}243$	$\binom{3}{2}(0{,}9)(0{,}1)^2$ $= 0{,}027$	$(0{,}1)^3$ $0{,}001$
X	0	5	50	500

our payoff ($)

$E(X) = 0 + 5 \cdot 0{,}243 + 50 \cdot 0{,}027 + 500 \cdot 0{,}001 = 3{,}065$

$\Rightarrow$ a lottery ticket should cost $> 3{,}065$ $.