Cantor's Paradise

Mariam Mousa
Department of Mathematics and Actuarial Sciences in AUC

Introduction

Set theory is one of the most fundamental theories in mathematics. It was founded by the German mathematician Georg Cantor. In this poster, we will introduce Cantor's very interesting, but also quite sad life. We will also prove Cantor's Theorem, which says that the cardinality of the power set of a set is greater than the cardinality of the set itself. We will also discuss two corollaries of this theorem.

Cantor's Life

Georg Cantor is one of the most influential mathematicians of the 19th century. He is mostly known for founding the set theory and introducing the continuum hypothesis. He was born in Saint Petersburg in 1845 in rich family. Since his childhood, Cantor had been a prodigy. His skills in playing the violin and mathematics were outstanding. In Höheren Gewerbeschule, he studied engineering for two years then joined ETH Zurich to study mathematics. After his father's death, Cantor joined the University of Berlin where he was hugely influenced by his professors: Ernst Kummer, Leopold Kronecker and Karl Weierstrass. Upon receiving his Ph.D, he left Berlin to work as a lecturer in Halle University under Eduard Heine.

Weierstrass and Heine's influence appeared in Cantor's first paper that discussed the trigonometric series. Then, in 1872, Cantor published another work generalizing the theorem in his first paper. These early works of Cantor gave foundation to the work he is most known for, the transfinite sets. In 1872, Cantor met Richard Dedekind who became one of the greatest supporters of Cantor's ideas about infinite sets. Dedekind and Cantor exchanged a lot of letters. Since 1873, we can see, in these letters, Cantor's interest in the countability of sets growing. On the 9th of December, Cantor sends a letter to Dedekind of a proof of why there can be no bijection from the set of real numbers to the set of natural numbers. Then, in 1874, Cantor publishes a five-page paper with two of his most important achievements: proof of the countability of the set of algebraic numbers and the uncountability of the real numbers. Cantor proved the uncountability of the real numbers again in 1891, but with a simpler argument known as Cantor's Diagonal Argument. Cantor's astonishing work showed that there are different sizes of infinity.

Georg Cantor

Another one of Cantor's most important contributions to the set theory is the continuum hypothesis. It appears in his works for the first time in 1878 where he suggested that the next cardinal number after $\aleph_0$, the size of the natural numbers' infinity, is definitely $\aleph_1$, the size of the real numbers' infinity, and there is no other sizes between them. Cantor spent almost the rest of his life trying to prove the continuum hypothesis. This continuous disappointment of his failed trials caused him severe depression that lasted until he died.In addition, Leopold Kronecker's intense and continuous opposition to Cantor's set theory was another main cause of Cantor's mental breakdown. Cantor was always taking Kronecker's opposition into consideration.

He left out some parts of his work in his 1874 paper due to Kronecker's influence. In all of the letters Cantor sent to the Swedish mathematician Mittag-Leffler, Kronecker is mentioned by name. In the year 1899, Cantor almost lost all his passion for mathematics after his youngest son died. Then, in 1903, a mathematician called Julius König tried to disprove Cantor's transfinite set theory. Cantor considered this humiliation, though König's proof was disproved less than a day later. All of these incidents were the reason for Cantor's mental health to get worse and consequently he was hospitalized many times over his last 20 years. The last hospitalization was in 1917, where he eventually died of a heart attack.
To be a mathematician means that one might spend his whole life trying to find a solution to one problem. They might reach that solution or die looking for it. Even someone as brilliant as Georg Cantor can die before solving that one problem. However, Cantor's life teaches us that the road to a problem leads to solutions for others that we probably have not thought of. Cantor did not solve the continuum hypothesis, but his work created a whole mathematical paradise for those who came after him.

Cantor's Theorem

We start by defining a set.

Definition 0.1. A *set* is a collection of objects.

Definition 0.2. Let A and B be sets. We say that A is a *subset* of B, and write $A \subseteq B$ if every element in A is also an element in B.

Example 0.3. The following are examples of subsets.

1. The set of natural numbers is a subset of the set of integers.
2. The set of rational numbers is a subset of the set of real numbers.
3. The set of odd integers is a subset of the set of integers.

We are now ready to introduce the notion of a power set.

Definition 0.4. The *power set* $\mathcal{P}(S)$ of a set S is the set of all the subsets of S.

Example 0.5. 1. Let A be the set $\{1, 2\}$, then the power set of A is $\{\{1\},\{2\},\{1,2\},\phi\}$.

2. Let S be the set $\{a, b\}$, then the power set of S is $\{\{a\}, \{b\}, \{a, b\}, \phi\}$.

Definition 0.6. Let A and B be sets.

- We say that the cardinality of A is equal to the cardinality of B, and write $|A| = |B|$, if there is a bijective function from A to B.
- We say that the cardinality of A is less than or equal to the cardinality of B, and write $|A| \leq |B|$, if there is an injective function from A to B.
- We say that the cardinality of A is strictly less than the cardinality of B, and write $|A| < |B|$, if $|A| \leq |B|$ and $|A| \neq |B|$ i.e. there is an injective function from A to B, but there is no bijection from A to B.
- We say a set is *countably infinite* if there is a bijection from this set to the set of natural numbers $\mathbb{N}$.

We will now show that the set of integers is a countably infinite set.

Theorem 0.7. *The set of integers $\mathbb{Z}$ is countably infinite.*

Proof. To prove $\mathbb{Z}$ is countably infinite we need to find a bijection from $\mathbb{Z}$ to $\mathbb{N}$. Take $f : \mathbb{N} \to \mathbb{Z}$, such that:
$f(x) = 0$, if $x = 0$.
$f(x) = \frac{n+1}{2}$, if x is odd.
$f(x) = \frac{-n}{2}$, if x is even.
Now, we can see that the elements of $\mathbb{Z}$ can be listed in terms of the elements of $\mathbb{N}$ such that:
$0 \to 0$
$1 \to 1$
$2 \to -1$
$3 \to 2$
and so on
This proves that $\mathbb{Z}$ is countably infinite. □

We will now present the main theorem of the article, Cantor's Theorem, which states that it is impossible to have a bijection between any set and its powerset.

Theorem 0.8 (Cantor's Theorem)**.** *Let S be any (finite or infinite) set. Then $|S| < |\mathcal{P}(S)|$.*

Proof. Let S be any set.
First, we will show that $S \leq \mathcal{P}(S)$ by constructing an injective function $g : S \to \mathcal{P}(S)$. Let f be the function that assigns to every element in S the subset that contains this element. So, $f(x) = \{x\}$.
Now, we need to prove that f is injective. Let $f(x) = f(y)$.
This means $\{x\} = \{y\}$ and so, $x = y$. Hence, f is injective.
Second, we will prove that $|S| \neq |\mathcal{P}(S)|$. For the sake of contradiction, assume that there is a bijection $h : S \to \mathcal{P}(S)$.
Let $A = \{x \in S \mid x \notin h(x)\}$, which means $A \subseteq S$ and $A \in \mathcal{P}(\mathcal{S})$, and since h is surjective, A must be an image of some preimage x in S. Now, we have two cases either $x \in A$ or $x \notin A$. If $x \in A$, then $x \notin h(x)$, but since $h(x)$=A, $x \notin A$, which is a contradiction. If $x \notin A$, then $x \in h(x)$, but since $h(x) = A$, $x \in A$, which is also a contradiction. Hence, h is not surjective.
Therefore, we have shown that $|S| < |\mathcal{P}(S)|$. □

From Cantor's theorem we can deduce the following consequences.

Corollary 0.9. *The power set of the natural numbers is uncountable.*

Proof. By Cantor's Theorem, $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$. So, there is no bijection from $\mathbb{N}$ to its power set. Hence, the power set of the natural numbers is uncountable. □

Corollary 0.10. *There is an infinite sequence $A_0, A_1, A_2, \ldots$ of infinite sets such that $|A_i| < |A_{i+1}|$ for all $i \in \mathbb{N}$. In other words, there is an infinity hierarchy of infinities.*

Proof. By Cantor's Theorem, we know that for any set S, $|\mathcal{P}(S)| > |S|$. Then this means that $|\mathcal{P}(\mathcal{P}(S))| > |\mathcal{P}(S)|$. So, this can be done for infinitely many times for any infinite set. Take for example the set of natural numbers $\mathbb{N}$. By Cantor's Theorem, we know that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ and $|\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N})|$ and so on. Hence, there is an infinity hierarchy of infinities.

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