# Kuratowski's Theorem

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**Abstract.** In this paper, we use a number of lemmas and theorems to prove a very important result about the planarity of graphs that is known as Kuratowski's theorem.

## 1 Introduction

In graph theory, the way graphs are drawn , or embedded, is of great importance. Some graphs can be drawn in a way that their edges only intersect at vertices. These graphs are called planar graphs. Kazimierz Kuratowski is one of the mathematicians who succeeded at characterizing non-planar graphs. Three other mathematicians were also able to prove the same theorem, but eventually it was named after Kuratowski[2].

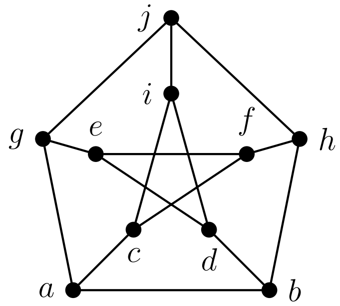
**Theorem 1.1.** *(Kuratowski's Theorem) A graph $G$ is nonplanar if and only if $G$ contains a subgraph that is a subdivision of either $K_{3,3}$ or $K_5$.*

## 2 Preparation

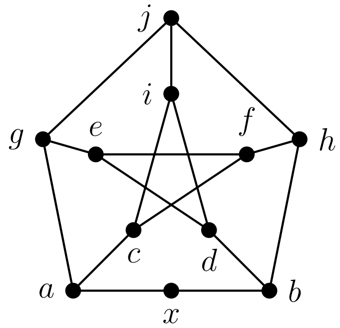
Before proving the theorem, we need to define some of the terms that we are going to use in the proof.

**Definition 2.1.** *A graph $H$ is called a subdivision of $G$ if $H$ can be obtained* *by replacing some edge $e$ in $G$ by a path that starts at an end vertex of $e$ and ends at the other.*

For example, consider Petersen graph:



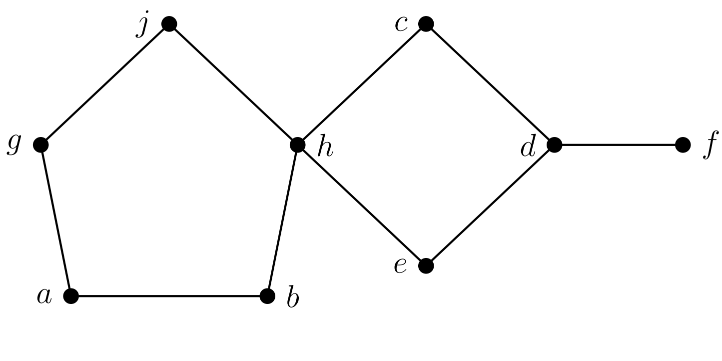

One possible subdivision of it is the following graph:



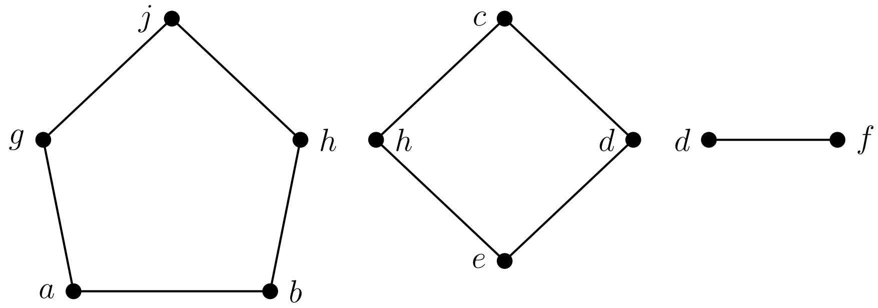

Here, we have replaced the edge $ab$ with the path $a - x - b$.

**Definition 2.2.** *A block in a graph $G$ is a maximal induced subgraph that does not have a cut vertex.*

For example, consider the following graph:



We can see that its blocks are the following subgraphs:




Now, we will need to prepare for Kuratowski's theorem by proving some easier theorems.

**Theorem 2.3.** *A graph $G$ is planar if and only if each block of $G$ is planar.*

*Proof.* First, we will consider the forward direction. Let G be a planar graph. For the sake of contradiction assume there is a block of G that is not planar. This means there is no planar representation of this block, which means there is no planar representation of G itself. Therefore, each block of G has to be planar.

For the backward direction, we will proceed with induction on the number of blocks $k$. For the base case, let $k = 1$, hence, $G$ itself is a block, which clearly satisfies the theorem. For the induction step, assume every graph with $k - 1$ blocks satisfies the theorem. We will show that the theorem is true for all graphs with k blocks. Delete a block, say $H$, that intersects with the rest of the graph at one vertex only. Notice that such a block must exist because if all blocks have more than one intersection with the graph then the graph itself will be a block which takes us back to the base case. Now, the obtained graph, say $G'$, has $k - 1$ blocks, and therefore, by the induction hypothesis, $G'$ is planar. Now, if we patch $H$, a planar subgraph, back to $G'$, we must obtain a planar graph, since $H$ intersects with $G'$ in one vertex and both $G'$ and $H$ are planar.[3] ■

Also, we need to keep in my mind these two trivial lemmas:

**Lemma 2.4.** *A graph $G$ is planar if and only if every subdivision of it is planar.*

**Lemma 2.5.** *If $G$ is a planar graph, then every subgraph of it is planar.*

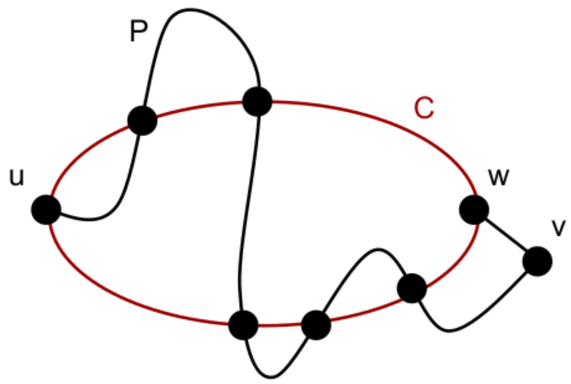
We will also show the following two lemmas and use it in the proof:

**Lemma 2.6.** *$G$ is 2-connected if and only if for every $u$,$v \in V(G)$, there exists a cycle that includes both $u$ and $v$.*

*Proof.* For the backward direction, assume there is a cycle containing every two vertices. Hence, there are two paths connecting every two vertices. If we delete any vertex $x$, there will still be a path connecting every two vertices. Therefore, $G$ will still be connected. Hence, $G$ does not have a cut vertex and it is 2-connected.

To prove the forward direction, we will proceed by induction on the distance from u to v. For the base case, let $d(u, v) = 1$, i.e. $u$ and $v$ are adjacent. Notice that $u$ cannot be of degree 1 because if this is the case, then $v$ would be a cut vertex, but we know $G$ is 2-connected, hence, $u$ has more than one neighbour. Let $w$ be one of its neighbours. Now, delete $u$ from the graph. Notice that the new graph is still connected since $G$ is 2-connected. Therefore, there must be a path from $v$ to $w$. Now, if we return $u$ back to the graph, adding it to the two ends of the $v - w$ path, we will get a cycle containing $u$ and $v$.

For the induction step, suppose that for any u, v $\in V(G)$ with distance $d-1$, there is a cycle that includes v and u. We will prove the same thing for any u and v of distance $d$. Let $P$ be a path between $u$ and $v$ of length d. Let another vertex, $w$, be on $P$ and adjacent to $v$. Notice that $d(u, w) = d - 1$ and therefore, there is a cycle that includes both $u$ and $w$. If $v$ is also on that cycle then we are done. Assume $v$ is not on the cycle. Since $G$ is 2-connected, $G - w$ is connected. Hence, there is path from $u$ to $v$ that does not have $w$. Now, we have a cycle that includes $u$ and $v$. To see the cycle, trace the exterior of the following graph: [4]




■

Now, we are ready to delve into the proof.

# 3 Kuratowski's Theorem Proof

**Theorem 3.1.** *A graph $G$ is nonplanar if and only if $G$ contains a subgraph that is a subdivision of either $K_{3,3}$ or $K_5$.*

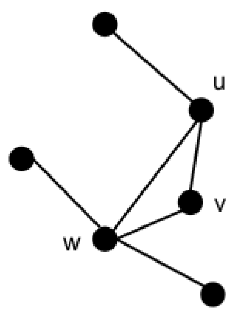
*Proof.* The backward direction can be easily shown by **Lemma 2.4** and **2.5**. It is clear that if G contains a subgraph that is a subdivision of nonplanar graph, $K_{3,3}$ and $K_5$, then G must be nonplanar.
For the forward direction, assume, for the sake of contradiction, that there is a graph $G$ that is nonplanar and does not contain any subdivisions of $K_{3,3}$ or $K_5$ as subgraphs. Moreover, assume $G$ is the smallest such graph. By smallest, we mean the one with the least number of vertices and edges. Therefore, any graph with fewer vertices or edges is planar and does not contain any subdivision of $K_{3,3}$ or $K_5$ as subgraphs.

**Claim 1.** *$G$ is 2-connected.*

*Proof.* By **Theorem 2.3**, a graph is planar if and only if each block of it is planar. Since $G$ is nonplanar, it must contain a nonplanar block. However, since a block of $G$ is smaller than $G$ and does not contain any subdivisions of $K_{3,3}$ or $K_5$ as subgraphs, this contradicts the minimality of $G$. Therefore, $G$ must be a block itself and hence $G$ is 2-connected. [4] ■

**Claim 2.** *$G$ does not contain any vertices of degree 2.*

*Proof.* For the sake of contradiction, assume there is a vertex $v$ of degree 2 in $G$. Let $w$ and $u$ be the two neighbours of $v$. Now, we have two possibilities either $u$ and $w$ are adjacent or they are not adjacent. Suppose they are adjacent, then obtain the graph $H$ by deleting $v$. Since $H$ is smaller than $G$, by the minimality of $G$, $H$ is planar. Obtain a planar representation of $H$ and insert in it the path $uvw$ as shown in the figure below. Now, this would be a planar representation of $G$, but this contradicts that $G$ is nonplanar.




For the second case, let $u$ and $w$ be nonadjacent. Delete $v$ and replace it with the edge $uw$ to obtain the graph $H$. Now, create a subdivision of $H$ by inserting $v$ on the edge $uw$. Now, we have a planar representation of $G$, but again this contradicts that $G$ is nonplanar. Hence, $G$ does not contain any vertices of degree 2. [4] ■

**Claim 3.** *$G$ is 3-connected*

*Proof.* Assume, for the sake of contradiction, that $G$ is not 3-connected, so $\kappa(G) = 2$. Hence, there are two vertices, $u$ and $v$, such that $G - u, v$ is disconnected. Let $H_1$ be one connected component of $G - u, v$ and let $H_2$ be the union of the other components. Also, let $G_1$ be $F_1 \cup \{u, v\}$ and $G_2$ be $F_2 \cup \{u, v\}]$, with the addition of the edge $uv$ to both $G_i$'s. We claim that at least one of the $G_i$'s is nonplanar. For contradiction, assume that both are planar, i.e. there is a planar embedding of each $G_i$. If we merge the two embeddings, we construct a planar embedding of $G + uv$. Hence, $G$ itself is planar, but we know $G$ is nonplanar. Therefore, at least one of the $G_i$'s is nonplanar, say $G_1$. Since $G_1$ is smaller than $G$, by the minimality of $G$, $G_1$ must contain a subdivision of $K_{3,3}$ or $K_5$. Moreover since $G$ does not contain any $K_5$ or $K_{3,3}$ subdivisions, $G_1$ is not a subgraph of $G$ and therefore $uv$ is not an edge in $G$. Now, we combine $G_1 - uv$ and $G_2 - uv$ by merging $u$ and $v$ to obtain $G$. Since both $G_1$ is connected, there is a $uv$-path, which is a subdivision of the edge $uv$. This means that $G$ contains a subdivision of $K_5$ or $K_{3,3}$, but this contradicts that $G$ does not contain any $K_5$ or $K_{3,3}$ subdivisions. Therefore, our original assumption is wrong and $G$ is 3-connected. [5] ■

**Claim 4.** *$G$ must have an edge $uv$ such that $G - uv$ is still 2-connected.*

*Proof.* By **Lemma 2.6**, we need to show that for any two vertices, $a, b$ in $G - uv$, there is a cycle that includes both of them. Now, we have three cases:
(1) Without loss of generality, $a = u$ and $b = v$. Let $c$ and $d$ be other two vertices in $G$. Since G is 2-connected, $G - cu$ is still connected. Hence there is an dv-path, $P_1$. Also, $G - cv$ is connected, so there is a ud-path, say $P_2$. Similarly, in $G - dv$, there is a cu-path, call it $P_3$. Again, in $G - ud$, there is a cv-path, $P_4$. But, now $u$ and $v$ lie on the same cycle that is $u, P_3, c, P_4, v, P_1, d, P_2, u$.
(2) Without loss of generality, only $a = u$ and $b \neq v$. Using the same argument as in the first case, we can find a ub-path, $P_1$, a cb-path, $P_2$, and a cu-path $P_3$. Now, we have the cycle u, $P_1$, b, $P_2$, c, $P_3$ u, which contains both $u$ and $b$.
(3) Neither $a = u$ nor $b = v$. Again, using the same method, we find an ab-path, $P_1$, a vb-path, $P_2$, an av-path, $P_3$, and now we can form an ab-cycle as follows $a, P_1, b, P_2, v, P_3, a$ Hence, for any two vertices, $a, b$, in $G - uv$, there is a cycle that includes both of them. Therefore, $G - uv$ is 2-connected.[5] ■

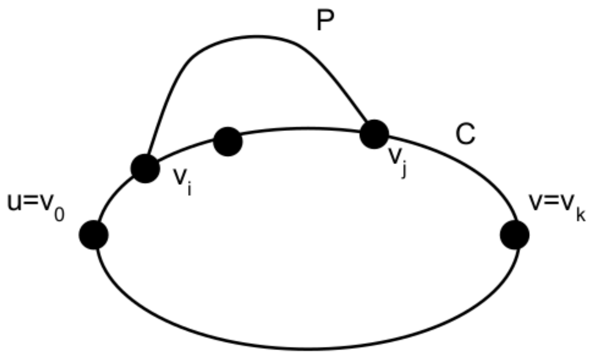
Now, let $H$ be the graph obtained by deleting the edge $uv$ from $G$. By minimality of $G$, $H$ is planar. Also, $H$ is 2-connected. Hence, there is a cycle that contains both $u$ and $v$ in $H$. Now, from a planar representation of $H$, choose a cycle $C$ that has the following properties:
(1) $C$ contains both $u$ and $v$.

(2) The number of regions inside $C$ is maximal among all other representations of $H$. In other words, if there is another cycle $C^{'}$ in any planar representation of $H$ that contains $u$ and $v$, then the number of regions inside $C^{'}$ is less than or equal to the number of regions inside $C$.

We will write $C$ as $u = v_0, v_1, v_2, ..., v_k = v, v_{k+1}, v_{k+2}, ..., v_l, v_0$. Notice that $k \geq 2$, since $u$ and $v$ are not adjacent in $H$. Also, notice that:

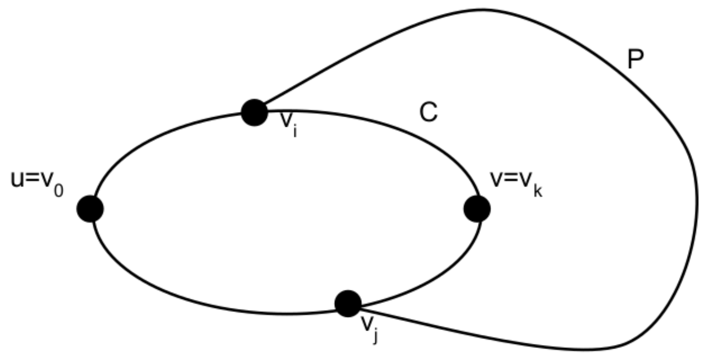
- There is no path connecting two vertices in the set $v_0, v_1, ..., v_k$ that lies outside $C$.
- There is no path connecting two vertices in the set $v_k, v_{k+1}, ..., v_l, v_0$ that lies outside $C$.

To prove this, assume, for contradiction, that there is a path connecting two vertices, $v_i$ and $v_j$, in the set $v_0, v_1, ..., v_k$ that lies outside $C$, call it $P$. Now, we can construct a cycle $C^{'}$ as $v_0, v_1, ..., v_i, P, v_j, ..., v_k, v_{k+1}, ..., v_0$, but this cycle has greater number of regions than $C$, as shown in the figure below:




This contradicts the maximality of $C$. Hence, there is no such path. The same proof can be used to show the second observation.

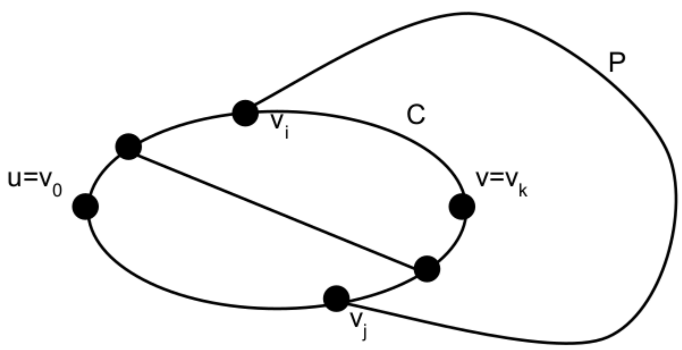
Notice that we cannot add the edge $uv$, since $G$ is non-planar. Hence, there must be a $v_iv_j$ path from a vertex, $v_i$, in the set $v_0, v_1, ..., v_k$ to a vertex, $v_j$, in the set $v_k, v_{k+1}, ..., v_l, v_0$ exterior to $C$ that prevents us from adding $uv$ exterior to $C$, as shown in the figure below:

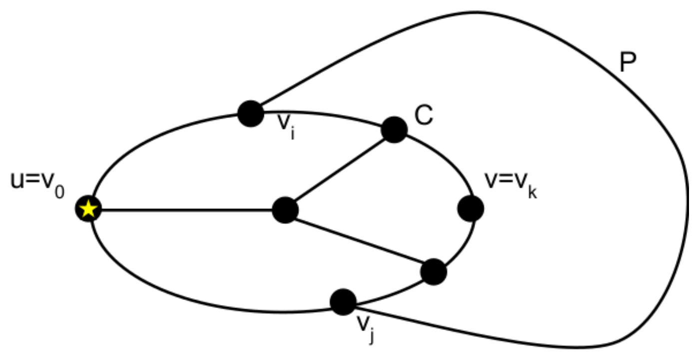


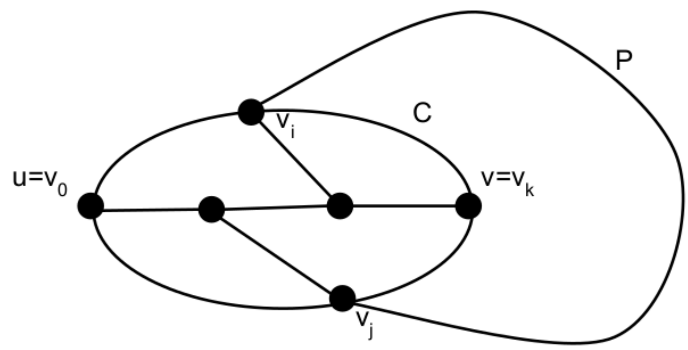

Also, there must be a structure inside $C$ that also prevents us from adding $uv$ inside $C$. This structure can take four forms:

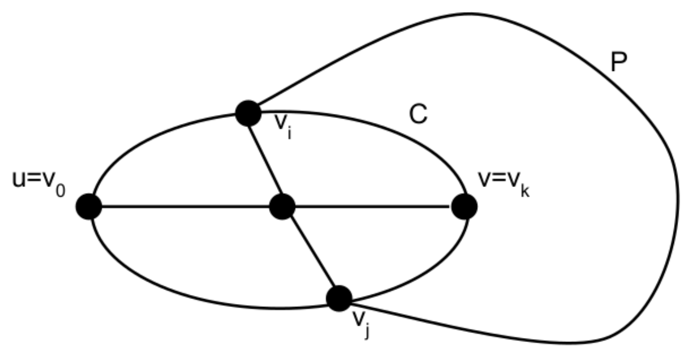
1. A $v_a v_b$ path such that $0 < a < i$, $j < b < k$ and only $v_a$ and $v_b$ belong to the cycle $C$.
2. A vertex $w$ not on $C$ is connected to $C$ via three disjoint paths that each of their terminals is one of $v_0, v_i, v_k$ or $v_j$. For example, if $v_0$ is one terminal, then the other two terminals are $v_a$ and $v_b$ such that $i \le a < k$ and $k \le b < j$.
3. Two vertices $x$ and $w$ not on $C$ with an $xw$ path and two disjoint paths connecting $w$ with $v_0$ and $v_i$ and $x$ with $v_k$ and $v_j$.
4. A vertex $w$ connected to each of $v_0$, $v_k$, $v_i$, and $v_j$ through a path.

The four forms are shown below:






Now, we can see that for the first three cases, $G$ would have a subdivision of $K_{3,3}$ and in the last case $G$ contains a subdivision of $K_5$. This contradicts that $G$ does not contain a subdivision of either $K_{3,3}$ or $K_5$. Hence, our assumption that there is a nonplanar that does not contain a subdivision of either $K_{3,3}$ or $K_5$ is wrong and the theorem must be true. ■

# References


[1] John Harris and Jeffry Hirst and Michael Mossinghoff. *Combinatorics and Graph Theory*. Second Edition, Springer, 2008.

[2] Daoud Siniora. *Graph Theory*. 2020.

[3] Gregkow. *Stack Exchange*. $9^{th}$ of July 2014. https://math.stackexchange.com/questions/840276/planar-graph-block/840292

[4] Mary Radcliffe. *Math 228: Kuratowski's Theorem*, 2016.

[5] Yifan Xu. *Kuratowski's Theorem*, $28^{th}$ of August 2017.