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Some New Notions of Continuity in Generalized Primal Topological Space

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Abstract: This study analyzes the characteristics and functioning of S_g^*-functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms in generalized topological spaces ($\mathcal{GTS}$). A few important points to emphasize are S_g^*-continuous functions, S_g^*-irresolute functions, perfectly S_g^*-continuous, and strongly S_g^*-continuous functions in $\mathcal{GTS}$ and generalized primal topological spaces ($\mathcal{GPTS}$). Some specific kinds of S_g^* functions, such as S_g^*-open mappings and S_g^*-closed mappings, are discussed. We also analyze the $\mathcal{GPTS}$, providing a thorough look at the way these functions work in this specific context. The goal here is to emphasize the concrete implications of S_g^* functions and to further the theoretical understanding of them by merging different viewpoints. This work advances the area of topological research by providing new perspectives on the behavior of S_g^* functions and their applicability in various topological settings. The outcomes reported here contribute to our theoretical understanding and establish a foundation for further research.

Keywords: generalized primal topological space; τ_g–S_g^*-continuous function; τ_g–S_g^*-homeomorphism; τ_g–$S_g^{*\#}$-homeomorphism; $\mathcal{P}_g$–S_g^*-continuous function; $\mathcal{P}_g$–S_g^*-homeomorphism; $\mathcal{P}_g$–$S_g^{*\#}$-homeomorphism

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1. Introduction

Studying different kinds of functions and their characteristics is essential to expanding our knowledge of topological structures and how they relate to each other in the framework of $\mathcal{GTS}$. In topological space, Levine [1] initiated the generalized set. Numerous authors [2–10] have examined generalized sets, their generalizations, and related notions that are comparable. General topology is crucial in many domains of mathematics and the applied sciences. In actuality, it is utilized in particle physics, quantum physics, computer-aided geometric design, digital topology, computational topology for computer-aided design, data mining, and computational topology for geometric and molecular design.

Csaszar established the idea of generalized topological spaces in [10–13] and investigated these classes' elementary nature. Refs. [14–16] provided a generalized continuous map in topological space. In particular, refs. [6,15,17–19] explored the notion of generalized continuous functions and proposed the ideas of continuous functions on generalized topological spaces and generalized primal topological spaces. We revisit certain concepts delineated in [19]. Of these functions, the S_g^* function has drawn a great deal of attention because of its prospective applications and capacity for merging ideas from other topological frameworks. Several important topics are covered in this paper's exploration of the S_g^* function in the context of $\mathcal{GTS}$, including S_g^*-continuous, S_g^*-irresolute, perfectly S_g^*-continuous, and strongly S_g^*-continuous functions. We also explore more specialized forms such as functions that are S_g^*-open and S_g^*-closed mappings. The objective of this study is to expand current theories and gain new insights by investigating these qualities in $\mathcal{GTS}$. Moreover, this study broadens the scope of our analysis to the $\mathcal{GPTS}$, offering a thorough explanation of the behavior of these S_g^*-functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms in this more focused context. With its distinct structure and properties, the $\mathcal{GPTS}$ provides an extensive framework for studying the subtleties of S_g^* functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms along with their interactions.

This paper is organized as follows. In the following Preliminaries section, we present the definitions and key terms required to comprehend S_g^* functions. The Main Results section is split into four subsections. The first of these explores S_g^*-continuous, S_g^*-irresolute, perfectly S_g^*-continuous, and strongly S_g^*-continuous functions along with S_g^*-open and S_g^*-closed mappings and other features of S_g^* functions in generalized topological spaces. The second subsection deal with the preservation of topological structures under S_g^*-homeomorphisms and $S_g^{*\#}$-homeomorphisms in a generalized topology. The analysis is expanded to S_g^* functions in generalized primal topological spaces in the third part, encompassing the same features as in the generalized case within the more challenging primal framework. The discussion in the fourth and final subsection involves generalized primal topological spaces, namely, S_g^*-homeomorphisms and $S_g^{*\#}$-homeomorphisms, discusses the behavior of these functions in the primal context, and contrasts its characteristics with those of the generalized topology. In conclusion, we provide a summary of our research and suggest future possibilities for investigation. By providing a new perspective on the behavior of these functions in diverse topological spaces, our research aims to advance knowledge of topological functions' features and consequences.

2. Preliminaries

The purpose of this section is to explain the main results through a discussion of various definitions and results from the literature.

Definition 1 ([20])**.** *Consider that $\mathcal{Y} \neq \phi$. A generalized topology on $\mathcal{Y}$ is defined as a collection $\tau_g \subseteq 2^{\mathcal{Y}}$ if $\phi \in \tau_g$ and if all unions of non-empty subsets of τ_g are components of τ_g. The $\mathcal{GTS}$ is the pair $(\mathcal{Y}, \tau_g)$.*

Remark 1 ([10])**.** *The members of τ_g are indicated as τ_g-open. We consider $\mathcal{E}$ to be any subset of $(\mathcal{Y}, \tau_g)$. $\mathcal{E}$ is considered as τ_g-closed if $(\mathcal{Y} \setminus \mathcal{E})$ is τ_g-open. τ_g-closure of $\mathcal{E}$ is indicated by Cl_g. $(\mathcal{E})$ is the intersection of all τ_g-closed sets that contain $\mathcal{E}$. τ_g-interior is represented as Int_g. Finally, $(\mathcal{E})$ is the union of all τ_g-open sets contained in $\mathcal{E}$.*

Definition 2 ([21–24])**.**

1. *Assume that* $(\mathcal{W}, \tau_g)$ *is a* $\mathcal{GTS}$. *Let* $\mathcal{E}$ *be any subset of* $(\mathcal{W}.\ \tau_g)$ *is generalized semi-open if there is a* τ_g*-open set* $\mathcal{U}_\alpha$ *in* $\mathcal{W}$ *such that* $\mathcal{U}_\alpha \subseteq \mathcal{E} \subseteq Cl(\mathcal{U}_\alpha)$*, or, appropriately, if* $\mathcal{E} \subseteq Cl(Int(\mathcal{E}))$.
2. *A generalized semi-closed set is the complement of a generalized semi-open set. The family of all generalized semi-open sets in* $(\mathcal{W}.\ \tau_g)$ *is indicated as* τ_g*-SO.*
3. *The union of all generalized semi-open sets of* $\mathcal{W}$ *contained in* $\mathcal{E}$ *is the generalized semi-interior of* $\mathcal{E}$ *(briefly,* $s_g Int(\mathcal{E})$*).*
4. *Generalized semi-closure of* $\mathcal{E}$ *(briefly,* $s_g Cl(\mathcal{E})$*) is the intersection of all generalized semi-closed sets of* $\mathcal{W}$ *containing* $\mathcal{E}$.

Definition 3.

1. *Consider* $(\mathcal{W}, \tau_g)$ *as a* $\mathcal{GTS}$. *A subset* $\mathcal{E}$ *of* $\mathcal{W}$ *is generalized semi-generalized closed (briefly,* $\tau_g - Sg$*-closed) if* $s_g Cl(\mathcal{E}) \subseteq \mathcal{F}_\alpha$ *whenever* $\mathcal{E} \subseteq \mathcal{F}_\alpha$ *and* $\mathcal{F}_\alpha$ *is generalized semi-open in* $\mathcal{W}$*. Similarly,* $(\tau_g - Sg\text{-closed})^c$ *is generalized semi-generalized open, and is indicated as* $\tau_g - Sg$*-open.*
2. *The generalized semi-generalized interior (briefly,* $s_g Int^*(\mathcal{E})$*) of* $\mathcal{E}$ *is specified as the union of all* $\tau_g - Sg$*-open sets of* $\mathcal{W}$ *contained in* $\mathcal{E}$.
3. *Assuming a subset* $\mathcal{E}$ *of* $(\mathcal{W}, \tau_g)$*, the generalized semi-generalized closure (briefly,* $s_g Cl^*(\mathcal{E})$*) of* $\mathcal{E}$ *is termed as the intersection of all* $\tau_g - Sg$*-closed sets in* $\mathcal{W}$ *containing* $\mathcal{E}$.

Definition 4. *Assume* $(\mathcal{W}, \tau_g)$ *to be a* $\mathcal{GTS}$*; any subset* $\mathcal{E}$ *of generalized topological space is termed as generalized* S_g^**-open set (briefly,* τ_g*-*S_g^**-open) if there exists a* τ_g*-open set* $\mathcal{F}_\alpha$ *in* $\mathcal{W}$ *such that* $\mathcal{F}_\alpha \subseteq \mathcal{E} \subseteq s_g Cl^*(\mathcal{F}_\alpha)$*. The collection of all* τ_g*-*S_g^**-open sets in* $(\mathcal{W}, \tau_g)$ *is represented by* $S_g^*O(\mathcal{W}, \tau_g)$.

Definition 5 ([8,25])**.** *Consider* $\mathcal{W} \neq \phi$*. A collection* $\mathcal{P}$ *of* $2^{\mathcal{W}}$ *is indicated as primal on* $\mathcal{W}$ *if the following criteria are met:*

1. $\mathcal{W} \notin \mathcal{P}$.
2. *For* $\mathcal{D}, \mathcal{E} \subseteq \mathcal{W}$ *has* $\mathcal{E} \subseteq \mathcal{D}$ *such that* $\mathcal{D} \notin \mathcal{P}$ *if* $\mathcal{E} \notin \mathcal{P}$.
3. *For* $\mathcal{D}, \mathcal{E} \subseteq \mathcal{W}$*; then,* $\mathcal{D} \cap \mathcal{E} \notin \mathcal{P}$ *whenever* $\mathcal{D} \notin \mathcal{P}$ *and* $\mathcal{E} \notin \mathcal{P}$.

A pair $(\mathcal{W}, \tau_g)$ *with a primal* $\mathcal{P}$ *on* $\mathcal{W}$ *is termed generalized primal topological space* $(\mathcal{GPTS})$*, symbolized as* $(\mathcal{W}, \tau_g, \mathcal{P})$*. The members of* $(\mathcal{W}, \tau_g, \mathcal{P})$ *are known as* $(\tau_g, \mathcal{P})$*-open sets, while their complements are considered* $(\tau_g, \mathcal{P})$*-closed sets.*

Definition 6 ([26])**.** *Consider* $(\mathcal{W}, \tau_g, \mathcal{P})$ *to be a* $(\mathcal{GPTS})$*. Suppose an operator* $Cl^\circ{:}2^{\mathcal{W}} \to 2^{\mathcal{W}}$*, defined as* $Cl^\circ\ (\mathcal{E}) = \mathcal{E} \cup \mathcal{E}^{\ \circ}$*, for a subset* $\mathcal{E}$ *of* $\mathcal{W}$*. Here,* Cl° *is a Kuratowskis closure operator.*

Definition 7 ([27])**.**

1. *Assume that* $(\mathcal{W}, \tau_g, \mathcal{P})$ *is a* $\mathcal{GPTS}$. *A subset* $\mathcal{E}$ *of* $(\mathcal{W}, \tau_g, \mathcal{P})$ *is generalized primal semi-open if there exists a* τ_g*-open set* $\mathcal{U}_\alpha$ *in* $\mathcal{W}$ *such that* $\mathcal{U}_\alpha \subseteq \mathcal{E} \subseteq Cl^\circ\ (\mathcal{U}_\alpha)$*, or equivalently if* $\mathcal{E} \subseteq Cl^\circ(Int(\mathcal{E}))$.
2. *The generalized primal semi-closed set is the complement of the generalized primal semi-open set. Thus,* $(\mathcal{W}, \tau_g, \mathcal{P})$ *is the collection of all generalized primal semi-open sets in* $(\mathcal{W}, \tau_g, \mathcal{P})$.
3. *The* $(\tau_g, \mathcal{P})$ *semi-interior of* $\mathcal{E}$ *(briefly,* $(\tau_g, \mathcal{P})sInt(\mathcal{E})$ *is the union of all generalized primal semi-open sets of* $\mathcal{W}$ *contained in* $\mathcal{E}$.
4. *The intersection of all generalized primal semi-closed sets of* $\mathcal{W}$ *containing* $\mathcal{E}$ *is* $(\tau_g, \mathcal{P})$ *(semi-closure of* $\mathcal{E}$*, briefly* $(\tau_g, \mathcal{P})sCl(\mathcal{E})$*).*

Definition 8 ([28])**.** *Assume* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. The mapping* $\mathrm{j} : \mathcal{W} \to \mathcal{Y}$ *is termed a* τ_g*-continuous function if there is an inverse image that is* τ_g*-open in* $(\mathcal{W}, \tau_{g_1})$ *for every* τ_g*-open set in* $(\mathcal{Y}, \tau_{g_2})$*.*

3. Main Results

This instance summarizes the main results of our study on S_g^* functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms, emphasizing how they behave in generalized primal topological spaces as well as generalized topological spaces. Four subsections make up this section, each focusing on particular characteristics of S_g^* functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms in various topological frameworks.

3.1. S_g^* Functions in $\mathcal{GTS}$

The fundamental characteristics and definitions of τ_g-S_g^* functions in $\mathcal{GTS}$ are covered in this section. We first define τ_g-S_g^*-continuous and τ_g-S_g^*-irresolute functions, then examine their properties and how they go beyond the ideas of standard continuity. After that, the topic of τ_g-S_g^*-open and τ_g-S_g^*-closed mappings are discussed, along with their functions and ramifications in these generalized frameworks. Strongly τ_g-S_g^*-continuous and perfectly τ_g-S_g^*-continuous functions are examined at the section's conclusion, emphasizing their significance for comprehending advanced topological aspects.

3.1.1. τ_g-S_g^*-Continuous Functions

Definition 9. *Assume* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. If* j^{-1} $(\mathcal{O}_\gamma)$ *is a* τ_g*-semi-open set in* $(\mathcal{W}, \tau_{g_1})$ *for all* τ_g*-open sets* $\mathcal{O}_\gamma \in (\mathcal{Y}, \tau_{g_2})$*, then the mapping* $\mathrm{j} : \mathcal{W} \to \mathcal{Y}$ *is indicated as* τ_g*-semi-continuous.*

Definition 10. *Assume* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. If* $\mathrm{j}^{-1}(\mathcal{O}_\gamma)$ *is a* τ_g*–Sg-closed set in* $(\mathcal{W}, \tau_{g_1})$ *for all* τ_g*-closed sets* $\mathcal{O}_\gamma \in (\mathcal{Y}, \tau_{g_2})$*, then the mapping* $\mathrm{j} : \mathcal{W} \to \mathcal{Y}$ *is indicated as* τ_g*–Sg-continuous.*

Definition 11. *Assume* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. The function* $\mathrm{j} : \mathcal{W} \to \mathcal{Y}$ *is indicated as* τ_g*–*S_g^**-continuous if there is an inverse image that is* τ_g*–*S_g^**-open in* $(\mathcal{W}, \tau_{g_1})$ *for every* τ_g*-open set in* $(\mathcal{Y}, \tau_{g_2})$*.*

Theorem 1. *Every* τ_g*-continuous function is* τ_g*–*S_g^**-continuous.*

Proof. Consider the mapping $\mathrm{j}:\mathcal{W} \to \mathcal{Y}$ to be a τ_g-continuous function; then, according to Definition 8, $\forall\ \mathcal{O}_\gamma \in (\mathcal{Y}, \tau_{g_2})$, $\mathrm{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-open in $(\mathcal{W}, \tau_{g_1})$. As every τ_g-open is τ_g–S_g^*-open $\Rightarrow \mathrm{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g–S_g^*-open in $(\mathcal{W}, \tau_{g_1})$; thus, $\mathrm{j} : \mathcal{W} \to \mathcal{Y}$ is τ_g–S_g^*-continuous. $\square$

Remark 2. *The subsequent illustration indicates that the converse of Theorem 1 is not valid.*

Example 1. *Suppose* $\mathcal{W} = \{d, k, q\}$*,* $\mathcal{Y} = \{w, m\}$ *and* $\tau_{g_1} = \{\phi, \mathcal{W}, \{d\}\}$ *along with* $S_g^*O(\mathcal{W}, \tau_{g_1}) = \{\phi, \mathcal{W}, \{d\}, \{d, k\}, \{d, q\}\}$ *and* $\tau_{g_2} = \{\phi, \mathcal{Y}, \{w\}\}$*. Define* $\mathrm{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *by* $\mathrm{j}(d) = \mathrm{j}(k) = w$ *and* $\mathrm{j}(q) = m$*. Here,* $\mathrm{j}^{-1}\{w\} = \{d, k\}$*, which is* τ_g*–*S_g^*O *but not* τ_g*-open. Hence,* j *is* τ_g*–*S_g^**-continuous but not* τ_g*-continuous.*

Theorem 2. *Every* τ_g*–*S_g^**-continuous function is* τ_g*-semi-continuous.*

Proof. Assume that $\mathrm{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is a τ_g–S_g^*-continuous mapping. Consider $\mathcal{O}_\gamma$ as a τ_g-open set in $(\mathcal{Y}, \tau_{g_2})$. Because j is a τ_g–S_g^*-continuous function, $\Rightarrow \mathrm{j}^{-1}(\mathcal{O}_\gamma)$ is a

τ_g–S_g^*-open set in $\mathcal{W}$. Notably, every τ_g–S_g^*-open set is τ_g-semi-open; consequently, $\mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is a τ_g-semi-open set in $\mathcal{W} \Rightarrow \mathfrak{j}$ and is τ_g-semi-continuous. □

Remark 3. *As illustrated below, the converse of Theorem 2 is false.*

Example 2. *Suppose* $\mathcal{W} = \{d, k, q\}$, $\mathcal{Y} = \{w, m, v\}$ *and* $\tau_{g_1} = \{\phi, \{d\}, \{q\}, \{d, q\}\} = S_g^*O(\mathcal{W}, \tau_{g_1})$. $\tau_{g_2} = \{\phi, \{w, m\}\}$. *Define* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *by* $\mathfrak{j}(d) = w$, $\mathfrak{j}(k) = m$ *and* $\mathfrak{j}(q) = v$. *Here,* $\mathfrak{j}^{-1}\{w, m\} = \{d, k\}$, *which is* τ_g*-semi-open but not* τ_g–S_g^*O. *Hence,* $\mathfrak{j}$ *is* τ_g*-semi-continuous but not* τ_g–S_g^**-continuous.*

Theorem 3. *Every* τ_g–S_g^**-continuous function is* τ_g*–Sg-continuous.*

Proof. Assume $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is a τ_g–S_g^*-continuous mapping. Consider $\mathcal{O}_\gamma$ as a τ_g-open set in $(\mathcal{Y}, \tau_{g_2})$. Because $\mathfrak{j}$ is τ_g–S_g^*-continuous $\Rightarrow$, $\mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-S_g^*-open in $\mathcal{W}$. As every τ_g–S_g^*-open set is τ_g–Sg-open $\Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g–Sg-open in $\mathcal{W}$. Thus, $\mathfrak{j}$ is τ_g–Sg-continuous. □

Remark 4. *The subsequent illustration demonstrates that the converse of Theorem 3 is not valid.*

Example 3. *Suppose that* $\mathcal{W} = \{d, k, q\}$, $\mathcal{Y} = \{w, m, v\}$ *and* $\tau_{g_1} = \{\phi, \{d\}, \{q\}, \{d, q\}\} = S_g^*O(\mathcal{W}, \tau_{g_1})$. $\tau_{g_2} = \{\phi, \{w, m\}\}$. *Define* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *by* $\mathfrak{j}(d) = w$, $\mathfrak{j}(k) = m$ *and* $\mathfrak{j}(q) = v$. *Here,* $\mathfrak{j}^{-1}\{w, m\} = \{d, k\}$ *which is* τ_g*–Sg-open but not* τ_g-S_g^*O. *Hence,* $\mathfrak{j}$ *is* τ_g*–Sg-continuous but not* τ_g–S_g^**-continuous.*

Theorem 4. *If* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is* τ_g–S_g^**-continuous and* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g*-continuous, then* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g–S_g^**-continuous.*

Proof. Consider $\mathcal{O}_\gamma$ as τ_g-open in $\mathcal{Z}$. Because $\mathfrak{j}$ is τ_g-continuous $\Rightarrow$, $\mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-open in $\mathcal{Y}$. Furthermore, $\mathfrak{f}$ is τ_g–S_g^*-continuous $\Rightarrow \mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{O}_\gamma))$ is τ_g-S_g^*-open in $\mathcal{W}$. Consequently, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{O}_\gamma)) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{O}_\gamma)$ is τ_g–S_g^*-open in $\mathcal{W} \Rightarrow \mathfrak{j}o\mathfrak{f}$ is τ_g–S_g^*-continuous. □

Theorem 5. *Assume* $\mathcal{W}$ *and* $\mathcal{Y}$ *as* $\mathcal{GTS}$ *and a mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *from* $\mathcal{W}$ *to* $\mathcal{Y}$. *Then,* $\mathfrak{j}$ *is* τ_g-S_g^**-continuous iff every* τ_g*-closed set in* $\mathcal{Y}$ *has an inverse image that is* τ_g–S_g^**-closed in* $\mathcal{W}$.

Proof. Necessity: Assume $\mathcal{E}$ to be any τ_g-closed set in $\mathcal{Y} \Rightarrow \mathcal{Y} \setminus \mathcal{E}$ is τ_g-open in $\mathcal{W}$. Because $\mathfrak{j}$ is τ_g–S_g^*-continuous $\Rightarrow$, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{E})$ is τ_g-S_g^*-open in $\mathcal{W}$. Consequently, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{E}) = \mathcal{W} \setminus (\mathfrak{j})^{-1}(\mathcal{E})$ is τ_g-S_g^*-open in $\mathcal{W}$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-S_g^*-closed in $\mathcal{W}$.
Sufficiency: Assume $\mathcal{O}_\gamma$ is τ_g-open set in $\mathcal{Y} \Rightarrow \mathcal{Y} \setminus \mathcal{O}_\gamma$ is τ_g-closed in $\mathcal{W}$. According to the presumptions, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{O}_\gamma)$ is τ_g-S_g^*-closed in $\mathcal{W}$. Consequently, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{O}_\gamma) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-S_g^*-closed in $\mathcal{W} \Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-S_g^*-open in $\mathcal{W}$. Thus, $\mathfrak{j}$ is τ_g-S_g^*-continuous. □

Theorem 6. *If* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is* τ_g-S_g^**-continuous, then* $\mathfrak{j}(\tau_g\text{-}S_g^*Cl(\mathcal{E})) \subseteq Cl_g(\mathfrak{j}\,(\mathcal{E}))$.

Proof. Because $\mathfrak{j}\,(\mathcal{E}) \subseteq Cl_g(\mathfrak{j}\,(\mathcal{E}))$, $\mathcal{E} \subseteq \mathfrak{j}^{-1}\,(Cl_g(\mathfrak{j}(\mathcal{E})))$. As $\mathfrak{j}$ is τ_g–S_g^*-continuous and $Cl_g(\mathfrak{j}(\mathcal{E}))$ is a τ_g-closed set in $\mathcal{Y} \Rightarrow \mathfrak{j}^{-1}\,(Cl_g(\mathfrak{j}(\mathcal{E})))$ is τ_g-S_g^*-closed set in $\mathcal{W}$. Consequently, τ_g-$S_g^*\mathrm{Cl}(\mathcal{E}) \subseteq \mathfrak{j}^{-1}\,(Cl_g(\mathfrak{j}(\mathcal{E}))) \Rightarrow \mathfrak{j}(\tau_g\text{-}S_g^*\mathrm{Cl}(\mathcal{E})) \subseteq Cl_g(\mathfrak{j}(\mathcal{E}))$. □

Theorem 7. *Assume two generalized topological spaces* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$. *If* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$, *then the subsequent illustrations are equivalent.*

1. $\mathfrak{j}$ *is* τ_g-S_g^**-continuous.*

2. *For every τ_g-closed set in $\mathcal{Y}$, there is an inverse image that is τ_g-S_g^*-closed in $\mathcal{W}$.*
3. τ_g-$S_g^*Cl(\jmath^{-1}(\mathcal{E})) \subseteq \jmath^{-1}\,(Cl_g(\mathcal{E}))\ \forall\, \mathcal{E} \in \mathcal{Y}$.
4. $\jmath(\tau_g$-$S_g^*Cl(\mathcal{E})) \subseteq Cl_g(\jmath(\mathcal{E}))\ \forall\, \mathcal{E}$ *in* $\mathcal{W}$.
5. $\jmath^{-1}\,(Int_g(\mathcal{D})) \subseteq \tau_g$-$S_g^*Int(\jmath^{-1}\,(\mathcal{D}))$ *for every* $\mathcal{D}$ *in* $\mathcal{Y}$.

Proof. (1) $\Rightarrow$ (2) indicated by Theorem 5.

(2) $\Rightarrow$ (3) assume any subset $\mathcal{E}$ of $\mathcal{Y}$. Then, $\mathrm{Cl}_g(\mathcal{E})$ is $\tau_g - closed$ in $\mathcal{Y}$. Consequently, according to (2), $\jmath^{-1}\,(Cl_g(\mathcal{E}))$ is τ_g-S_g^*-closed in $\mathcal{W}$. Hence, $\jmath^{-1}\,(Cl_g(\mathcal{E})) = \tau_g$-$S_g^*\mathrm{Cl}(\jmath^{-1}\,(Cl_g(\mathcal{E}))) \supseteq \tau_g$-$S_g^*\mathrm{Cl}(\jmath^{-1}\,(\mathcal{E}))$.

(3) $\Rightarrow$ (4) let $\mathcal{E}$ be any τ_g-open subset of $\mathcal{W}$. By (3), $\jmath^{-1}\,(Cl_g(\mathcal{E})) \supseteq \tau_g$-$S_g^*\mathrm{Cl}(\jmath^{-1}\,(\mathcal{E})) \supseteq \tau_g$-$S_g^*\mathrm{Cl}((\mathcal{E})$. Hence, $\jmath\,(\tau_g$-$S_g^*\mathrm{Cl}(\mathcal{E})) \subseteq Cl_g(\jmath(\mathcal{E}))$.

(4) $\Rightarrow$ (5) assume $\jmath(\tau_g$-$S_g^*\mathrm{Cl}\,(\mathcal{E})) \subseteq Cl_g(\jmath(\mathcal{E}))$ for every $\mathcal{E} \in \mathcal{W}$. Then, τ_g-$S_g^*\mathrm{Cl}\,(\mathcal{E}) \subseteq \jmath^{-1}\,(Cl_g(\jmath(\mathcal{E}))) \Rightarrow \mathcal{W} - (\tau_g$-$S_g^*\mathrm{Cl}\,(\mathcal{E})) \supseteq \mathcal{W} - (\jmath^{-1}(Cl_g(\jmath(\mathcal{E})))) \Rightarrow \tau_g$–$S_g^*\mathrm{Int}(\mathcal{W} - \mathcal{E}) \supseteq \jmath^{-1}(Int_g(\mathcal{Y} - \jmath(\mathcal{E})))$. Then, τ_g–$S_g^*\mathrm{Int}(\jmath^{-1}(\mathcal{D})) \supseteq \jmath^{-1}(Int_g(\mathcal{D}))$ for every set $\mathcal{D} = \mathcal{Y} - \jmath(\mathcal{E})$ in $\mathcal{Y}$.

(5) $\Rightarrow$ (1) consider $\mathcal{E}$ to be any τ_g-open in $\mathcal{Y} \Rightarrow \jmath^{-1}\,(Int_g(\mathcal{E})) \subseteq \tau_g$-$S_g^*\mathrm{Int}(\jmath^{-1}(\mathcal{E})) \Rightarrow \jmath^{-1}(\mathcal{E}) \subseteq \tau_g$-$S_g^*\mathrm{Int}(\jmath^{-1}(\mathcal{E}))$. Additionally, $\jmath^{-1}(\mathcal{E}) \supseteq \tau_g$-$S_g^*\mathrm{Int}(\jmath^{-1}(\mathcal{E}))$. Hence, $\jmath^{-1}(\mathcal{E}) = \tau_g$-$S_g^*\mathrm{Int}(\jmath^{-1}(\mathcal{E}))$. Consequently, $\jmath^{-1}\,(\mathcal{E})$ is τ_g-S_g^*-open in $\mathcal{W} \Rightarrow$ (1). $\square$

Now we provide the relationship of the above theorems in Figure 1.

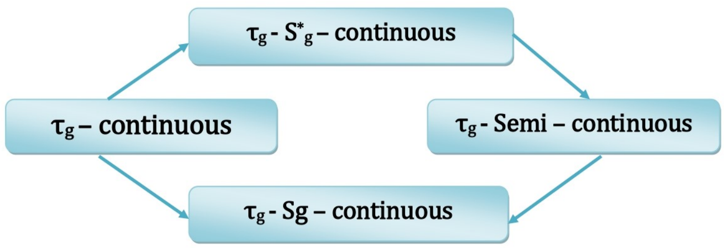

Figure 1. The relations of all theorems above concerning mappings.

3.1.2. τ_g-S_g^*-Irresolute Function

Definition 12. *Assume $(\mathcal{W}, \tau_{g_1})$ and $(\mathcal{Y}, \tau_{g_2})$ as $\mathcal{GTS}$. A map $\jmath : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is τ_g-S_g^*-irresolute if for every τ_g-S_g^*-open set in $(\mathcal{Y}, \tau_{g_2})$ there is an inverse image that is τ_g-S_g^*-open in $\mathcal{W}$.*

Similarly, a map $\jmath : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is τ_g-S_g^*-irresolute if for every τ_g-S_g^*-closed set in $(\mathcal{Y}, \tau_{g_2})$ there is an inverse image that is τ_g-S_g^*-closed in $\mathcal{W}$.

Theorem 8. *If $\jmath : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is a τ_g-S_g^*-irresolute map then $\jmath$ is τ_g–S_g^*-continuous.*

Proof. Assume $\mathcal{O}_\gamma$ to be an τ_g-open set in $\mathcal{Y}$. Subsequently, every τ_g-open set is τ_g-S_g^*-open $\Rightarrow \mathcal{O}_\gamma$ is τ_g-S_g^*-open in $\mathcal{Y}$. Because $\jmath$ is τ_g-S_g^*-irresolute $\Rightarrow$, $\jmath^{-1}\,(\mathcal{O}_\gamma)$ is τ_g-S_g^*-open in $\mathcal{W}$. Thus, $\jmath$ is τ_g-S_g^*-continuous. However, the converse of this statement is invalid. $\square$

Example 8. *Suppose $\mathcal{W} = \{d, k, q\} = \mathcal{Y}$, $\tau_{g_1} = \{\phi, \{d\}, \{q\}, \{d, q\}, \mathcal{W}\} = S_g^*O(\mathcal{W}, \tau_{g_1})$, $\tau_{g_2} = \{\phi, \{d\}, \{d, k\}, \mathcal{Y}\}$ and $(\mathcal{Y}, \tau_{g_1})S_g^*O = \{\phi, \{d\}, \{d, k\}, \{d, q\}, \mathcal{Y}\}$. Define $\jmath : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ by $\jmath(d) = k$, $\jmath(k) = q$ and $\jmath(q) = d$. Here, $\jmath^{-1}\{d, q\} = \{k, q\}$, which is not τ_g-S_g^*O in $\mathcal{W}$. Hence, $\jmath$ is not τ_g–S_g^*-irresolute.*

Theorem 9. *Assume $(\mathcal{W}, \tau_{g_1})$, $(\mathcal{Y}, \tau_{g_2})$ and $(\mathcal{Z}, \tau_{g_3})$ as $\mathcal{GTS}$. If $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is τ_g-S_g^*-irresolute and $\jmath : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ is τ_g-S_g^*-irresolute, then $\jmath \circ \mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ is τ_g-S_g^*-irresolute.*

Proof. Assume $\mathcal{C}$ is τ_g-S_g^*-open in $\mathcal{Z}$. As $\mathfrak{j}$ is τ_g-S_g^*-irresolute, $\mathfrak{j}^{-1}$ $(\mathcal{C})$ is τ_g-S_g^*-open in $\mathcal{Y}$. Additionally, $\mathfrak{f}$ is τ_g-S_g^*-irresolute, and $\mathfrak{f}^{-1}$ $(\mathfrak{j}^{-1}(\mathcal{C}))$ is τ_g-S_g^*-open in $\mathcal{W}$. Consequently, $\mathfrak{f}^{-1}$ $(\mathfrak{j}^{-1}(\mathcal{C}))$ = $(\mathfrak{j}\mathfrak{o}\mathfrak{f})^{-1}$ $(\mathcal{C})$ is τ_g-S_g^*-open in $\mathcal{W}$. Hence, $\mathfrak{j}\mathfrak{o}\mathfrak{f}$ is τ_g-S_g^*-irresolute. □

Theorem 10. *If* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is* τ_g-S_g^**-irresolute and* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g-S_g^**-continuous, then* $\mathfrak{j}\mathfrak{o}\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g-S_g^**-continuous.*

Proof. Assume $\mathcal{E}$ to be a τ_g-open set and $\mathcal{C}$ to be τ_g-S_g^*-open in $\mathcal{Z}$. As $\mathfrak{j}$ is τ_g-S_g^*-continuous, $\mathfrak{j}^{-1}$ $(\mathcal{E})$ is τ_g-S_g^*-open in $\mathcal{Y}$. Additionally, $\mathfrak{f}$ is τ_g-S_g^*-irresolute, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{C}))$ is τ_g-S_g^*-open in $\mathcal{W}$. However, $\mathfrak{f}^{-1}$ $(\mathfrak{j}^{-1}(\mathcal{C}))$ = $(\mathfrak{j}\mathfrak{o}\mathfrak{f})^{-1}(\mathcal{C})$. Hence, $\mathfrak{j}\mathfrak{o}\mathfrak{f}$ is τ_g-S_g^*-continuous. □

3.1.3. Strongly τ_g–S_g^*-Continuous Function

Definition 13. *Consider* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *to be any two* $\mathcal{GTS}$*. A mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is called strongly* τ_g–S_g^**-continuous if for every* τ_g–S_g^**-open set in* $\mathcal{Y}$ *there is an inverse image that is* τ_g*-open in* $\mathcal{W}$*.*

Theorem 11. *If* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is strongly* τ_g-S_g^**-continuous, then* $\mathfrak{j}$ *is a* τ_g*-continuous function.*

Proof. Assume $\mathcal{E}$ to be any τ_g-open set in $\mathcal{Y}$. Because every τ_g-open set is τ_g–S_g^*-open $\Rightarrow$, $\mathcal{E}$ is τ_g–S_g^*-open in $\mathcal{Y}$. Because $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is strongly τ_g-S_g^*-continuous $\Rightarrow$ $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}$ is τ_g-continuous. However, the converse does not hold, as demonstrated by the illustration below. □

Example 11. *Assume* $\mathcal{W} = \{d, k, q\} = \mathcal{Y}$, $\tau_{g_1} = \{\phi, \{d\}, \{q\}, \{d, q\}, \mathcal{W}\}$ *and* $\tau_{g_2} = \{\phi, \{d\}, \{d, k\}, \mathcal{Y}\}$*. Define* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *by* $\mathfrak{j}(d) = k$, $\mathfrak{j}(k) = q$ *and* $\mathfrak{j}(q) = d$*. Here,* $S_g^*O(\mathcal{W}, \tau_{g_1}) = \{\phi, \{d\}, \{q\}, \{d, q\}, \mathcal{W}\}$ *and* $(\mathcal{Y}, \tau_{g_2})S_g^*O = \{\phi, \{d\}, \{d, k\}, \{d, q\}, \mathcal{Y}\} \Rightarrow \mathfrak{j}$ *is* τ_g*-continuous but not strongly* τ_g–S_g^**-continuous, as* $\mathfrak{j}^{-1}\{d, q\} = \{k, q\}$*, which is not* τ_g*-open in* $\mathcal{W}$*.*

Theorem 12. *The mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is strongly* τ_g–S_g^**-continuous if and only if for every* τ_g–S_g^**-closed set in* $\mathcal{Y}$ *there is an inverse image that is* τ_g*-closed in* $\mathcal{W}$*.*

Proof. Consider $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ as strongly τ_g–S_g^*-continuous. Assume $\mathcal{E}$ to be any τ_g–S_g^*-closed set in $\mathcal{Y}$. Then, $\mathcal{E}^c$ is τ_g–S_g^*-open in $\mathcal{Y}$. Because $\mathfrak{j}$ is strongly τ_g–S_g^*-continuous $\Rightarrow$, $\mathfrak{j}^{-1}(\mathcal{E}^c)$ is τ_g-open in $\mathcal{W}$. However, $\mathfrak{j}^{-1}(\mathcal{E}^{\,c}) = \mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{E}) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-closed in $\mathcal{W}$.

Conversely, for every τ_g–S_g^*-closed set in $\mathcal{Y}$, consider an inverse image that is τ_g-closed in $\mathcal{W}$. Assume $\mathcal{E}$ to be any τ_g–S_g^*-open set in $\mathcal{Y} \Rightarrow \mathcal{E}^c$ is τ_g–S_g^*-closed set in $\mathcal{Y}$. Now, $\mathfrak{j}^{-1}(\mathcal{E}^c)$ is τ_g-closed in $\mathcal{W}$ but $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W} \Rightarrow \mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is strongly τ_g-S_g^*-continuous. □

Theorem 13. *Consider* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *as strongly* τ_g–S_g^**-continuous and the mapping* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ *as* τ_g–S_g^**-continuous; then,* $\mathfrak{j}\mathfrak{o}\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g*-continuous.*

Proof. Assume $\mathcal{E}$ to be any τ_g-open set in $\mathcal{Z}$. As $\mathfrak{j}$ is τ_g–S_g^*-continuous $\Rightarrow$ $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-S_g^*-open in $\mathcal{Y}$. Furthermore, $\mathfrak{f}$ is strongly τ_g–S_g^*-continuous, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ is τ_g-open in $\mathcal{W}$. Therefore, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ = $(\mathfrak{j}\mathfrak{o}\mathfrak{f})^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}\mathfrak{o}\mathfrak{f}$ is τ_g-continuous. □

Theorem 14. *Consider* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is strongly* τ_g–S_g^**-continuous and the mapping* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g–S_g^**-irresolute. Then,* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ *is strongly* τ_g-S_g^**-continuous.*

Proof. Assume $\mathcal{E}$ to be any τ_g–S_g^*-open set in $\mathcal{Z}$. As $\mathfrak{j}$ is τ_g–S_g^*-irresolute $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-S_g^*-open in $\mathcal{Y}$. Furthermore, $\mathfrak{f}$ is strongly τ_g–S_g^*-continuous, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ is τ_g-open in $\mathcal{W}$. Therefore, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E})) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ is strongly τ_g–S_g^*-continuous. □

Theorem 15. *Consider* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *as* τ_g–S_g^**-continuous and the mapping* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ *as strongly* τ_g–S_g^**-continuous; then,* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g–S_g^**-irresolute.*

Proof. Let $\mathcal{E}$ be any τ_g–S_g^*-open set in $\mathcal{Z}$. As $\mathfrak{j}$ is strongly τ_g–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{Y}$. Furthermore, $\mathfrak{f}$ is τ_g–S_g^*-continuous, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ is τ_g–S_g^*-open in $\mathcal{W}$. However, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E})) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ is τ_g–S_g^*-irresolute. □

3.1.4. Perfectly τ_g–S_g^*-Continuous Function

Definition 14. *Consider* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. A mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is called perfectly* τ_g–S_g^**-continuous if for every* τ_g–S_g^**-open set in* $\mathcal{Y}$ *there is an inverse image that is* τ_g*-open and* τ_g*-closed in* $\mathcal{W}$*.*

Theorem 16. *If a mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is perfectly* τ_g–S_g^**-continuous, then* $\mathfrak{j}$ *is strongly* τ_g–S_g^**-continuous.*

Proof. Assume $\mathcal{E}$ to be any τ_g–S_g^*-open set in $\mathcal{Y}$. Because $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is perfectly τ_g–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}$ is strongly τ_g-S_g^*-continuous. The subsequent illustration demonstrates why the contrary of this theorem need not be true. □

Example 16. *Assume* $\mathcal{W} = \{d,k,q\} = \mathcal{Y}$, $\tau_{g_1} = \{\phi, \{d\}, \{d,k\}, \{d,q\}, \mathcal{W}\} = \tau_{g_2}$*. Define an identity map* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$*. Thus,* $\mathfrak{j}$ *is* τ_g–S_g^**-continuous but not perfectly* τ_g–S_g^**-continuous, as* $\mathfrak{j}^{-1}\{d\} = \{d\}$ *is* τ_g*-open in* $\mathcal{W}$ *but not* τ_g*-closed in* $\mathcal{W}$*.*

Theorem 17. *A mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is perfectly* τ_g–S_g^**-continuous iff for every* τ_g-S_g^**-closed set* $\mathcal{E}$ *in* $\mathcal{Y}$ *there is an inverse image* $\mathfrak{j}^{-1}(\mathcal{E})$ *that is both* τ_g*-open and* τ_g*-closed in* $\mathcal{W}$*.*

Proof. Consider $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ as perfectly τ_g-S_g^*-continuous. Let $\mathcal{E}$ be any τ_g-S_g^*-closed set in $\mathcal{Y}$. Then, $\mathcal{E}^c$ is a τ_g–S_g^*-open set in $\mathcal{Y}$. Because $\mathfrak{j}$ is perfectly τ_g–S_g^*-continuous, $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E}^c)$ is both τ_g-open and τ_g-closed in $\mathcal{W}$. However, $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathcal{W} \backslash \mathfrak{j}^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is both τ_g-open and τ_g-closed in $\mathcal{W}$.

Conversely, consider that for every τ_g–S_g^*-closed set in $\mathcal{Y}$ there is an inverse image that is both τ_g-open and τ_g-closed in $\mathcal{W}$. Assume $\mathcal{E}$ to be any τ_g-S_g^*-open set in $\mathcal{Y} \Rightarrow \mathcal{E}^c$ is τ_g-S_g^*-closed set in $\mathcal{Y}$. Because $\mathfrak{j}^{-1}(\mathcal{E}^c)$ is both τ_g-open and τ_g-closed in $\mathcal{W}$. But, $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathcal{W} \backslash \mathfrak{j}^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is both τ_g-open and τ_g-closed in $\mathcal{W} \Rightarrow \mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is perfectly τ_g-S_g^*-continuous. □

Now we provide a relationship of the above theorems in the below Figure 2.

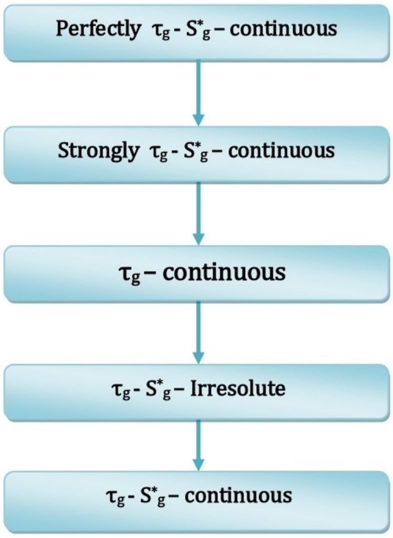

Figure 2. The relations of all the above theorems concerning mappings.

3.1.5. τ_g–S_g^*-Open Map

Definition 15. *Assume* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. A map* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is a* τ_g–S_g^**-open map if for every* τ_g*-open set* $\mathcal{E}$ *in* $\mathcal{W}, \mathfrak{j}(\mathcal{E})$ *is* τ_g-S_g^**-open in* $\mathcal{Y}$*.*

Theorem 18. *If* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is a* τ_g*-open map, then* $\mathfrak{j}$ *is a* τ_g-S_g^**-open map.*

Proof. Assume $\mathcal{E}$ to be any τ_g-open set in $\mathcal{W}$; then, $\mathfrak{j}(\mathcal{E})$ is τ_g-open in $\mathcal{Y}$. Because every τ_g-open set is τ_g–S_g^*-open, $\Rightarrow \mathfrak{j}(\mathcal{E})$ is τ_g–S_g^*-open in $\mathcal{Y}$. Hence, $\mathfrak{j}$ is a τ_g–S_g^*-open map. However, the converse is invalid. □

Example 18. *Assume* $\mathcal{W} = \{d, k, q\} = \mathcal{Y}, \tau_{g_1} = \{\phi, \{d\}, \{d, k\}, \mathcal{W}\}$ *and* $\tau_{g_2} = \{\phi, \{d\}, \mathcal{Y}\}$*. Define* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *by* $\mathfrak{j}(d) = d, \mathfrak{j}(k) = k$ *and* $\mathfrak{j}(q) = q$*. Here,* $\mathfrak{j}^{-1}\{d, q\} = \{d, q\}$*, which belongs to* τ_g-S_g^*O *in* $\mathcal{Y}$ *but does not belong to* $\tau_{g_2} \Rightarrow$*. Hence, the inverse image is* τ_g–S_g^*O *but not* τ_g*-open.*

3.1.6. τ_g–S_g^*-Closed Map

Definition 16. *Assume* $(\mathcal{W}, \tau_{g_1})$ *and* $(\mathcal{Y}, \tau_{g_2})$ *as* $\mathcal{GTS}$*. A map* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is said to be a* τ_g–S_g^**-closed map if for every* τ_g*-closed set* $\mathcal{E}$ *in* $\mathcal{W}, \mathfrak{j}(\mathcal{E})$ *is* τ_g-S_g^**-closed in* $\mathcal{Y}$*.*

Theorem 19. *If* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is a* τ_g*-closed map, then* $\mathfrak{j}$ *is a* τ_g-S_g^**-closed map.*

Proof. Assume $\mathcal{E}$ to be any τ_g-closed set in $\mathcal{W}$; then, $\mathfrak{j}(\mathcal{E})$ is τ_g-closed in $\mathcal{Y}$. Because every τ_g-closed set is τ_g–S_g^*-closed $\Rightarrow \mathfrak{j}(\mathcal{E})$ is τ_g–S_g^*-closed in $\mathcal{Y}$. Hence, $\mathfrak{j}$ is τ_g–S_g^*-closed map. The subsequent example demonstrates that the converse is invalid. □

Example 19. *Assume* $\mathcal{W} = \{d, k, q\} = \mathcal{Y}, \tau_{g_1} = \{\phi, \{d\}, \{d, k\}, \mathcal{W}\}$ *and* $\tau_{g_2} = \{\phi, \{d\}, \mathcal{Y}\}$*. Define* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *by* $\mathfrak{j}(d) = d, \mathfrak{j}(k) = k$ *and* $\mathfrak{j}(q) = q$*. Here,* $\mathfrak{j}^{-1}\{d, q\} = \{d, q\}$*, which belongs to* τ_g-S_g^*C *in* $\mathcal{Y}$ *but does not belong to* τ_{g_2}*-closed* $\Rightarrow$ *inverse image is* τ_g-S_g^*C *but not* τ_g*-closed.*

Theorem 20. *If* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ *is* τ_g*-closed and* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g–S_g^**-closed, then* $\mathfrak{j} \circ \mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ *is* τ_g–S_g^**-closed.*

Proof. Assume $\mathcal{D}$ to be any τ_g-closed in $\mathcal{W}$. Because $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is a τ_g-closed map, $\Rightarrow \mathfrak{f}(\mathcal{D})$ is τ_g-closed in $\mathcal{Y}$. Furthermore, $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ is τ_g–S_g^*-closed $\Rightarrow \mathfrak{j}(\mathfrak{f}(\mathcal{D}))$ is τ_g–S_g^*-closed in $\mathcal{Z}$. Hence, $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ is a τ_g–S_g^*-closed map. □

3.2. *τ_g–S_g^*-Homeomorphism and τ_g–$S_g^{*\#}$-Homeomorphism*

The idea of a τ_g–S_g^*-homeomorphisms is introduced and analyzed in this part, with an emphasis on how to maintain the structure of S_g^*-open and S_g^*-closed sets as we proceed between generalized topological spaces. We have discovered a new type of mapping that is weaker than τ_g–S_g^*-homeomorphism, known as τ_g–$S_g^{*\#}$-homeomorphism. According to the composition of mappings, the τ_g–$S_g^{*\#}$-homeomorphism subclass of the τ_g–S_g^*-homeomorphism class is closed.

3.2.1. τ_g–S_g^*-Homeomorphism

Definition 17 ([10])**.** *Assume $(\mathcal{W}, \tau_g)$ to be a generalized topological space and $\mathcal{E}$ to be any subset of $\mathcal{W}$; then, $\mathcal{E}$ is known as:*

1. *Generalized closed (g-closed), if for any open set $\mathfrak{U}_\alpha$ of $\mathcal{W}$ we have $Cl(\mathcal{E}) \subseteq \mathfrak{U}_\alpha$ whenever $\mathcal{E} \subseteq \mathfrak{U}_\alpha$.*
2. *Generalized semi-closed (gs-closed), if for any open set $\mathfrak{U}_\alpha$ of $\mathcal{W}$ we have $sCl(\mathcal{E}) \subseteq \mathfrak{U}_\alpha$ whenever $\mathcal{E} \subseteq \mathfrak{U}_\alpha$.*
3. *Semi-generalized closed (sg-closed), if for any semi-open set $\mathfrak{U}_\alpha$ of $\mathcal{W}$ we have $sCl(\mathcal{E}) \subseteq \mathfrak{U}_\alpha$ whenever $\mathcal{E} \subseteq \mathfrak{U}_\alpha$.*

Definition 18. *Assume that a mapping $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$ is called:*

1. *Generalized continuous (briefly, G_{τ_g}-continuous) if for each $\mathfrak{U}_\alpha$ closed in $(\mathcal{Y}, \tau_{g_2})\mathfrak{j}^{-1}(\mathfrak{U}_\alpha)$ is g-closed in $(\mathcal{W}, \tau_{g_1})$.*
2. *Generalized semi-continuous (briefly, GS_{τ_g}-continuous) if for each $\mathfrak{U}_\alpha$ closed in $(\mathcal{Y}, \tau_{g_2})\mathfrak{j}^{-1}(\mathfrak{U}_\alpha)$ is gs-closed in $(\mathcal{W}, \tau_{g_1})$.*
3. *Semi-generalized continuous (briefly, sG_{τ_g}-continuous) if for each $\mathfrak{U}_\alpha$ closed in $(\mathcal{Y}, \tau_{g_2})\mathfrak{j}^{-1}(\mathfrak{U}_\alpha)$ is sg-closed in $(\mathcal{W}, \tau_{g_1})$.*

Definition 19. *Assume that a bijective mapping $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$ is called:*

1. *A G_{τ_g}-homeomorphism if $\mathfrak{j}$ and $\mathfrak{j}^{-1}$ are both G_{τ_g}-continuous.*
2. *A GS_{τ_g}-homeomorphism if $\mathfrak{j}$ and $\mathfrak{j}^{-1}$ are both GS_{τ_g}-continuous.*
3. *A sG_{τ_g}-homeomorphism if $\mathfrak{j}$ and $\mathfrak{j}^{-1}$ are both sG_{τ_g}-continuous.*

Definition 20. *A bijection function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$ is called a τ_g-homeomorphism if $\mathfrak{j}$ is both τ_g-continuous and τ_g-open.*

Theorem 21. *For any bijection $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$, the following are equivalent:*

1. *The inverse function $\mathfrak{j}^{-1}$ is τ_g–S_g^*-continuous.*
2. *$\mathfrak{j}$ is a τ_g–S_g^*-open function.*
3. *$\mathfrak{j}$ is a τ_g–S_g^*-closed function*

Proof. Because $\mathfrak{j}$ is a bijection $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$, 1 ⇒ 2, as $\mathfrak{j}^{-1}$ is τ_g-continuous. Moreover, $\mathfrak{j}(\mathcal{E})$ is τ_g-open in $\mathcal{Y}$ whenever $\mathcal{E}$ is τ_g-open in $\mathcal{W} \Rightarrow \mathfrak{j}$ is τ_g-open.

2 ⇒ 3, as $\mathfrak{j}$ is τ_g-S_g^*-open. If $\mathcal{E}$ is τ_g-closed, $\Rightarrow \mathcal{E}^c$ is τ_g-open; then, $\mathfrak{j}(\mathcal{E}^c)$ is τ_g-S_g^*-open, that is, $\mathcal{Y}\backslash\mathfrak{j}(\mathcal{E})$ is τ_g-S_g^*-open $\Rightarrow \mathfrak{j}(\mathcal{E})$ is τ_g-S_g^*-closed, that is, $\mathfrak{j}$ is τ_g-S_g^*-closed.

3 ⇒ 1. If $\mathcal{E}$ is τ_g-open in $\mathcal{W}$, $\mathcal{E}^c$ is τ_g-closed $\Rightarrow \mathfrak{j}(\mathcal{E}^c)$ is τ_g-S_g^*-closed, that is, $\mathcal{Y}\backslash\mathfrak{j}(\mathcal{E})$ is τ_g-S_g^*-closed in $\mathcal{E} \Rightarrow \mathfrak{j}(\mathcal{E})$ is τ_g–S_g^*–τ_g-open in $\mathcal{E}$. Hence, $\mathfrak{j}^{-1}$ is τ_g–S_g^*-continuous. □

Definition 21. *A bijection function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$ is called a τ_g–S_g^*-homeomorphism if $\mathfrak{j}$ is both τ_g–S_g^*-continuous and τ_g–S_g^*-open.*

Theorem 22. *Any bijection function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$ which is τ_g–S_g^*-continuous has the analogous statements listed below:*

1. *The function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is τ_g–S_g^*-open.*
2. *$\mathfrak{j}$ is τ_g–S_g^*-homeomorphism.*
3. *$\mathfrak{j}$ is a τ_g–S_g^*-closed function*

Proof. 1 ⇒ 2 Since $\mathfrak{j}$ is both τ_g-S_g^*-continuous and τ_g-S_g^*-open, $\mathfrak{j}$ is τ_g-S_g^*-homeomorphism.

2 ⇒ 3 $\mathfrak{j}$ is τ_g–S_g^*-continuous and $\mathfrak{j}$ is τ_g–S_g^*-homeomorphism $\Rightarrow \mathfrak{j}$ is τ_g–S_g^*-open; hence, it is τ_g–S_g^*-closed by Theorem 21.

3 ⇒ 1 $\mathfrak{j}$ is a τ_g-S_g^*-closed $\Rightarrow \mathfrak{j}$ is a τ_g-S_g^*-open by the Theorem 21. □

Remark 5. *From the subsequent illustrations, a composition of two τ_g–S_g^*-homeomorphisms need not be a τ_g–S_g^*-homeomorphism.*

Example 22. *Let $\mathcal{W} = \{d, k, q, w\}$ and $\tau_g = \{\mathcal{W}, \{k\}, \{q\}, \{d, k\}, \{k, q\}, \{d, k, q\}\}$. Consider $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{W}, \tau_{g_1})$ and $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{W}, \tau_{g_1})$ defined by $\mathfrak{f}(d) = d$, $\mathfrak{f}(k) = q$, $\mathfrak{f}(q) = k$ and $\mathfrak{f}(w) = w$ and $\mathfrak{j}(d) = q$, $\mathfrak{j}(k) = d$, $\mathfrak{j}(q) = w$ and $\mathfrak{j}(w) = k$, respectively. Then, $\mathfrak{f}$ and $\mathfrak{j}$ are τ_g–S_g^*-homeomorphisms; however, the composition $\mathfrak{j}o\mathfrak{f}$ is not a τ_g–S_g^*-homeomorphism.*

Example 22. *Let $\mathcal{W} = \mathcal{Y} = \mathcal{Z} = \{d, k, q, w\}$, $\tau_{g_1} = \{\phi, \{d, k\}, \{q, w\}, \mathcal{W}\}$, $\tau_{g_2} = \{\phi, \{d\}, \{k\}, \{d, k\}, \{d, k, q\}, \mathcal{Y}\}$ and $\tau_{g_3} = \{\phi, \{d\}, \{d, k\}, \{d, k, q\}, \mathcal{Z}\}$ be generalized topologies on $\mathcal{W}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Then, the mappings $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ and $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ are defined by $\mathfrak{f}(d) = w$, $\mathfrak{f}(k) = k$ and $\mathfrak{f}(q) = q$, $\mathfrak{f}(w) = d$, $\mathfrak{j}(d) = k$, $\mathfrak{j}(k) = d$, $\mathfrak{j}(q) = w$, $\mathfrak{j}(w) = q$. Here, $\mathfrak{f}$ and $\mathfrak{j}$ are both τ_g–S_g^*-homeomorphisms, but $\mathfrak{j}o\mathfrak{f}$ is not a τ_g–S_g^*-homeomorphism.*

Theorem 23. *Every τ_g-homeomorphism is a τ_g–S_g^*-homeomorphism.*

Proof. Assume $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ to be a τ_g-homeomorphism. If $\mathcal{O}_\gamma$ is an open set in $\mathcal{W}$, then $\mathfrak{j}(\mathcal{O}_\gamma)$ is τ_g-open in $\mathcal{Y}$, as $\mathfrak{j}$ is a τ_g-open function. Hence, $\mathfrak{j}(\mathcal{O}_\gamma)$ is τ_g-S_g^*-open and $\mathfrak{j}$ is a τ_g–S_g^*-open function. Further, the τ_g-continuity $\Rightarrow \mathfrak{j}$ is τ_g–S_g^*-continuous, that is, $\mathfrak{j}$ is a τ_g–S_g^*-homeomorphism. □

Remark 6. *The converse of the above Theorem 23 is invalid, as demonstrated by the subsequent illustrations.*

Example 23. *Suppose $\mathcal{W} = \{d, k, q\}$, $\mathcal{Y} = \{w, m\}$ and $\tau_{g_1} = \{\phi, \mathcal{W}, \{d\}\}$ and $S_g^*O(\mathcal{W}, \tau_{g_1}) = \{\phi, \mathcal{W}, \{d\}, \{d, k\}, \{d, q\}\}$. $\tau_{g_2} = \{\phi, \mathcal{Y}, \{w\}\}$. Define $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ by $\mathfrak{j}(d) = \mathfrak{j}(k) = w$ and $\mathfrak{j}(q) = m$. Here, $\mathfrak{j}^{-1}\{w\} = \{d, k\}$ which is τ_g–S_g^*O but not τ_g-open. Hence, $\mathfrak{j}$ is a τ_g–S_g^*-homeomorphism but not a τ_g-homeomorphism.*

Example 23. *Consider $\mathcal{W} = \mathcal{Y} = \{d, k\}$, $\tau_{g_1} = \{\phi, \{d\}, \mathcal{W}\}$ and $\tau_{g_2} = \{\phi, \{k\}, \{d, k\}, \mathcal{Y}\}$. Assuming $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ as the identity function, $\Rightarrow$ $\mathfrak{j}$ is a τ_g–S_g^*-homeomorphism, but not a τ_g-homeomorphism.*

Theorem 24. *Every G_{τ_g}-homeomorphism is a τ_g–S_g^*-homeomorphism.*

Proof. Assume $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ to be a G_{τ_g}-homeomorphism. As every G_{τ_g}-continuous map is τ_g–S_g^*-continuous and any G_{τ_g}-open map is also τ_g–S_g^*-open, any G_{τ_g}-homeomorphism is a τ_g–S_g^*-homeomorphism. □

Remark 7. *The converse of the above Theorem 24 is invalid, as demonstrated by the subsequent illustrations.*

Example 24. *Suppose $\mathcal{W} = \{d, k, q\}$, $\mathcal{Y} = \{w, m\}$ and $\tau_{g_1} = \{\phi, \mathcal{W}, \{d\}\}$ and $S_g^*O(\mathcal{W}, \tau_{g_1}) = \{\phi, \mathcal{W}, \{d\}, \{d, k\}, \{d, q\}\}$. $\tau_{g_2} = \{\phi, \mathcal{Y}, \{w\}\}$. Define $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ by $\mathfrak{j}(d) = \mathfrak{j}(k) = w$ and $\mathfrak{j}(q) = m$. Here, $\mathfrak{j}^{-1}\{w\} = \{d, k\}$ which is τ_g–S_g^*O but not τ_g-open. Hence, $\mathfrak{j}$ is a τ_g-S_g^*-homeomorphism but not a G_{τ_g}-homeomorphism.*

3.2.2. τ_g–$S_g^{*\#}$-Homeomorphism

Definition 22. *A bijective mapping $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ from a generalized topological space $\mathcal{W}$ into a generalized topological space $\mathcal{Y}$ is known as a τ_g–$S_g^{*\#}$-homeomorphism if $\mathfrak{j}$ and $(\mathfrak{j})^{-1}$ are both τ_g–S_g^*-irresolute.*

Theorem 25. *Any τ_g–$S_g^{*\#}$-homeomorphism is a τ_g–S_g^*-homeomorphism, but not conversely. A τ_g–$S_g^{*\#}$-homeomorphism is a subclass of the class of τ_g–S_g^*-homeomorphisms.*

Proof. Because any τ_g–$S_g^{*\#}$-open map is a τ_g–S_g^*-open map [29], the proof follows from Theorem 3.32 [29]. In Example 22, the mapping $\mathfrak{j}$ is a τ_g–S_g^*-homeomorphism but not a τ_g–$S_g^{*\#}$-homeomorphism. Because $\{\mathrm{k}, \mathrm{q}\}$ is a τ_g–S_g^*-closed set in $\mathcal{Y}$ but $((\mathfrak{j})^{-1})^{-1}(\{\mathrm{k}, \mathrm{q}\}) = \{\mathrm{d}, \mathrm{w}\}$ is not τ_g–S_g^*-closed in $\mathcal{Z}$, it is the case that $(\mathfrak{j})^{-1}$ is not τ_g–S_g^*-irresolute. Hence, $\mathfrak{j}$ is not a τ_g–$S_g^{*\#}$-homeomorphism. □

Theorem 26. *If $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ and $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}) \to (\mathcal{Z}, \tau_{g_3})$ are τ_g–$S_g^{*\#}$-homeomorphisms, then the composition $\mathfrak{j}o\mathfrak{f}$:$(\mathcal{W}, \tau_{g_1}) \to (\mathcal{Z}, \tau_{g_3})$ is a τ_g–$S_g^{*\#}$-homeomorphism.*

Proof. Assume $\mathcal{E}$ to be a τ_g–S_g^*-open set in $\mathcal{Z}$. Now, $(\mathfrak{j}o\mathfrak{f})^{-1}$ $(\mathcal{E}) = \mathfrak{f}^{-1}((\mathfrak{j})^{-1}(\mathcal{E}))$ = $(\mathfrak{f})^{-1}$ $(\mathcal{D})$, where $\mathcal{D} = (\mathfrak{j})^{-1}$ $(\mathcal{E})$. By hypothesis, $\mathcal{D}$ is a τ_g–S_g^*-open set in $\mathcal{Y}$; again by hypothesis, $(\mathfrak{f})^{-1}$ $(\mathcal{D})$ is a τ_g–S_g^*-open set in $\mathcal{W}$. Thus, $\mathfrak{j}o\mathfrak{f}$ is τ_g–S_g^*-irresolute. Moreover, for any τ_g–S_g^*-open set $\mathcal{K}$ in $(\mathcal{W}, \tau_{g_1})$, $\mathfrak{j}o\mathfrak{f}(\mathcal{K}) = \mathfrak{j}(\mathfrak{f}(\mathcal{K}))$ = $\mathfrak{j}(\mathcal{Q})$, where $\mathcal{Q} = \mathfrak{f}(\mathcal{K})$. By hypothesis, $\mathfrak{f}(\mathcal{K})$ is a τ_g–S_g^*-open set in $(\mathcal{Y}, \tau_{g_2})$ and $\mathfrak{j}(\mathfrak{f}(\mathcal{K}))$ = $\mathfrak{j}(\mathcal{Q})$ is a τ_g–S_g^*-open set in $(\mathcal{Z}, \tau_{g_3})$; thus, $(\mathfrak{j}o\mathfrak{f})^{-1}$ is τ_g–S_g^*-irresolute, proving that $\mathfrak{j}o\mathfrak{f}$ is a τ_g–$S_g^{*\#}$-homeomorphism. □

Corollary 27. *In the collection of all topological spaces, a τ_g–$S_g^{*\#}$-homeomorphism is an equivalence relation.*

Proof. Because the composition of a bijective mapping is an equivalence relation $\Rightarrow$ τ_g–$S_g^{*\#}$-homeomorphism is an equivalence relation. □

3.3. S_g^* Functions in $\mathcal{GPTS}$

This section builds on the preceding one, but analyzes $\mathcal{GPTS}$. With its distinct open set structures, $\mathcal{GPTS}$ provides an innovative perspective on S_g^* functions. In this particular context, we study how the features of $\mathcal{P}_g$–S_g^*-continuous, $\mathcal{P}_g$–S_g^*-irresolute, and other related functions appear. Furthermore, the behavior of perfectly $\mathcal{P}_g$–S_g^*-continuous and strongly $\mathcal{P}_g$–S_g^*-continuous functions in $\mathcal{GPTS}$ is examined, and additional aspects or challenges are highlighted.

3.3.1. $\mathcal{P}_g$–S_g^*-Continuous Functions

Definition 23. *Assume $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ and $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ to be two $\mathcal{GPTS}$. The mapping $\mathfrak{j} : \mathcal{W} \to \mathcal{Y}$ is termed as a $\mathcal{P}_g$-continuous function if every τ_g-open set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ has an inverse image that is τ_g-open in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$.*

Definition 24. *Assume $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ and $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ to be two $\mathcal{GPTS}$. If $\mathfrak{f}^{-1}(\mathcal{O}_\gamma)$ is a $\mathcal{P}_g$-semi-open set in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ for all τ_g-open sets $\mathcal{O}_\gamma \in (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$, then the mapping $\mathfrak{j} : \mathcal{W} \to \mathcal{Y}$ is indicated as $\mathcal{P}_g$-semi-continuous.*

Definition 25. *Assume $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ and $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ as $\mathcal{GPTS}$. If $\mathfrak{f}^{-1}(\mathcal{O}_\gamma)$ is a $(\mathcal{Y}, \tau_g)Sg$-closed set in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ for all τ_g-closed set $\mathcal{O}_\gamma \in (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$, then the mapping $\mathfrak{j} : \mathcal{W} \to \mathcal{Y}$ is indicated as $\mathcal{P}_g$–Sg-continuous.*

Definition 26. *The function $\mathfrak{j} : \mathcal{W} \to \mathcal{Y}$ is indicated as $\mathcal{P}_g$–S_g^*-continuous if every τ_g-open-set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ has an inverse image that is $\mathcal{P}_g$–S_g^*-open in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$.*

Theorem 28. *Every $\mathcal{P}_g$-continuous function is $\mathcal{P}_g$–S_g^*-continuous.*

Proof. Consider the mapping $\mathfrak{j}$: $\mathcal{W} \to \mathcal{Y}$ to be $\mathcal{P}_g$-continuous; then, according to Definition 23, $\forall\ \mathcal{O}_\gamma \in (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$, $\mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-open in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$. As every τ_g-open is $\mathcal{P}_g$–S_g^*-open $\Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-open in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$. Thus, $\mathfrak{j} : \mathcal{W} \to \mathcal{Y}$ is $\mathcal{P}_g$–S_g^*-continuous. □

Remark 8. *The subsequent illustration demonstrates that the converse of Theorem 28 is not valid.*

Example 28. *Let $\mathcal{W} = \{d,k,q\}$, τ_{g_1} =$\{\phi, \{d\}, \{k\}, \{d,k\}\}$, $\mathcal{Y} = \{w, m\}$, τ_{g_2} =$\{\phi, \{w\}\}$ and $\mathcal{P}_\alpha = \{\phi, \{d\}, \{q\}, \{d,q\}\}$. Define $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ by $\mathfrak{j}(k) = \mathfrak{j}(q) = \{w\}$ and $\mathfrak{j}(d) = \{m\}$. Here, $\mathfrak{j}^{-1}\{w\} = \{k, q\}$, which is $\mathcal{P}_g$–S_g^*O but not τ_g-open. Hence, $\mathfrak{j}$ is $\mathcal{P}_g$-=S_g^*-continuous but not $\mathcal{P}_g$-continuous.*

Theorem 29. *Every $\mathcal{P}_g$=-S_g^*-continuous function is $\mathcal{P}_g$-semi-continuous.*

Proof. Assume that $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is $\mathcal{P}_g$-S_g^*-continuous. Consider $\mathcal{O}_\gamma$ as a τ_g-open set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$. Because $\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-continuous function $\Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is a $\mathcal{P}_g$-S_g^*-open in $\mathcal{W}$. But, every $\mathcal{P}_g$–S_g^*-open set is $\mathcal{P}_g$-semi-open. Consequently, $\mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is a $\mathcal{P}_g$-semi-open set in $\mathcal{W} \Rightarrow \mathfrak{j}$ is $\mathcal{P}_g$-semi-continuous. □

Remark 9. *The subsequent illustration indicates that the converse of Theorem 29 is not valid.*

Example 29. *Consider $\mathcal{W} = \mathcal{Y} = \{d,k,q\}$, $\tau_{g_1} = \tau_{g_2} = \{\phi, \{d\}, \{k\}, \{d,k\}\}$ and p=$\{\phi, \{d\}, \{q\}, \{d,q\}\}$. Here $\mathcal{P}_\alpha$-$S_g^*O = \{\phi, \mathcal{Y}, \{d\}, \{k\}, \{d,k\}\}$ and $\mathcal{P}_\alpha$-SO=$\{\phi, \{d\}, \{k\}, \{d,k\}, \{d,q\}, \{k,q\}, \mathcal{Y}\}$. Define $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ by identity map. Here, $\mathfrak{j}^{-1}\{d, q\} = \{d, q\}$, which is $\mathcal{P}_g$-semi-continuous but not $\mathcal{P}_g$–S_g^*-continuous.*

Theorem 30. *Every $\mathcal{P}_g$–S_g^*-continuous function is $\mathcal{P}_g$–Sg-continuous.*

Proof. Assume that $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is a $\mathcal{P}_g$–S_g^*-continuous mapping. Consider $\mathcal{O}_\gamma$ as a τ_g-open set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$. Because $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. As every $\mathcal{P}_g$–S_g^*-open is $\mathcal{P}_g$–Sg-open $\Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–Sg-open in $\mathcal{W}$. Thus, $\mathfrak{j}$ is $\mathcal{P}_g$–Sg-continuous. □

Remark 10. *The subsequent illustration demonstrates that the converse of Theorem 30 is not valid.*

Example 30. *Assume $\mathcal{W} = \{d, k, q\} = \mathcal{Y}$ and $\tau_{g_1} = \tau_{g_2} = \{\phi, \{k\}\}$ and $p = \{\phi, \{k\}, \{q\}, \{k, q\}\}$. In this space $\mathcal{P}_\alpha$-$S_g^*O = \{\phi, \{k\}\}$ and $\mathcal{P}_\alpha$-$SgO = \{\phi, \{d\}, \{k\}, \{q\}, \{k, q\}, \{d, k\}\}$. Define $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ by identity map. Here, $\mathfrak{j}^{-1}\{d\} = \{d\}$, which is $\mathcal{P}_g$–Sg-continuous but not $\mathcal{P}_g$–S_g^*-continuous.*

Theorem 31. *If $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is $\mathcal{P}_g$–S_g^*-continuous and $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is $\mathcal{P}_g$-continuous, then $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is $\mathcal{P}_g$–S_g^*-continuous.*

Proof. Consider $\mathcal{O}_\gamma$ as τ_g-open in $\mathcal{Z}$. Because $\mathfrak{j}$ is $\mathcal{P}_g$-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is τ_g-open in $\mathcal{Y}$. Furthermore, $\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{O}_\gamma))$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Consequently, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{O}_\gamma)) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W} \Rightarrow \mathfrak{j}o\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-continuous. □

Theorem 32. *Assume a mapping $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ from $\mathcal{GPTS}$ $\mathcal{W}$ into a $\mathcal{GPTS}$ $\mathcal{Y}$. Then, $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous iff every τ_g-closed set in $\mathcal{Y}$ has an inverse image that is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W}$.*

Proof. Necessity: Assume $\mathcal{E}$ to be any τ_g-closed set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \Rightarrow \mathcal{Y} \setminus \mathcal{E}$ is τ_g-open in $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$. Because $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Consequently, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{E}) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W}$.

Sufficiency: Assume $\mathcal{O}_\gamma$ to be a τ_g-open set in $\mathcal{Y} \Rightarrow \mathcal{Y} \setminus \mathcal{O}_\gamma$ is τ_g-closed in $\mathcal{W}$. According to the presumptions, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W}$. Consequently, $\mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{O}_\gamma) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W} \Rightarrow \mathfrak{j}^{-1}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Thus, $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous. □

Theorem 33. *If $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is $\mathcal{P}_g$-continuous; then, $\mathfrak{j}(\mathcal{P}_g\text{-}S_g^*Cl(\mathcal{E})) \subseteq Cl_{gp}(\mathfrak{j}(\mathcal{E}))$.*

Proof. Because $\mathfrak{j}(\mathcal{E}) \subseteq Cl_{gp}(\mathfrak{j}(\mathcal{E}))$, $\mathcal{E} \subseteq \mathfrak{j}^{-1}(Cl_{gp}(\mathfrak{j}(\mathcal{E})))$. As $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous and $Cl_{gp}(\mathfrak{j}(\mathcal{E}))$ is a τ_g-closed set in $\mathcal{Y} \Rightarrow \mathfrak{j}^{-1}(Cl_{gp}(\mathfrak{j}(\mathcal{E})))$ is a $\mathcal{P}_g$–S_g^*-closed set in $\mathcal{W}$. Consequently, $\mathcal{P}_g\text{-}S_g^*\mathrm{Cl}(\mathcal{E}) \subseteq \mathfrak{j}^{-1}(Cl_{gp}(\mathfrak{j}(\mathcal{E}))) \Rightarrow \mathfrak{j}(\mathcal{P}_g\text{-}S_g^*\mathrm{Cl}(\mathcal{E})) \subseteq Cl_{gp}(\mathfrak{j}(\mathcal{E}))$. □

Theorem 34. *Assume any two generalized primal topological spaces $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ and $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$. If $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$, then the following illustrations are equivalent:*

1. *$\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous.*
2. *For every τ_g-closed set in $\mathcal{Y}$, there is an inverse image that is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W}$.*
3. *$\mathcal{P}_g\text{-}S_g^*Cl(\mathfrak{j}^{-1}(\mathcal{E})) \subseteq \mathfrak{j}^{-1}(Cl_{gp}(\mathcal{E})) \ \forall\, \mathcal{E} \in \mathcal{Y}$.*
4. *$\mathfrak{j}(\mathcal{P}_g\text{-}S_g^*Cl(\mathcal{E})) \subseteq Cl_{gp}(\mathfrak{j}(\mathcal{E})) \ \forall\, \mathcal{E}$ in $\mathcal{W}$.*
5. *$\mathfrak{j}^{-1}(Int_{gp}(\mathcal{D})) \subseteq \mathcal{P}_g\text{-}S_g^*Int(\mathfrak{j}^{-1}(\mathcal{D}))$ for every $\mathcal{D}$ in $\mathcal{Y}$.*

Proof. (1) $\Rightarrow$ (2) indicated by Theorem 32.

(2) $\Rightarrow$ (3) assume any subset $\mathcal{E}$ of $\mathcal{Y}$. Then, $Cl_{gp}(\mathcal{E})$ is $\tau_g - closed$ in $\mathcal{Y}$. Consequently, according to (2), $\mathfrak{j}^{-1}(Cl_{gp}(\mathcal{E}))$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W}$. Hence, $\mathfrak{j}^{-1}(Cl_{gp}(\mathcal{E})) = \mathcal{P}_g\text{-}S_g^*\mathrm{Cl}(\mathfrak{j}^{-1}(Cl_{gp}(\mathcal{E}))) \supseteq \mathcal{P}_g\text{-}S_g^*\mathrm{Cl}(\mathfrak{j}^{-1}(\mathcal{E}))$.

(3) $\Rightarrow$ (4) let $\mathcal{E}$ be any τ_g-open subset of $\mathcal{W}$. By (3), j^{-1} $(Cl_{gp}(\mathcal{E})) \supseteq \mathcal{P}_g$-$S_g^*$Cl($\mathrm{j}^{-1}$ $(\mathcal{E})$) $\supseteq \mathcal{P}_g$-S_g^*Cl(($\mathcal{E}$). Hence, j ($\mathcal{P}_g$-S_g^*Cl($\mathcal{E}$)) $\subseteq Cl_{gp}(\mathrm{j}(\mathcal{E}))$.

(4) $\Rightarrow$ (5) assume $\mathrm{j}(\mathcal{P}_g$-S_g^*Cl $(\mathcal{E})) \subseteq Cl_{gp}(\mathrm{j}(\mathcal{E}))$ for every $\mathcal{E} \in \mathcal{W}$. Then, $\mathcal{P}_g$-S_g^*Cl $(\mathcal{E}) \subseteq \mathrm{j}^{-1}$ $(Cl_{gp}(\mathrm{j}(\mathcal{E}))) \Rightarrow \mathcal{W}$ - ($\mathcal{P}_g$-S_g^*Cl $(\mathcal{E})) \supseteq \mathcal{W}$ - (j^{-1} $(Cl_g(\mathrm{j}(\mathcal{E}))))$ $\Rightarrow \mathcal{P}_g$-$S_g^*$Int($\mathcal{W}$ - $\mathcal{E}$) $\supseteq \mathrm{j}^{-1}$ $(Int_{gp}(\mathcal{Y}$ - $\mathrm{j}(\mathcal{E})))$. Then, $\mathcal{P}_g$-S_g^*Int($\mathrm{j}^{-1}(\mathcal{D})$) $\supseteq \mathrm{j}^{-1}(Int_{gp}(\mathcal{D}))$ for every set $\mathcal{D} = \mathcal{Y}$ - $\mathrm{j}(\mathcal{E})$ in $\mathcal{Y}$.

(5) $\Rightarrow$ (1) consider $\mathcal{E}$ to be any τ_g-open in $\mathcal{Y} \Rightarrow \mathrm{j}^{-1}$ $(Int_{gp}(\mathcal{E})) \subseteq \mathcal{P}_g$-$S_g^*$Int($\mathrm{j}^{-1}(\mathcal{E})$) $\Rightarrow$ $\mathrm{j}^{-1}(\mathcal{E}) \subseteq \mathcal{P}_g$-$S_g^*$Int($\mathrm{j}^{-1}(\mathcal{E})$). Additionally, $\mathrm{j}^{-1}(\mathcal{E}) \supseteq \mathcal{P}_g$-$S_g^*$Int($\mathrm{j}^{-1}(\mathcal{E})$). Hence, $\mathrm{j}^{-1}(\mathcal{E}) = \mathcal{P}_g$-$S_g^*$Int($\mathrm{j}^{-1}(\mathcal{E})$). Consequently, j^{-1} $(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W} \Rightarrow$ (1). □

Now, we provide the relationship of the above theorems in Figure 3.

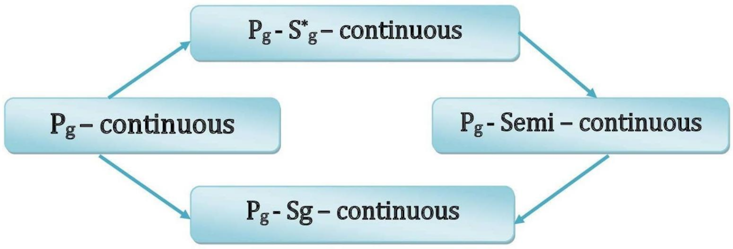

Figure 3. The relations of all the above theorems concerning mappings.

3.3.2. $\mathcal{P}_g$–S_g^*-Irresolute Function

Definition 27. *Assume $\mathcal{W}$ and $\mathcal{Y}$ as $\mathcal{GPTS}$ and that a mapping $\mathrm{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ from $\mathcal{W}$ to $\mathcal{Y}$ is $\mathcal{P}_g$-S_g^*-irresolute if for every $\mathcal{P}_g$–S_g^*-open set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ there is an inverse image that is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$.*

Remark 11. *A mapping $\mathrm{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is $\mathcal{P}_g$-S_g^*-irresolute if for every $\mathcal{P}_g$–S_g^*-closed set in $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ there is an inverse image that is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{W}$.*

Theorem 35. *If $\mathrm{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is a $\mathcal{P}_g$–S_g^*-irresolute map, then j is $\mathcal{P}_g$–S_g^*-continuous.*

Proof. Assume $\mathcal{O}_\gamma$ to be an τ_g-open set in $\mathcal{Y}$. Subsequently, every τ_g-open set is $\mathcal{P}_g$–S_g^*-open $\Rightarrow \mathcal{O}_\gamma$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Y}$. Because j is $\mathcal{P}_g$–S_g^*-irresolute $\Rightarrow \mathrm{j}^{-1}$ $(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Thus, j is $\mathcal{P}_g$–S_g^*-continuous. □

Remark 12. *The subsequent illustration demonstrates that above theorem's converse is invalid.*

Example 35. *Assuming $\mathcal{W} = \{d, k, q\}$, $\tau_{g_1} = \{\phi, \{k\}\} = (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)S_g^*O$ and $\mathcal{P}_\alpha = \{\phi, \{k\}, \{q\}, \{k, q\}\}$ and $\mathcal{Y} = \{d, k, q\}$, $\tau_{g_2} = \{\phi, \{d\}, \{k\}, \{d, k\}\}$ and $\mathcal{P}_\beta = \{\phi, \{d\}, \{q\}, \{d, q\}\}$, in this generalized primal topological space $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ open sets $= \{\phi, \{d\}, \{k\}, \{d, k\}\}$ and $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ $S_g^*O = \{\phi, \{d\}, \{k\}, \{d, k\}, \{d, k, q\}\}$. Defining $\mathrm{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ by j(k) = $\{k\} = \{d\}$ and j(q) = $\{q\}$ $\Rightarrow$ j is $\mathcal{P}_g$-S_g^*-continuous but not $\mathcal{P}_g$-S_g^*-irresolute.*

Theorem 36. *Assume $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$, $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ and $(\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ as $\mathcal{GPTS}$. If $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is $\mathcal{P}_g$–S_g^*-irresolute and $\mathrm{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is $\mathcal{P}_g$–S_g^*-irresolute, then $\mathrm{j}\mathfrak{o}\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is $\mathcal{P}_g$–S_g^*-irresolute.*

Proof. Assume $\mathcal{C}$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Z}$. As j is $\mathcal{P}_g$–S_g^*-irresolute, j^{-1} $(\mathcal{C})$ is $\mathcal{P}_g$-S_g^*-open in $\mathcal{Y}$. Additionally $\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-irresolute, $\mathfrak{f}^{-1}$ $(\mathrm{j}^{-1}(\mathcal{C}))$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Consequently, $\mathfrak{f}^{-1}$ $(\mathrm{j}^{-1}(\mathcal{C}))$ = $(\mathrm{j}\mathfrak{o}\mathfrak{f})^{-1}$ $(\mathcal{C})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. Hence, $\mathrm{j}\mathfrak{o}\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-irresolute. □

Theorem 37. *If* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is* $\mathcal{P}_g$–S_g^**-irresolute and* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$–S_g^**-continuous, then* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$–S_g^**-continuous.*

Proof. Assume $\mathcal{E}$ to be a τ_g-open set and $\mathcal{C}$ to be $\mathcal{P}_g$–S_g^*-open in $\mathcal{Z}$. As $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous, $\mathfrak{j}^{-1}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Y}$. Additionally $\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-irresolute and $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{C}))$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$. However, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{C})) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{C})$. Hence, $\mathfrak{j}o\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-continuous. □

3.3.3. Strongly $\mathcal{P}_g$–S_g^*-Continuous Function

Definition 28. *Consider* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *as* $\mathcal{GPTS}$*. A mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is called strongly* $\mathcal{P}_g$–S_g^**-continuous if for every* $\mathcal{P}_g$–S_g^**-open set in* $\mathcal{Y}$ *there is an inverse image that is* τ_g*-open in* $\mathcal{W}$*.*

Theorem 38. *If* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is strongly* $\mathcal{P}_g$-S_g^**-continuous; then,* $\mathfrak{j}$ *is a* $\mathcal{P}_g$*-continuous function.*

Proof. Assume $\mathcal{E}$ to be any τ_g-open set in $\mathcal{Y}$. Because every τ_g-open set is $\mathcal{P}_g$–S_g^*-open $\Rightarrow \mathcal{E}$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Y}$. Because $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is strongly $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}$ is $\mathcal{P}_g$-continuous. □

Remark 13. *The subsequent illustration demonstrates why the contrary of the preceding theorem need not be true.*

Example 38. *Assume* $\mathcal{W} = \{d, k, q\}$*,* $\tau_{g_1} = \{\phi, \{k\}\} = (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)S_g^*O$ *and* $\mathcal{P}_\alpha = \{\phi, \{k\}, \{q\}, \{k, q\}\}$ *and* $\mathcal{Y} = \{d, k, q\}$*,* $\tau_{g_2} = \{\phi, \{d\}, \{k\}, \{d, k\}\}$ *and* $\mathcal{P}_\beta = \{\phi, \{d\}, \{q\}, \{d, q\}\}$*. In this generalized primal topological space,* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *Open sets =* $\{\phi, \{d\}, \{k\}, \{d, k\}\}$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ $S_g^*O = \{\phi, \{d\}, \{k\}, \{d, k\}, \{d, k, q\}\}$*. Defining* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *by* $\mathfrak{j}(k) = \{k\} = \{d\}$ *and* $\mathfrak{j}(q) = \{q\} \Rightarrow \mathfrak{j}$ *is a* $\mathcal{P}_g$*-continuous function but not strongly* $\mathcal{P}_g$–S_g^**-continuous.*

Theorem 39. *The mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is strongly* $\mathcal{P}_g$–S_g^**-continuous if and only if for every* $\mathcal{P}_g$–S_g^**-closed set in* $\mathcal{Y}$ *there is an inverse image that is* τ_g*-closed in* $\mathcal{W}$*.*

Proof. Consider $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ as strongly $\mathcal{P}_g$–S_g^*-continuous. Assume $\mathcal{E}$ to be any $\mathcal{P}_g$–S_g^*-closed set in $\mathcal{Y}$. Then, $\mathcal{E}^c$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Y}$. Because $\mathfrak{j}$ is strongly $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E}^c)$ is τ_g-open in $\mathcal{W}$. But, $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathfrak{j}^{-1}(\mathcal{Y} \setminus \mathcal{E}) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-closed in $\mathcal{W}$.

Conversely, consider that for every $\mathcal{P}_g$–S_g^*-closed set in $\mathcal{Y}$ there is an inverse image that is τ_g-closed in $\mathcal{W}$. Assume $\mathcal{E}$ to be any $\mathcal{P}_g$–S_g^*-open set in $\mathcal{Y} \Rightarrow \mathcal{E}^c$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{Y}$. Because $\mathfrak{j}^{-1}(\mathcal{E}^c)$ is τ_g-closed in $\mathcal{W}$ but $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$, it is the case that $\mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W} \Rightarrow \mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is strongly $\mathcal{P}_g$–S_g^*-continuous. □

Theorem 40. *Consider* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is strongly* $\mathcal{P}_g$–S_g^**-continuous and the mapping* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$–S_g^**-continuous; then,* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$*-continuous.*

Proof. Assume $\mathcal{E}$ to be any τ_g-open set in $\mathcal{Z}$. As $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Y}$. Furthermore, as $\mathfrak{f}$ is strongly $\mathcal{P}_g$–S_g^*-continuous, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ is τ_g-open in $\mathcal{W}$. Therefore, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E})) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}o\mathfrak{f}$ is $\mathcal{P}_g$-continuous. □

Theorem 41. *Consider* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *as strongly* $\mathcal{P}_g$*–*S_g^**-continuous and the mapping* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *as* $\mathcal{P}_g$*–*S_g^**-irresolute; then,* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is strongly* $\mathcal{P}_g$*–*S_g^**-continuous.*

Proof. Assume $\mathcal{E}$ to be any $\mathcal{P}_g$-S_g^*-open set in $\mathcal{Z}$. As $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-irresolute, $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{Y}$. Furthermore, as $\mathfrak{f}$ is strongly $\mathcal{P}_g$–S_g^*-continuous, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ is τ_g-open in $\mathcal{W}$. Therefore, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E})) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is strongly $\mathcal{P}_g$–S_g^*-continuous. □

Theorem 42. *Consider* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *as* $\mathcal{P}_g$*–*S_g^**-continuous and the mapping* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *as strongly* $\mathcal{P}_g$*–*S_g^**-continuous; then,* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$*–*S_g^**-irresolute.*

Proof. Let $\mathcal{E}$ be any $\mathcal{P}_g$–S_g^*-open set in $\mathcal{Z}$. As $\mathfrak{j}$ is strongly $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{Y}$. Furthermore, $\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-continuous, $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E}))$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{W}$, but $\mathfrak{f}^{-1}(\mathfrak{j}^{-1}(\mathcal{E})) = (\mathfrak{j}o\mathfrak{f})^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is $\mathcal{P}_g$–S_g^*-irresolute. □

3.3.4. Perfectly $\mathcal{P}_g$–S_g^*-Continuous Function

Definition 29. *Consider* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *as* $\mathcal{GPTS}$*. A mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is called perfectly* $\mathcal{P}_g$*–*S_g^**-continuous if for every* $\mathcal{P}_g$*–*S_g^**-open set in* $\mathcal{Y}$ *there is an inverse image that is* τ_g*-open and* τ_g*-closed in* $\mathcal{W}$*.*

Theorem 43. *If a mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is perfectly* $\mathcal{P}_g$*–*S_g^**-continuous, then* $\mathfrak{j}$ *is strongly* $\mathcal{P}_g$*–*S_g^**-continuous.*

Proof. Assume $\mathcal{E}$ to be any $\mathcal{P}_g$-S_g^*-open set in $\mathcal{Y}$. Because $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$, is perfectly $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E})$ is τ_g-open in $\mathcal{W}$. Hence, $\mathfrak{j}$ is strongly $\mathcal{P}_g$–S_g^*-continuous. □

Remark 14. *The subsequent illustration demonstrates why the contrary of the preceding theorem need not be true.*

Example 43. *Assuming* $\mathcal{W} = \mathcal{Y} = \{d, k, q\}$, $\tau_{g_1} = \tau_{g_2} = \{\phi, \{d\}, \{k\}, \{d, k\}\}$ *and* $\mathcal{P}_\beta = \{\phi, \{d\}, \{q\}, \{d, q\}\}$*, in this generalized primal topological space* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *open sets* $= \{\phi, \{d\}, \{k\}, \{d, k\}\}$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ $S_g^*O = \{\phi, \{d\}, \{k\}, \{d, k\}, \{d, k, q\}\}$*. Defining* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *by* $\mathfrak{j}(k) = \{k\} = \{q\}$ *and* $\mathfrak{j}(d) = \{d\} \Rightarrow \mathfrak{j}$ *is strongly* $\mathcal{P}_g$*–*S_g^**-continuous but not perfectly* $\mathcal{P}_g$*–*S_g^**-continuous.*

Theorem 44. *A mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is perfectly* $\mathcal{P}_g$*–*S_g^**-continuous iff for every* $\mathcal{P}_g$*–*S_g^**-closed set* $\mathcal{E}$ *in* $\mathcal{Y}$ *there is an inverse image* $\mathfrak{j}^{-1}(\mathcal{E})$ *that is both* τ_g*-open and* τ_g*-closed in* $\mathcal{W}$*.*

Proof. Consider that $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is perfectly $\mathcal{P}_g$-S_g^*-continuous. Let $\mathcal{E}$ be any $\mathcal{P}_g$–S_g^*-closed set in $\mathcal{Y}$. Then, $\mathcal{E}^c$ is a $\mathcal{P}_g$–S_g^*-open set in $\mathcal{Y}$. Because $\mathfrak{j}$ is perfectly $\mathcal{P}_g$–S_g^*-continuous $\Rightarrow \mathfrak{j}^{-1}(\mathcal{E}^c)$ is both τ_g-open and τ_g-closed in $\mathcal{W}$. But, $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$. Hence, $\mathfrak{j}^{-1}(\mathcal{E})$ is both τ_g-open and τ_g-closed in $\mathcal{W}$.

Conversely, consider that for every $\mathcal{P}_g$-S_g^*-closed set in $\mathcal{Y}$ there is an inverse image that is both τ_g-open and τ_g-closed in $\mathcal{W}$. Assume $\mathcal{E}$ to be any $\mathcal{P}_g$–S_g^*-open set in $\mathcal{Y} \Rightarrow \mathcal{E}^c$ is $\mathcal{P}_g$–S_g^*-closed set in $\mathcal{Y}$. Because $\mathfrak{j}^{-1}(\mathcal{E}^c)$ is both τ_g-open and τ_g-closed in $\mathcal{W}$, but $\mathfrak{j}^{-1}(\mathcal{E}^c) = \mathcal{W} \setminus \mathfrak{j}^{-1}(\mathcal{E})$, we have that $\mathfrak{j}^{-1}(\mathcal{E})$ is both τ_g-open and τ_g-closed in $\mathcal{W} \Rightarrow \mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is perfectly $\mathcal{P}_g$–S_g^*-continuous. □

Now we provide the relationship of the above theorems in Figure 4.

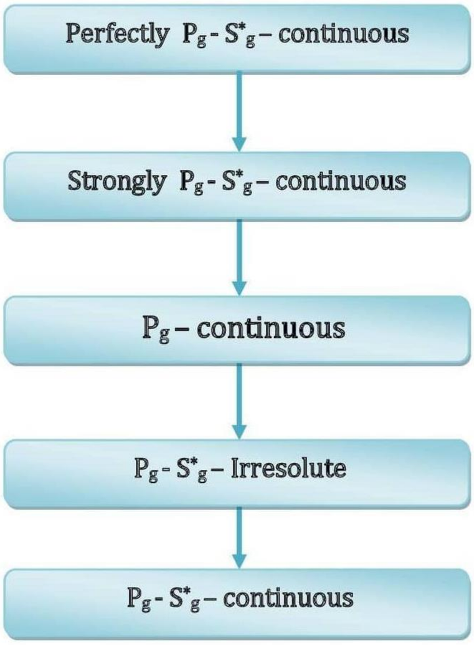

Figure 4. The relations of all above theorems concerning mappings.

3.3.5. $\mathcal{P}_g$–S^*_g-Open Map

Definition 30. *Assume* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *as* $\mathcal{GPTS}$. *A map* $\jmath : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \rightarrow (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is a* $\mathcal{P}_g$–S^*_g*-open map if for every* τ_g*-open set* $\mathcal{E}$ *in* $\mathcal{W}$, $\jmath(\mathcal{E})$ *is* $\mathcal{P}_g$–S^*_g*-open in* $\mathcal{Y}$.

Theorem 45. *If* $\jmath : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \rightarrow (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is* τ_g*-open, then* $\jmath$ *is a* $\mathcal{P}_g$–S^*_g*-open map.*

Proof. Assume $\mathcal{E}$ to be any τ_g-open set in $\mathcal{W}$; then, $\jmath(\mathcal{E})$ is τ_g-open in $\mathcal{Y}$. Because every τ_g-open set is $\mathcal{P}_g$–S^*_g-open $\Rightarrow$ $\jmath(\mathcal{E})$ is $\mathcal{P}_g$–S^*_g-open in $\mathcal{Y}$. Hence, $\jmath$ is a $\mathcal{P}_g$–S^*_g-open map. □

Remark 15. *The subsequent example demonstrates that the above theorem's converse is invalid.*

Example 45. *Let* $\mathcal{W} = \{d,k,q\}$, $\tau_{g_1} = \{\phi, \{d\}, \{k\}, \{d,k\}\}$, $\mathcal{Y} = \{w, m\}$, $\tau_{g_2} = \{\phi, \{w\}\}$ *and* $\mathcal{P}_\alpha = \{\phi, \{d\}, \{q\}, \{d,q\}\}$. *Define* $\jmath : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \rightarrow (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *by* $\jmath(k) = \jmath(q) = \{w\}$ *and* $\jmath(d) = \{m\}$. *Here,* $\jmath^{-1}\{w\} = \{k, q\}$, *which is* $\mathcal{P}_g$-S^*_gO *but not* τ_g*-open. Hence,* $\jmath$ *is a* $\mathcal{P}_g$-S^*_g*-open map but not a* τ_g*-open map.*

3.3.6. $\mathcal{P}_g$–S^*_g-Closed Map

Definition 31. *Assume* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *as* $\mathcal{GPTS}$. *A map* $\jmath : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \rightarrow$ *and* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is said to be* $\mathcal{P}_g$–S^*_g*-closed if for every* τ_g*-closed set* $\mathcal{E}$ *in* $\mathcal{W}$, $\jmath(\mathcal{E})$ *is* $\mathcal{P}_g$–S^*_g*-closed in* $\mathcal{Y}$.

Theorem 46. *If* $\jmath : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \rightarrow (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is a* τ_g*-closed map, then* $\jmath$ *is a* $\mathcal{P}_g$–S^*_g*-closed map.*

Proof. Assume $\mathcal{E}$ to be any τ_g-closed set in $\mathcal{W}$; then, $\jmath(\mathcal{E})$ is τ_g-closed in $\mathcal{Y}$. Because every τ_g-closed set is $\mathcal{P}_g$–S^*_g-closed $\Rightarrow$ $\jmath(\mathcal{E})$ is $\mathcal{P}_g$–S^*_g-closed in $\mathcal{Y}$. Hence, $\jmath$ is a $\mathcal{P}_g$–S^*_g-closed map. □

Remark 16. *The subsequent example demonstrates that the above theorem's converse is invalid.*

Example 46. *Similar to the illustration in Example 45.*

Theorem 47. *If* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *is* τ_g*-closed and* $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$–S_g^**-closed, then* $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is* $\mathcal{P}_g$–S_g^**-closed.*

Proof. Assume $\mathcal{D}$ to be any τ_g-closed in $\mathcal{W}$. Because $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is a τ_g-closed map $\Rightarrow \mathfrak{f}(\mathcal{D})$ is τ_g-closed in $\mathcal{Y}$. Furthermore, $\mathfrak{j} : (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is $\mathcal{P}_g$–S_g^*-closed $\Rightarrow \mathfrak{j}(\mathfrak{f}(\mathcal{D}))$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{Z}$. Hence, $\mathfrak{j}o\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ is a $\mathcal{P}_g$–S_g^*-closed map. □

3.4. $\mathcal{P}_g$–S_g^*-Homeomorphism and $\mathcal{P}_g - S_g^{*\#}$-Homeomorphism

The notions of $\mathcal{P}_g$–S_g^*-homeomorphism and $\mathcal{P}_g - S_g^{*\#}$-homeomorphism are examined in this section in the context of generalized primal topological spaces. Primal topology imposes extra limitations that are not presented in generalized topological spaces, which makes the study of homeomorphisms more complex in this context.

3.4.1. $\mathcal{P}_g$–S_g^*-Homeomorphism

Definition 32 ([8]). *Assume* $(\mathcal{W}, \tau_g, \mathcal{P}_g)$ *to be a generalized primal topological space and* $\mathcal{E}$ *to be any subset of* $\mathcal{W}$*; then,* $\mathcal{E}$ *is known as:*

1. *A generalized primal closed (gp-closed) set if for any open set* $\mathfrak{U}_\alpha$ *of* $\mathcal{W}$*,* $Cl^\circ(\mathcal{E}) \subseteq \mathfrak{U}_\alpha$ *whenever* $\mathcal{E} \subseteq \mathfrak{U}_\alpha$*.*
2. *A generalized primal semi-closed (gps-closed) set if for any open set* $\mathfrak{U}_\alpha$ *of* $\mathcal{W}$*,* $sCl^\circ(\mathcal{E}) \subseteq \mathfrak{U}_\alpha$ *whenever* $\mathcal{E} \subseteq \mathfrak{U}_\alpha$*.*
3. *A semi-generalized primal closed (sgp-closed) set if for any semi-open set* $\mathfrak{U}_\alpha$ *of* $\mathcal{W}$*,* $sCl^\circ(\mathcal{E}) \subseteq \mathfrak{U}_\alpha$ *whenever* $\mathcal{E} \subseteq \mathfrak{U}_\alpha$*.*

Definition 33. *Assume that a mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *from a generalized primal topological space* $\mathcal{W}$ *into a generalized primal topological space* $\mathcal{Y}$ *is called:*

1. *Generalized primal continuous (briefly,* $G_{\mathcal{P}_g}$*-continuous) if for each* $\mathfrak{U}_\alpha$ *closed in* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$*,* $\mathfrak{j}^{-1}(\mathfrak{U}_\alpha)$ *is gp-closed in* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$*.*
2. *Generalized primal semi-continuous (briefly,* $GS_{\mathcal{P}_g}$*-continuous) if for each* $\mathfrak{U}_\alpha$ *closed in* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$*,* $\mathfrak{j}^{-1}(\mathfrak{U}_\alpha)$ *is gps-closed in* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$*.*
3. *Semi-generalized primal continuous (briefly,* $sG_{\mathcal{P}_g}$*-continuous) if for each* $\mathfrak{U}_\alpha$ *closed in* $(\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *,* $\mathfrak{j}^{-1}(\mathfrak{U}_\alpha)$ *is sgp-closed in* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha)$*.*

Definition 34. *Assume that a bijective mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *from a generalized primal topological space* $\mathcal{W}$ *into a generalized primal topological space* $\mathcal{Y}$ *is called:*

1. *A* $G_{\mathcal{P}_g}$*-homeomorphism if* $\mathfrak{j}$ *and* $\mathfrak{j}^{-1}$ *are both* $G_{\mathcal{P}_g}$*-continuous.*
2. *A* $GS_{\mathcal{P}_g}$*-homeomorphism if* $\mathfrak{j}$ *and* $\mathfrak{j}^{-1}$ *are both* $GS_{\mathcal{P}_g}$*-continuous.*
3. *A* $sG_{\mathcal{P}_g}$*-homeomorphism if* $\mathfrak{j}$ *and* $\mathfrak{j}^{-1}$ *are both* $sG_{\mathcal{P}_g}$*-continuous.*

Definition 35. *A bijection function* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *from a generalized primal topological space* $\mathcal{W}$ *into a generalized primal topological space* $\mathcal{Y}$ *is called a* $\mathcal{P}_g$*-homeomorphism if* $\mathfrak{j}$ *is both* $\mathcal{P}_g$*-continuous and* τ_g*-open.*

Theorem 48. *For any bijection* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *from a generalized primal topological space* $\mathcal{W}$ *into a generalized primal topological space* $\mathcal{Y}$*, the following are equivalent:*

1. *The inverse function* $\mathfrak{j}^{-1}$ *is* $\mathcal{P}_g$–S_g^**-continuous.*
2. $\mathfrak{j}$ *is a* $\mathcal{P}_g$–S_g^**-open function.*

3. *$\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-closed function*

Proof. Because $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ from a generalized primal topological space $\mathcal{W}$ into a generalized primal topological space $\mathcal{Y}$.

1 $\Rightarrow$ 2, as $\mathfrak{j}^{-1}$ is $\mathcal{P}_g$-continuous. $\mathfrak{j}(\mathcal{E})$ is τ_g-open in $\mathcal{Y}$ whenever $\mathcal{E}$ is τ_g-open in $\mathcal{W} \Rightarrow \mathfrak{j}$ is τ_g-open.

2 $\Rightarrow$ 3, as $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-open. If $\mathcal{E}$ is τ_g-closed $\Rightarrow \mathcal{E}^c$ is τ_g-open, then $\mathfrak{j}(\mathcal{E}^c)$ is $\mathcal{P}_g$–S_g^*-open, that is, $\mathcal{Y} \setminus \mathfrak{j}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open $\Rightarrow \mathfrak{j}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-closed, that is, $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-closed.

3 $\Rightarrow$ 1. If $\mathcal{E}$ is τ_g-open in $\mathcal{W}$, $\mathcal{E}^c$ is τ_g-closed $\Rightarrow \mathfrak{j}(\mathcal{E}^c)$ is $\mathcal{P}_g$–S_g^*-closed, that is, $\mathcal{Y} \setminus \mathfrak{j}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-closed in $\mathcal{E} \Rightarrow \mathfrak{j}(\mathcal{E})$ is $\mathcal{P}_g$–S_g^*-open in $\mathcal{E}$; hence, $\mathfrak{j}^{-1}$ is $\mathcal{P}_g$–S_g^*-continuous. □

Definition 36. *A bijection function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ from a generalized primal topological space $\mathcal{W}$ into a generalized primal topological space $\mathcal{Y}$ is called a $\mathcal{P}_g$–S_g^*-homeomorphism if $\mathfrak{j}$ is both $\mathcal{P}_g$–S_g^*-continuous and $\mathcal{P}_g$–S_g^*-open.*

Theorem 49. *Any bijection function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ from a generalized primal topological space $\mathcal{W}$ into a generalized primal topological space $\mathcal{Y}$ which is $\mathcal{P}_g$–S_g^*-continuous has the analogous statements listed below:*

1. *The function $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is $\mathcal{P}_g$–S_g^*-open.*
2. *$\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-homeomorphism.*
3. *$\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-closed function*

Proof. 1 $\Rightarrow$ 2 Because $\mathfrak{j}$ is both $\mathcal{P}_g$–S_g^*-continuous and $\mathcal{P}_g$–S_g^*-open, $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-homeomorphism.

2 $\Rightarrow$ 3 $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous and $\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-homeomorphism $\Rightarrow \mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-open; hence, it is $\mathcal{P}_g$–S_g^*-closed by Theorem 48.

3 $\Rightarrow$ 1 $\mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-closed $\Rightarrow \mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-open, by Theorem 48. □

Theorem 50. *Every $\mathcal{P}_g$-homeomorphism is a $\mathcal{P}_g$–S_g^*-homeomorphism.*

Proof. Assume $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ to be a $\mathcal{P}_g$-homeomorphism. If $\mathcal{O}_\gamma$ is an open set in $\mathcal{W}$, then $\mathfrak{j}(\mathcal{O}_\gamma)$ is τ_g-open in $\mathcal{Y}$, as $\mathfrak{j}$ is a τ_g-open function; hence, $\mathfrak{j}(\mathcal{O}_\gamma)$ is $\mathcal{P}_g$–S_g^*-open, meaning that $\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-open function. Further, the $\mathcal{P}_g$-continuity $\Rightarrow \mathfrak{j}$ is $\mathcal{P}_g$–S_g^*-continuous, that is, $\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-homeomorphism. □

Remark 17. *The converse of the above Theorem 50 is invalid, as demonstrated by the subsequent illustration.*

Example 50. *Consider $\mathcal{W} = \mathcal{Y} = \{d, k\}$, $\mathcal{P}_\alpha = \mathcal{P}_\beta = \{\phi, \{k\}, \{d, k\}\}$, $\tau_{g_1} = \{\phi, \{d\}, \mathcal{W}\}$ and $\tau_{g_2} = \{\phi, \{k\}, \{d, k\}, \mathcal{Y}\}$. Assuming that $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ is the identity function $\Rightarrow \mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-homeomorphism, but not a $\mathcal{P}_g$-homeomorphism.*

Remark 18. *From the subsequent illustration, a composition of two $\mathcal{P}_g$–S_g^*-homeomorphisms need not be a $\mathcal{P}_g$–S_g^*-homeomorphism.*

Example 50. *Let $\mathcal{W} = \mathcal{Y} = \mathcal{Z} = \{d, k, q, w\}$, $\mathcal{P}_\alpha = \mathcal{P}_\beta = \mathcal{P}_\gamma = \{\phi, \{d, k\}, \{d\}, \{k\}, \{q, w\}, \{q\}, \{w\}\}$, $\tau_{g_1} = \{\phi, \{d, k\}, \{q, w\}, \mathcal{W}\}$, $\tau_{g_2} = \{\phi, \{d\}, \{k\}, \{d, k\}, \{d, k, q\}, \mathcal{Y}\}$ and $\tau_{g_3} = \{\phi, \{d\}, \{d, k\}, \{d, k, q\}, \mathcal{Z}\}$ be generalized primal topologies on $\mathcal{W}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Then, the mappings $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ and $\mathfrak{j} : (\mathcal{Y}, \tau_{g2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ are defined by $\mathfrak{f}(d) = w, \mathfrak{f}(k) = k$ and $\mathfrak{f}(q) = q, \mathfrak{f}(w) = d$, $\mathfrak{j}(d) = k, \mathfrak{j}(k) = d, \mathfrak{j}(q) = w, \mathfrak{j}(w) = q$. Here, $\mathfrak{f}$ and $\mathfrak{j}$ are both $\mathcal{P}_g$–S_g^*-homeomorphisms, but $\mathfrak{j}o\mathfrak{f}$ is not a $\mathcal{P}_g$–S_g^*-homeomorphism.*

3.4.2. $\mathcal{P}_g$–$S_g^{*\#}$-Homeomorphism

Definition 37. *A bijective mapping* $\mathfrak{j} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *from a generalized primal topological space* $\mathcal{W}$ *into a generalized primal topological space* $\mathcal{Y}$ *is known as a* $\mathcal{P}_g$–$S_g^{*\#}$*-homeomorphism if* $\mathfrak{j}$ *and* $(\mathfrak{j})^{-1}$ *are both* $\mathcal{P}_g$–S_g^**-irresolute.*

Theorem 51. *Any* $\mathcal{P}_g$-$S_g^{*\#}$*-homeomorphism is a* $\mathcal{P}_g$–S_g^**-homeomorphism, but not conversely. The* τ_g–$S_g^{*\#}$*-homeomorphism is a subclass of the class of* $\mathcal{P}_g$–S_g^**-homeomorphisms.*

Proof. Because any $\mathcal{P}_g$–$S_g^{*\#}$-open map is a τ_g–S_g^*-open map [27], the proof follows from Theorem 3.32 [29]. In Example 50, the mapping $\mathfrak{j}$ is a $\mathcal{P}_g$–S_g^*-homeomorphism but not a $\mathcal{P}_g$–$S_g^{*\#}$-homeomorphism. Because $\{$k, q$\}$ is a $\mathcal{P}_g$–S_g^*-closed set in $\mathcal{Y}$ but $((\mathfrak{j})^{-1})^{-1}(\{k, q\}) = \{d, w\}$ is not $\mathcal{P}_g$–S_g^*-closed in $\mathcal{Z}$, it is the case that $(\mathfrak{j})^{-1}$ is not $\mathcal{P}_g$–S_g^*-irresolute. Hence, $\mathfrak{j}$ is not a $\mathcal{P}_g$–$S_g^{*\#}$-homeomorphism. □

Theorem 52. *If* $\mathfrak{f} : (\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Y}, \tau_{g_2}, \mathcal{P}_\beta)$ *and* $\mathfrak{j} : (\mathcal{Y}, \tau_{g2}, \mathcal{P}_\beta) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *are* $\mathcal{P}_g$–$S_g^{*\#}$*-homeomorphisms, then the composition* $\mathfrak{j}o\mathfrak{f}$*:* $(\mathcal{W}, \tau_{g_1}, \mathcal{P}_\alpha) \to (\mathcal{Z}, \tau_{g_3}, \mathcal{P}_\gamma)$ *is a* $\mathcal{P}_g$–$S_g^{*\#}$*-homeomorphism.*

Proof. Assume $\mathcal{E}$ to be a $\mathcal{P}_g$–S_g^*-open set in $\mathcal{Z}$. Now, $(\mathfrak{j}o\mathfrak{f})^{-1}$ $(\mathcal{E}) = \mathfrak{f}^{-1}((\mathfrak{j})^{-1}(\mathcal{E})) = (\mathfrak{f})^{-1}$ $(\mathcal{D})$, where $\mathcal{D} = (\mathfrak{j})^{-1}$ $(\mathcal{E})$. By hypothesis, $\mathcal{D}$ is a $\mathcal{P}_g$–S_g^*-open set in $\mathcal{Y}$; again by hypothesis, $(\mathfrak{f})^{-1}$ $(\mathcal{D})$ is a $\mathcal{P}_g$–S_g^*-open set in $\mathcal{W}$. Thus, $\mathfrak{j}o\mathfrak{f}$ is $\mathcal{P}_g$–S_g^*-irresolute. In addition, for any $\mathcal{P}_g$–S_g^*-open set $\mathcal{K}$ in $(\mathcal{W}, \tau_{g_1})$, $\mathfrak{j}o\mathfrak{f}(\mathcal{K}) = \mathfrak{j}(\mathfrak{f}(\mathcal{K})) = \mathfrak{j}(\mathcal{Q})$, where $\mathcal{Q} = \mathfrak{f}(\mathcal{K})$. By hypothesis, $\mathfrak{f}(\mathcal{K})$ is a $\mathcal{P}_g$–S_g^*-open set in $(\mathcal{Y}, \tau_{g_2})$ and $\mathfrak{j}(\mathfrak{f}(\mathcal{K})) = \mathfrak{j}(\mathcal{Q})$ is a $\mathcal{P}_g$–S_g^*-open set in $(\mathcal{Z}, \tau_{g_3})$; thus, $(\mathfrak{j}o\mathfrak{f})^{-1}$ is $\mathcal{P}_g$–S_g^*-irresolute, proving that $\mathfrak{j}o\mathfrak{f}$ is a $\mathcal{P}_g$–$S_g^{*\#}$-homeomorphism. □

Corollary 53. *In the collection of all topological spaces, a* $\mathcal{P}_g$–$S_g^{*\#}$*-homeomorphism is an equivalence relation.*

Proof. Because the composition of a bijective mapping is an equivalence relation $\Rightarrow$ a $\mathcal{P}_g$–$S_g^{*\#}$-homeomorphism is an equivalence relation. □

4. Methodology

The research technique utilized in this study aims to provide a thorough analysis of the S_g^* function, S_g^*-homeomorphism, and $S_g^{*\#}$-homeomorphism in both $\mathcal{GTS}$ and $\mathcal{GPTS}$. To do this, a methodological approach must be used to define, prove, and examine the characteristics of these functions in many contexts. The stages involved in this research are outlined below:

- **Review of Existing Literature:** The study starts with a thorough analysis of the existing literature and prior research on generalized topological spaces, generalized primal topological spaces, and τ_g–S_g^* functions. Important basic ideas and earlier research on open maps, continuity, and both irresolute and closed mappings in topological spaces are noted. The definitions and theorems established later are based on this review.
- **Definition of τ_g–S_g^* functions:** Here, the τ_g–S_g^*-function and all of its associated characteristics are indicated, such as:
  - **τ_g-S_g^*-continuous mapping:** A function that maintains τ_g–S_g^*-open sets under continuous mappings.
  - **τ_g–S_g^*-irresolute:** A function such that each τ_g–S_g^*-open set's inverse image is also τ_g–S_g^*-open.
  - **Strongly τ_g–S_g^*-continuous mapping:** A mapping in which for every τ_g–S_g^*-open set there is an inverse image that is open.

- **Perfectly τ_g–S_g^*-continuous mapping:** A mapping in which for every τ_g–S_g^*-open set there is an inverse image that is both open and closed.
- **τ_g–S_g^*-open and τ_g–S_g^*-closed maps:** Functions that respectively map τ_g–S_g^*-open sets to τ_g-open sets and τ_g–S_g^*-closed sets to τ_g-closed sets.

- **Development of Propositions and Theorems:** After defining each function type, we proceed to develop associated theorems and propositions. This comprises the following aspects:
  - Theorems that specify the circumstances in which a function is either τ_g–S_g^*-irresolute or τ_g–S_g^*-continuous.
  - The links between τ_g–S_g^*-open and τ_g–S_g^*-closed maps, along with how they affect the topological structure.
  - Theoretical analysis of the behavior of strongly and perfectly τ_g–S_g^*-continuous functions in generalized topological spaces.
- **Extension to Generalized Primal Topological Spaces:** Subsequently, we broaden the scope of our analysis by including these functions in the context of $\mathcal{GPTS}$. Within this framework, we redefine τ_g–S_g^* functions to emphasize the fundamental character of the space. This allows for analysis of these functions under additional conditions and limitations, as the generalized primal topological space provides a more constrained framework.
- **Analysis of S_g^*- and $S_g^{*\#}$-Homeomorphisms:** The fourth stage explores the ways in which these homeomorphisms maintain the S_g^*-closed and S_g^*-open sets in various spaces. τ_g–S_g^*-homeomorphisms in generalized topological spaces and $\mathcal{P}_g$-S_g^*-homeomorphisms in generalized primal topological spaces make up this section. Through the use of formal arguments and comparative analysis, the characteristics and behavior of homeomorphisms in both contexts are examined.
- **Proofs and Verification:** Precise mathematical proofs support every theorem and statement. To confirm the accuracy of our findings, we utilize conventional methods in topology such as continuity rules and arguments from set theory. This stage guarantees that the conclusions are sound both mathematically and logically.
- **Comparative Study:** We carry out a comparative study of the behaviors of S_g^* functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms in generalized topological spaces and generalized primal topological spaces to gain a deeper understanding of their relevance. This comparison demonstrates the differences and similarities in their behavior under various topological contexts.
- **Conclusions and Upcoming Work:** The last phase entails a summary of the main study's findings and suggestions for future work paths. We explain the consequences of our findings and point out possible directions for future research that might provide fresh insights, especially in more specialized or applied topological situations.

5. Conclusions

A thorough analysis of S_g^* functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms in the frameworks of both generalized primal topological spaces and generalized topological spaces is presented in this research. We have looked at several important characteristics of these functions, such as strongly S_g^*-continuous and perfectly S_g^*-continuous functions, S_g^*-open and S_g^*-closed maps, and S_g^*-continuous and S_g^*-irresolute functions. We have uncovered new findings about the properties and behaviors of these functions in generalized topological spaces while providing thorough definitions, theorems, and assertions. Our research gains greater depth by extension to generalized primal topological spaces, which enables us to study the behavior of S_g^*-functions in a more specific context. The results of this study show that although many of the features remain the same, there are distinct differences in the behavior of these functions due to the more profound constraints of primal topological spaces. The outcomes advance the theoretical knowledge of S_g^*-functions, S_g^*-homeomorphisms, and $S_g^{*\#}$-homeomorphisms in primal and generalized topological spaces. This study establishes connections between various types of functions and provides rigorous proofs, paving the way for future research into topological function theory. Subsequent research endeavors might explore more precise uses of S_g^*-functions or expand these ideas to encompass other specialized topological configurations. To sum up, the study of S_g^*-functions extends the reach of topological studies by providing fresh ideas about function characteristics in many topological contexts. These findings not only support existing theories but also create opportunities for more study in this area in the future.

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