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Some Categories of Compactness and Connectedness in Generalized Topological Spaces

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Abstract: The presented work enhances the study of topological characteristics in non-classical circumstances by examining S_g^*-compact and S_g^*-connected spaces in the context of generalized topological spaces. The notions of S_g^*-compactness and S_g^*-connectedness are explored better to understand their characteristics and behavior in these generalized topologies. Further, the study implements the same extended contexts to investigate the separation axioms, particularly T_0, T_1, and T_2. The structural consequences of these axioms, which are essential for categorizing topological spaces according to the distinctness of points and sets, are examined. Since the premises of S_g^*-compactness and S_g^*-connectedness are not directly related to the separation axioms, exploring them independently enhances the analysis of generalized topology. This study serves as theoretical insights and establishes the foundation for future research in generalized topological spaces, contributing to their continued evolution.

Keywords: τ_g-connected; τ_g-compact; τ_g-S_g^*-connected; τ_g-S_g^*-compact

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1. Introduction

Topology, as a cornerstone of modern mathematics, offers a framework for analyzing continuity, convergence, and separation across various spaces. Among its core concepts, compactness and connectedness stand out for their theoretical significance and practical applications. The study of generalized topological spaces has seen significant advancements through the works of various researchers. Levine's [1] concept of generalized closed sets is important in general topology and has been thoroughly studied by many topologists in recent years. The analysis of generalized closed sets has produced intriguing new ideas, such as new covering properties and separation axioms. Császár [2] introduced the concept of generalized open sets, a foundational element that has greatly influenced subsequent studies in this area. Khayyeri and Mohamadian [3] exploration provides a critical framework for understanding the structural aspects of these spaces. Building on this, Sarsak [4] examined key properties of generalized open sets, shedding light on their behavior and further enriching the theoretical landscape of generalized topologies. These contributions are essential to the ongoing efforts to extend classical topological concepts into more generalized and versatile frameworks. The work of Maki, Balachandran and Devi [5], Maki, Rao, and Gani [6], and Navalagi and Page [7] has been instrumental in advancing the understanding of semi-generalized closed sets, generalized semi-closed sets, and their generalizations, such as semi-open and pre-open sets, which lay the foundation for further explorations into topological structures respectively. Advancements in generalized topological spaces have significantly enriched the field of topology. The concept of S_g^*-open sets was recently introduced by Missier and Jesti [8], including several additional functions in S_g^*-open sets. The idea of μ-compactness in $\mathcal{GTS}$, introduced by Thomas and John [9], provided a refined understanding of compactness in a generalized framework. Extending these notions to generalized topological spaces has enabled mathematicians to address more complex structures and solve broader classes of problems.

To classify topological spaces according to the distinguishability of points and sets, this study also focuses on the separation axioms, namely T_0, T_1, and T_2. Urysohn [10] began the systematic exploration of separation axioms in topology. This work was later expanded by Freudenthal and Est [11] and offered a more comprehensive discussion of these concepts. In the intermediate space between T_0 and T_1, significant contributions were made by Youngs [12]. These studies laid the groundwork for further innovations, including Stone's [13] result, which showed that any topological space can be transformed into a T_0 space by identifying indistinguishable points. These foundational works continue to inform and inspire contemporary research in the field.

In this context, the S_g^*-compact and S_g^*-connected spaces offer a new dimension to the study of generalized topological spaces. By examining the relationship between separation axioms such as T_0, T_1, and T_2, this work explores how these axioms influence the structural properties of these spaces. The study delves into the categorization of S_g^*-T_0, S_g^*-T_1, and S_g^*-T_2 axioms, which contribute to the overall behavior of these extended topological frameworks. This work aims to further clarify the connections between separation axioms and the unique properties of S_g^*-sets, advancing our understanding of these spaces.

2. Preliminaries

This section explains the main section by discussing various definitions and outcomes from the literature.

Definition 1 ([3]). *Assume a set $\mathcal{W} \neq \phi$. A collection $\tau_g \subseteq 2^{\mathcal{W}}$ ($2^{\mathcal{W}}$ is the power set of $\mathcal{W}$) is referred to as a generalized topology ($\mathcal{GT}$) on $\mathcal{W}$ if $\phi \in \tau_g$ and all unions of non-empty sub-classes of τ_g are part of τ_g. The pair $(\mathcal{W}, \tau_g)$ is known as generalized topological space ($\mathcal{GTS}$).*

Remark 1 ([2]). *The members of τ_g are indicated as open. Perceive $\mathcal{E}$ to be any subset of $(\mathcal{W}, \tau_g)$. $\mathcal{E}$ is considered as closed if $(\mathcal{W} \setminus \mathcal{E})$ is open. Closure of $\mathcal{E}$ is indicated by $Cl_g(\mathcal{E})$ is the intersection of all closed set that contains $\mathcal{E}$. The interior is represented as $Int_g(\mathcal{E})$, which is the union of all open sets that are contained in $\mathcal{E}$.*

Definition 2 ([2]). *Assume that an operator $\psi{:}\mathcal{W} \to 2^{2^{\mathcal{W}}}$ in $\mathcal{GTS}$ satisfies $x \in \mathcal{F}$ for $\mathcal{F} \in \psi(x)$. Therefore, $\mathcal{F} \in \psi(x)$ is referred to a generalized neighbourhood of a point x in $\mathcal{W}$. The collection of all generalized neighbourhood of $\mathcal{W}$ is termed as $\Psi(\mathcal{W})$.*

Definition 3 ([4–7]). *1. Consider $(\mathcal{W}, \tau_g)$ as a $\mathcal{GTS}$. Assume that $\mathcal{E}$ as a subset of $(\mathcal{W}, \tau_g)$ is generalized τ_g-semi-open if there is an open set $\mathcal{F}$ in $\mathcal{W}$ such that $\mathcal{F} \subseteq \mathcal{E} \subseteq Cl(\mathcal{F})$ or appropriately, if $\mathcal{E} \subseteq Cl(Int(\mathcal{E}))$.*

2. *The complement of a generalized τ_g-semi-open set is a generalized τ_g-semi-closed set. $SO(\mathcal{W}, \tau_g)$ is the family of all generalized τ_g-semi-open sets in $(\mathcal{W}, \tau_g)$.*
3. *The union of all generalized τ_g-semi-open sets of $\mathcal{W}$ contained in $\mathcal{E}$ is generalized semi-interior of $\mathcal{E}$ (shortly, $s_gInt(\mathcal{E})$).*
4. *The intersection of all generalized semi−closed sets of $\mathcal{W}$ containing $\mathcal{E}$ is the generalized semi-closure of $\mathcal{E}$ (briefly $s_gCl(\mathcal{E})$).*

Definition 4 ([9]). *A generalized topological space $(\mathcal{W}, \tau_g)$ is called τ_g-compact if every open cover of $(\mathcal{W}, \tau_g)$ has a finite sub-cover.*

Definition 5 ([14]). *Let $(\mathcal{W}, \tau_{g_1})$ and $(\mathcal{Y}, \tau_{g_2})$ be a $\mathcal{GTS}$. A mapping $\jmath:(\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ is called τ_g-S_g^*-irresolute if the inverse image of every τ_g-S_g^*-open set in $(\mathcal{Y}, \tau_{g_2})$ is a τ_g-S_g^*-open in $(\mathcal{W}, \tau_{g_1})$.*

3. Main Results

The main section of this article systematically explores the compactness, connectedness, and separation axioms within generalized topological frameworks. This part focuses on generalized topology, where the concepts of S_g^*-compactness, S_g^*-connectedness, and separation axioms T_0, T_1, and T_2 are introduced, analyzed, and supported with theoretical insights and examples.

3.1. τ_g-S_g^*-Compact Space in $\mathcal{GTS}$

Definition 6.

1. *Consider $(\mathcal{W}, \tau_g)$ to be $\mathcal{GTS}$. A subset $\mathcal{E}$ of $\mathcal{W}$ is generalized semi-generalized closed (briefly τ_g-Sg-closed) if $s_gCl(\mathcal{E}) \subseteq \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{F}$ is generalized τ_g-semi-open in $\mathcal{W}$.*
2. *The complement of a generalized semi-generalized closed set is a generalized semi-generalized open. It is indicated as τ_g-Sg-open.*
3. *The generalized semi-generalized interior (briefly $s_gInt^*(\mathcal{E})$) of $\mathcal{E}$ is specified as the union of all τ_g-Sg-open sets of $\mathcal{W}$ contained in $\mathcal{E}$.*
4. *Assume a subset $\mathcal{E}$ of a space $\mathcal{W}$. The generalized semi-generalized closure (briefly $s_gCl^*(\mathcal{E})$) of $\mathcal{E}$ is termed as the intersection of all τ_g-Sg-closed sets in $\mathcal{W}$ containing $\mathcal{E}$.*

Definition 7. *A subset $\mathcal{E}$ of a generalized topological space $(\mathcal{W}, \tau_g)$ is called generalized S_g^*-open (τ_g-S_g^*-open shortly) if there is an open set $\mathcal{F} \in \tau_g$ such that $\mathcal{F} \subseteq \mathcal{E} \subseteq s_gCl^*(\mathcal{F})$. The collection of all τ_g-S_g^*-open sets in $(\mathcal{W}, \tau_g)$ is represented by $S_g^*O(\mathcal{W}, \tau_g)$.*

Definition 8. *A generalized topological space $(\mathcal{W}, \tau_g)$ is called τ_g-semi-compact if every semi-open cover of $(\mathcal{W}, \tau_g)$ has a finite subcover.*

Definition 9. *Assume a generalized topological space $(\mathcal{W}, \tau_g)$ and a subset $\mathcal{E}$ of $(\mathcal{W}, \tau_g)$. A collection $\{\mathcal{E}_i : i \in \wedge\}$ of τ_g-S_g^*-open set in $(\mathcal{W}, \tau_g)$ is referred to as τ_g-S_g^*-open cover of $\mathcal{E}$ if $\mathcal{E} \subset \cup_{i \in \wedge} \mathcal{E}_i$.*

Definition 10. *If every τ_g-S_g^*-open cover of $(\mathcal{W}, \tau_g)$ has a finite subcover, then the generalized topological space $(\mathcal{W}, \tau_g)$ is called τ_g-S_g^*-compact.*

Definition 11. *A subset $\mathcal{E}$ of a generalized topological space $(\mathcal{W}, \tau_g)$ is called τ_g-S_g^*-compact relative to $\mathcal{W}$ if, for every collection $\{O_i{:}i \in \wedge\}$ of τ_g-S_g^*-open subsets of $\mathcal{W}$ such that $\mathcal{E} \subseteq \cup \{O_i{:}i \in \wedge\}$, there is a finite subset $\wedge_o \subseteq \wedge$ such that $\mathcal{E} \subseteq \cup \{O_i{:}i \in \wedge_o\}$.*

Definition 12. *Assume $\mathcal{E}$ to be a subset of generalized topological space $(\mathcal{W}, \tau_g)$. If $\mathcal{E}$ is τ_g-S_g^*-compact as a subspace of $\mathcal{W}$, then $\mathcal{E}$ is called τ_g-S_g^*-compact.*

Theorem 1.

1. *Every τ_g-semi-compact space is τ_g-S_g^*-compact.*
2. *Every τ_g-S_g^*-compact space is τ_g-compact.*

Proof. 1 and 2 originates from Definition 4, Definition 8, and Definition 10. □

Example 1. *Let $\mathcal{W}$ = $\mathbb{N}$ (the set of natural numbers), and let τ_g be the generalized topology on $\mathbb{N}$ defined as follows:*
$\tau_g = \{\phi, \{n + 2, n + 3, n + 4, \dots\} \mid n \in \mathbb{N}\}$.
Now, consider the subset $\mathcal{E} = \{1, 2, 3\}$. To check if $\mathcal{E}$ is τ_g-semi-compact, let $\{O_i{:}i \in \wedge\}$ be a collection of τ_g-S_g^*-open sets such that $\mathcal{E} \subseteq \cup \{O_i{:}i \in \wedge\}$. For any such cover, we can always find a finite subcover since $\mathcal{E}$ is finite and hence τ_g-S_g^*-semi-compact.
Now, let $\{O_i{:}i \in \wedge\}$ be a collection of τ_g-S_g^*-open sets such that $\mathcal{E} \subseteq \cup \{O_i{:}i \in \wedge\}$. Since $\mathcal{E}$ is finite and every point in $\mathcal{E}$ must be covered by some O_i, a finite subset $\wedge_o \subseteq \wedge$ will suffice to cover $\mathcal{E}$. Therefore, $\mathcal{E}$ is τ_g-S_g^*-compact.
For $\mathcal{E}$, any open cover in the generalized topology τ_g will also have a finite subcover since $\mathcal{E}$ is finite. Thus, $\mathcal{E}$ is τ_g-compact.

Theorem 2. *Every τ_g-S_g^*-closed subset of a τ_g-S_g^*-compact space is τ_g-S_g^*-compact relative to $(\mathcal{W}, \tau_g)$.*

Proof. Assume that $\mathcal{E}$ as a τ_g-S_g^*-closed subset of τ_g-S_g^*-compact space $(\mathcal{W}, \tau_g) \Rightarrow \mathcal{E}^c$ is τ_g-S_g^*-open in $(\mathcal{W}, \tau_g)$. Consider $\{O_i{:}i \in \wedge\}$ be a cover of $\mathcal{E}$ by τ_g-S_g^*-open subset of $\mathcal{W}$ such that $\mathcal{E} \subset \cup \{O_i{:}i \in \wedge\} \Rightarrow \mathcal{E}^c \subset \cup \{O_i{:}i \in \wedge\} = \mathcal{W}$. Consequently, $(\mathcal{W}, \tau_g)$ is τ_g-S_g^*-compact, so there is a finite subset $\mathcal{E}_o$ of $\mathcal{E}$ such that $\mathcal{E} \subset \mathcal{E}^c \cup \{O_i{:}i \in \wedge\} = \mathcal{W}$. Then $\mathcal{E} \subset \cup \{O_i{:}i \in \wedge\}$ and therefore $\mathcal{E}$ is τ_g-S_g^*-compact relative to $\mathcal{W}$. □

Theorem 3. *Assume a surjective τ_g-S_g^*-continuous map $\mathfrak{h}$: $(\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$. If $(\mathcal{W}, \tau_{g_1})$ is τ_g-S_g^*-compact, then $(\mathcal{Y}, \tau_{g_2})$ is τ_g-compact.*

Proof. Consider an open cover $\{\mathcal{E}_i{:}i \in \wedge\}$ of $\mathcal{Y}$. $\mathfrak{h}$ is τ_g-S_g^*-continuous $\Rightarrow \{\mathfrak{h}^{-1}(\mathcal{E}_i){:}i \in \wedge\}$ is a τ_g-S_g^*-open cover of $\mathcal{W}$. Furthermore, there is a finite τ_g-subcover $\{\mathfrak{h}^{-1}(\mathcal{E}_1), \mathfrak{h}^{-1}(\mathcal{E}_2), \mathfrak{h}^{-1}(\mathcal{E}_3), ..., \mathfrak{h}^{-1}(\mathcal{E}_n)\}$, as $\mathcal{W}$ is τ_g-S_g^*-compact. The surjectiveness of $\mathcal{W} \Rightarrow \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n\}$ is a finite τ_g-subcover of $\mathcal{Y} \Rightarrow \mathcal{Y}$ is τ_g-compact. □

Theorem 4. *Assume a surjective τ_g-S_g^*-irresolute map $\mathfrak{h}$: $(\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$. If $(\mathcal{W}, \tau_{g_1})$ is τ_g-S_g^*-compact, then $(\mathcal{Y}, \tau_{g_2})$ is τ_g-S_g^*-compact.*

Proof. Consider a τ_g-S_g^*-open cover $\{\mathcal{E}_i{:}i \in \wedge\}$ of $\mathcal{Y}$. $\mathfrak{h}$ is τ_g-S_g^*-irresolute $\Rightarrow \{\mathfrak{h}^{-1}(\mathcal{E}_i){:}i \in \wedge\}$ is a τ_g-S_g^*-open cover of $\mathcal{W}$. Furthermore, there is a finite τ_g-subcover $\{\mathfrak{h}^{-1}(\mathcal{E}_1), \mathfrak{h}^{-1}(\mathcal{E}_2), \mathfrak{h}^{-1}(\mathcal{E}_3), ..., \mathfrak{h}^{-1}(\mathcal{E}_n)\}$, as $\mathcal{W}$ is τ_g-S_g^*-compact. Now $\mathfrak{h}$ implies that $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n\}$ is a finite τ_g-subcover of $\mathcal{Y} \Rightarrow \mathcal{Y}$ is τ_g-S_g^*-compact. □

Theorem 5. *If a τ_g-S_g^*-irresolute map $\mathfrak{h}$: $(\mathcal{W}, \tau_{g_1}) \to (\mathcal{Y}, \tau_{g_2})$ and a subset $\mathcal{D}$ of $(\mathcal{W}, \tau_{g_1})$ is τ_g-S_g^*-compact relative to $\mathcal{W}$, then the image $\mathfrak{h}(\mathcal{D})$ is τ_g-S_g^*-compact relative to $\mathcal{Y}$.*

Proof. Let $\{\mathcal{E}_i\text{:i} \in \wedge\}$ be any collection of τ_g-S_g^*-open subsets of $\mathcal{Y}$ such that $\mathfrak{h}(\mathcal{D}) \subset \cup \{\mathcal{E}_i\text{:i} \in \wedge\}$. Then $\mathcal{D} \subset \cup \{\mathfrak{h}^{-1}(\mathcal{E}_i)\text{:i} \in \wedge\}$ holds. $\mathcal{D}$ is τ_g-S_g^*-compact relative to $\mathcal{W}$ by hypothesis, so there is a $\wedge_o$, a finite subset of $\wedge$, such that $\mathcal{D} \subset \cup \{\mathfrak{h}^{-1}(\mathcal{E}_i)\text{:i} \in \wedge_o\} \Rightarrow \mathfrak{h}(\mathcal{D}) \subset \cup \{\mathcal{E}_i\text{:i} \in \wedge_o\}$. Therefore, $\mathfrak{h}(\mathcal{D})$ is τ_g-S_g^*-compact relative to $\mathcal{Y}$. □

Theorem 6. *If a surjective map* $\mathfrak{h}$*: (*$\mathcal{W}$*,* τ_{g_1}*)* $\rightarrow$ *(*$\mathcal{Y}$*,* τ_{g_2}*) is strongly* τ_g-S_g^**-continuous, and (*$\mathcal{W}$*,* τ_{g_1}*) is* τ_g*-compact, then (*$\mathcal{Y}$*,* τ_{g_2}*) is* τ_g-S_g^**-compact.*

Proof. Consider a τ_g-S_g^*-open cover $\{\mathcal{E}_i\text{:i} \in \wedge\}$ of $\mathcal{Y}$. $\mathfrak{h}$ is strongly τ_g-S_g^*-continuous $\Rightarrow$ $\{\mathfrak{h}^{-1}(\mathcal{E}_i)\text{:i} \in \wedge\}$ is an open cover of $\mathcal{W}$. Furthermore, there is a finite τ_g-subcover $\{\mathfrak{h}^{-1}(\mathcal{E}_1)$, $\mathfrak{h}^{-1}(\mathcal{E}_2)$, $\mathfrak{h}^{-1}(\mathcal{E}_3)$, ..., $\mathfrak{h}^{-1}(\mathcal{E}_n)\}$, as $\mathcal{W}$ is τ_g-compact. The surjectiveness of $\mathfrak{h} \Rightarrow \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, ..., \mathcal{E}_n\}$ is a finite subcover of $\mathcal{Y} \Rightarrow \mathcal{Y}$ is τ_g-S_g^*-compact. □

Theorem 7. *If a surjective map* $\mathfrak{h}$*: (*$\mathcal{W}$*,* τ_{g_1}*)* $\rightarrow$ *(*$\mathcal{Y}$*,* τ_{g_2}*) is perfectly* τ_g-S_g^**-continuous and (*$\mathcal{W}$*,* τ_{g_1}*) is* τ_g*-compact, then (*$\mathcal{Y}$*,* τ_{g_2}*) is* τ_g-S_g^**-compact.*

Proof. Every perfectly τ_g-S_g^*-continuous is strongly τ_g-S_g^*-continuous. Consequently, the result is derived from Theorem 6. □

3.2. τ_g-S_g^*-Connected Space

Definition 13. *Assume a generalized topological space (*$\mathcal{W}$*,* τ_g*) and two disjoint non-empty open sets* $\mathcal{D}$ *and* $\mathcal{E}$ *in* $\mathcal{W}$*. Then (*$\mathcal{W}$*,* τ_g*) is known as* τ_g*-disconnection if* $\mathcal{D} \cup \mathcal{E} = \mathcal{W}$*.*

Definition 14. *Consider a generalized topological space (*$\mathcal{W}$*,* τ_g*) and two disjoint non-empty open sets* $\mathcal{D}$ *and* $\mathcal{E}$ *in* $\mathcal{W}$*. Then (*$\mathcal{W}$*,* τ_g*) is known as a* τ_g*-connected space if (*$\mathcal{W}$*,* τ_g*) has no* τ_g*-disconnection* $\{\mathcal{D}, \mathcal{E}\}$*.*

Definition 15. *Assume a generalized topological space (*$\mathcal{W}$*,* τ_g*) and two disjoint non-empty* τ_g-S_g^**-open sets* $\mathcal{D}$*, and* $\mathcal{E}$ *in (*$\mathcal{W}$*,* τ_g*) does not exist as* $\mathcal{D} \cup \mathcal{E} = \mathcal{W}$*. Then space (*$\mathcal{W}$*,* τ_g*) is known as* τ_g-S_g^**-connected space, otherwise called* τ_g-S_g^**-disconnected space if* $\mathcal{D} \cup \mathcal{E} = \mathcal{W}$*.*

Example 7. *Consider* $\mathcal{W} = \{e_1, f_1, d_1\}$ *and* $\tau_g = \{\phi, \{f_1\}, \{d_1\}, \{f_1, d_1\}\}$*. In this space (*$\mathcal{W}$*,* τ_g*)*$S_g^*O = \{\phi, \{f_1\}, \{d_1\}, \{f_1, d_1\}\}$*. Thus, there does not exist any non-empty* τ_g-S_g^**-open sets* $\mathcal{D}$ *and* $\mathcal{E}$ *such that* $\mathcal{D} \cup \mathcal{E} = \mathcal{W} \Rightarrow$ *(*$\mathcal{W}$*,* τ_g*) is* τ_g-S_g^**-connected space.*

Theorem 8. *Every* τ_g-S_g^**-connected space is* τ_g*-connected space.*

Proof. Let ($\mathcal{W}$, τ_g) be a τ_g-S_g^*-connected space. Assume ($\mathcal{W}$, τ_g) is not a τ_g-connected space. Then for any two disjoint non-empty open subsets $\mathcal{D}$ and $\mathcal{E}$ in $\mathcal{W}$, $\mathcal{D} \cup \mathcal{E} = \mathcal{W}$. Every open set is τ_g-S_g^*-open set. Thus, $\mathcal{D}$ and $\mathcal{E}$ are also τ_g-S_g^*-open sets, and $\mathcal{D} \cup \mathcal{E} = \mathcal{W}$. Then ($\mathcal{W}$, τ_g) is not τ_g-S_g^*-connected space, which implies a contradiction. Therefore, ($\mathcal{W}$, τ_g) is a τ_g-connected space. □

Remark 2. *The converse of the above theorem is invalid as demonstrated by the following illustration.*

Example 8. *Consider that* $\mathcal{W} = \{e_1, f_1, d_1, k_1\}$ *and* $\tau_g = \{\phi, \mathcal{W}, \{e_1\}, \{f_1, d_1\}, \{e_1, f_1, d_1\}\}$*. In (*$\mathcal{W}$*,* τ_g*)*$O = \{\phi, \mathcal{W}, \{e_1\}, \{f_1, d_1\}, \{e_1, f_1, d_1\}\}$ *and (*$\mathcal{W}$*,* τ_g*)*S_g^* $O = \{\phi, \mathcal{W}, \{e_1\}, \{f_1, d_1\}, \{e_1, f_1, d_1\}, \{f_1, d_1, k_1\}\}$*, there does not exist any non-empty open sets* $\mathcal{D}$ *and* $\mathcal{E}$ *such that* $\mathcal{D} \cup \mathcal{E} = \mathcal{W} \Rightarrow$ *(*$\mathcal{W}$*,* τ_g*) is a* τ_g*-connected space. However, there are two non-empty disjoint* τ_g-S_g^**-open sets* $\{e_1\}$ *and* $\{f_1, d_1, k_1\}$ *such that* $\{e_1\} \cup \{f_1, d_1, k_1\} = \mathcal{W} \Rightarrow$ *(*$\mathcal{W}$*,* τ_g*) is not a* τ_g-S_g^**-connected space.*

Theorem 9. *Assume a generalized topological space ($\mathcal{W}$, τ_g). Then the subsequent is equivalent.*

i. $\mathcal{W}$ is τ_g-S_g^*-connected.
ii. ϕ and $\mathcal{W}$ are only a τ_g-S_g^*-open set and a τ_g-S_g^*-closed set in $\mathcal{W}$.
iii. Any τ_g-S_g^*-continuous mapping $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) is a constant map where ($\mathcal{Y}$, τ_{g_2}) is at least a two-point discrete space.

Proof. (i) then (ii): A τ_g-S_g^*-open set and τ_g-S_g^*-closed set $\mathcal{E}$ in ($\mathcal{W}$, τ_g) $\Rightarrow$ $\mathcal{E}^c$ is also a τ_g-S_g^*-open set and a τ_g-S_g^*-closed set in ($\mathcal{W}$, τ_g). Then $\mathcal{E} \cup \mathcal{E}^c = \mathcal{W}$, as $\mathcal{E}$ and $\mathcal{E}^c$ are both disjoint τ_g-S_g^*-open sets, which implies a contradiction. ($\mathcal{W}$, τ_g) is a τ_g-S_g^*-connected space, so $\mathcal{W}$ is ϕ or $\mathcal{W}$.

(ii) then (i): Consider two disjoint τ_g-S_g^*-open sets $\mathcal{E}$ and $\mathcal{D}$ in ($\mathcal{W}$, τ_g) and $\mathcal{E} \cup \mathcal{D} = \mathcal{W}$. Since $\mathcal{E}^c = \mathcal{D}$, $\mathcal{E}$ is a τ_g-S_g^*-open set and a τ_g-S_g^*-closed set. By hypothesis, $\mathcal{E}$ is ϕ or $\mathcal{W}$, which is a contradiction. Hence, ($\mathcal{W}$, τ_g) is a τ_g-S_g^*-connected space.

(ii) then (iii): Assume a τ_g-S_g^*-continuous mapping $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) is a constant map where ($\mathcal{Y}$, τ_{g_2}) is at least a two-point discrete space. Then $\mathfrak{h}^{-1}$ ({x}) is a τ_g-S_g^*-open and a τ_g-S_g^*-closed set for all elements of $\mathcal{Y}$. $\mathcal{W} = \cup\, \mathfrak{h}^{-1}$ ({x}), where x $\in \mathcal{Y}$ and $\mathfrak{h}^{-1}$ ({x}) is τ_g-S_g^*-open and τ_g-S_g^*-closed in $\mathcal{W}$. By hypothesis, $\mathfrak{h}^{-1}$ ({x}) is ϕ or $\mathcal{W}$, $\forall$ x $\in \mathcal{Y}$, then $\mathfrak{h}$ fails to map. There is, then, at least one point for x elements of $\mathcal{Y}$ where $\mathfrak{h}^{-1}$ ({x}) is not empty $\Rightarrow \mathfrak{h}^{-1}$ ({x}) = $\mathcal{W} \Rightarrow \mathfrak{h}$ is a constant function.

(iii) then (ii): Let $\mathcal{E}$ be τ_g-S_g^*-open and τ_g-S_g^*-closed in a generalized topological space ($\mathcal{W}$, τ_g) where $\mathcal{E}$ is a non-empty set. Consider a τ_g-S_g^*-continuous mapping $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) defined as $\mathfrak{h}$ ($\mathcal{E}$) = {x} and $\mathfrak{h}$ ($\mathcal{E}^c$) = {y}, where x, y $\in \mathcal{Y}$ and x $\neq$ y. By hypothesis, $\mathfrak{h}$ is a constant map and $\mathcal{E} = \mathcal{W}$. □

Theorem 10. *Assume $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) is a surjective, τ_g-S_g^*-continuous mapping and ($\mathcal{W}$, τ_{g_1}) is a τ_g-S_g^*-connected space. Then ($\mathcal{Y}$, τ_{g_2}) is a τ_g-connected space.*

Proof. Assume ($\mathcal{Y}$, τ_{g_2}) is not a τ_g-connected space. Consider two non-empty disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{Y}$. For example, $\mathcal{E} \cup \mathcal{D} = \mathcal{Y}$. Because $\mathfrak{h}$ is τ_g-S_g^*-continuous, $\mathfrak{h}^{-1}(\mathcal{E}) \cup \mathfrak{h}^{-1}(\mathcal{D})$ = $\mathcal{W}$ and $\mathfrak{h}^{-1}(\mathcal{E})$. $\mathfrak{h}^{-1}(\mathcal{D})$ is a τ_g-S_g^*-open disjoint set in $\mathcal{W}$, which implies a contradiction. Thus, $\mathcal{W}$ is τ_g-S_g^*-connected. Hence, $\mathcal{Y}$ is a τ_g-connected space. □

Theorem 11. *Consider $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) is a surjective, τ_g-S_g^*-irresolute function, and ($\mathcal{W}$, τ_{g_1}) is a τ_g-S_g^*-connected space, so ($\mathcal{Y}$, τ_{g_2}) is a τ_g-connected space.*

Proof. Assume ($\mathcal{Y}$, τ_{g_2}) is a τ_g-S_g^*-connected space. Consider two non-empty τ_g-S_g^*-open sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{Y}$, such as $\mathcal{E} \cup \mathcal{D} = \mathcal{Y}$. $\mathfrak{h}$ is surjective, τ_g-S_g^*-continuous, so $\mathfrak{h}^{-1}(\mathcal{E}) \cup \mathfrak{h}^{-1}(\mathcal{D})$ = $\mathcal{W}$ and $\mathfrak{h}^{-1}(\mathcal{E})$. $\mathfrak{h}^{-1}(\mathcal{D})$ is a τ_g-S_g^*-open disjoint subset in $\mathcal{W}$, which implies a contradiction. Thus, $\mathcal{W}$ is τ_g-S_g^*-connected. Hence, $\mathcal{Y}$ is also a τ_g-S_g^*-connected space. □

Theorem 12. *Consider $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) is a strongly τ_g-S_g^*-continuous function, and ($\mathcal{W}$, τ_{g_1}) is a τ_g-connected space. Then its image is a τ_g-S_g^*-connected space.*

Proof. Consider $\mathfrak{h}$: ($\mathcal{W}$, τ_{g_1}) $\rightarrow$ ($\mathcal{Y}$, τ_{g_2}) is a strongly τ_g-S_g^*-continuous map, and ($\mathcal{W}$, τ_{g_1}) is a τ_g-connected space. Assume ($\mathcal{Y}$, τ_{g_2}) is not a τ_g-S_g^*-connected space for two non-empty disjoint τ_g-S_g^*-open sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{Y}$, such as $\mathcal{E} \cup \mathcal{D} = \mathcal{Y}$. $\mathfrak{h}$ is strongly τ_g-S_g^*-continuous, so $\mathfrak{h}^{-1}(\mathcal{E}) \cup \mathfrak{h}^{-1}(\mathcal{D}) = \mathcal{W}$ and $\mathfrak{h}^{-1}(\mathcal{E})$, $\mathfrak{h}^{-1}(\mathcal{D})$ are open disjoint sets in $\mathcal{W}$, which implies a contradiction. Thus, $\mathcal{W}$ is τ_g-connected; hence, $\mathcal{Y}$ is a τ_g-S_g^*-connected space. □

3.3. τ_g-S_g^*-Separation Axioms

Definition 16. *A generalized topological space ($\mathcal{W}$, τ_g) is called a τ_g-S_g^*Tc space if every τ_g-S_g^*-closed set is closed.*

Definition 17. *Consider that $\mathcal{E}$, to be a subset of $\mathcal{W}$, is said to be a τ_g-T_0 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ or $\mathfrak{r} \notin \mathcal{M}_1$, $\mathfrak{s} \in \mathcal{M}_1$, where $\mathcal{M}_1$ is an open set of $\mathcal{W}$.*

Definition 18. *Let $\mathcal{E} \subseteq \mathcal{W}$. $\mathcal{E}$ is said to be a τ_g-T_1 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ and $\mathfrak{r} \notin \mathcal{N}_1$, $\mathfrak{s} \in \mathcal{N}_1$, where $\mathcal{M}_1$ and $\mathcal{N}_1$ are open sets of $\mathcal{W}$.*

Definition 19. *Assume $\mathcal{E} \subseteq \mathcal{W}$. $\mathcal{E}$ is said to be a τ_g-T_2 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ and $\mathfrak{r} \notin \mathcal{N}_1$, $\mathfrak{s} \in \mathcal{N}_1$, where $\mathcal{M}_1$ and $\mathcal{N}_1$ are open sets of $\mathcal{W}$ and $\mathcal{M}_1 \neq \mathcal{N}_1$.*

Definition 20. *The function $\jmath$:$\mathcal{W} \to \mathcal{Y}$ is indicated as τ_g-S_g^*-continuous if there is an inverse image that is τ_g-S_g^*-open in ($\mathcal{W}$, τ_{g_1}) for every open set in ($\mathcal{Y}$, τ_{g_2}).*

3.3.1. τ_g-S_g^*-T_0, τ_g-S_g^*-T_1, τ_g-S_g^*-T_2 Spaces

Definition 21. *Consider that $\mathcal{E}$, to be a subset of $\mathcal{W}$, is said to be a τ_g-S_g^*-T_0 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ or $\mathfrak{r} \notin \mathcal{M}_1$, $\mathfrak{s} \in \mathcal{M}_1$, where $\mathcal{M}_1$ is a τ_g-S_g^*-open set of $\mathcal{W}$.*

Definition 22. *Let $\mathcal{E} \subseteq \mathcal{W}$. $\mathcal{E}$ is said to be a τ_g-S_g^*-T_1 space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ and $\mathfrak{r} \notin \mathcal{N}_1$, $\mathfrak{s} \in \mathcal{N}_1$, where $\mathcal{M}_1$ and $\mathcal{N}_1$ are τ_g-S_g^*-open sets of $\mathcal{W}$.*

Definition 23. *Assume $\mathcal{E} \subseteq \mathcal{W}$. $\mathcal{E}$ is said to be a τ_g-S_g^*-T_2(τ_g-S_g^*-Housdorff) space if for every explicit point $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ and $\mathfrak{r} \notin \mathcal{N}_1$, $\mathfrak{s} \in \mathcal{N}_1$, where $\mathcal{M}_1$ and $\mathcal{N}_1$ are τ_g-S_g^*-open sets of $\mathcal{W}$ and $\mathcal{M}_1 \neq \mathcal{N}_1$.*

Theorem 13.

- *i.* *If $\mathcal{W}$ is τ_g-T_0, then $\mathcal{W}$ is τ_g-S_g^*-T_0.*
- *ii.* *If $\mathcal{W}$ is τ_g-T_1, then $\mathcal{W}$ is τ_g-S_g^*-T_0 and τ_g-S_g^*-T_1.*
- *iii.* *If $\mathcal{W}$ is τ_g-T_2, then $\mathcal{W}$ is τ_g-S_g^*-T_2.*
- *iv.* *If $\mathcal{W}$ is τ_g-S_g^*-T_2, then $\mathcal{W}$ is τ_g-S_g^*-T_0.*
- *v.* *If $\mathcal{W}$ is τ_g-S_g^*-T_2, then $\mathcal{W}$ is τ_g-S_g^*-T_1.*

Proof. i) Assume $\mathcal{W}$ is a τ_g-T_0 space. There is an open set $\mathcal{F}$ for distinct points of every pair $\mathfrak{r}$ $\mathfrak{s}$ of $\mathcal{W}$ where $\mathfrak{s} \notin \mathcal{F}$, $\mathfrak{r} \in \mathcal{F}$, or $\mathfrak{r} \notin \mathcal{F}$, $\mathfrak{s} \in \mathcal{F}$. Then $\mathcal{F} \in \tau_g$-S_g^*O($\mathcal{W}$) with $\mathfrak{s} \notin \mathcal{F}$, $\mathfrak{r} \in \mathcal{F}$, or $\mathfrak{r} \notin \mathcal{F}$, $\mathfrak{s} \in \mathcal{F}$. Therefore, $\mathcal{W}$ is a τ_g-S_g^*-T_0 space. In the same way, we can demonstrate (ii), (iii), (iv), and (v). □

See the below Figure 1 for the relation between the above theorems.

Remark 3. *The illustration below demonstrates that the converse of the above theorem is invalid.*

Example 13. *Let $\mathcal{W} = \{e_1, f_1, d_1\}$ and $\tau_g = \{\phi, \mathcal{W}, \{d_1\}, \{e_1, f_1\}\}$. In ($\mathcal{W}$, τ_g)O = $\{\phi, \mathcal{W}, \{d_1\}, \{e_1, f_1\}\}$ and τ_g-S_g^*O = P($\mathcal{W}$). Hence, ($\mathcal{W}$, τ_g) is as follows:*

- *τ_g-S_g^*-T_0 but not τ_g-T_0. For explicit points $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, no open set exists with $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ or $\mathfrak{r} \notin \mathcal{M}_1$, $\mathfrak{s} \in \mathcal{M}_1$, where $\mathcal{M}_1$ is an open set of $\mathcal{W}$.*
- *τ_g-S_g^*-T_1 space but not τ_g-T_1 space. For explicit points $\mathfrak{r}$ and $\mathfrak{s}$ of $\mathcal{W}$, no two open sets $\mathcal{M}_1$ and $\mathcal{N}_1$ exists with $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ and $\mathfrak{r} \notin \mathcal{N}_1$, $\mathfrak{s} \in \mathcal{N}_1$.*

- τ_g-S_g^*-T_2 *space but not* τ_g-T_2 *space. For explicit points* $\mathfrak{r}$ *and* $\mathfrak{s}$ *of* $\mathcal{W}$*, no two distinct open sets* $\mathcal{M}_1$ *and* $\mathcal{N}_1$ *exists with* $\mathfrak{s} \notin \mathcal{M}_1$, $\mathfrak{r} \in \mathcal{M}_1$ *and* $\mathfrak{r} \notin \mathcal{N}_1$, $\mathfrak{s} \in \mathcal{N}_1$.

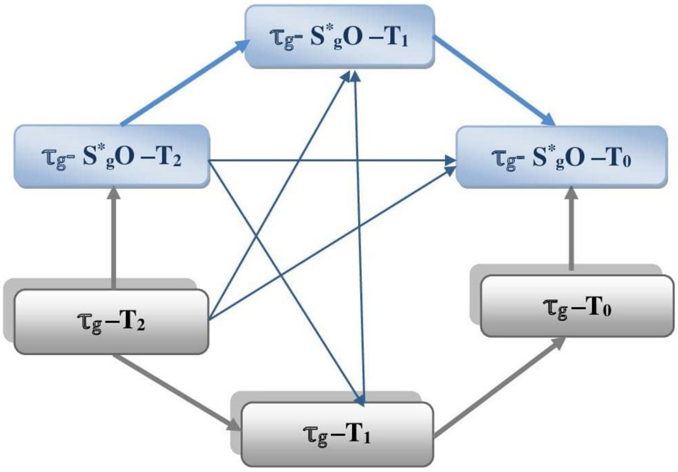

Figure 1. An illustration of the above theorems and considerations.

Theorem 14. *If* $\mathfrak{h}$ *is bijective and strongly* τ_g-S_g^**-open, and* $\mathcal{W}$ *is* τ_g-S_g^*-T_0*, then* $\mathcal{Y}$ *is a* τ_g-S_g^*-T_0 *space.*

Proof. Take $\mathfrak{r}_2$ and $\mathfrak{s}_2$ of $\mathcal{Y}$ with $\mathfrak{r}_2 \neq \mathfrak{s}_2$. By hypothesis, $\mathfrak{r}_2 = \mathfrak{h}(\mathfrak{r}_1)$ and $\mathfrak{s}_2 = \mathfrak{h}(\mathfrak{s}_1)$ where $\mathfrak{r}_1$ and $\mathfrak{s}_1$ are the explicit points of $\mathcal{W}$. By hypothesis, $\mathcal{M}_1 \in \tau_g$-S_g^*O($\mathcal{W}$) with $\mathfrak{r}_1 \in \mathcal{M}_1$ and $\mathfrak{s}_1 \notin \mathcal{M}_1$. Therefore, $\mathfrak{h}(\mathfrak{r}_1) \in \mathfrak{h}(\mathcal{M}_1)$ and $\mathfrak{h}(\mathfrak{s}_1) \notin \mathfrak{h}(\mathcal{M}_1)$. $\mathfrak{h}(\mathcal{M}_1) \in \tau_g$-$S_g^*$O($\mathcal{Y}$) as $\mathcal{W}$ is strongly τ_g-S_g^*-open. Thus, $\mathfrak{h}(\mathcal{M}_1)$ is a τ_g-S_g^*-open set in $\mathcal{Y}$ with $\mathfrak{r}_2 \in \mathfrak{h}(\mathcal{M}_1)$ and $\mathfrak{s}_2 \notin \mathfrak{h}(\mathcal{M}_1)$. Therefore, $\mathcal{Y}$ is a τ_g-S_g^*-T_0 space. □

Theorem 15. *If* $\mathfrak{h}$ *is injective and* τ_g-S_g^**-irresolute, and* $\mathcal{Y}$ *is* τ_g-S_g^*-T_0*, then* $\mathcal{X}$ *is a* τ_g-S_g^*-T_0 *space.*

Proof. Take $\mathfrak{r}_1$ and $\mathfrak{s}_1$ explicit points of $\mathcal{W}$. $\mathfrak{h}$ is injective $\Rightarrow \mathfrak{h}(\mathfrak{s}_1) \neq \mathfrak{h}(\mathfrak{r}_1)$. As $\mathcal{Y}$ is τ_g-S_g^*-T_0, there is a $\mathcal{F} \in \tau_g$-S_g^*O($\mathcal{Y}$) such that $\mathfrak{h}(\mathfrak{r}_1) \in \mathcal{F}$, $\mathfrak{h}(\mathfrak{s}_1) \notin \mathcal{F}$, or there is a $\mathcal{F}_\beta \in \tau_g$-$S_g^*$O($\mathcal{Y}$) such that $\mathfrak{h}(\mathfrak{s}_1) \in \mathcal{F}_\beta$, $\mathfrak{h}(\mathfrak{r}_1) \notin \mathcal{F}_\beta$ with $\mathfrak{h}(\mathfrak{s}_1) \neq \mathfrak{h}(\mathfrak{r}_1)$. As $\mathfrak{h}$ is τ_g-S_g^*-irresolute, then $\mathfrak{h}^{-1}(\mathcal{F}) \in \tau_g$-$S_g^*$O($\mathcal{W}$), and there is a $\mathfrak{h}^{-1}(\mathfrak{r}_1) \in \mathcal{F}$, $\mathfrak{h}^{-1}(\mathfrak{s}_1) \notin \mathcal{F}$ or $\mathfrak{h}^{-1}(\mathcal{F}_\beta) \in \tau_g$-$S_g^*$O($\mathcal{W}$) $\Rightarrow \mathfrak{h}^{-1}(\mathfrak{s}_1) \in \mathcal{F}_\beta$, $\mathfrak{h}^{-1}(\mathfrak{r}_1) \notin \mathcal{F}_\beta$. Hence, $\mathcal{W}$ is a τ_g-S_g^*-T_0 space. □

Theorem 16. *Assume* $\mathcal{W}$ *is* τ_g-S_g^*-T_1 *iff* $\mathfrak{s}_1 \in \mathcal{W}$ *singleton* $\{\mathfrak{s}_1\} \in \tau_g$-$S_g^*C(\mathcal{W})$.

Proof. Assume $\mathcal{W}$ is τ_g-S_g^*-T_1, $\mathfrak{r}_1 \in \mathcal{W}$. Then $\mathfrak{s}_1 \in \mathcal{W}$ - $\{\mathfrak{r}_1\} \Rightarrow \mathfrak{r}_1 \neq \mathfrak{s}_1 \in \mathcal{W}$. However, $\mathcal{W}$ is a τ_g-S_g^*-T_1 space $\Rightarrow$ there is a $\mathcal{F}$, $\mathcal{F}_\beta \in \tau_g$-$S_g^*$O($\mathcal{W}$) $\Rightarrow \mathfrak{r}_1 \notin \mathcal{F}$, $\mathfrak{s}_1 \in \mathcal{F}_\beta \subseteq (\mathcal{W}$ - $\{\mathfrak{r}_1\})$. Furthermore, $\mathfrak{s}_1 \in \mathcal{F}_\beta \subseteq (\mathcal{W}$ - $\{\mathfrak{r}_1\}) \Rightarrow (\mathcal{W}$ - $\{\mathfrak{r}_1\}) \in \tau_g$-$S_g^*$O($\mathcal{W}$). Thus, $\{\mathfrak{r}_1\}$ is τ_g-S_g^*-closed. Conversely, consider that $\mathfrak{r}_1 \neq \mathfrak{s}_1 \in \mathcal{W}$. Then $\{\mathfrak{r}_1\}$ and $\{\mathfrak{s}_1\}$ are τ_g-S_g^*-closed sets, and $\{\mathfrak{r}_1\}^c$ is τ_g-S_g^*-open. Certainly, $\{\mathfrak{r}_1\} \notin \{\mathfrak{r}_1\}^c$ and $\{\mathfrak{s}_1\} \in \{\mathfrak{r}_1\}^c$. Similarly $\{\mathfrak{s}_1\}^c$ is τ_g-S_g^*-open, $\{\mathfrak{s}_1\} \notin \{\mathfrak{s}_1\}^c$ and $\{\mathfrak{r}_1\} \in \{\mathfrak{s}_1\}^c$. Thus $\mathcal{W}$ is τ_g-S_g^*-T_1 space. □

Theorem 17. *Let* $\mathfrak{h}$:$\mathcal{W} \to \mathcal{Y}$*. Then the subsequent findings are valid:*

(i) *If* $\mathfrak{h}$ *is* τ_g-S_g^**-continuous and injective,* $\mathcal{Y}$ *is* τ_g-$T_1 \Rightarrow \mathcal{W}$ *is* τ_g-S_g^*-T_1.
(ii) *If* $\mathfrak{h}$ *is* τ_g-S_g^**-continuous and injective,* $\mathcal{Y}$ *is* τ_g-$T_2 \Rightarrow \mathcal{W}$ *is* τ_g-S_g^*-T_2.
(iii) *If* $\mathfrak{h}$ *is* τ_g-S_g^**-irresolute and injective,* $\mathcal{Y}$ *is* τ_g-S_g^*-$T_2 \Rightarrow \mathcal{W}$ *is* τ_g-S_g^*-T_2.

Proof. (i) Assume $\mathfrak{r}_1 \neq \mathfrak{s}_1$, where $\mathfrak{r}_1, \mathfrak{s}_1 \in \mathcal{W}$. Then $\mathfrak{h}(\mathfrak{r}_1) = \mathfrak{r}_2$ and $\mathfrak{h}(\mathfrak{s}_1) = \mathfrak{s}_2$. Additionally, $\mathfrak{h}(\mathfrak{r}_1) \neq \mathfrak{h}(\mathfrak{s}_1)$. $(\mathcal{Y}, \tau_{g_2})$ τ_g-$T_1 \Rightarrow \mathfrak{r}_2 \in \mathcal{M}_1$, $\mathfrak{s}_2 \notin \mathcal{M}_1$ and $\mathfrak{s}_2 \in \mathcal{N}_1$, $\mathfrak{r}_2 \notin \mathcal{N}_1$. Then $\mathfrak{r}_1 \in \mathfrak{h}^{-1}(\mathcal{M}_1)$, $\mathfrak{r}_1 \notin \mathfrak{h}^{-1}(\mathcal{N}_1)$ and $\mathfrak{s}_1 \in \mathfrak{h}^{-1}(\mathcal{N}_1)$, $\mathfrak{s}_1 \notin \mathfrak{h}^{-1}(\mathcal{M}_1)$. According to the definition of τ_g-S_g^*-continuity, $\mathfrak{h}^{-1}(\mathcal{M}_1)$ and $\mathfrak{h}^{-1}(\mathcal{N}_1) \in \tau_g$-$S_g^*O(\mathcal{W})$. For $\mathfrak{r}_1 \neq \mathfrak{s}_1$, $\mathfrak{r}_1, \mathfrak{s}_1 \in \mathcal{W} \Rightarrow \mathfrak{r}_1 \in \mathfrak{h}^{-1}(\mathcal{M}_1)$, $\mathfrak{r}_1 \notin \mathfrak{h}^{-1}(\mathcal{N}_1)$ and $\mathfrak{s}_1 \in \mathfrak{h}^{-1}(\mathcal{N}_1)$, $\mathfrak{s}_1 \notin \mathfrak{h}^{-1}(\mathcal{M}_1)$. Thus, $(\mathcal{W}, \tau_{g_1})$ is a τ_g-S_g^*-T_1 space. In the same way, (ii) and (iii) can be proven. □

Theorem 18. *The subsequent statements are equivalent.*

1. $\mathcal{W}$ *is* τ_g-S_g^*-T_2.
2. *If* $\mathfrak{r}_1 \in \mathcal{W}$, *then* $\mathfrak{r}_1 \neq \mathfrak{s}_1$, *there is a* $\mathcal{U}$ *containing* $\mathfrak{r}_1$ *and* $\mathfrak{s}_1 \notin \tau_g$-$S_g^*$*Cl*$(\mathcal{U})$.

Proof. (1) ⇒ (2) Take $\mathfrak{r}_1 \in \mathcal{W}$ and $\mathfrak{s}_1 \in \mathcal{W}$ with $\mathfrak{r}_1 \neq \mathfrak{s}_1$. There is a disjoint set $\mathcal{U}$ and $\mathcal{V} \in \tau_g$-S_g^*O$(\mathcal{W})$ such that $\mathfrak{r}_1 \in \mathcal{U}$ and $\mathfrak{s}_1 \in \mathcal{V}$. Then $\mathfrak{r}_1 \in \mathcal{U} \subseteq \mathcal{V}^c$ and $\mathcal{V}^c \in \tau_g$-S_g^*C$(\mathcal{W})$ and $\mathfrak{s}_1 \notin \mathcal{V}^c \Rightarrow \mathfrak{s}_1 \notin \tau_g$-$S_g^*Cl(\mathcal{U})$. (2) ⇒ (1) Consider $\mathfrak{r}_1 \in \mathcal{W}$ and $\mathfrak{s}_1 \in \mathcal{W}$ with $\mathfrak{r}_1 \neq \mathfrak{s}_1 \Rightarrow$ there is a τ_g-S_g^*-open $\mathcal{U}$ containing $\mathfrak{r}_1$ such that $\mathfrak{s}_1 \notin \tau_g$-$S_g^*Cl(\mathcal{U}) \Rightarrow \mathfrak{s}_1 \in (\mathcal{W}$ - $(\tau_g$-S_g^*Cl$(\mathcal{U})))$. $(\mathcal{W}$ - $(\tau_g$-S_g^*Cl$(\mathcal{U}))) \in \tau_g$-$S_g^*O(\mathcal{W})$ and $\mathfrak{r}_1 \notin (\mathcal{W}$ - $(\tau_g$-S_g^*Cl$(\mathcal{U})))$. Additionally, $\mathcal{U} \cap (\mathcal{W}$ - $(\tau_g$-S_g^*Cl$(\mathcal{U})))$ = ϕ. Therefore, $\mathcal{W}$ is τ_g-S_g^*-T_2. □

3.3.2. τ_g-S_g^*-Regular Space

Definition 24. *If* $\forall$ $\mathcal{F} \in \tau_g$-S_g^**C*$(\mathcal{W})$ *and* $\mathfrak{r} \notin \mathcal{F}$, *there are disjoint open sets* $\mathcal{E}$ *and* $\mathcal{D}$ *such that* $\mathcal{F} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$.

The relationship between τ_g-S_g^**-regularity and* τ_g*-regularity is as follows.*

Theorem 19. *Every* τ_g-S_g^**-regular space is* τ_g*-regular.*

Proof. Considering $\mathcal{W}$ to be a τ_g-S_g^*-regular space. Take $\mathcal{F} \in \tau_g$-S_g^*C$(\mathcal{W})$ and $\mathfrak{r} \notin \mathcal{F}$. As $\mathcal{W}$ is τ_g-S_g^*-regular, there is a pair of disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ such that $\mathcal{F} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$. Hence, $\mathcal{W}$ is τ_g-regular. □

Remark 4. *Every* τ_g*-regular space is not a* τ_g-S_g^**-regular space.*

Example 19. *Let* $\mathcal{W} = \{e_1, f_1, d_1\}$ *and* $\tau_g = \{\phi, \mathcal{W}, \{d_1\}, \{e_1, f_1\}\}$. *Hence,* $(\mathcal{W}, \tau_g)$ *is* τ_g*-regular but not a* τ_g-S_g^**-regular space. For* $\{f_1\} \in \tau_g$-S_g^**C*$(\mathcal{W})$ *and* $\mathfrak{r} \notin \{f_1\}$, *there are no disjoint open sets* $\mathcal{E}$ *and* $\mathcal{D}$ *such that* $\{f_1\} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$.

Theorem 20. *Every* τ_g*-regular space with a* τ_g-S_g^**Tc space is* τ_g-S_g^**-regular.*

Proof. Considering $\mathcal{W}$ to be τ_g-regular, τ_g-S_g^*Tc. Assume $\mathcal{F} \in \tau_g$-S_g^*C$(\mathcal{W})$ and $\mathfrak{r} \in \mathcal{W}$ such that $\mathfrak{r} \notin \mathcal{F}$. As $\mathcal{W}$ is a τ_g-S_g^*Tc space, $\mathcal{F}$ is closed, and $\mathfrak{r} \notin \mathcal{F}$. Since $\mathcal{W}$ is τ_g-regular, there is a pair of disjoint open sets $\mathcal{E}$ and $\mathcal{D}$ such that $\mathcal{F} \subseteq \mathcal{E}$, $\mathfrak{r} \in \mathcal{D}$. Hence, $\mathcal{W}$ is τ_g-S_g^*-regular. □

Theorem 21. *If* $\mathcal{W}$ *is* τ_g-S_g^**-regular, then it is a* τ_g*-regular space.*

Proof. According to this fact, every closed set belongs to τ_g-S_g^*C$(\mathcal{W})$. □

Theorem 22. *The subsequent statements are equivalent.*

1. $\mathcal{W}$ *is* τ_g-S_g^**-regular.*
2. $\forall$ $\mathfrak{r} \in \mathcal{W}$ *and each* τ_g-S_g^**-open nbhd* $\mathcal{U}$, *there is an open nbhd* $\mathcal{N}_1$ *of* $\mathcal{W}$ *such that Cl*$(\mathcal{N}_1) \subseteq \mathcal{U}$.

Proof. (1) ⇒ (2) Assume $\mathcal{U}$ is a τ_g-S_g^*-neighbourhood of $\mathfrak{r}$. There is a $\mathcal{E} \in \tau_g$-S_g^*O$(\mathcal{W})$ such that $\mathfrak{r} \in \mathcal{E} \subseteq \mathcal{U}$. Now $\mathcal{E}^c \in \tau_g$-S_g^*C$(\mathcal{W})$ and $\mathfrak{r} \notin \mathcal{E}^c$. From (1), there is a $\mathcal{R}$, $\mathcal{S}$ such that $\mathcal{E}^c \subseteq$

$\mathcal{R}$, $\mathfrak{r} \in \mathcal{S}$, $\mathcal{R} \cap \mathcal{S} = \phi$. Thus, $\mathcal{S} \subseteq {\mathcal{M}_1}^c$. Now $\mathrm{Cl}(\mathcal{S}) \subseteq \mathrm{Cl}(\mathcal{R}^c) = \mathcal{E}^c$ and $\mathcal{E}^c \subseteq \mathcal{R} \Rightarrow \mathcal{R}^c \subseteq \mathcal{E} \subseteq \mathcal{U}$. Thus, $\mathrm{Cl}(\mathcal{S}) \subseteq \mathcal{U}$. (2) $\Rightarrow$ (1) Consider a τ_g-S_g^*-closed $\mathcal{F}$ in $\mathcal{W}$ and $\mathfrak{r} \notin \mathcal{F}$ or $\mathfrak{r} \in (\mathcal{F})^c$ and $\mathcal{W}$ is τ_g-S_g^*-open $\Rightarrow (\mathcal{F})^c$ is τ_g-S_g^*-nbhd of $\mathfrak{r}$. By hypothesis, there is an open nbhd $\mathcal{N}_1$ such that $\mathfrak{r} \in \mathcal{N}_1$, $\mathrm{Cl}(\mathcal{N}_1) \subseteq (\mathcal{F})^c \Rightarrow \mathcal{F} \subseteq \{\mathcal{W}$ - $\mathrm{Cl}(\mathcal{N}_1)\}$ and $\mathcal{N}_1 \cap \{\mathcal{W}$ - $\mathrm{Cl}(\mathcal{N}_1)\} = \phi$. Thus, $\mathcal{W}$ is τ_g-S_g^*-regular. □

Theorem 23. *Assume $\mathcal{W}$ is τ_g-S_g^*-regular iff for every $\mathcal{E} \in \tau_g$-$S_g^*C(\mathcal{W})$ and point $\mathfrak{p} \in (\mathcal{W}$ - $\mathcal{E})$, $\mathfrak{r} \in \mathcal{U}$, $\mathcal{E} \subseteq \mathcal{N}_1$ and $Cl(\mathcal{N}_1) \cap Cl(\mathcal{U}) = \phi$, where $\mathcal{U}$ and $\mathcal{N}_1$ are open sets.*

Proof. Given that $\mathcal{W}$ is τ_g-S_g^*-regular. Assume $\mathcal{E} \in \tau_g$-$S_g^*\mathrm{C}(\mathcal{W})$ and $\mathfrak{r} \notin \mathcal{E}$. Then $\mathfrak{p}$ $\mathcal{M}_1$ and $\mathcal{E} \subseteq \mathcal{N}_1$ and $\mathcal{M}_1 \cap \mathcal{N}_1 = \phi$ where $\mathcal{M}_1$ and $\mathcal{N}_1$ are open sets $\Rightarrow \mathcal{M}_1 \cap \mathrm{Cl}(\mathcal{N}_1) = \phi$. As $\mathcal{W}$ is τ_g-S_g^*-regular, $\mathfrak{p} \in \mathcal{R}$ and $\mathrm{Cl}(\mathcal{N}_1) \subseteq \mathcal{S}$, $\mathcal{R} \cap \mathcal{N}_1 = \phi$, where $\mathcal{R}$ and $\mathcal{S}$ are open. Furthermore, $\mathrm{Cl}(\mathcal{R}) \cap \mathcal{S} = \phi$. $\mathcal{U} = \mathcal{M}_1 \cap \mathcal{R} \Rightarrow \mathfrak{p} \in \mathcal{U}$, $\mathcal{E} \subseteq \mathcal{N}_1$ and $\mathrm{Cl}(\mathcal{N}_1) \cap \mathrm{Cl}(\mathcal{U}) = \phi$, where $\mathcal{N}_1$ and $\mathcal{U}$ are open in $\mathcal{W}$. On the other hand, consider $\mathcal{N}_1$ and $\mathcal{U}$ are open sets. $\mathfrak{p} \in \mathcal{U}$, $\mathcal{E} \subseteq \mathcal{N}_1$ and $\mathrm{Cl}(\mathcal{N}_1) \cap \mathrm{Cl}(\mathcal{U}) = \phi$ $\forall$ $\mathcal{E} \in \tau_g$-$S_g^*\mathrm{C}(\mathcal{W})$ and $\mathfrak{p} \in (\mathcal{W}$ - $\mathcal{E}) \Rightarrow \mathfrak{p} \in \mathcal{U}$, $\mathcal{E} \subseteq \mathcal{N}_1$ and $\mathcal{U} \cap \mathcal{N}_1 = \phi$. Thus, $\mathcal{W}$ is τ_g-S_g^*-regular. □

Theorem 24. *A subspace $\mathcal{Y}$ of τ_g-S_g^*-regular ($\mathcal{W}$, τ_g) is τ_g-S_g^*-regular.*

Proof. Obvious. □

Theorem 25. *Consider that $\mathfrak{h}$ is bijective and τ_g-S_g^*-irresolute and an open map from τ_g-S_g^*-regular $\mathcal{W}$ into $\mathcal{Y}$. Then $\mathcal{Y}$ is τ_g-S_g^*-regular.*

Proof. Let $\mathfrak{r}_1 \in \mathcal{Y}$ and $\mathcal{F} \in \tau_g$-$S_g^*\mathrm{C}(\mathcal{W})$ and $\mathfrak{r}_1 \notin \mathcal{F}$. Additionally, $\mathfrak{h}$ is τ_g-S_g^*-irresolute. Then $\mathfrak{h}^{-1}(\mathcal{F}) \in \tau_g$-$S_g^*\mathrm{C}(\mathcal{W})$. Now assume $\mathfrak{r}_1 = \mathfrak{h}(\mathfrak{r})$. Then $\mathfrak{h}^{-1}(\mathfrak{r}_1) = \mathfrak{r}$ and $\mathfrak{r} \notin \mathfrak{h}^{-1}(\mathcal{F})$. If $\mathcal{W}$ is τ_g-S_g^*-regular, then there is a $\mathcal{R}$ and $\mathcal{S}$ such that $\mathfrak{r} \in \mathcal{R}$ and $\mathfrak{h}^{-1}(\mathcal{F}) \subseteq \mathcal{S}$, $\mathcal{R} \cap \mathcal{S} = \phi$. $\mathfrak{h}$ is open and bijective $\Rightarrow \mathfrak{r}_1 \in \mathfrak{h}(\mathcal{R})$, $\mathcal{F} \subseteq \mathfrak{h}(\mathcal{S})$ and $\mathfrak{h}(\mathcal{R} \cap \mathcal{S}) = \mathfrak{h}(\phi) = \phi \Rightarrow \mathcal{Y}$ is τ_g-S_g^*-regular. □

3.3.3. τ_g-S_g^*-Normal Space

Definition 25. *Assume $\mathcal{W}$ is τ_g-S_g^*-normal if for each pair $\mathcal{E}$, $\mathcal{D} \in \tau_g$-$S_g^*C(\mathcal{W})$, there are open sets $\mathcal{R}$ and $\mathcal{S}$ in $\mathcal{W}$ such that $\mathcal{D} \subseteq \mathcal{R}$ and $\mathcal{E} \subseteq \mathcal{S}$.*

Theorem 26. *Every τ_g-S_g^*-normal is τ_g-normal.*

Proof. As $\mathcal{W}$ is a τ_g-S_g^*-normal. Assume disjoint sets $\mathcal{E}$ and $\mathcal{D}$ in $\mathcal{W}$. Therefore, $\mathcal{E}$, $\mathcal{D} \in \tau_g$-$S_g^*\mathrm{C}(\mathcal{W})$. $\mathcal{W}$ is τ_g-S_g^*-normal $\Rightarrow$ there is a pair $\mathcal{F}$, $\mathcal{H}$ such that $\mathcal{D} \subseteq \mathcal{F}$, $\mathcal{E} \subseteq \mathcal{H}$. Thus, $\mathcal{W}$ is τ_g-normal. □

Remark 5. *The subsequent illustration demonstrates that the converse of the above theorem is invalid.*

Example 26. *Consider $\mathcal{W} = \{e_1, f_1, d_1\}$ and $\tau_g = \{\phi, \mathcal{W}, \{f_1\}, \{d_1\}, \{f_1, d_1\}, \{e_1, f_1\}\}$. Here ($\mathcal{W}$, τ_g) is τ_g-normal but not a τ_g-S_g^*-normal space. For disjoint sets $\{e_1\}$, $\{f_1, d_1\} \in \tau_g$-$S_g^*C(\mathcal{W})$, there are no open sets $\mathcal{R}$ and $\mathcal{S}$ in $\mathcal{W}$.*

Theorem 27. *If $\mathcal{Y}$ is τ_g-normal and a τ_g-S_g^*Tc space, then $\mathcal{Y}$ is τ_g-S_g^*-normal.*

Proof. Assume $\mathcal{Y}$ is τ_g-normal. Consider disjoint set $\mathcal{E}$, $\mathcal{D} \in \tau_g$-$S_g^*\mathrm{C}(\mathcal{Y})$. As τ_g-S_g^*Tc space, $\mathcal{E}$ and $\mathcal{D}$ are closed. Since $\mathcal{Y}$ is τ_g-normal, there are disjoint open sets $\mathcal{R}$ and $\mathcal{S}$ in $\mathcal{Y}$ such that $\mathcal{E} \subseteq \mathcal{R}$ and $\mathcal{D} \subseteq \mathcal{S}$. Thus, $\mathcal{Y}$ is τ_g-S_g^*-normal. □

Theorem 28. *Every τ_g-S_g^*-normal is τ_g-g-normal.*

Proof. Since $\mathcal{W}$ is τ_g-S_g^*-normal. Disjoint set $\mathcal{E}, \mathcal{D} \in \tau_g$-$S_g^*$C($\mathcal{Y}$) $\Rightarrow$ there is a disjoint $\mathcal{R}$, $\mathcal{S}$ such that $\mathcal{E} \subseteq \mathcal{R}$ and $\mathcal{D} \subseteq \mathcal{S}$. Thus, $\mathcal{W}$ is τ_g-g-normal. □

Remark 6. *Every τ_g-g-normal space is not τ_g-S_g^*-normal.*

Example 28. *Let $\mathcal{W} = \{e_1, f_1, d_1\}$ and $\tau_g = \{\phi, \mathcal{W}, \{f_1\}, \{d_1\}, \{f_1, d_1\}, \{e_1, d_1\}\}$. Here ($\mathcal{W}$, τ_g) is τ_g-g-normal but not a τ_g-S_g^*-normal space. For disjoint τ_g-S_g^*-closed sets $\{e_1\}$, $\{f_1, d_1\}$, there are no open sets $\mathcal{R}$ and $\mathcal{S}$ in $\mathcal{W}$.*

Theorem 29. *Every τ_g-S_g^*-normal space is τ_g-w-normal.*

Proof. Similar to Theorem 28. □

Remark 7. *The converse of the above theorem is invalid, as demonstrated by subsequent illustration.*

Example 29. *Let $\mathcal{W} = \{e_1, f_1, d_1\}$ and $\tau_g = \{\phi, \mathcal{W}, \{f_1\}, \{d_1\}, \{f_1, d_1\}, \{e_1, d_1\}\}$. Here ($\mathcal{W}$, τ_g) is τ_g-w-normal but not a τ_g-S_g^*-normal space. For disjoint τ_g-S_g^*-closed sets $\{e_1\}$, $\{f_1, d_1\}$, there are no open sets $\mathcal{R}$ and $\mathcal{S}$ in $\mathcal{W}$.*

Theorem 30. *If $\mathcal{Y}$ is a τ_g-S_g^*-closed subspace of τ_g-S_g^*-normal $\mathcal{W}$, then $\mathcal{Y}$ is τ_g-S_g^*-normal.*

Proof. Assume $\mathcal{W}$ is τ_g-S_g^*-normal and $\mathcal{Y}$ is a τ_g-S_g^*-closed subspace. A pair of disjoint sets $\mathcal{E}$ and $\mathcal{D} \in \tau_g$-S_g^*C($\mathcal{Y}$) $\Rightarrow$ there is a $\mathcal{F}, \mathcal{H} \in \mathcal{W}$ such that $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{D} \subseteq \mathcal{H} \Rightarrow \mathcal{F} \cap \mathcal{Y}$ and $\mathcal{H} \cap \mathcal{Y}$ are open in $\mathcal{Y}$. Furthermore, $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{D} \subseteq \mathcal{H} \Rightarrow \mathcal{E} \cap \mathcal{Y} \subseteq \mathcal{Y} \cap \mathcal{F}$ and $\mathcal{Y} \cap \mathcal{D} \subseteq \mathcal{Y} \cap \mathcal{H}$ and $(\mathcal{F} \cap \mathcal{Y}) \cap (\mathcal{Y} \cap \mathcal{H}) = \mathcal{Y} \cap (\mathcal{F} \cap \mathcal{H}) = \phi$. Thus, $\mathcal{Y}$ is τ_g-S_g^*-normal. □

Theorem 31. *The subsequent statements in ($\mathcal{W}$, τ_g) are equivalent.*

(1) *$\mathcal{W}$ is τ_g-S_g^*-normal.*
(2) *$\mathcal{E} \subseteq \mathcal{U}_1 \subseteq Cl(\mathcal{U}_1) \subseteq \mathcal{U}_2$ for each $\mathcal{E} \in \tau_g$-$S_g^*C(\mathcal{W})$ and each $\mathcal{U}_2 \in \tau_g$-$S_g^*O(\mathcal{W})$ with $\mathcal{E} \subseteq \mathcal{U}_2$, where $\mathcal{U}_1$ is an open set.*
(3) *For any disjoint set $\mathcal{E}, \mathcal{D} \in \tau_g$-$S_g^*C(\mathcal{W})$, there is an open set $\mathcal{U}_1$ such that $\mathcal{E} \subseteq \mathcal{U}_1$ and $Cl(\mathcal{U}_1) \cap \mathcal{D} = \phi$.*
(4) *For each disjoint set $\mathcal{E}, \mathcal{D} \in \tau_g$-$S_g^*C(\mathcal{W})$, there are open sets $\mathcal{U}_1, \mathcal{U}_2$ such that $\mathcal{E} \subseteq \mathcal{U}_2$, $\mathcal{D} \subseteq \mathcal{U}_1$ and $Cl(\mathcal{U}_2) \cap Cl(\mathcal{U}_1) = \phi$.*

Proof. (1) $\Rightarrow$ (2) Take $\mathcal{E} \in \tau_g$-S_g^*C($\mathcal{W}$). $\mathcal{U}_2 \in \tau_g$-S_g^*C($\mathcal{W}$) such that $\mathcal{E} \in \mathcal{U}_2 \Rightarrow \mathcal{E}$ and ($\mathcal{W}$ - $\mathcal{U}_2$) are two disjoint sets of τ_g-S_g^*C($\mathcal{W}$). According to the τ_g-S_g^*-normal definition, $\mathcal{E} \subseteq \mathcal{U}_1$ and ($\mathcal{W}$ - $\mathcal{U}_2$) $\subseteq \mathcal{U}_3$, where $\mathcal{U}_1$ and $\mathcal{U}_3$ are open disjoint sets. Therefore, $\mathcal{U}_1 \subseteq$ ($\mathcal{W}$ - $\mathcal{U}_3$) and $\mathcal{U}_1 \cap \mathcal{U}_3 = \phi \Rightarrow \mathcal{U}_1 \subseteq$ ($\mathcal{W}$ - $\mathcal{U}_3$) and Cl($\mathcal{U}_1$) $\subseteq$ ($\mathcal{W}$ - $\mathcal{U}_3$) $\subseteq \mathcal{U}_2 \Rightarrow$ Cl($\mathcal{U}_1$) $\subseteq \mathcal{U}_2$. Thus, $\mathcal{E} \subseteq \mathcal{U}_1 \subseteq$ Cl($\mathcal{U}_1$) $\subseteq \mathcal{U}_2$. (2) $\Rightarrow$ (3) Consider disjoint sets $\mathcal{E}, \mathcal{D} \in \tau_g$-$S_g^*$C($\mathcal{W}$), $\mathcal{E} \subseteq$ ($\mathcal{W}$ - $\mathcal{D}$) where $\mathcal{E}$ and ($\mathcal{W}$ - $\mathcal{D}$) are τ_g-S_g^*-closed and τ_g-S_g^*-open sets in $\mathcal{W}$. From (2), there is an open set $\mathcal{U}_1$ and $\mathcal{E} \subseteq \mathcal{U}_1 \subseteq$ Cl($\mathcal{U}_1$) $\subseteq$ ($\mathcal{W}$ - $\mathcal{D}$). Cl($\mathcal{U}_1$) $\subseteq$ ($\mathcal{W}$ - $\mathcal{D}$) $\Rightarrow \mathcal{D}$) $\cap$ Cl($\mathcal{U}_1$) = ϕ. Thus, $\mathcal{E} \subseteq \mathcal{U}_1$ and Cl($\mathcal{U}_1$) $\subseteq \mathcal{D} = \phi$. (3) $\Rightarrow$ (4) Consider disjoint sets $\mathcal{E}, \mathcal{D} \in \tau_g$-$S_g^*$C($\mathcal{W}$). By (3), there is an open set $\mathcal{U}_2$ such that $\mathcal{E} \subseteq \mathcal{U}_2$ and Cl($\mathcal{U}_2$) $\cap \mathcal{D} = \phi$. However, Cl($\mathcal{U}_2$) is closed $\Rightarrow$ a τ_g-S_g^*-closed set in $\mathcal{W}$. As Cl($\mathcal{U}_2$), $\mathcal{D} \in \tau_g$-S_g^*C($\mathcal{W}$). Then, by (3), $\mathcal{D} \subseteq \mathcal{U}_1$ and Cl($\mathcal{U}_2$) $\cap$ Cl($\mathcal{U}_1$) = ϕ, where $\mathcal{U}_1$ is an open set in $\mathcal{W}$. (4) $\Rightarrow$ (1) Assume disjoint sets $\mathcal{E}, \mathcal{D} \in \tau_g$-$S_g^*$C($\mathcal{W}$). Then by hypothesis there are open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathcal{E} \subseteq \mathcal{U}_2$ and $\mathcal{D} \subseteq \mathcal{U}_1$, Cl($\mathcal{U}_2$) $\cap$ Cl($\mathcal{U}_1$) = $\phi \Rightarrow \mathcal{E} \subseteq \mathcal{U}_2$ and $\mathcal{D} \subseteq \mathcal{U}_1$. Thus, $\mathcal{W}$ is τ_g-S_g^*-normal. □

Theorem 32. *Assume a mapping $\mathfrak{h}:\mathcal{W} \to \mathcal{Y}$. If $\mathfrak{h}$ is bijective, open, and τ_g-S_g^*-irresolute from a τ_g-S_g^*-normal $\mathcal{W}$ onto $\mathcal{Y}$, then $\mathcal{Y}$ is τ_g-S_g^*-normal.*

Proof. Assume disjoint sets $\mathcal{E}$, $\mathcal{D} \in \tau_g$-S_g^*C($\mathcal{W}$). Since $\mathfrak{h}$ is τ_g-S_g^*-irresolute, $\mathfrak{h}^{-1}(\mathcal{E})$ and $\mathfrak{h}^{-1}(\mathcal{D})$ are in τ_g-S_g^*C($\mathcal{W}$). $\mathcal{W}$ is τ_g-S_g^*-normal $\Rightarrow \mathfrak{h}^{-1}(\mathcal{E}) \subseteq \mathcal{U}_2$ and $\mathfrak{h}^{-1}(\mathcal{D}) \subseteq \mathcal{U}_1$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are open in $\mathcal{W}$. Also, as $\mathfrak{h}$ is bijective and open, $\mathfrak{h}(\mathcal{U}_2)$ and $\mathfrak{h}(\mathcal{U}_1)$ are open and $\mathcal{E} \subseteq \mathfrak{h}(\mathcal{U}_2)$, $\mathcal{D} \subseteq \mathfrak{h}(\mathcal{U}_1)$. Thus, $\mathcal{Y}$ is τ_g-S_g^*-normal. □

Theorem 33. *The subsequent statements in ($\mathcal{W}$, τ_g) are equivalent.*

(1) $\mathcal{W}$ is τ_g-g-normal.
(2) There are disjoint open sets $\mathcal{U}_1, \mathcal{U}_2 \in \tau_g$-$S_g^*$O($\mathcal{W}$) such that $\mathcal{E} \subseteq \mathcal{U}_2$, $\mathcal{D} \subseteq \mathcal{U}_1$ for each disjoint set $\mathcal{E}$ and $\mathcal{D}$.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{W}$ is τ_g-g-normal. Assume disjoint sets $\mathcal{E}$, $\mathcal{D}$ of $\mathcal{W}$. Since ($\mathcal{W}$, τ_g) is τ_g-g-normal, there are disjoint τ_g-g-open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathcal{E} \subseteq \mathcal{U}_2$, $\mathcal{D} \subseteq \mathcal{U}_1 \Rightarrow \mathcal{U}_1$, $\mathcal{U}_2 \in \tau_g$-S_g^*O($\mathcal{W}$) with $\mathcal{E} \subseteq \mathcal{U}_2$, $\mathcal{D} \subseteq \mathcal{U}_1$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$. (2) $\Rightarrow$ (1) Assume two disjoint τ_g-S_g^*-closed sets $\mathcal{E}$, $\mathcal{D} \in \tau_g$-S_g^*C($\mathcal{W}$). By hypothesis, $\mathcal{E} \subseteq \mathcal{U}_2$, $\mathcal{D} \subseteq \mathcal{U}_1$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint sets in τ_g-S_g^*O($\mathcal{W}$). Since $\mathcal{E} \subseteq \tau_g$-gInt($\mathcal{U}_2$), $\mathcal{D} \subseteq \tau_g$-gInt($\mathcal{U}_1$) and τ_g-gInt($\mathcal{U}_2$) $\cap$ τ_g-gInt($\mathcal{U}_1$) = ϕ. Thus, ($\mathcal{W}$, τ_g) is τ_g-g-normal. □

4. Methodology

A theoretical method is used in this study to examine the separation axioms, connectedness, and compactness in generalized topological spaces. The following is the structure of the methodology:

- **Literature Review:** Existing research on the separation axioms, connectedness, and compactness in generalized topological spaces is thoroughly reviewed. The fundamental framework for applying these ideas to the S_g^*-compact and S_g^*-connected spaces in more comprehensive frameworks is laid forth in this review.
- **Definition and Evolution:** In the generalized topological contexts, new definitions and characteristics of connectedness and compactness are put forward. To make sure that these definitions align with the fundamental structure of generalized topologies, they are thoroughly examined.
- **Analysis of the Separation Axioms:** Within the same generalized frames, the separation axioms T_0, T_1, and T_2 are examined.
- **Theoretical Proofs:** The specified properties and definitions are supported by formal proofs, demonstrating their applicability and validity. The associations between compactness, connectedness, and separation properties are analyzed.
- **Comparative Analysis:** To demonstrate the differences, analogies, and advantages of the generalized approach, a comparison between these expanded notions and their classical equivalents is analyzed.
- **Conclusions and Consequences:** The outcome findings are synthesized to provide a comprehensive understanding of the behavior of S_g^*-compact space, S_g^*-connected space, and separation axioms within generalized topological contexts. The implications of these results for future research in generalized topology are discussed.

5. Conclusions

This study provides a comprehensive exploration of compactness, connectedness, and separation axioms within the frameworks of generalized topological spaces. By extending classical notions of compactness and connectedness to S_g^*-compact and S_g^*-connected spaces, the research highlights the versatility and depth of these properties in generalized settings. The definitions and properties introduced are validated through rigorous theoretical analysis, offering new insights into the structural characteristics of these spaces. In addition, the investigation of separation axioms S_g^*-T_0, S_g^*-T_1, and S_g^*-T_2 within generalized topology underscores their critical role in classifying spaces based on point–set distinguishability. Although no direct relationships between S_g^*-compactness, S_g^*-connectedness, and separation axioms were established, their independent study provides a clearer understanding of the topological landscape in generalized frameworks. This work contributes to the growing body of research in generalized topology, laying the groundwork for future studies that may further refine these concepts or explore their applications in advanced mathematical theories and related disciplines. It emphasizes the importance of extending classical topological ideas to accommodate more complex and generalized structures.

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