Problem 7: Optional Challenge Problem

Bloch Sphere

So far we've talked about complex numbers and vectors. It turns out that in quantum mechanics, quantum states are described by vectors with complex components. The state of a qubit in a quantum computer is given by a two dimensional vector of the form

$$\vec{v} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

These qubit states live on the surface of the **Bloch Sphere**, where θ is the latitudinal angle (ranging from 0 to π) and ϕ is the azimuth angle (ranging from 0 to 2π).

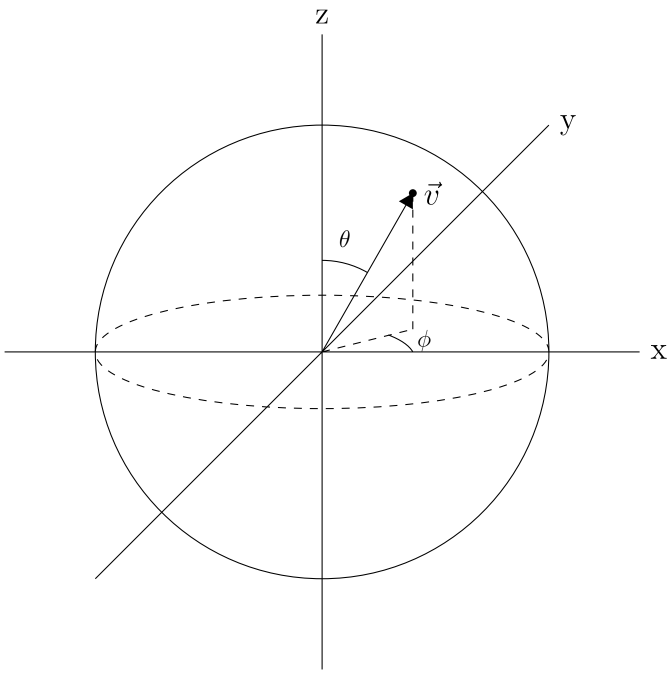


Figure 3.2: The Bloch sphere showing a general Bloch vector with coordinates described by θ and ϕ.

Understanding the Bloch sphere is at the heart of manipulating quantum states in quantum computing, so let's take some time to practice with Bloch vectors.

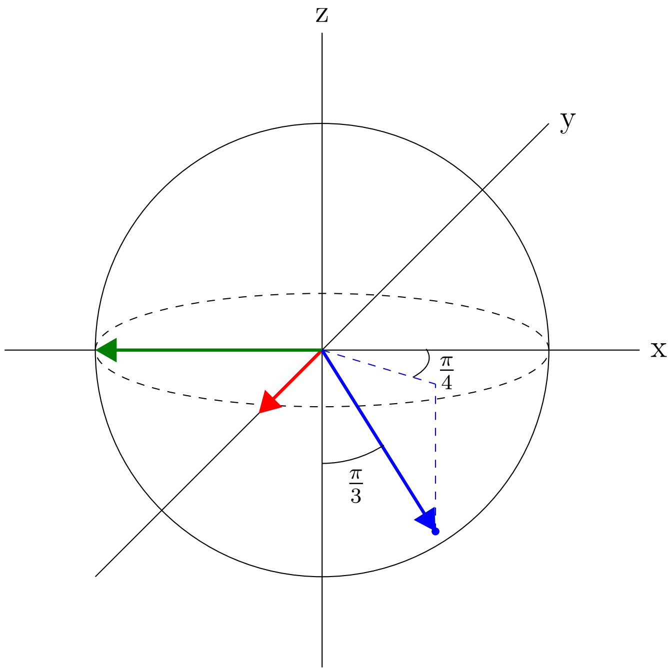
a) Where on the Bloch sphere do these vectors point?

i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

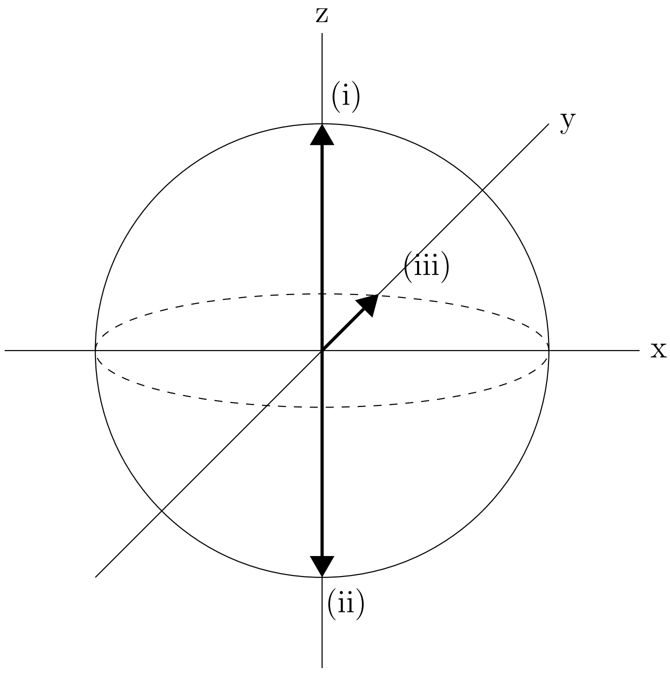
iii) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$

b) Write the vector form of the red, green and blue vectors on the following Bloch sphere.



Problem 7 Solutions

Part (a): Bloch vectors (i), (ii) and (iii):



Part (b):

- Red: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$
- Green: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$
- Blue: be careful! The angle θ is always measured with respect to the positive z axis, so $\theta = \pi - \frac{pi}{3} = \frac{2\pi}{3}$ for this one. The vector is $\begin{pmatrix} \frac{1}{2} \\ e^{-i\frac{\pi}{4}} \frac{\sqrt{3}}{2} \end{pmatrix}$

Problem 8: Optional Challenge Problem

Note: This problem is very difficult! If you attempt this problem, we highly encourage working together with your classmates to solve it. Good luck!

Complex Impedance

Because Euler's formula links complex numbers so closely to the periodic functions sin and cos, we often use complex numbers to describe periodic processes. For example, this is done when analyzing voltages and currents in a circuit that uses alternating current (AC) sources.

In any electric circuit, there are three major quantities: Voltage (V), Current (I) and Resistance (R), which are related via *Ohm's law*:

$$\frac{V}{I} = R \tag{3.1}$$

Which tells us the relationship between an input voltage and a current response in a circuit. This equation works quite well when V, I and R do not change in time. However, certain components in a circuit may have very different responses to an AC source, which Ohm's law cannot easily address. To analyze these time-varying voltages and currents, we must use complex numbers.

An AC source is a voltage source which oscillates in time, and usually takes the form:

$$V(t) = V_0 \cos(\omega t + \phi_V)$$

V_0 the magnitude of the signal, ω is the *driving frequency*, and ϕ_V is the *phase*. We often represent this oscillating voltage using a *complex voltage*:

$$\tilde{V}(t) = V_0 e^{i(\omega t + \phi_V)}$$

Part (a): Show that the complex AC voltage and real AC voltage are related in the following way:

$$V(t) = \mathrm{Re}[\tilde{V}(t)]$$

(Note: $\mathrm{Re}[z]$ denotes the operation of taking the real part of a complex number z. So for $z = a + bi$, $\mathrm{Re}[z] = a$.)

Now suppose we have a circuit to which we apply an oscillating voltage source represented by our complex $\tilde{V}(t)$. In general we may get an oscillating current response in the circuit that can also be represented as a complex number:

$$\tilde{I}(t) = I_0 e^{i(\omega t + \phi_I)}$$

Where again, the real current measured is given by $I(t) = \operatorname{Re} \tilde{I}(t)$.

Part (b): Using the complex voltage $\tilde{V}(t)$ and complex current $\tilde{I}(t)$, find the *complex impedance* of a circuit Z, defined as:

$$Z = \frac{\tilde{V}(t)}{\tilde{I}(t)} \tag{3.2}$$

in terms of variables V_0, I_0, ϕ_V and ϕ_I.

Note how the complex impedance is very similar to Ohm's Law (Eq.3.1), except we have replaced the real voltage and current with their complex counterparts. Physically, Z represents something very similar to resistance, except it is a complex number, and takes into account how the components of the circuit react to time-varying voltages and currents.

Part (c):
Suppose we have a circuit with complex impedance given by:

$$Z = \frac{1 - \omega^2 LC}{i\omega C}$$

Where L and C are constants. We drive the circuit with a time-varying complex voltage:

$$\tilde{V}(t) = V_0 e^{i\omega t}$$

i) Using Eq.3.2, find the current response for the circuit $\tilde{I}(t)$ in terms of V_0, ω, L and C.

ii) Write $\tilde{I}(t)$ in the form $\tilde{I}(t) = I_0 e^{i(\omega t + \phi_I)}$. What is I_0 and ϕ_I in terms of V_0, ω, L and C?

iii) If we were to measure the current in this circuit, what would the actual current measured $I(t)$? (Recall that the actual current is the real part of the complex current)

Even more challenge: What happens to the measured current as the driving frequency approaches the value $\frac{1}{\sqrt{LC}}$?

Problem 8 Solution

Part (a):
We find the real part of the complex voltage using Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$):

$$\tilde{V}(t) = V_0 e^{i(\omega t+\phi_V)}$$
$$\tilde{V}(t) = V_0(\cos(\omega t + \phi_V) + i\sin(\omega t + \phi_V))$$

Taking the real part of this means that we simply disregard the part that has a factor i:

$$\mathrm{Re}[\tilde{V}(t)] = V_0\cos(\omega t + \phi_V) = V(t)$$

Which does agree with our original oscillating voltage $V(t)$. This means that we can safely work with complex voltages, as long as we take the real part of it to get the actual voltage in the end.

Part (b):
Given $\tilde{V}(t) = V_0 e^{i(\omega t+\phi_V)}$ and $\tilde{I}(t) = I_0 e^{i(\omega t+\phi_I)}$, we use Eq.3.2 to find Z:

$$Z = \frac{\tilde{V}(t)}{\tilde{I}(t)}$$
$$Z = \frac{V_0 e^{i(\omega t+\phi_V)}}{I_0 e^{i(\omega t+\phi_I)}}$$

Further simplify using exponent combination rules:

$$Z = \frac{V_0}{I_0} e^{i(\omega t+\phi_V)-i(\omega t-\phi_I)}$$
$$Z = \frac{V_0}{I_0} e^{i(\phi_V-\phi_I)}$$

Part (c):

i) Given $Z = \frac{1-\omega^2 LC}{i\omega C}$ and $\tilde{V}(t) = V_0 e^{i\omega t}$ we calculate $\tilde{I}(t)$ using Eq.3.2:

$$\tilde{I}(t) = \frac{\tilde{V}(t)}{Z}$$
$$\tilde{I}(t) = \left(\frac{i\omega C}{1-\omega^2 LC}\right) V_0 e^{i\omega t}$$

We can further simplify this by making the connection that $i = e^{i\frac{\pi}{2}}$ and combining it with the other exponential term:

$$\tilde{I}(t) = \frac{\omega C V_0}{1-\omega^2 LC} e^{i\omega t} e^{i\frac{\pi}{2}}$$
$$\tilde{I}(t) = \frac{\omega C V_0}{1-\omega^2 LC} e^{i(\omega t+\frac{\pi}{2})}$$

ii) If we simplify by putting i into the exponent, the solution of part (i) is actually already in the form $\tilde{I}(t) = I_0 e^{i(\omega t+\phi_I)}$. We can read off the quantities of interest:

$$I_0 = \frac{\omega C V_0}{1-\omega^2 LC} \qquad\qquad \phi_V = \frac{\pi}{2}$$

iii) The actual current we measure will be the real part of $\tilde{I}(t)$, which can find by using Euler's formula:

$$I(t) = \mathrm{Re}[\tilde{I}(t)] = \frac{\omega C V_0}{1-\omega^2 LC} \cos\Big(\omega t + \frac{\pi}{2}\Big)$$

We can further simplify this using the trigonometric identity $\cos\big(\theta + \frac{\pi}{2}\big) = -\sin\theta$:

$$I(t) = -\frac{\omega C V_0}{1-\omega^2 LC} \sin(\omega t)$$

Even more challenge: If we look at the magnitude of the current $I_0 = \frac{\omega C V_0}{1-\omega^2 LC}$, as $\omega \to \frac{1}{\sqrt{LC}}$, $\omega^2 LC \to 1$ which makes the denominator in I_0 go toward 0. This means the magnitude of the current actually diverges toward ∞! What this means is that if we have a circuit with this particular impedance Z and we drive it at this special frequency $\omega = \frac{1}{\sqrt{LC}}$, then we can get very large current responses. In reality, the current does not actually go to ∞ because there are always losses in our circuit. However, we do observe these large increases in the current response, and this phenomenon is known as *resonance*. A circuit with this particular form of impedance is called a *resonant circuit*.