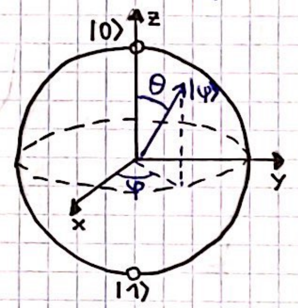
vectors and intro to matrices

by Maria Delgado

MORE Vectors

What Do vectors Mean for quantum computing?



<u>Qubits</u> are two-level quantum systems that lie in the Bloch Sphere and their states can be represented as vectors.

$$\vec{\psi} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi} \sin(\theta/2) \end{pmatrix}$$

Vector review

* vector representation $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$
* vector addition $\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$
* vector-scalar Mult. $c * \vec{v} = \begin{pmatrix} c * v_1 \\ \vdots \\ c * v_n \end{pmatrix}$
* vector magnitude: $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 \cdots + v_n^2}$

Shapes(vectors)

rows columns

vector shape (#rows x #columns)

* column vector: (n×1)
* row vector: (1×M)

Vector Traspose

* The traspose is an operation which flips the shape of a vector

▸ If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ then its traspose is $\vec{v}^T = (v_1 \ v_2 \cdots v_n)$

▸ If $\vec{w} = (w_1 \ w_2 \cdots w_n)$ then its aspose is $\vec{w}^T = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

It does not change anything about the vector geometically. Just changes the shape

THE INNER PRODUCT

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}\,\vec{w}^T = \sum_{i=1}^{n} v_i w_i$$

where $\vec{v}, \vec{w} \in \mathbb{R}^n$ are row vectors (scalar product/ dot product)

It is a:

1. vector x vector to scalar mapping
2. tool for calculate vector magnitude
3. tool for vector normalization
4. tool for geometrically comparing vectors
5. tool for determining vector orthogonality

VECTOR TO SCALAR MAPPING

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$$

$$= (v_1 \ v_2 \ v_3) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$= \sum_{i=1}^{3} v_i w_i$$

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CALCULATING VECTOR MAGNITUDE

$$\langle \vec{v}, \vec{v} \rangle = \vec{v}\vec{v}^T = \sum_{i=1}^{n} v_i v_i = \sum_{i=1}^{n} v_i^2 = \|\vec{v}\|^2 \quad \text{where } \vec{v} \in \mathbb{R}^n$$

$$\therefore \|\vec{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

The inner product of a vector with itself gives us the magnitude squared of the vector.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \|\vec{v}\| ?$$

$$= \sqrt{\vec{v}^T\vec{v}} = \sqrt{1^2+2^2+3^2}$$

$$= \sqrt{1+4+9} = \sqrt{14}$$

VECTOR NORMALIZATION

- A vector is normalized if it has a magnitude of 1 (unit vector)

$$\frac{\vec{v}}{\sqrt{\langle \vec{v}, \vec{v} \rangle}} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \|\vec{v}\| = \sqrt{\vec{v}^T\vec{v}} = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\rightarrow \vec{w} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$\|\vec{w}\| = \sqrt{\vec{w}^T\vec{w}} = \sqrt{\frac{1}{5} + \frac{4}{5}}$$

$$= 1$$

GEOMETRICALLY COMPARING VECTORS

$$\theta = \cos^{-1}\left(\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \, \|\vec{y}\|}\right)$$

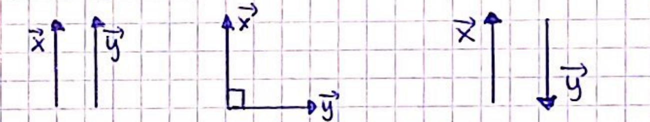
$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta$$

where $\theta = \angle(\vec{x}, \vec{y})$ is the angle between $\vec{x}$ and $\vec{y}$

VECTOR ORTHOGONALITY

- Let's consider some possibilities when $\vec{x}$ and $\vec{y}$ are unit vectors ($\|\vec{x}\| = \|\vec{y}\| = 1$) . . .

PARALLEL — ORTHOGONAL — ANTI-PARALLEL



PARALLEL	ORTHOGONAL	ANTI-PARALLEL
$\theta = 0°$	$\theta = 90°$	$\theta = 180°$
$\langle \vec{x}, \vec{y} \rangle = 1$	$\langle \vec{x}, \vec{y} \rangle = 0$	$\langle \vec{x}, \vec{y} \rangle = -1$

Conjugate Traspose

$$\vec{v}^{\dagger} = (\vec{v}^T)^* = (\vec{v}^*)^T$$

$$\vec{w} = \begin{pmatrix} 3e^{-i\theta} \\ 4+2i \end{pmatrix} \quad \vec{w}^{\dagger} = \begin{pmatrix} 3e^{i\theta} & 4-2i \end{pmatrix}$$

The complex conjugate, also called the "Hermitian conjugate" $\vec{v}^H$

- $\bar{z} = z^*$, $\bar{\vec{v}} = \vec{v}^*$

THE COMPLEX INNER PRODUCT

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^{\dagger}\vec{w} = \vec{v}_1^*\vec{w}_1 + \cdots + \vec{v}_n^*\vec{w}_n = \sum_{i=1}^{n} v_i^* w_i \quad \text{where } \vec{v}, \vec{w} \in \mathbb{C}^n \text{ (column vectors)}$$

$$\vec{w} = \begin{pmatrix} 5e^{i\theta/3} \\ 2+i \end{pmatrix} \quad \langle \vec{w}, \vec{w} \rangle = \vec{w}^{\dagger}\vec{w} = \begin{pmatrix} 5e^{-i\theta/3} & 2-i \end{pmatrix}\begin{pmatrix} 5e^{i\theta/3} \\ 2+i \end{pmatrix} = (5e^{-i\theta/3})(5e^{i\theta/3})\,(2-i)(2+i)$$

$$= 5 \cdot 5 + 4 - 2i + 2i - i^2 = 25 + 4 + 1 = 30$$

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LINEAR COMBINATIONS

- A linear combination of a set terms is simply the addition of those terms multiplied by scalar coefficients
- In the case of vectors, a linear combination is simply a weighted sum of vectors

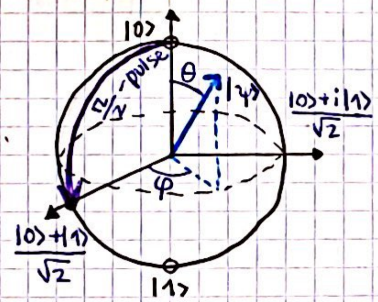
$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \sum_{i=1}^{n} a_i \vec{v}_i$$

- In the case of quantum states, a superposition is simply a linear combination of quantum states.

$$\frac{1}{\sqrt{2}}\left|\text{alive}\right\rangle + \frac{1}{\sqrt{2}}\left|\text{dead}\right\rangle$$

INTRO TO MATRICES

What do matrices mean for q.comp?



As we saw, we represent q.states as vectors
↓
Our goal is to manipulate these states → q.algorithms
↓
Q.gates → MATRICES

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Matrix

~ We can think of a matrix as a collection of row vectors or a collection of column v.

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$

geometrically, they are transformations that allow us to both rotate and scale vectors

Matrix notation and shape

An $(n \times m)$ matrix is written as

$$X = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$

MATRIX SHAPE : (#rows × #cols)

Solving linear systems of equations

$$\begin{cases} x - 3y + z = 2 \\ 3x - 4y + z = 0 \\ 2y - z = 1 \end{cases}$$

system of equations can be represented and solved with vectors and matrices

$$\hookrightarrow \begin{pmatrix} 1 & -3 & 1 \\ 3 & -4 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ -17 \end{pmatrix}$$

MATRIX OPERATIONS

Matrix addition

general equation (only add matrices of the same shape)

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

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Matrix-scalar Multiplication

general equation

$$c*A = c*\begin{pmatrix} a_{11} & \cdots & a_{12} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} c*a_{11} & c*a_{12} & \cdots & c*a_{1m} \\ c*a_{21} & c*a_{22} & \cdots & c*a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c*a_{n1} & c*a_{n2} & \cdots & c*a_{nm} \end{pmatrix}$$

Matrix-vector Multiplication

general equation → (the vector height must match the matrix width)

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11}*x_1 + a_{12}*x_2 + \cdots + a_{1m}*x_m \\ a_{21}*x_1 + a_{22}*x_2 + \cdots + a_{2m}*x_m \\ \vdots \\ a_{n1}*x_1 + a_{n2}*x_2 + \cdots + a_{nm}*x_m \end{pmatrix} = \begin{pmatrix} \langle \vec{a}_1, \vec{x} \rangle \\ \langle \vec{a}_2, \vec{x} \rangle \\ \vdots \\ \langle \vec{a}_n, \vec{x} \rangle \end{pmatrix}$$

Matrix-matrix Multiplication

general equation → (the 1st matrix width must match the second matrix height)

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{pmatrix} = \begin{pmatrix} \langle \vec{a}_1, \vec{b}_1 \rangle & \langle \vec{a}_1, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_1, \vec{b}_k \rangle \\ \langle \vec{a}_2, \vec{b}_1 \rangle & \langle \vec{a}_2, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_2, \vec{b}_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{a}_n, \vec{b}_1 \rangle & \langle \vec{a}_n, \vec{b}_2 \rangle & \cdots & \langle \vec{a}_n, \vec{b}_k \rangle \end{pmatrix}$$

Matrix traspose

if $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$, then $X^T = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1m} & x_{2m} & \cdots & x_{nm} \end{pmatrix}$

Remember $(n \times m)^T = (m \times n)$

Flip the matrix about its diagonals

Matrix conjugate traspose

if $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$, then $X^\dagger = \begin{pmatrix} x_{11}^* & x_{21}^* & \cdots & x_{n1}^* \\ x_{12}^* & x_{22}^* & \cdots & x_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ x_{1m}^* & x_{2m}^* & \cdots & x_{nm}^* \end{pmatrix}$

The same as traspose but conjugate any complex numbers in the matrix

SOLVING LINEAR SYSTEMS OF EQUATIONS

The identity matrix

It's defined as

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$a*1 = a$

$A\mathbb{I} = \mathbb{I}A = A$

$\vec{v}^T\mathbb{I} = \vec{v}^T \quad \mathbb{I}\vec{v} = \vec{v}$

Matrix inversion

$XX^{-1} = X^{-1}X = \mathbb{I}$

X^{-1} is thus the inverse of matrix X

~ Many matrices do not have inverses

(2x2) Matrix

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$X^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Now we can solve system of linear equations!