INTRO TO QUANTUM COMPUTING

LECTURE #13

QUANTUM MECHANICS 3

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ANNOUNCEMENTS

Lecture 1 - Principles of Quantum Mechanics

What is quantum and how do things behave on quantum length scales?

Lecture 2 - Quantum Two-Level Systems and Measurement

Objective - What are two-level systems and what can we do with them?

Lecture 3 - Postulates of Quantum Mechanics

Objective - What are the foundational rules of quantum mechanics?

TODAY'S LECTURE

➔ The six postulates of quantum mechanics

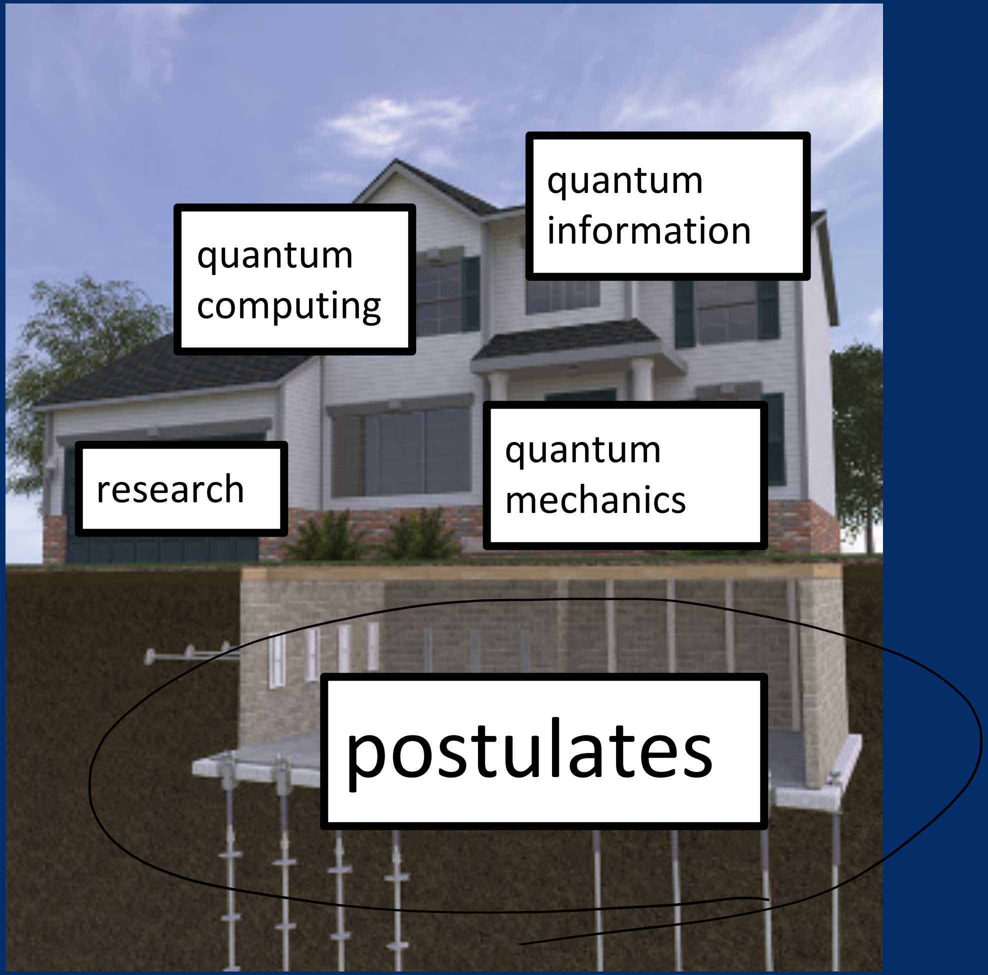
- 1 - Describing a quantum state
- 2 & 3 - Observing and measuring a quantum state
- 4 & 5 - What happens after a measurement
- 6 - How does a quantum state change with time

➔ How do these postulates relate to quantum computing?

Postulates of Quantum mechanics

Postulate:

an assumption used as a basis for mathematical reasoning



First Postulate:

A quantum state is represented by a ket $|\psi\rangle$ in the state space

Hilbert space

Classical analogy:

If I’m running, you can create a function that describes my path on a coordinate system.

First postulate

Consequence: The superposition of any two quantum states is also a quantum state

$$|\psi\rangle = \underline{a_1}\,|\psi_1\rangle + \underline{a_2}\,|\psi_2\rangle$$

$\langle\psi_1|\psi_1\rangle = 1$ ✓

$\langle\psi_2|\psi_2\rangle = 1$ ✓

$a_1, a_2 \in \mathbb{C} \qquad |a_1|^2 + |a_2|^2 = 1$

Example: (1) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

(2) $|\psi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = |0\rangle,$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}|0\rangle = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)|0\rangle + \frac{1}{2}|1\rangle$$

No matter what path I take while running, you can always represent my location as a point on the coordinate system. → Hilbert space

Second Postulate:

Classical observables are introduced into quantum mechanics using operators. Specifically, every observable (measurable property) of a physical system is described by an operator that acts on state kets

Classical analogy:

If I'm running, you can observe my:

- Speed
- Direction
- Position
- Energy
- Momentum
- ...

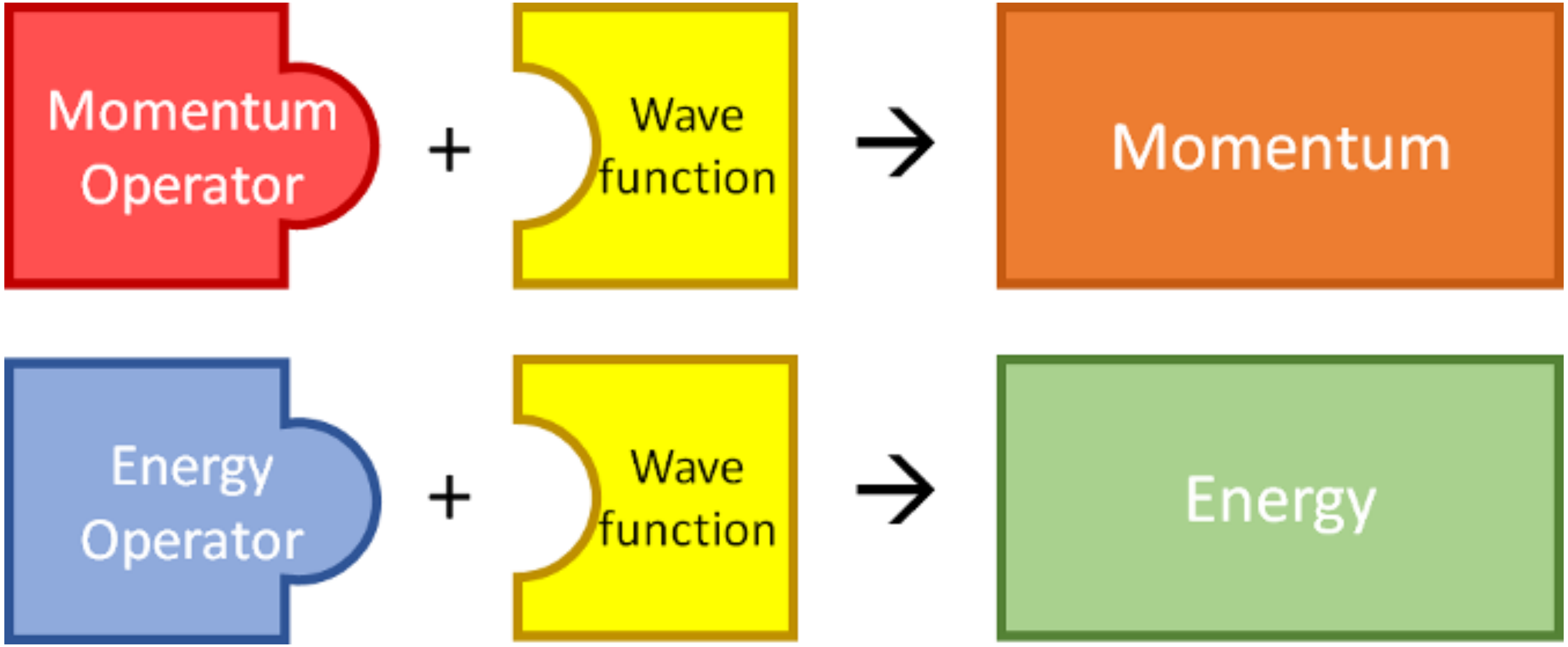
Classically, we can find out these values by measuring them. For example, we can measure energy.

In quantum, we apply an **operator** to the wavefunction to get these values:

Kinetic energy = Kinetic energy operator * $|\psi\rangle$

Second postulate

What's an operator?



Second postulate

Consequence: Relates observables that we can "measure" to the quantum world.

we can measure spin along the z-axis using $\sigma_z \equiv Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

spin along the x-axis using $\sigma_x \equiv X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

measurable operators have to be Hermitian:

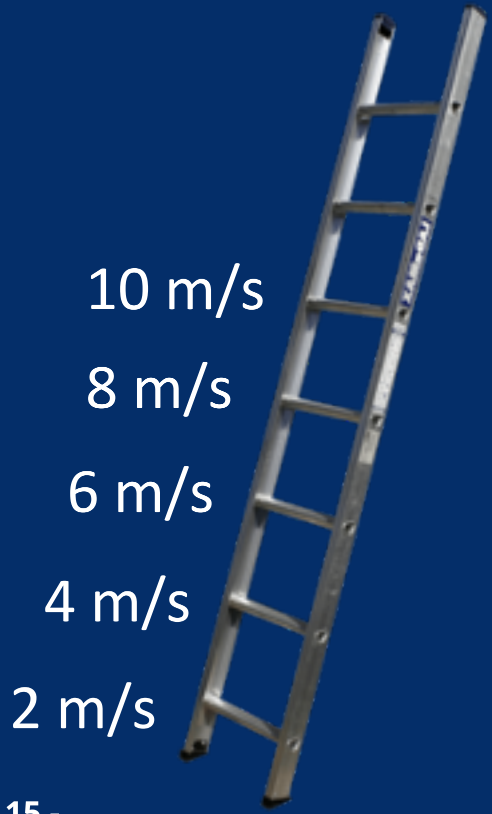
operator $\hat{A}$: $\hat{A} = \hat{A}^\dagger$

Third Postulate:

The result of a measurement of an observable with an operator $\hat{A}$ will only ever be an eigenvalue of $\hat{A}$.

Classical analogy:

We don’t really have a real-life example, because this is explicitly a quantum concept. The speed of a runner can only take on certain values.



Eigenvalue Review

Our matrix-vector multiplication simplifies to a scalar-vector multiplication!

$$\boxed{\hat{A}\vec{v} = \lambda\vec{v}}$$

eigenvalue

eigenstate

In German, “eigen” means *proper*, *characteristic*, or *own*.

Third postulate

Consequence: Determines the possible outcomes of an observable measurement

ex: measuring along z-axis for S-G experiment

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

eigenvectors of σ_z: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\sigma_z |0\rangle = (+1)\, |0\rangle$

$\sigma_z |1\rangle = (-1)\, |1\rangle$

possible outcomes: $|0\rangle, |1\rangle$

same thing

ex: measuring along x-axis for S-G experiment

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

eigenstates of σ_x: $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\sigma_x |+\rangle = (+1)\, |+\rangle$

$\sigma_x |-\rangle = (-1)\, |-\rangle$

possible outcomes are: $|+\rangle, |-\rangle$

All measurable operators are Hermitian

$\hat{A} = \hat{A}^\dagger \longrightarrow$ all eigenvalues of Hermitian operators are real

eigenvalue

$\hat{A}|a_i\rangle = a_i|a_i\rangle \rightarrow$ since $\hat{A}$ is Hermitian, all a_i s will be real

eigenvalues

eigenstates

eigenstates: $\langle a_i|a_i\rangle = 1$

$\langle a_i|a_j\rangle$ st. $i \neq j$ $\langle a_i|a_j\rangle = 0$ $\Big\}$ $\langle a_i|a_j\rangle = \delta_{ij}$ → 1 if $i=j$, → 0 if $i \neq j$

expected value of a measurement: $\langle a_i|\hat{A}|a_i\rangle = \langle a_i|(a_i)|a_i\rangle$

$= a_i \underbrace{\langle a_i|a_i\rangle}_{1} = a_i$

10 MIN BREAK!

Operators: They map a quantum state to another state

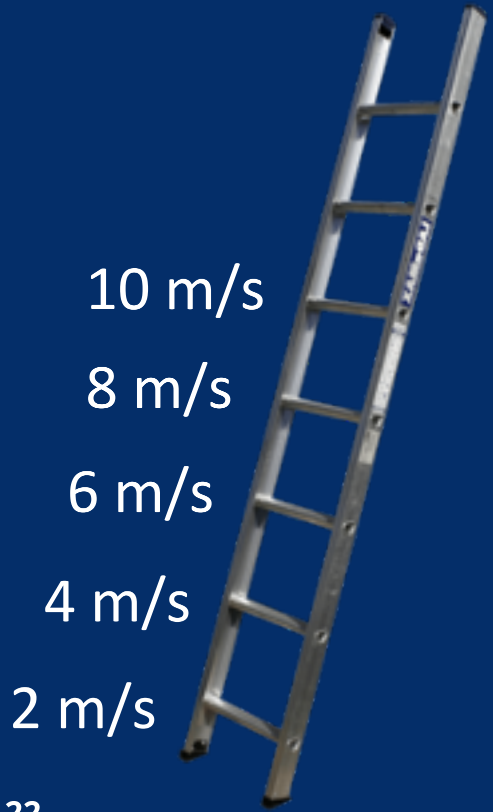
Observables: Things that we can measure

Fourth Postulate:

When a measurement of an observable with operator $\hat{A}$ is made on a generic state $|\psi\rangle$, the probability of obtaining a certain eigenvalue a_i is given by the square of the inner product of $|\psi\rangle$ with the corresponding eigenstate: $|\langle a_i|\psi\rangle|^2$

Analogy:

This would be like saying that there's a certain probability that you will be running at a certain speed. If your coach was measuring your speed, there is a certain probability that they would measure different values.



Fourth postulate

Consequence: Gives the probabilities of measuring the different outcomes of an observable measurement

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

measure along the z-axis (σ_z)

$\to |0\rangle$ with probability: $|\langle 0|\psi\rangle|^2 = |\alpha|^2$

$\to |1\rangle$ with probability: $|\langle 1|\psi\rangle|^2 = |\beta|^2$

measure along the x-axis (σ_x)

$\to |+\rangle$ with prob.: $|\langle +|\psi\rangle|^2 = \left|\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right)(\alpha|0\rangle + \beta|1\rangle)\right|^2$

$= |\alpha/\sqrt{2} + \beta/\sqrt{2}|^2 = \frac{1}{2}|\alpha+\beta|^2$

$\to |-\rangle$ with prob.: $|\langle -|\psi\rangle|^2 = \left|\left(\frac{\langle 0| - \langle 1|}{\sqrt{2}}\right)(\alpha|0\rangle + \beta|1\rangle)\right|^2$

$= |\alpha/\sqrt{2} - \beta/\sqrt{2}|^2 = \frac{1}{2}|\alpha-\beta|^2$

Fifth Postulate:

immediately after the measurement of an observable $\hat{\mathbf{A}}$ with a value $\mathbf{a_n}$, the state of the system is the normalized eigenstate $|a_n\rangle$

Analogy:

This would be like saying that before your coach measured your speed, it was not well-defined (we don't know for sure what speed you are at). But after the measurement, the speed is well-defined (we know the value).

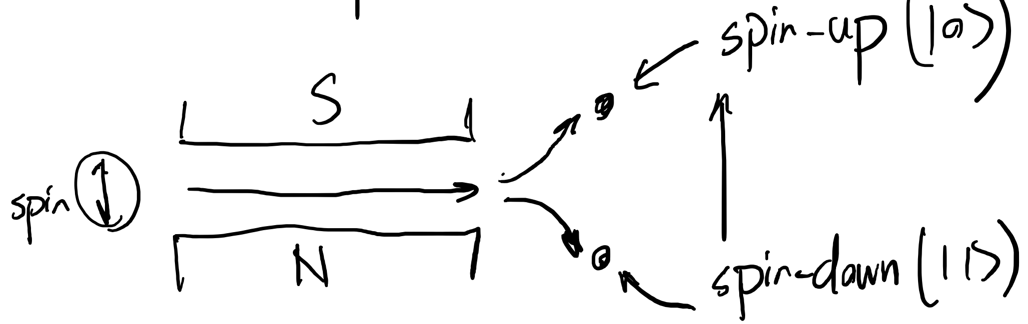
Fifth postulate

Consequence: The superposition collapses into one state after the measurement

→ Collapse of quantum superposition (wavefunction)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow[\text{get } 0]{\text{measure}} |\psi\rangle \rightarrow |0\rangle$$

example: S-G experiment:



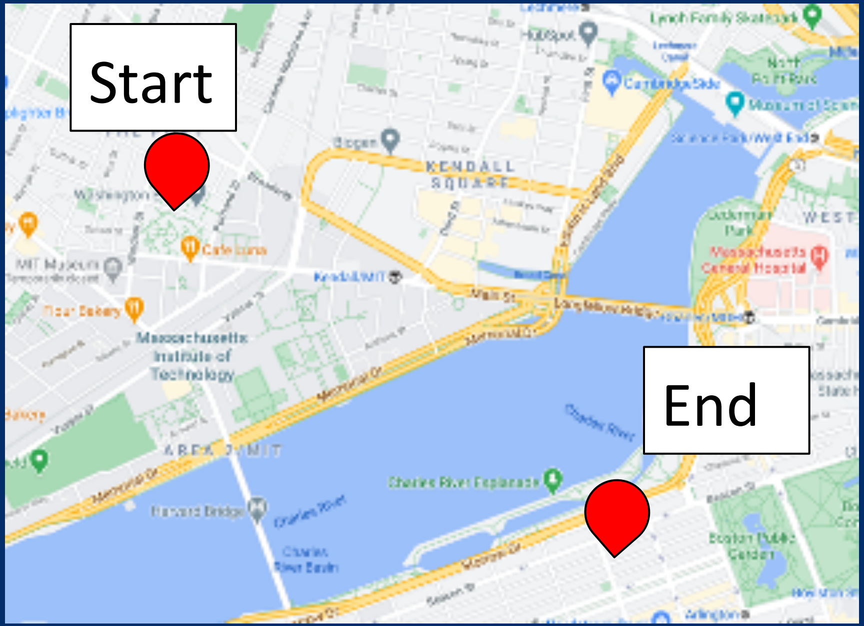
Sixth Postulate:

Quantum states generally change (evolve) with time. This time evolution preserves the normalization of the state. The time evolution of the state of a quantum system is described by:

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$$, for some unitary operator $\hat{U}$

Analogy:

This is like saying that if I went for a run and you knew where I ended up and the conditions I was running in (what I ate, weather, who I'm running with), you could figure out what my starting point was.



Sixth postulate

Consequence: The evolution of the quantum state with time is reversible.

Unitary operator: $U(t)$ $\quad\quad$ $\underline{U(t)\, U^\dagger(t) = U^\dagger(t)\, U(t) = \mathbb{I}}$

↳ unitary operators are "norm preserving": $\left.\begin{array}{l} |\psi\rangle \\ |\psi'\rangle = U(t)\,|\psi\rangle \end{array}\right\} \Rightarrow \langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle$

$U^\dagger(t) = U(-t)$ $\quad\quad$ $U(t)\, U(-t) = \mathbb{I}$

How do the postulates relate to quantum computing?

- Give us instructions on how to operate the quantum hardware
- Tells us what happens after we measure our quantum circuit
- Allow us to calculate the probability of getting the right answer based on the set of gates that we run in our algorithm