intro to vectors and complex numbers

by Maria Delgado

WHY ALL THE MATH?

1. The math is a necessary foundation and toolset for the quantum to come!

2. Linear algebra, probability, and complex numbers are extremely useful for every field of STEM, you will be better prepared for college

3. It is hard and you probably won't fully understand everything the 1st time. Practice!

INTRO TO VECTORS

scalars vs vectors

★ SCALARS

- A quantity having only magnitude (no direction)
- Written as: $a \in \mathbb{R}$

★ VECTORS

- A quantity with both magnitude and direction
- Written as $\vec{v} \in \mathbb{R}^n$
- ~ Can be described by a list of scalars (cartesian) or a radius and an angle (polar)

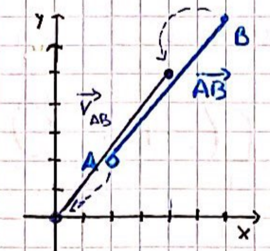
vector representation

★ General 2D vector notation

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{matrix} A = (2,2) \\ B = (6,7) \end{matrix}$$

$$\hookrightarrow \vec{v}_{AB} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

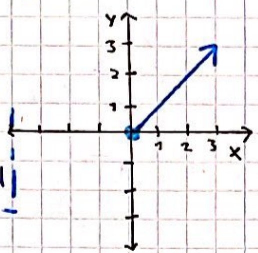


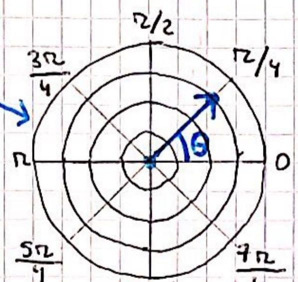
vector properties

$$\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$r = 3$

$\theta = \frac{\pi}{4}$ rad





vector magnitude

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$$

$$\vec{v} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\hookrightarrow \|\vec{v}\| = \sqrt{6^2 + 4^2}$$

$$= \sqrt{52}$$

$$\simeq 7.21$$

vector direction

$$\angle \vec{v} = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

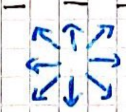
$$\vec{v} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\hookrightarrow \tan\theta = \frac{4}{6}$$

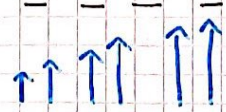
$$\therefore \theta = \tan^{-1}\left(\frac{4}{6}\right)$$

$$\approx 0.6 \text{ rad}$$

all vectors have



All vectors have the same magnitude but diff direction

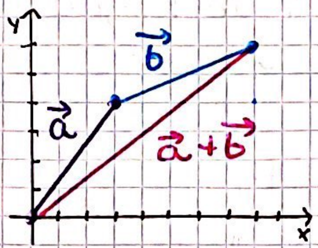


All vectors have the same direction but diff magnitude

vector operations

vector addition

$$\vec{a}+\vec{b}=\begin{pmatrix} a_x+b_x \\ a_y+b_y \end{pmatrix}$$

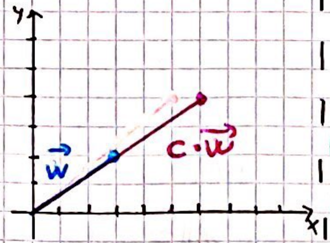


$\vec{a}=\begin{pmatrix}3\\4\end{pmatrix}, \vec{b}=\begin{pmatrix}5\\2\end{pmatrix}$

$\vec{a}+\vec{b}=\begin{pmatrix}3+5\\4+2\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}$

vector-scalar multip.

$$c\cdot\vec{w}=\begin{pmatrix} c\cdot w_x \\ c\cdot w_y \end{pmatrix}$$



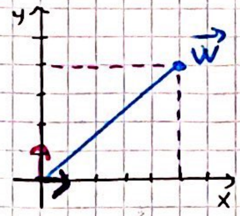
$c=2, \vec{w}=\begin{pmatrix}3\\2\end{pmatrix}$

$c\cdot\vec{w}=\begin{pmatrix}2\cdot 3\\2\cdot 2\end{pmatrix}=\begin{pmatrix}6\\4\end{pmatrix}$

vector decomposition

"Every vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\hat{x}$ and $\hat{y}$"

define vectors: $\hat{x}=\begin{pmatrix}1\\0\end{pmatrix}, \hat{y}=\begin{pmatrix}0\\1\end{pmatrix}$



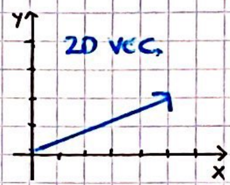
$\vec{w}=\begin{pmatrix}5\\4\end{pmatrix}=5\cdot\begin{pmatrix}1\\0\end{pmatrix}+4\begin{pmatrix}0\\1\end{pmatrix}$

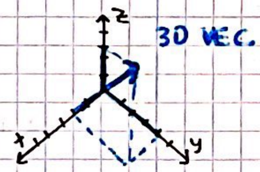
$=5\hat{x}+4\hat{y}$

$\vec{v}=\begin{pmatrix}v_x\\v_y\end{pmatrix}=v_x\hat{x}+v_y\hat{y}$

vector generalization

- ★ SCALARS: $a\in\mathbb{R}^1$
- ★ 2D VECTORS: $\vec{a}\in\mathbb{R}^2$
- ★ 3D VECTORS: $\vec{a}\in\mathbb{R}^3$
- ★ 4D VECTORS: $\vec{a}\in\mathbb{R}^4$
- ★ ND VECTORS: $\vec{a}\in\mathbb{R}^N$





Intro to complex numbers

why complex numbers

- Quadratic equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$\sqrt{-1}=???$

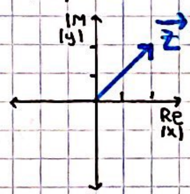
complex numbers definition

$$Z=\underbrace{a}_{\text{real component}}+\underbrace{\underbrace{i}_{\text{imag. unit}}\underbrace{b}_{\text{imag. coef}}}_{\text{Imaginary component}}$$

vector representation

$z=a+ib$

$\vec{z}=\begin{pmatrix}a\\b\end{pmatrix}$ Real component / Imag. component



IMAGINARY UNIT

$\underline{i=\sqrt{-1}}$ $\underline{i^2=-1}$

$(bi)^2=b^2i^2=-b^2$ ($b\in\mathbb{R}$)

$\sqrt{-a}=\sqrt{a}i$ ($a\in\mathbb{R}$)

$-1=i^2=\sqrt{-1}\sqrt{-1}\neq$

$\neq\sqrt{(-1)(-1)}=\sqrt{1}=1$

complex # addition

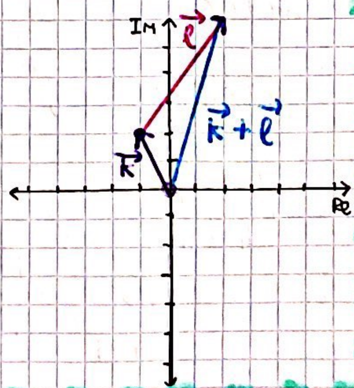
$(a+ib)+(c+id)=(a+c)+i(b+d)$

$k=-1+2i, \ l=3+4i$

$k+l=(-1+3)+i(2+4)=2+6i$

$\vec{k}=\begin{pmatrix}-1\\2\end{pmatrix}, \vec{l}=\begin{pmatrix}3\\4\end{pmatrix}$

$\vec{k}+\vec{l}=\begin{pmatrix}-1+3\\2+4\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}$



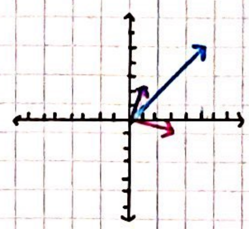
Complex # Multiplication

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= ac + i(ad+bc) + (-1)bd \\ &= (ac-bd) + i(ad+bc)\end{aligned}$$

Ex: $k = 1+2i$, $\ell = 3-i$

$$\begin{aligned}k \cdot \ell &= (1+2i)(3-i) \\ &= (3+2) + i(6-1) \\ &= 5+5i\end{aligned}$$

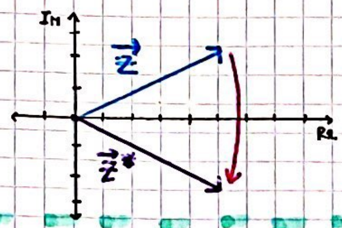


Complex # conjugation

$\overline{(a+ib)} = (a-ib)$

Ex: $z = 3+4i \rightarrow \overline{z} = 3-4i$

$w = 2-2i \rightarrow \overline{w} = 2+2i$

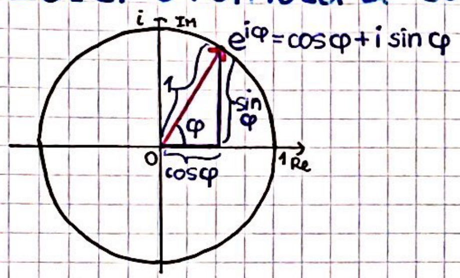


Complex # Modulus

$|a+ib| = \sqrt{a^2+b^2}$ $\qquad |a+ib|^2 = a^2+b^2$

$$\begin{aligned}|z| = |a+ib| = \sqrt{z \cdot \overline{z}} &= \sqrt{(a+ib)(a-ib)} \\ &= \sqrt{a^2 - iab + iab + i^2 b^2} = \sqrt{a^2+b^2}\end{aligned}$$

Euler's formula & complex exponentials

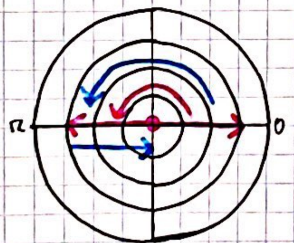


Euler's formula: $e^{i\varphi} = \cos\varphi + i\sin\varphi$

Polar representation: $z = x+iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi}$

$r = |z| = \sqrt{x^2+y^2}$ $\qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$

Euler's identity



$$\begin{aligned}e^{i\pi} + 1 &= 0 \\ &= \cos(\pi) + i\sin(\pi) + 1 \\ &= -1 + 1 = 0\end{aligned}$$

Complex exp. addition

"It's good to know the following two key identities"

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2}$$

Complex exp. conjugation

$z = x+iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi}$

$$\overline{e^{i\varphi}} = e^{-i\varphi}$$

$$\begin{aligned}z &= re^{i\varphi} = r(\cos\varphi + i\sin\varphi) \\ \overline{z} &= r(\cos\varphi - i\sin\varphi) = r[\cos(-\varphi) + i\sin(-\varphi)] \\ &= re^{-i\varphi}\end{aligned}$$

Complex exp. Multiplication

$e^{i\varphi} e^{i\theta} = e^{i(\varphi+\theta)}$ $\qquad e^{i\left(\frac{\pi}{2}\right)} e^{i\left(\frac{3\pi}{2}\right)} = e^{i2\pi}$

Complex exp. Modulus

$|z| = |re^{i\varphi}| = r$ $\qquad z = 3e^{i5\pi/2}$

$|z| = |3e^{i\frac{5\pi}{2}}| = 3$

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