HOMEWORK 8

MATHEMATICS FOR QUANTUM MECHANICS

Questions 1-7 refer to the following **normalized quantum states:**

$$|A\rangle = a\begin{pmatrix}5\\3i\end{pmatrix} \qquad |B\rangle = b\begin{pmatrix}-7\\4e^{-i\frac{\pi}{3}}\end{pmatrix} \qquad \langle C| = c\begin{pmatrix}1 & e^{i\frac{\pi}{6}}\end{pmatrix}$$

Where a, b, and c are real constants.

1. What is $\langle A|$?

   a) $a\begin{pmatrix}5 & 3i\end{pmatrix}$
   b) $a\begin{pmatrix}5 & -3i\end{pmatrix}$
   c) $a\begin{pmatrix}-5 & -3i\end{pmatrix}$
   d) $a\begin{pmatrix}5\\-3i\end{pmatrix}$

2. What is $\langle B|$?

   a) $b\begin{pmatrix}-7 & -4e^{-i\frac{\pi}{3}}\end{pmatrix}$
   b) $b\begin{pmatrix}-7 & 4e^{i\frac{\pi}{3}}\end{pmatrix}$
   c) $b\begin{pmatrix}7 & 4e^{-i\frac{\pi}{3}}\end{pmatrix}$
   d) $b\begin{pmatrix}7\\4e^{i\frac{\pi}{3}}\end{pmatrix}$

3. What is $|C\rangle$?

   a) $c\begin{pmatrix}1\\-e^{-i\frac{\pi}{6}}\end{pmatrix}$
   b) $c\begin{pmatrix}-1\\e^{-i\frac{\pi}{6}}\end{pmatrix}$
   c) $c\begin{pmatrix}1\\e^{-i\frac{\pi}{6}}\end{pmatrix}$
   d) $c\begin{pmatrix}1 & e^{-i\frac{\pi}{6}}\end{pmatrix}$

4. $|A\rangle$, $|B\rangle$ and $|C\rangle$ are all **normalized quantum states.** Which condition is satisfied for **any** normalized quantum state?

   a) $\langle\psi|\psi\rangle = |\psi\rangle$
   b) $\langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}$
   c) $\langle\psi|\psi\rangle = 1$
   d) $\langle\psi|\psi\rangle = 0$

5. By applying the normalization condition (from Question 4) to $|A\rangle$, determine the value of a.

   a) $a = 4$
   b) $a = 1$
   c) $a = \frac{1}{\sqrt{2}}$
   d) $a = \frac{1}{\sqrt{34}}$

6. By applying the normalization condition (from Question 4) to $|B\rangle$, determine the value of b.

   a) $b = \frac{1}{11}$
   b) $b = 0$
   c) $b = \frac{1}{\sqrt{65}}$
   d) $b = \frac{1}{\sqrt{33}}$

7. By applying the normalization condition (from Question 4) to $|C\rangle$, determine the value of c.

   a) $c = 0$
   b) $c = \frac{1}{2}$
   c) $c = 1$
   d) $c = \frac{1}{\sqrt{2}}$

Going forward it will be very important that we understand some of the conventions used in the field of quantum computing, Questions 8-13 we will be checking our understanding of them. These results are not very complex, but make future calculations much more manageable if we use them correctly.

8. What $|0\rangle$ in vector form?

   a) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

   b) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   c) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   d) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

9. What is $|1\rangle$ in vector form?

   a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   b) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   c) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

   d) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. Evaluate $\langle 0|0\rangle$

    a) $\frac{1}{2}$
    b) 0
    c) 2
    d) 1

11. Evaluate $\langle 0|1\rangle$

    a) 1
    b) 0
    c) $\frac{1}{2}$
    d) -1

12. Write the following superposition state in vector form:

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

13. Write the following vector in Dirac notation in terms of $|0\rangle$ and $|1\rangle$

$\begin{pmatrix} \sqrt{\frac{5}{7}} \\ \sqrt{\frac{2}{7}} \end{pmatrix}$

a) $\frac{1}{2}(|0\rangle + |1\rangle)$

b) $\sqrt{\frac{5}{7}}\,|0\rangle + \sqrt{\frac{2}{7}}\,|1\rangle$

c) $\sqrt{\frac{5}{7}}\,|1\rangle + \sqrt{\frac{2}{7}}\,|0\rangle$

d) $\frac{1}{\sqrt{2}}\left(\sqrt{\frac{5}{7}}\,|0\rangle + \sqrt{\frac{2}{7}}\,|1\rangle\right)$

Questions 14-21 refer to the following quantum states:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad |\phi\rangle = \frac{1}{\sqrt{13}}(2\,|0\rangle + 3e^{i\frac{\pi}{4}}\,|1\rangle)$$

14. What is the probability of measuring $|+\rangle$ in the $|0\rangle$ state?

    a) $\frac{1}{\sqrt{2}}$
    b) $\frac{1}{2}$
    c) 1
    d) 0

15. What is the probability of measuring $|-\rangle$ in the $|1\rangle$ state?

    a) $\frac{1}{2}$
    b) 1
    c) $\frac{1}{\sqrt{2}}$
    d) 0

16. What is the probability of measuring $|\phi\rangle$ in the $|0\rangle$ state?

    a) $\frac{1}{\sqrt{2}}$
    b) $\frac{4}{13}$
    c) $\frac{2}{\sqrt{13}}$
    d) $\frac{1}{2}$

17. What is the probability of measuring $|\phi\rangle$ in the $|1\rangle$ state?

    a) $\frac{1}{\sqrt{2}}$
    b) $\frac{3}{\sqrt{13}}$
    c) $\frac{1}{2}$
    d) $\frac{9}{13}$

Read before attempting Questions 18-21

While it is possible to evaluate the inner product for superposition states using the vector form of the state, it is often easier to use the values we know for $\langle 0|0\rangle$, $\langle 1|1\rangle$ and $\langle 0|1\rangle$ to make our calculations more efficient. Here is an example.

Example: Evaluate $\langle +|1\rangle$

1) First we find the bra form of the superposition state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

The bra form $\langle +|$ is simply the superposition of $\langle 0|$ and $\langle 1|$.

$$\langle +| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

Note: If the coefficients of $|0\rangle$ or $|1\rangle$ are complex, we must take the complex conjugate of the coefficiantes to get the correct result. This method is the same as taking the conjuage transpose of the vector.

2) Inner products follow distributive property! If we write out the expanded form of $\langle +|1\rangle$ we get

$$\langle +|1\rangle = \left[\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right]|1\rangle$$

To evalute this we use the distributive property, distributing $|1\rangle$ across terms in the parentheses.

$$\langle +|1\rangle = \frac{1}{\sqrt{2}}(\langle 0|1\rangle + \langle 1|1\rangle)$$

Now we use the values $\langle 1|1\rangle = 1$ and $\langle 0|1\rangle = 0$ to evaluate the rest. This gives us the final result:

$$\langle +|1\rangle = \frac{1}{\sqrt{2}}(0 + 1) = \frac{1}{\sqrt{2}}$$

Use this technique to solve Questions 18-21.

18. Evaluate $\langle +|+\rangle$

   a) 2
   b) 0
   c) 1
   d) $\frac{1}{2}$

19. Evaluate $\langle +|-\rangle$

   a) $\frac{1}{2}$
   b) 0
   c) 1
   d) 2

20. Evaluate $\langle \phi|+\rangle$

   a) $\frac{1}{\sqrt{26}}(2+3e^{-i\frac{\pi}{4}})$
   b) 0
   c) $\frac{1}{\sqrt{26}}(2-3e^{i\frac{\pi}{4}})$
   d) 1

21. Evaluate $\langle -|\phi\rangle$

   a) $\frac{1}{\sqrt{26}}(2-3e^{+i\frac{\pi}{4}})$
   b) $-\frac{1}{\sqrt{26}}(2+3e^{-i\frac{\pi}{4}})$
   c) 0
   d) $\frac{1}{\sqrt{26}}(-2+3e^{i\frac{\pi}{4}})$

Some very important operators in quantum computing are the *Pauli operators* (σ_x, σ_y, σ_z) and the *Hadamard* operator (H). The matrix forms of these operators are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Questions 22-35 referto these operators, and also uses the definitions of the following states:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

For Questions 22-29, use the matrix form of the operators (σ_x, σ_y, σ_z and H) and the vector form of $|0\rangle$ and $|1\rangle$ to evaluate the expressions through matrix multiplication.

22. Evaluate $\sigma_x |0\rangle$
    - a) $|1\rangle$
    - b) $|0\rangle$
    - c) $-|0\rangle$
    - d) $-|1\rangle$
23. Evaluate $\sigma_x |1\rangle$
    - a) $|1\rangle$
    - b) $|0\rangle$
    - c) $-|0\rangle$
    - d) $-|1\rangle$
24. Evaluate $\sigma_y |0\rangle$
    - a) $|0\rangle$
    - b) $i|0\rangle$
    - c) $-|1\rangle$
    - d) $i|1\rangle$
25. Evaluate $\sigma_y |1\rangle$
    - a) $i|0\rangle$
    - b) $-i|0\rangle$
    - c) $-|1\rangle$
    - d) $|0\rangle$

26. Evaluate $\sigma_z |0\rangle$

   a) $|1\rangle$
   b) $-|1\rangle$
   c) $|0\rangle$
   d) $-|0\rangle$

27. Evaluate $\sigma_z |1\rangle$

   a) $-|0\rangle$
   b) $|1\rangle$
   c) $-|1\rangle$
   d) $|0\rangle$

28. Evaluate $H |0\rangle$

   a) $|0\rangle$
   b) $\frac{1}{\sqrt{2}} |1\rangle$
   c) $|+\rangle$
   d) $|-\rangle$

29. Evaluate $H |1\rangle$

   a) $|0\rangle$
   b) $\frac{1}{\sqrt{2}} |1\rangle$
   c) $|+\rangle$
   d) $|-\rangle$

In Questions 30-35, we apply operators to superposition states. To solve these, **use the results from Questions 22-29 and the distributive property**.

30. Evaluate $\sigma_x \left|+\right\rangle$

    a) $\left|+\right\rangle$
    b) $\left|-\right\rangle$
    c) $\left|0\right\rangle$
    d) $\left|1\right\rangle$

31. Evaluate $\sigma_x \left|-\right\rangle$

    a) $\left|0\right\rangle$
    b) $\left|1\right\rangle$
    c) $-\left|-\right\rangle$
    d) $\left|+\right\rangle$

32. Evaluate $H \left|+\right\rangle$

    a) $\left|1\right\rangle$
    b) $\left|0\right\rangle$
    c) $\left|+\right\rangle$
    d) $-\left|+\right\rangle$

33. Evaluate $\sigma_z \left|+\right\rangle$

    a) $\left|+\right\rangle$
    b) $\left|-\right\rangle$
    c) $\left|0\right\rangle$
    d) $\left|1\right\rangle$

34. Evaluate $H\sigma_x \left|0\right\rangle$

    a) $\left|0\right\rangle$
    b) $\left|1\right\rangle$
    c) $\left|+\right\rangle$
    d) $\left|-\right\rangle$

35. Evaluate $\sigma_x H \left|0\right\rangle$

    a) $\left|0\right\rangle$
    b) $\left|1\right\rangle$
    c) $\left|+\right\rangle$
    d) $\left|-\right\rangle$