Problem 1: Optional Problem

Rotation Matrices

We will spend a lot of time talking about rotation matrices. As we will see later on, rotation matrices are extremely important in quantum mechanics, so let's take some time to study in detail how they work. As a reminder, a rotation matrix in 2 dimensions takes the form:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

When applied to a vector, this rotation matrix has the action of rotating the vector counter-clockwise by an angle θ. Rotation matrices have some important properties, some of which we will explore here.

a) Because R simply rotates a vector, it should not change it's magnitude. Show that:

$$|R(\theta)\vec{v}| = |\vec{v}|$$

Hint: Here are some steps to help with this

1. Start with a vector with variable elements: $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$
2. Calculate $R(\theta)\vec{v}$ in terms of x and y
3. Calculate the magnitude of both sides (recall: $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$)

It makes sense that if we rotate by an angle θ, then subsequently rotate by $-\theta$, we should be back to where we started. This can be stated in an equation

$$R(\theta)R(-\theta)\vec{v} = \vec{v}$$

Which implies

$$\begin{aligned} R(\theta)R(-\theta) &= I \\ R(-\theta) &= R^{-1}(\theta) \end{aligned} \tag{1}$$

b) Explicitly show that Eq.1 is true. (Recall: I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

Rotations also add to each other. If we rotate a vector by an angle θ_1, then subsequently rotate by angle θ_2, the total angle we have rotated the vector is $\theta_1 + \theta_2$. Our rotation matrix should properly reflect this.

c) Prove the equality:

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

Problem 1 Solutions

Part (a): We show that the equation holds by applying it to a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

First, let us calculate $|\vec{v}|^2$.

$$|\vec{v}|^2 = x^2 + y^2$$

Now we apply R to $\vec{v}$

$$R(\theta)\vec{v} = R(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$R(\theta)\vec{v} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Then we calculate the magnitude squared of this vector:

$$|R(\theta)\vec{v}|^2 = (R(\theta)\vec{v}) \cdot (R(\theta)\vec{v})$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} \cdot \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

$$|R(\theta)\vec{v}|^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2$$

Now we distribute to find the square of each of these binomials. Be careful, as it's easy to make mistakes here.

$$\begin{aligned} |R(\theta)\vec{v}|^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 &= x^2\cos^2\theta + y^2\sin^2\theta - 2xy\sin\theta\cos\theta \\ &+ x^2\sin^2\theta + y^2\cos^2\theta + 2xy\sin\theta\cos\theta \end{aligned}$$

$$|R(\theta)\vec{v}|^2 = x^2\cos^2\theta + x^2\sin^2\theta + y^2\cos^2\theta + y^2\sin^2\theta$$

Now we can factor x^2 from the terms with x^2 and y^2 from the terms with y^2.

$$x^2\cos^2\theta + x^2\sin^2\theta + y^2\cos^2\theta + y^2\sin^2\theta = x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta)$$

And finally we can use the trig identity: $\sin^2\theta + \cos^2\theta = 1$

$$|R(\theta)\vec{v}|^2 = x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) = x^2 + y^2$$

Which we see is exactly the same as $|\vec{v}|^2$, so mission accomplished!

Part (b): First we write what $R(-\theta)$ is, which is:

$$R(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

We need to use the rules for even and odd functions. sin is an odd function and cos is an even function so:

$$\cos(-\theta) = \cos\theta \qquad\qquad \sin(-\theta) = -\sin\theta$$

Using this we simplify

$$R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Now we do matrix multiplication with $R(\theta)$ to verify the equality.

$$R(\theta)R(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Part (c): Here we also explicitly do matrix multiplication

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) \\ \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{pmatrix}$$

To simplify this we use the trigonometric identites:

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 \qquad\qquad \cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

And we confirm that

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} = R(\theta_1 + \theta_2)$$

This is also consistent with Eq.1. If we set $\theta_1 = \theta$ and $\theta_2 = -\theta$ we see that:

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(\theta - \theta) = R(0)$$

A quick check will reveal that indeed $R(0) = I$, which is exactly what we expect.

Problem 2: Optional Challenge Problem

Matrix Exponential

We've learned about adding, multiplying, and even taking the inverse of a matrix. But how would one evaluate

$$e^M$$

for some **square matrix** M. At first glance, it doesn't seem to make much sense. However, there is a way to properly define this type of matrix operation, and it comes in very handy in quantum computation.

The definition of the matrix exponential e^M comes from the series definition of the exponential function. Any exponential function $f(x) = e^x$ can be represented by the series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ...$$

Where $n! = n \times (n-1) \times (n-2) \times ... \times 1$ is the factorial operation.

Using this understanding of the exponential, the matrix exponential e^M is defined to be

$$e^M \equiv \sum_{n=0}^{\infty} \frac{1}{n!} M^n = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + ...$$

There are a myriad of interesting properties of matrix exponentials, but in this problem we will focus on a specific case.

Matrix Exponential in Quantum

Consider a 2x2 matrix σ_y with elements:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

a) Verify that

$$\sigma_y^2 = I$$

where as usual I is the identity matrix.

b) Show that

$$e^{-i\frac{\theta}{2}\sigma_y} = I \cos\left(\frac{\theta}{2}\right) - i\sigma_y \sin\left(\frac{\theta}{2}\right)$$

Hint: the series expansion for sin and cos may be helpful

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ... \qquad \sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + ...$$

c) Write $e^{-i\frac{\theta}{2}\sigma_y}$ in matrix form (i.e. in the form $e^{-i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$).

This result should look awfully similar to a rotation matrix; and that's because it is a rotation matrix! Matrices that take the form $e^{-i\frac{\theta}{2}\sigma}$ actually perform rotations on Bloch vectors. $e^{-i\frac{\theta}{2}\sigma_y}$ specifically describes a rotation about the y-axis on the Bloch sphere by angle θ

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y}$$

There are other matrices σ_x and σ_z that do the same for x and z axes that we will learn more about.

d) Consider the Bloch vector:

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

i) In what direction on the Bloch sphere does this vector point?

ii) What is the matrix we should construct to rotate this vector about the y-axis by $\frac{\pi}{2}$?

iii) Perform the operation, what is the resulting vector?

Problem 2 Solutions

Part (a):

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Part (b):
Using the series expansion for the matrix exponential:

$$e^{-i\frac{\theta}{2}\sigma_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(-i\frac{\theta}{2}\sigma_y)^n$$

$\sigma_y^2 = I$, so $\sigma_y^n = I$ if n is even, and $\sigma_y^n = \sigma_y$ if n is odd. Using this, we write the series as

$$e^{-i\frac{\theta}{2}\sigma_y} = I - i\sigma_y\frac{\theta}{2} - I\frac{1}{2}\left(\frac{\theta}{2}\right)^2 + i\sigma_y\frac{1}{6}\left(\frac{\theta}{2}\right)^3 + I\frac{1}{24}\left(\frac{\theta}{2}\right)^4 - i\sigma_y\frac{1}{120}\left(\frac{\theta}{2}\right)^5 + ...$$

Separating terms that are multiplied by I and $i\sigma_y$ then factoring gives us

$$e^{-i\frac{\theta}{2}\sigma_y} = I(1 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 + \frac{1}{24}\left(\frac{\theta}{2}\right)^4 + ...) - i\sigma_y(\frac{\theta}{2} - \frac{1}{6}\left(\frac{\theta}{2}\right)^3 + \frac{1}{120}\left(\frac{\theta}{2}\right)^5 + ...)$$

The series in the parentheses multiplied by I and $i\sigma_y$ are $\cos\left(\frac{\theta}{2}\right)$ and $\sin\left(\frac{\theta}{2}\right)$ respectively. This leaves us with the desired result

$$e^{-i\frac{\theta}{2}\sigma_y} = I\cos\left(\frac{\theta}{2}\right) - i\sigma_y\sin\left(\frac{\theta}{2}\right)$$

Part (c):

$$e^{-i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Part (d):

i) $\vec{v}$ points along the positive z-axis.

ii) $e^{-i\frac{\pi}{4}\sigma_y} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

iii) $e^{-i\frac{\pi}{4}\sigma_y}\vec{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, this vector represents a superposition state!