intro to probability and maths for quantum 1

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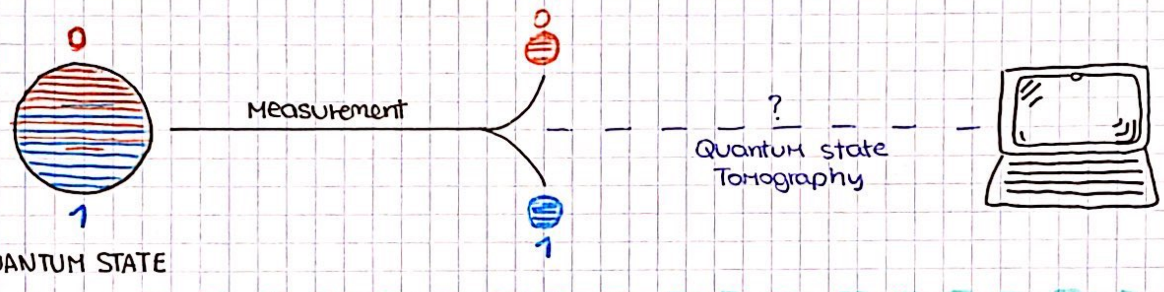
INTRO TO PROBABILITY

Why Probability?

* Probability is a branch of Mathematics that describes the "likelihood" of events occurring.
* Probability is used to make predictions in a variety of areas
  - Gambling and cards
  - Weather Forecasting
  - Sports prediction
  - Quantum physics!

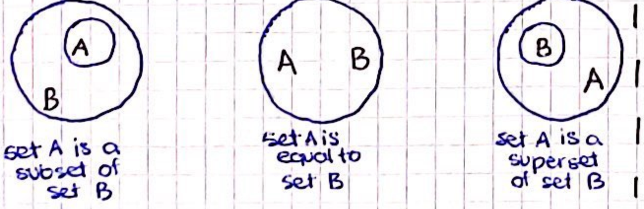


SET NOTATION

- A SET is a collection of distinct objects, which can be considered an object in its own right.
- The arrangement order of objects in a set does not matter
- The elements of a set can be anything from numbers to people to colors!

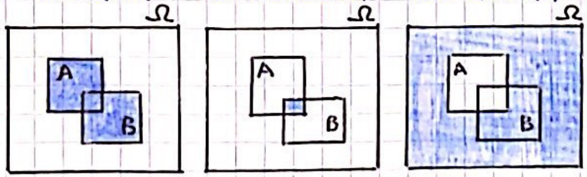
COMPARING SETS

$A \subset B$ $\quad A \subseteq B$ $\quad A = B$ $\quad A \supseteq B$ $\quad A \supset B$



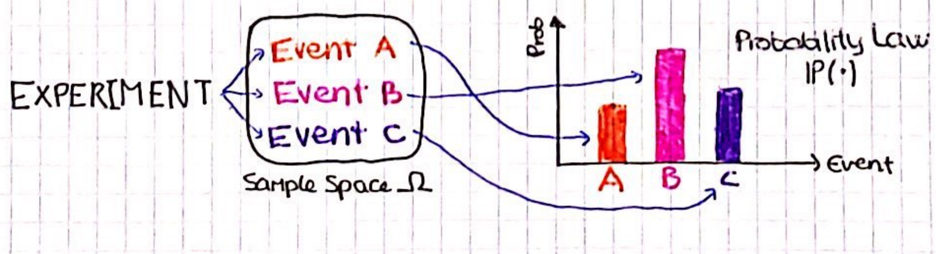
OPERATIONS ON SETS

$A \cup B$ UNION(OR) $\quad A \cap B$ INTERSECT(AND) $\quad A^c$ COMPLEMENT(-)



The Probabilistic Model

- A probabilistic model is a way to mathematically describe an unknown situation.

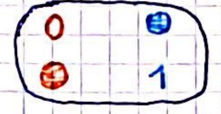


SAMPLE SPACE

★ The sample space (Ω) is the set of all possible outcomes in an experiment

$\Omega = \{A, B, C\}$

SAMPLE SPACE (QUANTUM)



$\Omega = \{|0\rangle, |1\rangle\}$

MORE

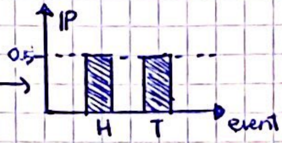
- Elements of the sample space (Ω) must be distinct and mutually exclusive. Unique outcome.
- The sample space must be collectively exhaustive, such that no matter what happens.

PROBABILITY LAW

★ A probability law assigns to each event, Y, a non-negative number, $\mathbb{P}(Y)$, that encodes our knowledge/belief about the collective likelihood of elements of Y.

Example:

$\Omega = \{H, T\} \to \mathbb{P}(H) = 1/2$, $\mathbb{P}(T) = 1/2$ →



← example was with normal coins (us)

FUNDAMENTAL AXIOMS OF PROBABILITY

1. NONNEGATIVE: *The probability of every event in the sample space must be greater than or equal to zero*

   $\mathbb{P}(A) \geq 0$, for every event A ~ there is no negative probability

2. ADDITIVITY: If A and B are two disjoint events, then the probability of their union is the sum of their individual probabilities

   $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

3. NORMALIZATION: The probability of the entire sample space (Ω) is equal to 1

   $\mathbb{P}(\Omega) = 1$

~ There are some further important properties of probability laws...

- PROBABILITY OF THE EMPTY SET:

  $\emptyset = \{\}$, $1 = \mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = 1 + \mathbb{P}(\emptyset) \Rightarrow \mathbb{P}(\emptyset) = 0$

- COMPLEMENT RULE:

  $\Omega = \{A, A^c\}$, $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

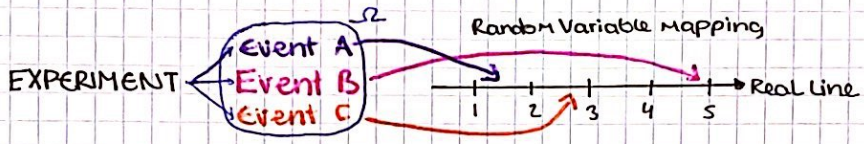
PROPERTIES OF PROBABILITY LAWS

* Every event A_i in our sample space Ω, has probability $0 \le \mathbb{P}(A_i) \le 1$ such that $\sum_i \mathbb{P}(A_i) = 1$
* If $A \subseteq B$, then $\mathbb{P}(A) \le \mathbb{P}(B)$
* Generally, $\mathbb{P}(F \cup S) = \mathbb{P}(F) + \mathbb{P}(S) - \mathbb{P}(F \cap S)$. Thus, $\mathbb{P}(F \cup S) \le \mathbb{P}(F) + \mathbb{P}(S)$
* $\mathbb{P}(F \cup S) = \mathbb{P}(F) + \mathbb{P}(F^c \cap S)$

DISCRETE RANDOM VARIABLES

- Frequently, outcomes of probabilistic models are numerical
- In other cases, the outcomes may not be numerical, but may be associated with numerical values of interest

* Random VARIABLE (RV): a function that assigns a numerical value to each possible outcome of the experiment



Example:

- Suppose we are interested in the age of diff family members

$$X(f) = \begin{cases} 10, & \text{if } f = \text{Joe} \\ 30, & \text{if } f = \text{Sam} \\ 32, & \text{if } f = \text{Alice} \end{cases}$$

~ Remember that even though it is called a random variable, it's actually a function which maps elements from the sample space Ω to $\mathbb{R}$

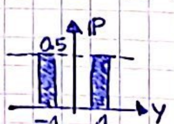
$$X : \Omega \to \mathbb{R}$$

PROBABILITY MASS FUNCTION

- Every discrete random variable has an associated probability mass function (PMF), which gives the probability of each numerical value that the random variable can take.
- It is a means of describing the discrete probability distribution.

Example.

(RV) $\to y = Y(c) = \begin{cases} +1, & \text{if } c = \text{heads} \\ -1, & \text{if } c = \text{tails} \end{cases}$ (PMF) $\to f_Y(y) = \begin{cases} 0.5, & \text{if } y = +1 \\ 0.5, & \text{if } y = -1 \end{cases}$ (PD) $\to$



EXPECTATION

* The expectation is the weighted average of the possible values of a random variable.

~ Sometimes refered to as the mean of a RV or center of gravity of the PMF.

$$\langle X \rangle = \mathbb{E}[X] = \sum_x x \, \mathbb{P}(X = x)$$

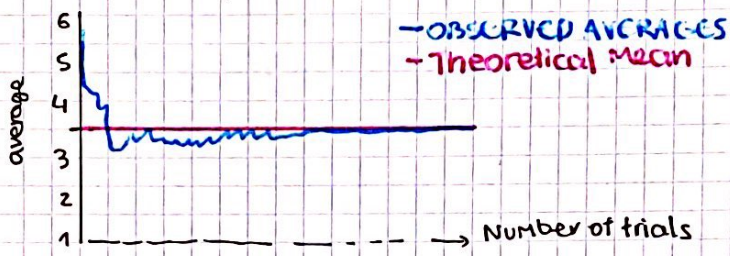
VARIANCE

* The Variance is expected value of the random variable $|X - E[X]|^2$ and provides a measure of the dispersion of X about its mean.

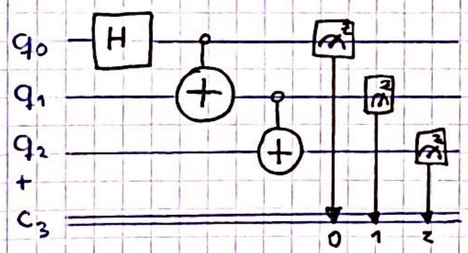
$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sum_x (x - \mathbb{E}[X])^2 \, \mathbb{P}(X = x)$$

LAW OF LARGE NUMBERS

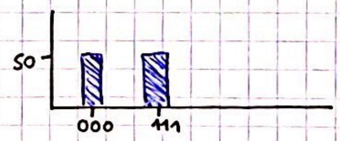
* The average of a large number of trials should be close to the expected value and will tend become closer to the expected value as the number of trials increase.



QUANTUM



ideally we expect a 50-50 distribution over the states |000⟩ and |111⟩



MATHS FOR QUANTUM PT.1.

DIRAC NOTATION

Bras and kets

* a ket is simply a column vector

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

* a bra is the conjugate tras. of a ket

$$\langle v| = |v\rangle^\dagger = (v_1 \; v_2 \cdots v_n)$$

QUANTUM STATES

* Ground state 0 $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
* Excited state 1 $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
* Superposition st. 0 1 $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\alpha, \beta \in \mathbb{C}$ are probability amplitudes, with $|\alpha|^2 + |\beta|^2 = 1$