INTRO TO QUANTUM COMPUTING

Week 5 Lab

# PROBABILITY AND RANDOM VARIABLES

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# PROGRAM FOR TODAY

- Logistics
- Attendance quiz
- Pre-lab zoom feedback
- Questions from last week
- Lab content
- Post-lab zoom feedback

# LOGISTICS

**Piazza** is a great resource for content-related questions!

- Post your questions from lecture, lab or homework
- Responses from instructors + TAs and fellow students
- Average response time – 15 minutes
- If you don't have access, email student@qubitbyqubit.org

# CANVAS ATTENDANCE QUIZ

- Please log into Canvas and answer your lab section's quiz (using the password posted below and in the chat).
  - This is lab number : **3**
  - Passcode: **3308**
- How many hours did you spend on last week's homework?
  - Less than 1 hour
  - 1-2 hours
  - 2-3 hours
  - More than 3 hours
  - I didn't do the homework
- **This quiz not graded, but counts for your lab attendance!**

# PRE-LAB ZOOM FEEDBACK

On a scale of 1 to 5, how would you rate your understanding of this week's content?

- 1 –Did not understand anything
- 2 – Understood some parts
- 3 – Understood most of the content
- 4 – Understood all of the content
- 5 – The content was easy for me/I already knew all of the content



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# LEARNING OBJECTIVES FOR LAB 5

- Understanding probability for a 6-sided dice
  - Events and normalization
  - Random variables
  - Probability mass function
  - Expectation and variance
- Joint probability for a dice and coin
- Relating probability to quantum computing
  - 1-qubit states and bra-ket notation
  - 2-qubit states and entanglement*

*Optional content




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# QUESTIONS FROM PAST WEEK

**Complex numbers (for our purposes) are scalars!**

$$\begin{pmatrix} 3+4i \\ 5-2i \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \underset{e^{i\frac{\pi}{2}}}{i} \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

# QUESTIONS FROM PAST WEEK

## Why do we take the transpose of one vector for inner product?

Dimensional consistency!

$$\vec{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \longrightarrow \text{Two } 2\times1 \text{ vectors}$$

$$\langle \vec{w}, \vec{v} \rangle = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 8$$

$1\times\underline{2}$ $\quad$ $\underline{2}\times1$

Inner dimensions match!

$$\overbrace{(3\times4)^T \times (3\times4)}^{\neq}$$

matrix

$$\underbrace{(4\times3) \times (3\times4)}$$



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# QUESTIONS FROM PAST WEEK

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 8$$

$$\underbrace{1 \times 2 \qquad 2 \times 1}_{\text{Inner dimensions match!}}$$




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# QUESTIONS FROM PAST WEEK

$$\begin{pmatrix} 3 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$2\times2$ $2\times1$

Inner dimensions match!



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# QUESTIONS FROM PAST WEEK

$$\begin{pmatrix} 3 & 2 \\ -2 & 5 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

$$3\times 2 \qquad 2\times 1$$

Inner dimensions match!

Taking the transpose for the inner product of vectors lets us use the "inner dimensions match" rule for vector as well as matrix multiplication




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# WHY PROBABILITY?

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow P = 1$$

Quantum mechanics (and therefore, quantum computing) is *inherently* probabilistic




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# PROBABILITY AROUND US

**Where have you seen probability in your daily lives?**

$[0, 1]$

$\Downarrow$

$[0\%, 100\%]$




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# PROBABILITY FOR A 6-SIDED DICE

| outcome | probability |
|---|---|
| 1 | equal |
| 2 | equal |
| 3 | equal |
| 4 | equal |
| 5 | equal |
| 6 | equal |



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# PROPERTIES OF PROBABILITY

- **Non-negativity:** The probabilities must be non-negative
- **Normalization:** The sum of the probabilities of all 6 sides must be 1

$\times \frac{1}{100\%}$ $100\% = 1$



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# PROBABILITY FOR A 6-SIDED DIE

| outcome | probability |
|---|---|
| 1 | equal ($p$) |
| 2 | equal ($p$) |
| 3 | equal ($p$) |
| 4 | equal ($p$) |
| 5 | equal ($p$) |
| 6 | equal ($p$) |

Let probability of any one outcome by $p$

$$p + p + p + p + p + p = 6p = 1$$

$$\boxed{p = \frac{1}{6}} = 0.167 = \underline{16.7\%}$$



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# PROBABILITY FOR A 6-SIDED DIE

| outcome | probability |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

Let probability of any one outcome by $p$

$$p + p + p + p + p + p = 1$$

$$p = \frac{1}{6}$$




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**Event**: A possible outcome of our experiment (rolling the dice)

| outcome | probability |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

- **Event A:** The outcome is 5. What is the probability of event A?

  $$\mathbb{P}(A) = \frac{1}{6}$$

- **Event B:** The outcome is more than 1 and less than 5. What is the probability of event B?

  $$\mathbb{P}(B) = P(\text{outcome}=2) + P(\text{outcome}=3) + P(\text{outcome}=4)$$

  $$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \boxed{\frac{1}{2}}$$



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# EVENTS

**Event**: A possible outcome of our experiment (rolling the dice)

| outcome | probability |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

- **Event C:** The outcome is an odd number less than 5. What is the probability of event C?

$$\mathbb{P}(C) = P(1) + P(3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

- **Event D:** The outcome is an even number that is not 2. What is the probability of event D?

$$\mathbb{P}(D) = P(4) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$P(\text{even} \cup !2) = P(\text{even}) + P(\text{not } 2) - P(\text{even} \cap \text{not } 2)$$

$$\mathbb{P}(S \cup T) = \mathbb{P}(S) + \mathbb{P}(T) - \mathbb{P}(S \cap T)$$



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# RANDOM VARIABLES

$\boldsymbol{X}$ : maps the outcome of the experiment to numbers

→ random variable

$P(X)$

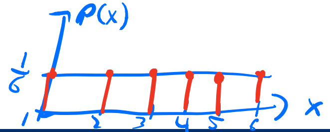
if the dice rolls 4 for an experiment, $X = 4$ for that roll

$P(4) = \frac{1}{6}$

if the dice rolls 2 for an experiment, $X = 2$ for that roll

If the dice rolls $x$ for an experiment, $X = x$ for that roll

$P(x) = \frac{1}{6}$







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# PROBABILITY MASS FUNCTION

$\boldsymbol{X}$ : maps the outcome of the experiment to numbers

random variable

$$\mathbb{P}(X = x) = f_X(x) = \begin{cases} 1/6, \text{if } x = 1 \\ 1/6, \text{if } x = 2 \\ 1/6, \text{if } x = 3 \\ 1/6, \text{if } x = 4 \\ 1/6, \text{if } x = 5 \\ 1/6, \text{if } x = 6 \end{cases}$$

Probability mass function
(PMF)



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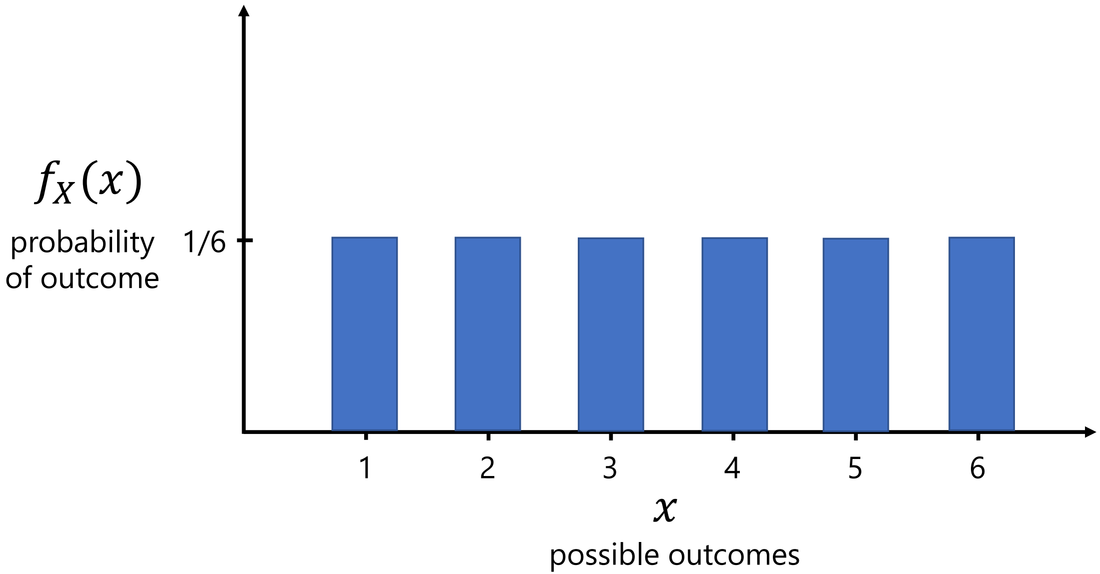
# PROBABILITY MASS FUNCTION



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# EXPECTATION AND VARIANCE

- Useful statistics to know about our pmf
- **Expectation:** What is the average value of $X$?

$N = \#$ times

$$\text{Avg} = \frac{\sum_N \text{values}}{N}$$

$$= \frac{3 \cdot 0 + 3 \cdot 1}{6}$$

$$\Rightarrow \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1$$

$$= \frac{1}{2}$$

$$\mathbb{E}[X] = \langle \underset{\#}{\overset{|\rangle\ \langle|}{X}} \rangle = \sum_x \underset{\text{value, outcome}}{x} \cdot \overset{\text{probability}}{\mathbb{P}(X = x)}$$

$$\Rightarrow \mathbb{E}[X] = \underline{1} \cdot \frac{1}{6} + \underline{2} \cdot \frac{1}{6} + \underline{3} \cdot \frac{1}{6} + \underline{4} \cdot \frac{1}{6} + \underline{5} \cdot \frac{1}{6} + \underline{6} \cdot \frac{1}{6}$$

$$\mathbb{E}[X] = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = \underline{3.5}$$



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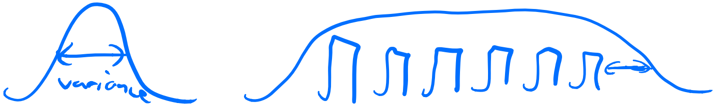
# EXPECTATION AND VARIANCE

- Useful statistics to know about our pmf
- **Variance:** How spread out are the different values of $X$?

$$var[X] = \mathbb{E}\underbrace{\left[\left(X - \mathbb{E}(X)\right)^2\right]} = \sum_x \overbrace{\left(x - \mathbb{E}(X)\right)^2}^{\text{value}} \cdot \overbrace{\mathbb{P}(X = x)}^{\text{prob}}$$

$$\begin{aligned} var[X] \\ &= (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} + (4-3.5)^2 \cdot \frac{1}{6} \\ &+ (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5)^2 \cdot \frac{1}{6} = 2.92 \end{aligned}$$

$3.5 + 2.92 = 6.42 > 6$



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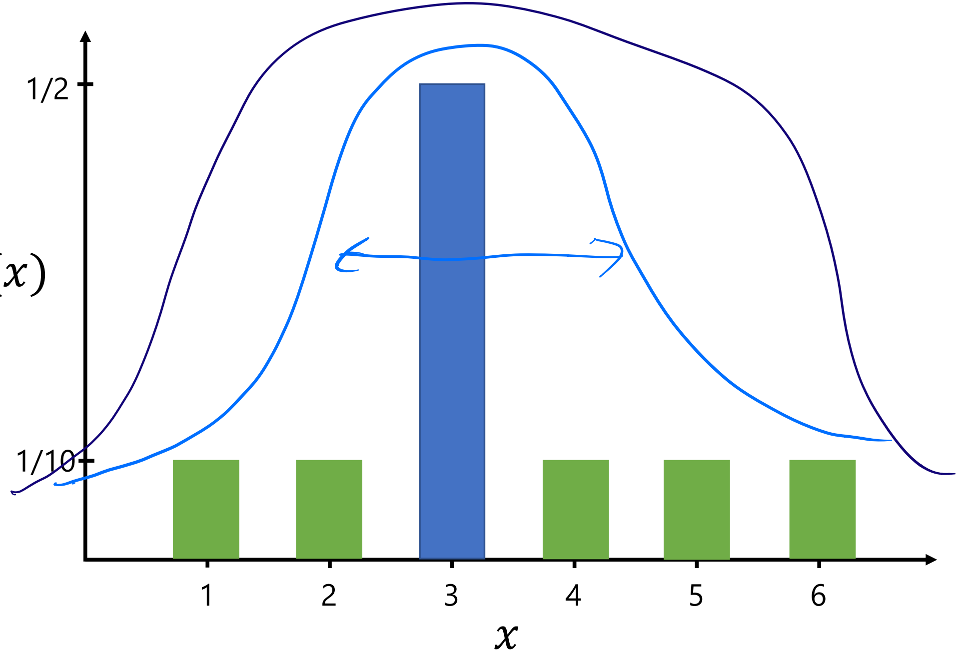
# PMF FOR AN UNFAIR DIE

pdf ↓ random variable

$$P(X = x) = f_X(x) = \begin{cases} 1/10, \text{ if } x = 1 \\ 1/10 \text{ if } x = 2 \\ 1/2, \text{ if } x = 3 \\ 1/10, \text{ if } x = 4 \\ 1/10 \text{ if } x = 5 \\ 1/10, \text{ if } x = 6 \end{cases}$$

pmf



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# EXPECTATION AND VARIANCE

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{10} + 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{10} + 5 \cdot \frac{1}{10} + 6 \cdot \frac{1}{10}$$

$$\mathbb{E}[X] = 3.3$$

$$var[X] = \mathbb{E}[(x - \mathbb{E}[X])^2] = \sum_x (x - \mathbb{E}[X])^2 \cdot \mathbb{P}(X = x)$$

$$var[X] = 1.81$$

lower than fair dice



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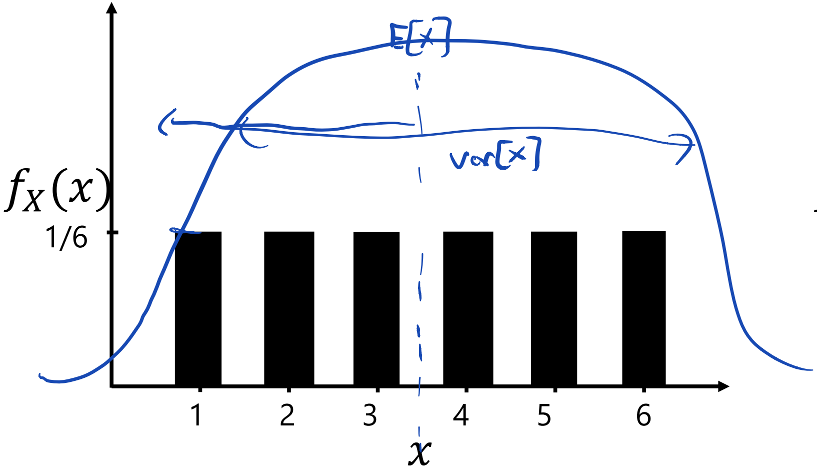
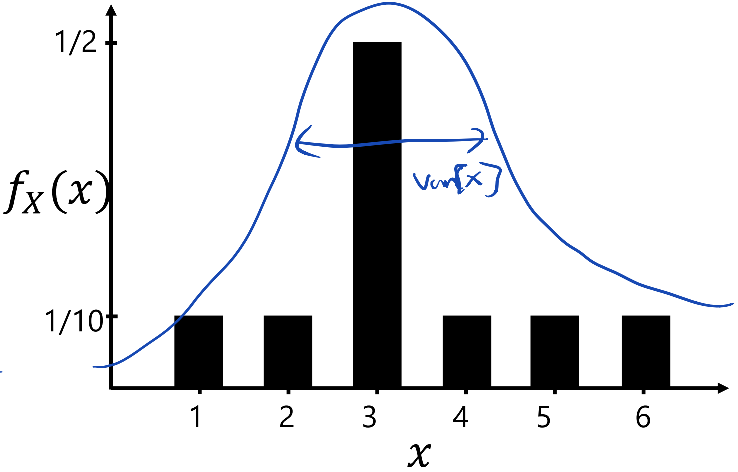
# HISTOGRAMS OF DICE

fair dice




$\mathbb{E}[X] = 3.5$
$var[X] = 2.92$

unfair dice




$\mathbb{E}[X] = 3.3$
$var[X] = 1.81$



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# QUESTIONS

**Questions about the content discussed so far?**



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# PROBABILITY OF A DICE AND A COIN

| dice \ coin | Heads | Tails |
|---|---|---|
| 1 | equal | equal |
| 2 | equal | equal |
| 3 | equal | equal |
| 4 | equal | equal |
| 5 | equal | equal |
| 6 | equal | equal |




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# PROBABILITY OF A DICE AND A COIN

| dice \ coin | H | T |
|---|---|---|
| 1 | 1/12 | 1/12 |
| 2 | 1/12 | 1/12 |
| 3 | 1/12 | 1/12 |
| 4 | 1/12 | 1/12 |
| 5 | 1/12 | 1/12 |
| 6 | 1/12 | 1/12 |

$12p = 1 \Rightarrow p = \frac{1}{12}$

$p(1,T) = p$


# RANDOM VARIABLES

$\boldsymbol{X}$ : maps outcome of a dice roll to numbers

$\boldsymbol{Y}$ : maps outcome of a coin toss to numbers (H=0, T=1)

if the dice rolls 4 and coin flips to H (i.e. 0) for an experiment,

$X = 4, Y = 0$ for that experiment

$$\mathbb{P}(X = 4, Y = 0) = \underbrace{\mathbb{P}(X = 4)}_{\frac{1}{6}} \cdot \underbrace{\mathbb{P}(Y = 0)}_{\frac{1}{2}} = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Independent



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# PROBABILITY OF A DICE AND A COIN

- **Event A:** The outcome of the dice is an even number, the coin gives H

- **Event B:** The outcome of the dice is less than 3, the coin gives T

$P(<3) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P(T) = \frac{1}{2}$

$\Rightarrow P(<3, T) = \boxed{\frac{1}{6}}$




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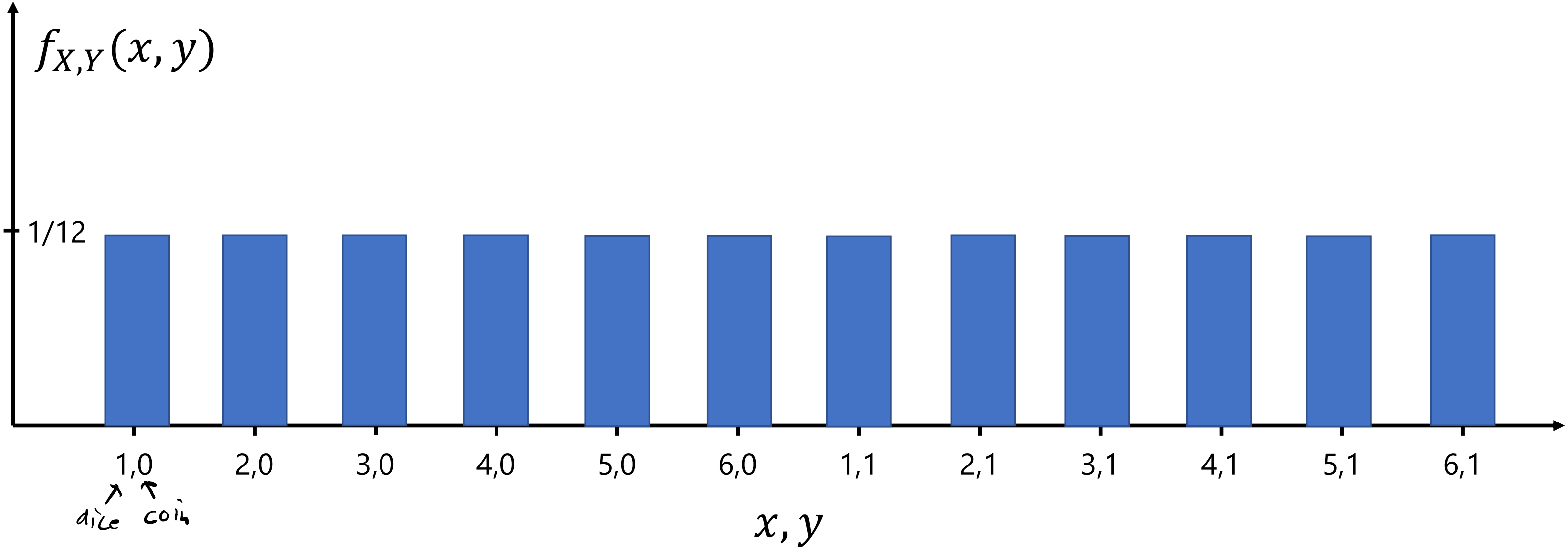
# PMF OF JOINT RANDOM VARIABLE



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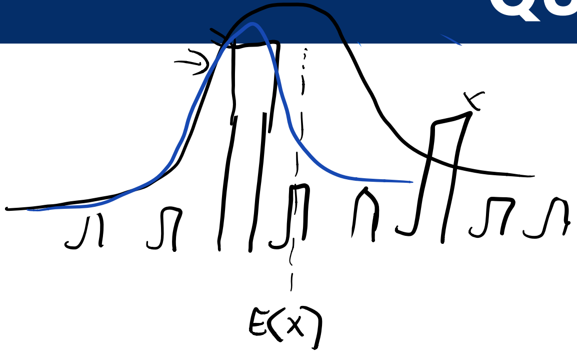
# QUESTIONS




**Questions about the content discussed so far?**



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# INTERLUDE: BRA-KET NOTATION

Bra-ket notation and vector notation:

$$\vec{0} = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{0}^\dagger = |0\rangle^\dagger = \langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\vec{1} = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{1}^\dagger = |1\rangle^\dagger = \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

ket column vector

bra row vector

Inner product notation:

Let $\vec{w} = |0\rangle$. We want to find $\langle \vec{w}, \vec{w} \rangle$

$$\langle \vec{w}, \vec{w} \rangle = \vec{w}^\dagger \vec{w}$$

$$= |0\rangle^\dagger |0\rangle = \langle 0|0\rangle$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$|-\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$

$\Rightarrow \langle -|-\rangle = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{2}$

$= (1 + 1) \frac{1}{2}$

$= 1$




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# 1-QUBIT STATES

Qubit state: $|\psi\rangle$

psi

| outcome | probability |
| --- | --- |
| $|0\rangle$ or $\begin{pmatrix}1\\0\end{pmatrix}$ | equal |
| $|1\rangle$ or $\begin{pmatrix}0\\1\end{pmatrix}$ | equal |



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# 1-QUBIT STATES

Qubit state: $|\psi\rangle$

| outcome | probability |
|---|---|
| $\lvert 0\rangle$ or $\binom{1}{0}$ | 1/2 |
| $\lvert 1\rangle$ or $\binom{0}{1}$ | 1/2 |

$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$

$\alpha$ & $\beta$ NOT probabilities

$$\cancel{|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle}$$

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}} = |+\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Prob amplitude

$\Sigma\,|\text{prob amp}|^2 = 1$

$$|\alpha|^2 = |\beta|^2 = \frac{1}{2}$$

$|\alpha|^2 + |\beta|^2 = \frac{1}{2} + \frac{1}{2} = 1$




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# 1-QUBIT STATES

Why do we care about $|\alpha|^2$ and $|\beta|^2$?

Remember normalization!

$$\langle\psi|\psi\rangle = 1$$

$$\langle 1|1\rangle = 1$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

$$\left(\frac{1}{\sqrt{2}}\right)^2$$

$$\langle +|+\rangle = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{2} = \frac{1+1}{2} = 1$$




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# 1-QUBIT STATES

Qubit state: $|\psi\rangle$

| outcome | probability |
|---|---|
| $\|0\rangle$ or $\begin{pmatrix}1\\0\end{pmatrix}$ | 4/5 |
| $\|1\rangle$ or $\begin{pmatrix}0\\1\end{pmatrix}$ | 1/5 |

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 = \frac{4}{5}, |\beta|^2 = \frac{1}{5}$$

$$|\psi\rangle = \sqrt{\frac{4}{5}}|0\rangle + \sqrt{\frac{1}{5}}|1\rangle$$




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# IMPORTANT TAKEAWAYS

- Random variable → Possible outcomes of the experiment

- Probability mass function → Probabilities of the different outcomes

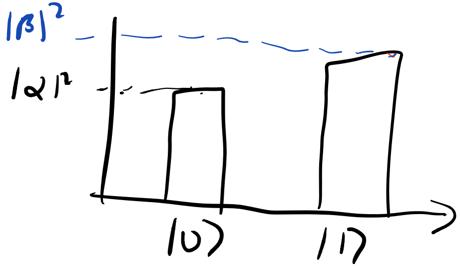
- Independent events → Probabilities multiply

- Bra → row vector ; ket → column vector

$H\Psi = E\Psi$

$\langle x| \Rightarrow (x_1 \;\; x_2)$ $\qquad |y\rangle \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

- Qubit states are represented with probability **amplitudes**




$P(X|Y)$

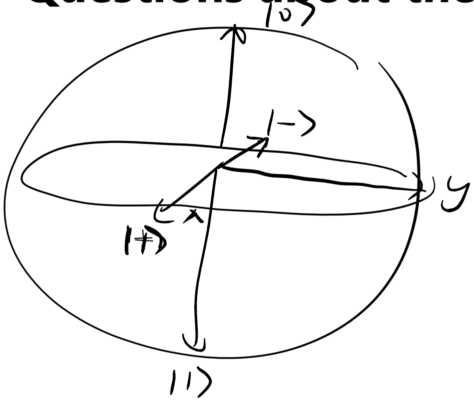
given

Conjugate variables:

$x$: position , $p$: momentum

$E$: energy , $t$: time

## Questions about the content discussed so far?




$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

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# POST-LAB ZOOM FEEDBACK

**After this lab,** on a scale of 1 to 5, how would you rate your understanding of this week's content?

- 1 –Did not understand anything
- 2 – Understood some parts
- 3 – Understood most of the content
- 4 – Understood all of the content
- 5 – The content was easy for me/I already knew all of the content



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# OPTIONAL CONTENT



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# TWO-QUBIT STATES

2-qubit state: $|\psi\rangle$

| qubit 2 / qubit 1 | $\|0\rangle$ | $\|1\rangle$ |
|---|---|---|
| $\|0\rangle$ | equal | equal |
| $\|1\rangle$ | equal | equal |



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# TWO-QUBIT STATES

2-qubit state: $|\psi\rangle$

| qubit 2 / qubit 1 | $\|0\rangle$ | $\|1\rangle$ |
|---|---|---|
| $\|0\rangle$ | 1/4 | 1/4 |
| $\|1\rangle$ | 1/4 | 1/4 |

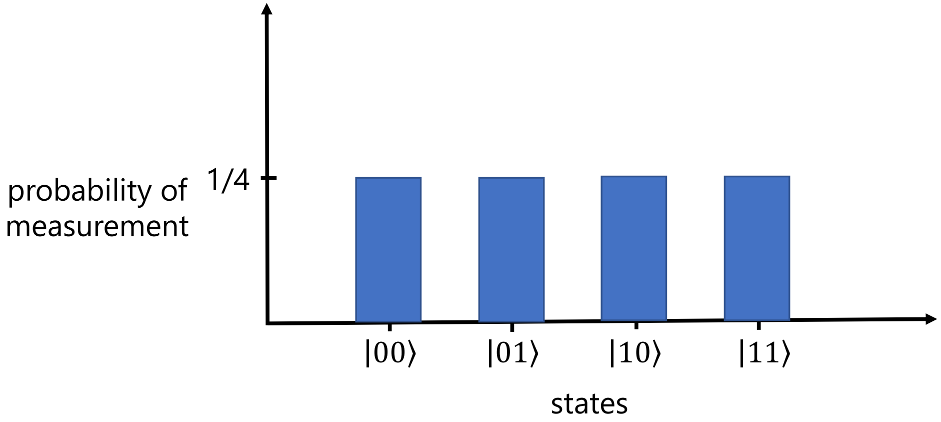
$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$|\alpha|^2 = |\beta|^2 = |\gamma|^2 = |\delta|^2 = \frac{1}{4}$$

# MEASUREMENT PROBABILITY FOR 2-QUBIT STATE

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$|\alpha|^2 = |\beta|^2 = |\gamma|^2 = |\delta|^2 = \frac{1}{4}$$



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# A LOADED TWO QUBIT STATE

2-qubit state: $|\psi\rangle$

| qubit 1 \ qubit 2 | $\|0\rangle$ | $\|1\rangle$ |
|---|---|---|
| $\|0\rangle$ | 1/2 | 0 |
| $\|1\rangle$ | 0 | 1/2 |

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

**Entanglement!**



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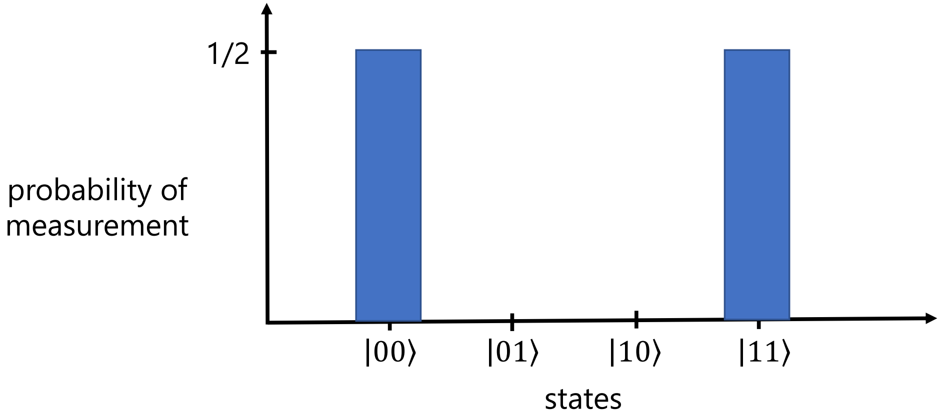
# THE 2-QUBIT ENTANGLED STATE

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$



# EXTRA PROBLEMS

# MORE INNER PRODUCTS

Inner product examples:

$$\text{Let } \vec{w} = |0\rangle \text{ and } \vec{v} = |1\rangle\text{. We want to find } \langle \vec{w}, \vec{v} \rangle$$

$$\begin{aligned}\langle \vec{w}, \vec{v} \rangle &= \vec{w}^{\dagger}\vec{v} \\ &= |0\rangle^{\dagger}|1\rangle = \langle 0|1\rangle \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\end{aligned}$$

$$\text{Let } \vec{w} = |1\rangle \text{ and } \vec{v} = |0\rangle\text{. We want to find } \langle \vec{w}, \vec{v} \rangle$$

$$\begin{aligned}\langle \vec{w}, \vec{v} \rangle &= \vec{w}^{\dagger}\vec{v} \\ &= |1\rangle^{\dagger}|0\rangle = \langle 1|0\rangle \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\end{aligned}$$