INTRO TO QUANTUM COMPUTING

Week 5 Lab

PROBABILITY AND RANDOM VARIABLES

<insert TA name>

<insert date>

© 2020 The Coding School

PROGRAM FOR TODAY

- Logistics
- Attendance quiz
- Pre-lab zoom feedback
- Questions from last week
- Lab content
- Post-lab zoom feedback

© 2020 The Coding School

LOGISTICS

Piazza is a great resource for content-related questions!

- Post your questions from lecture, lab or homework
- Responses from instructors + TAs and fellow students
- Average response time – 15 minutes
- If you don't have access, email student@qubitbyqubit.org

© 2020 The Coding School

CANVAS ATTENDANCE QUIZ

- Please log into Canvas and answer your lab section's quiz (using the password posted below and in the chat).
  - This is lab number :
  - Passcode:
- How many hours did you spend on last week's homework?
  - Less than 1 hour
  - 1-2 hours
  - 2-3 hours
  - More than 3 hours
  - I didn't do the homework
- **This quiz not graded, but counts for your lab attendance!**

© 2020 The Coding School
Canvas attendance quiz passcode:

PRE-LAB ZOOM FEEDBACK

On a scale of 1 to 5, how would you rate your understanding of this week's content?

- 1 –Did not understand anything
- 2 – Understood some parts
- 3 – Understood most of the content
- 4 – Understood all of the content
- 5 – The content was easy for me/I already knew all of the content

© 2020 The Coding School

LEARNING OBJECTIVES FOR LAB 5

- Understanding probability for a 6-sided dice
  - Events and normalization
  - Random variables
  - Probability mass function
  - Expectation and variance
- Joint probability for a dice and coin
- Relating probability to quantum computing
  - 1-qubit states and bra-ket notation
  - 2-qubit states and entanglement*

*Optional content

© 2020 The Coding School
Canvas attendance quiz passcode:

QUESTIONS FROM PAST WEEK

Complex numbers (for our purposes) are scalars!

© 2020 The Coding School

Canvas attendance quiz passcode:

QUESTIONS FROM PAST WEEK

Why do we take the transpose of one vector for inner product?

Dimensional consistency!

$$\vec{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \longrightarrow \text{Two } 2 \times 1 \text{ vectors}$$

$$\langle \vec{w}, \vec{v} \rangle = \underbrace{\begin{pmatrix} 3 & 2 \end{pmatrix}}_{1 \times 2} \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{2 \times 1} = 8$$

Inner dimensions match!

© 2020 The Coding School

Canvas attendance quiz passcode:

QUESTIONS FROM PAST WEEK

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 8$$

$$1 \times 2 \qquad 2 \times 1$$

Inner dimensions match!

© 2020 The Coding School

Canvas attendance quiz passcode:

QUESTIONS FROM PAST WEEK

$$\begin{pmatrix} 3 & 2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$2 \times 2 \qquad 2 \times 1$$

Inner dimensions match!

© 2020 The Coding School

Canvas attendance quiz passcode:

QUESTIONS FROM PAST WEEK

$$\begin{pmatrix} 3 & 2 \\ -2 & 5 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

$3 \times 2 \qquad 2 \times 1$

Inner dimensions match!

Taking the transpose for the inner product of vectors lets us use the "inner dimensions match" rule for vector as well as matrix multiplication

© 2020 The Coding School

WHY PROBABILITY?

Quantum mechanics (and therefore, quantum computing) is *inherently* probabilistic

© 2020 The Coding School

PROBABILITY AROUND US

Where have you seen probability in your daily lives?

© 2020 The Coding School

PROBABILITY FOR A 6-SIDED DICE

outcome	probability
1	equal
2	equal
3	equal
4	equal
5	equal
6	equal

© 2020 The Coding School

PROPERTIES OF PROBABILITY

- **Non-negativity:** The probabilities must be non-negative
- **Normalization:** The sum of the probabilities of all 6 sides must be 1

© 2020 The Coding School
Canvas attendance quiz passcode:

PROBABILITY FOR A 6-SIDED DIE

outcome	probability
1	equal
2	equal
3	equal
4	equal
5	equal
6	equal

Let probability of any one outcome by p

$$p + p + p + p + p + p = 1$$

$$p = \frac{1}{6}$$

© 2020 The Coding School
Canvas attendance quiz passcode:

PROBABILITY FOR A 6-SIDED DIE

outcome	probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Let probability of any one outcome by p

$$p + p + p + p + p + p = 1$$

$$p = \frac{1}{6}$$

© 2020 The Coding School

Canvas attendance quiz passcode:

EVENTS

Event: A possible outcome of our experiment (rolling the dice)

outcome	probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

- **Event A:** The outcome is 5. What is the probability of event A?

  $$\mathbb{P}(A) =$$

- **Event B:** The outcome is more than 1 and less than 5. What is the probability of event B?

  $$\mathbb{P}(B) =$$

© 2020 The Coding School
Canvas attendance quiz passcode:

EVENTS

Event: A possible outcome of our experiment (rolling the dice)

outcome	probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

- **Event C:** The outcome is an odd number less than 5. What is the probability of event C?

$$\mathbb{P}(C) =$$

- **Event D:** The outcome is an even number that is not 2. What is the probability of event D?

$$\mathbb{P}(D) =$$

$$\mathbb{P}\,(S \cup T) = \mathbb{P}(S) + \mathbb{P}(T) - \mathbb{P}(S \cap T)$$

© 2020 The Coding School

RANDOM VARIABLES

$\boldsymbol{X}$: maps the outcome of the experiment to numbers

random variable

if the dice rolls 4 for an experiment, $X = 4$ for that roll

if the dice rolls 2 for an experiment, $X = 2$ for that roll

If the dice rolls x for an experiment, $X = x$ for that roll

© 2020 The Coding School

PROBABILITY MASS FUNCTION

$\boldsymbol{X}$: maps the outcome of the experiment to numbers

random variable

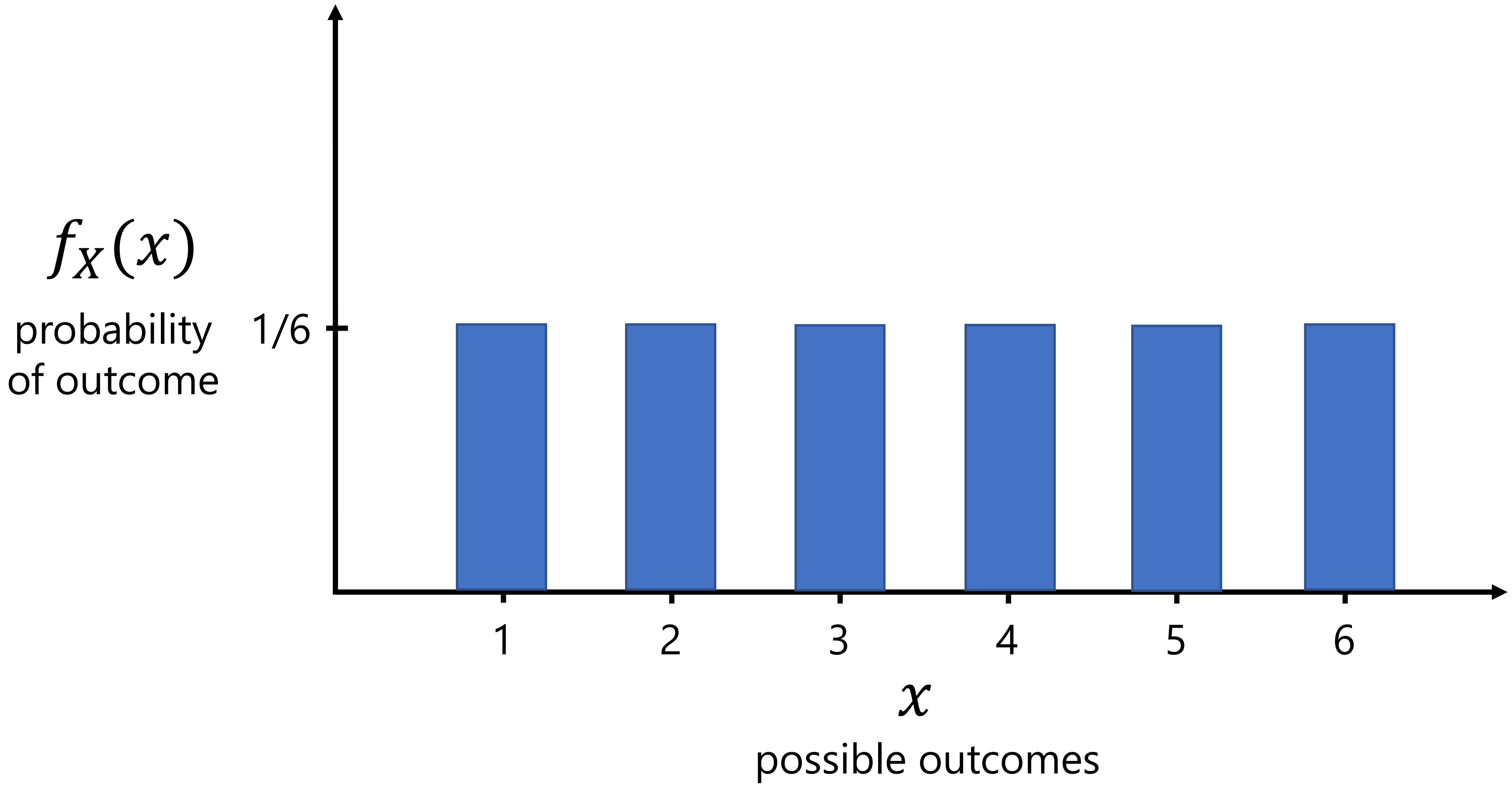
$$\mathbb{P}(X = x) = f_X(x) = \begin{cases} 1/6, \text{if } x = 1 \\ 1/6, \text{if } x = 2 \\ 1/6, \text{if } x = 3 \\ 1/6, \text{if } x = 4 \\ 1/6, \text{if } x = 5 \\ 1/6, \text{if } x = 6 \end{cases}$$

Probability mass function

© 2020 The Coding School

Canvas attendance quiz passcode:

PROBABILITY MASS FUNCTION



© 2020 The Coding School
Canvas attendance quiz passcode:

EXPECTATION AND VARIANCE

- Useful statistics to know about our pmf
- **Expectation:** What is the average value of X?

$$\mathbb{E}[X] = \langle X \rangle = \sum_x x \cdot \mathbb{P}(X = x)$$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$\mathbb{E}[X] = \frac{1+2+3+4+5+6}{6} = 3.5$$

© 2020 The Coding School

EXPECTATION AND VARIANCE

- Useful statistics to know about our pmf
- **Variance:** How spread out are the different values of X?

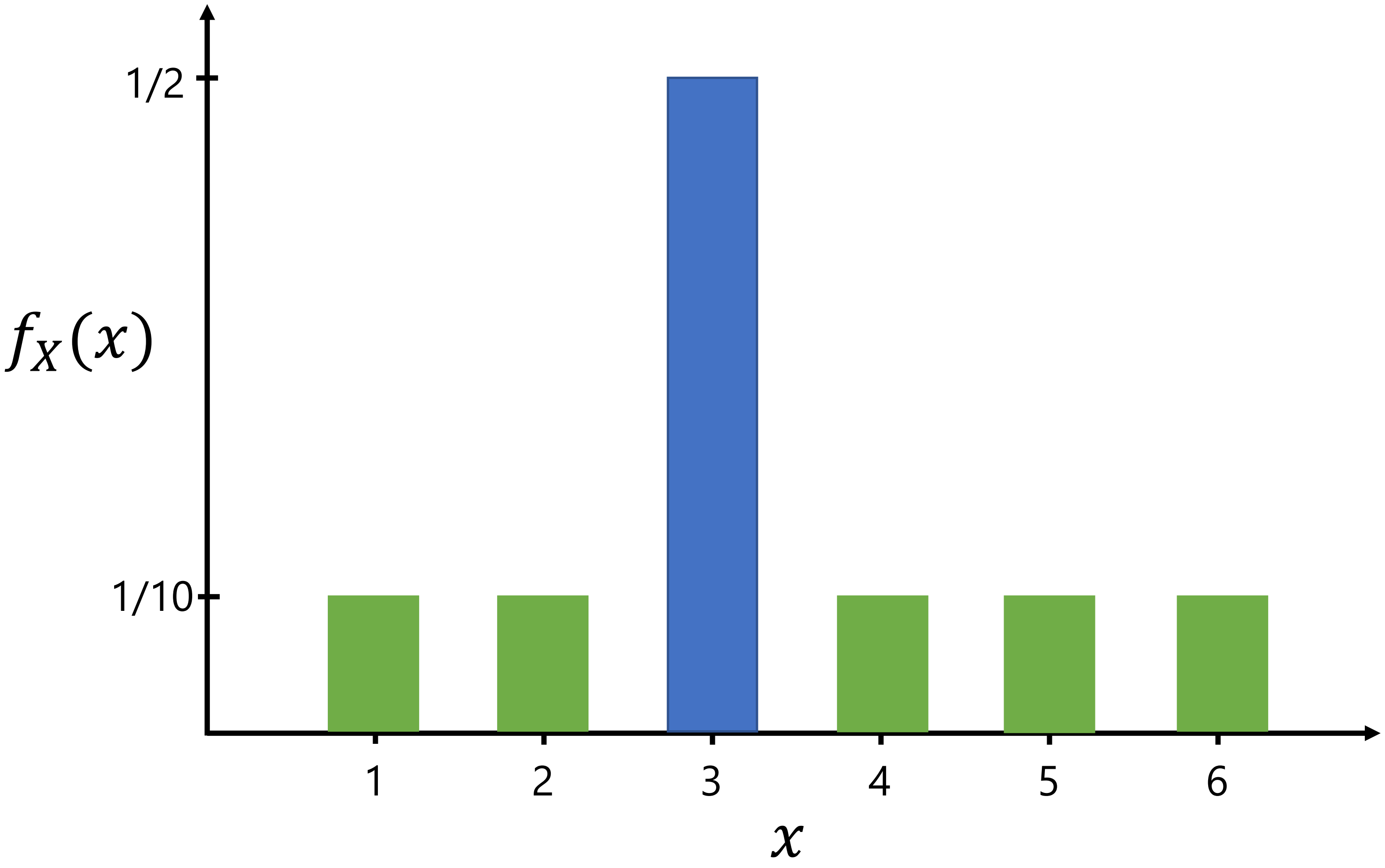
$$var[X] = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \sum_x \left(x - \mathbb{E}(X)\right)^2 \cdot \mathbb{P}(X = x)$$

$$var[X] = (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} + (4-3.5)^2 \cdot \frac{1}{6} + (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5)^2 \cdot \frac{1}{6} = 2.92$$

© 2020 The Coding School
Canvas attendance quiz passcode:

PMF FOR AN UNFAIR DIE

$$P(X = x) = f_X(x) = \begin{cases} 1/10, \text{if } x = 1 \\ 1/10 \text{ if } x = 2 \\ 1/2, \text{if } x = 3 \\ 1/10, \text{if } x = 4 \\ 1/10 \text{ if } x = 5 \\ 1/10, \text{if } x = 6 \end{cases}$$



© 2020 The Coding School

Canvas attendance quiz passcode:

EXPECTATION AND VARIANCE

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{10} + 3 \cdot \frac{1}{2} + 4 \cdot \frac{1}{10} + 5 \cdot \frac{1}{10} + 6 \cdot \frac{1}{10}$$

$$\mathbb{E}[X] = 3.3$$

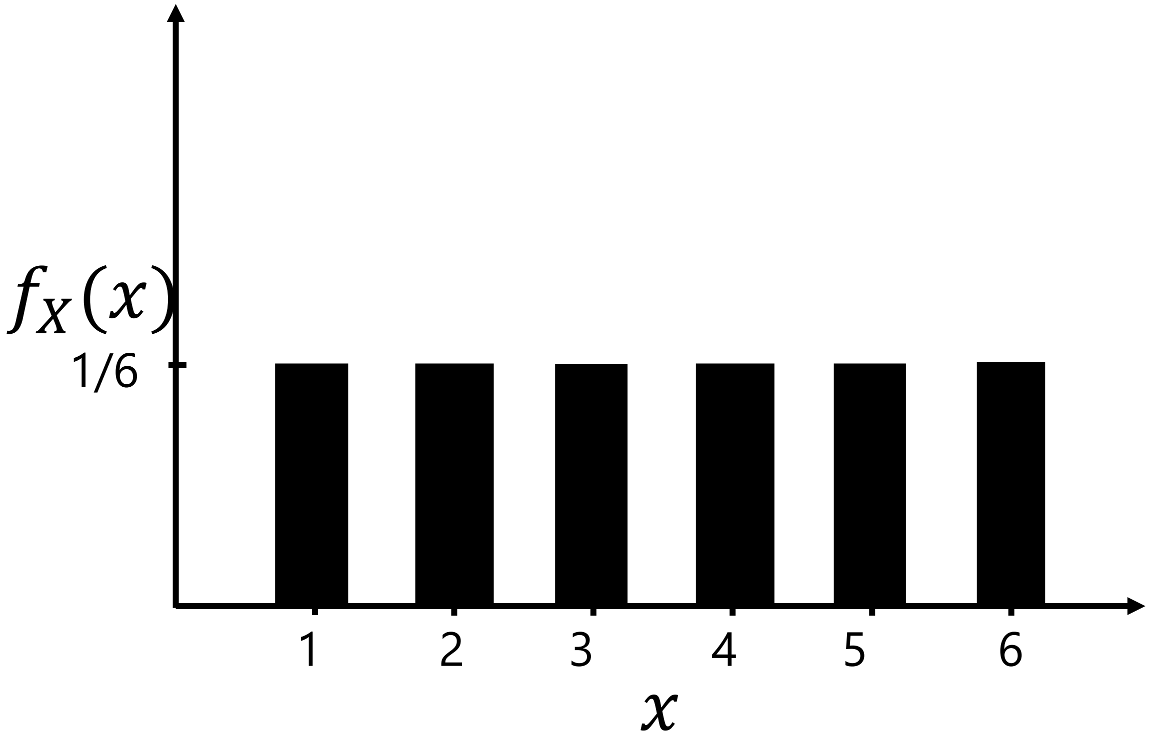
$$var[X] = \mathbb{E}[(x - \mathbb{E}[X])^2] = \sum_x (x - \mathbb{E}[X])^2 \cdot \mathbb{P}(X = x)$$

$$var[X] = 1.81$$

© 2020 The Coding School

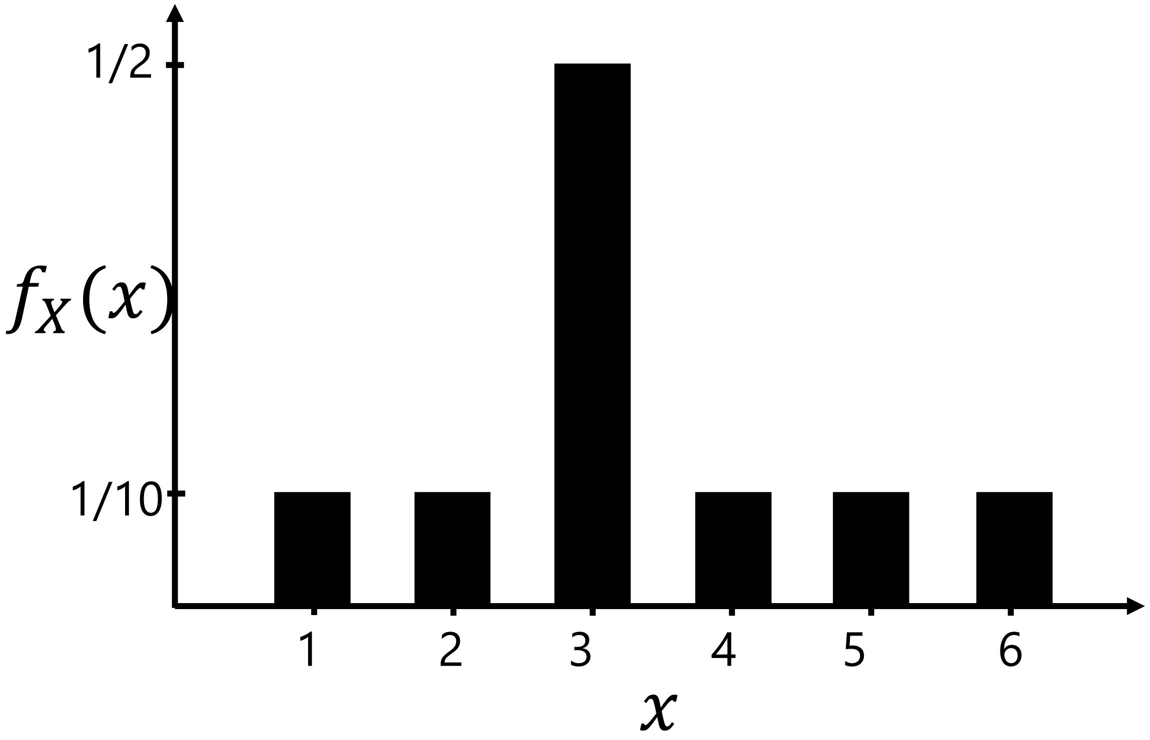
Canvas attendance quiz passcode:

fair dice



$$\mathbb{E}[X] = 3.5$$
$$var[X] = 2.92$$

unfair dice



$$\mathbb{E}[X] = 3.3$$
$$var[X] = 1.81$$

© 2020 The Coding School

Canvas attendance quiz passcode:

QUESTIONS

Questions about the content discussed so far?

© 2020 The Coding School

Canvas attendance quiz passcode:

PROBABILITY OF A DICE AND A COIN

dice \ coin	H	T
1	equal	equal
2	equal	equal
3	equal	equal
4	equal	equal
5	equal	equal
6	equal	equal

© 2020 The Coding School

PROBABILITY OF A DICE AND A COIN

dice \ coin	H	T
1	1/12	1/12
2	1/12	1/12
3	1/12	1/12
4	1/12	1/12
5	1/12	1/12
6	1/12	1/12

© 2020 The Coding School
Canvas attendance quiz passcode:

RANDOM VARIABLES

$\boldsymbol{X}$: maps outcome of a dice roll to numbers

$\boldsymbol{Y}$: maps outcome of a coin toss to numbers (H=0, T=1)

if the dice rolls 4 and coin flips to H (i.e. 0) for an experiment,

$X = 4, Y = 0$ for that experiment

$$\mathbb{P}(X = 4, Y = 0) = \mathbb{P}(X = 4) \cdot \mathbb{P}(Y = 0) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

© 2020 The Coding School

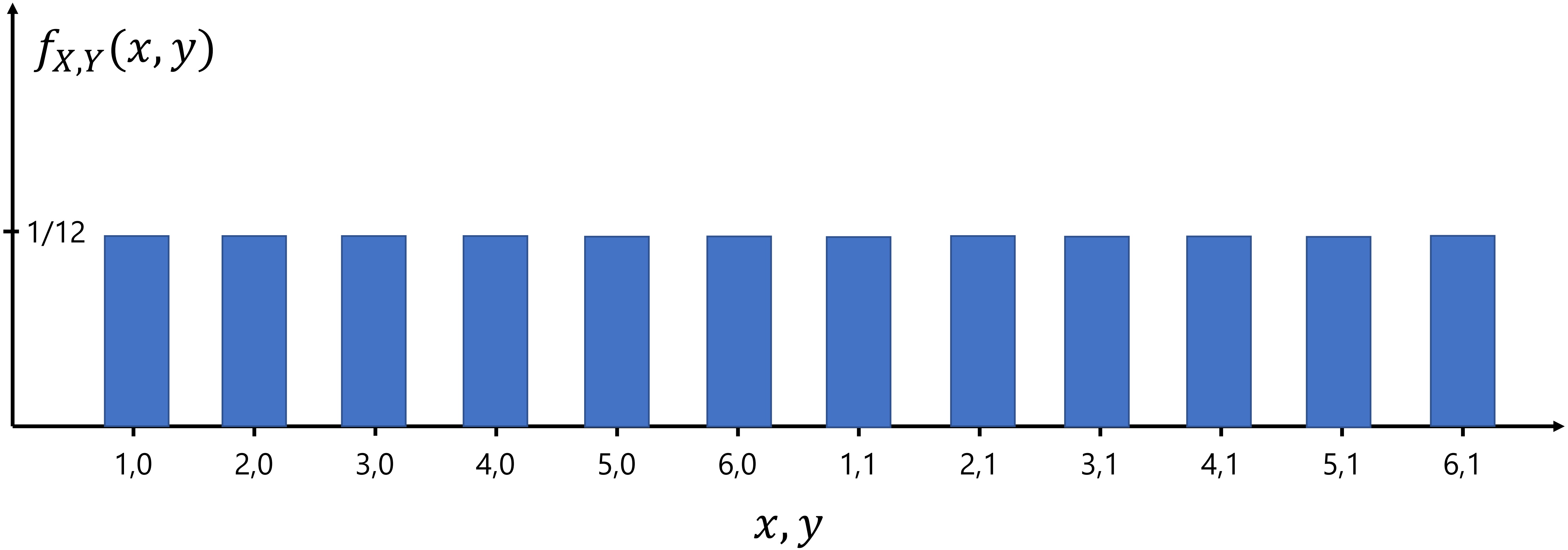
PROBABILITY OF A DICE AND A COIN

- **Event A:** The outcome of the dice is an even number, the coin gives H

- **Event B:** The outcome of the dice is less than 3, the coin gives T

© 2020 The Coding School
Canvas attendance quiz passcode:

PMF OF JOINT RANDOM VARIABLE



© 2020 The Coding School
Canvas attendance quiz passcode:

QUESTIONS

Questions about the content discussed so far?

© 2020 The Coding School
Canvas attendance quiz passcode:

INTERLUDE: BRA-KET NOTATION

Bra-ket notation and vector notation:

$$\vec{0} = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{0}^{\dagger} = |0\rangle^{\dagger} = \langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\vec{1} = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{1}^{\dagger} = |1\rangle^{\dagger} = \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

ket column vector

bra row vector

Inner product notation:

Let $\vec{w} = |0\rangle$. We want to find $\langle \vec{w}, \vec{w} \rangle$

$$\langle \vec{w}, \vec{w} \rangle = \vec{w}^{\dagger}\vec{w}$$

$$= |0\rangle^{\dagger}|0\rangle = \langle 0|0\rangle$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

© 2020 The Coding School

1-QUBIT STATES

Qubit state: $|\psi\rangle$

outcome	probability
$\|0\rangle$ or $\begin{pmatrix}1\\0\end{pmatrix}$	equal
$\|1\rangle$ or $\begin{pmatrix}0\\1\end{pmatrix}$	equal

© 2020 The Coding School

1-QUBIT STATES

Qubit state: $|\psi\rangle$

outcome	probability
$\|0\rangle$ or $\begin{pmatrix}1\\0\end{pmatrix}$	1/2
$\|1\rangle$ or $\begin{pmatrix}0\\1\end{pmatrix}$	1/2

$$\cancel{|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 = |\beta|^2 = \frac{1}{2}$$

© 2020 The Coding School
Canvas attendance quiz passcode:

1-QUBIT STATES

Why do we care about $|\alpha|^2$ and $|\beta|^2$?

Remember normalization!

$$\langle\psi|\psi\rangle = 1$$

© 2020 The Coding School

Canvas attendance quiz passcode:

1-QUBIT STATES

Qubit state: $|\psi\rangle$

outcome	probability
$|0\rangle$ or $\begin{pmatrix}1\\0\end{pmatrix}$	4/5
$|1\rangle$ or $\begin{pmatrix}0\\1\end{pmatrix}$	1/5

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 = \frac{4}{5}, |\beta|^2 = \frac{1}{5}$$

© 2020 The Coding School

Canvas attendance quiz passcode:

IMPORTANT TAKEAWAYS

- Random variable → Possible outcomes of the experiment
- Probability mass function → Probabilities of the different outcomes
- Independent events → Probabilities multiply
- Bra → row vector ; ket → column vector
- Qubit states are represented with probability **amplitudes**

© 2020 The Coding School

QUESTIONS?

Questions about the content discussed so far?

© 2020 The Coding School

Canvas attendance quiz passcode:

POST-LAB ZOOM FEEDBACK

After this lab, on a scale of 1 to 5, how would you rate your understanding of this week's content?

- 1 –Did not understand anything
- 2 – Understood some parts
- 3 – Understood most of the content
- 4 – Understood all of the content
- 5 – The content was easy for me/I already knew all of the content

© 2020 The Coding School

Canvas attendance quiz passcode:

OPTIONAL CONTENT

© 2020 The Coding School

Canvas attendance quiz passcode:

TWO-QUBIT STATES

2-qubit state: $|\psi\rangle$

qubit 2 / qubit 1	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	equal	equal
$\|1\rangle$	equal	equal

© 2020 The Coding School

Canvas attendance quiz passcode:

TWO-QUBIT STATES

2-qubit state: $|\psi\rangle$

qubit 1 \ qubit 2	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	1/4	1/4
$\|1\rangle$	1/4	1/4

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

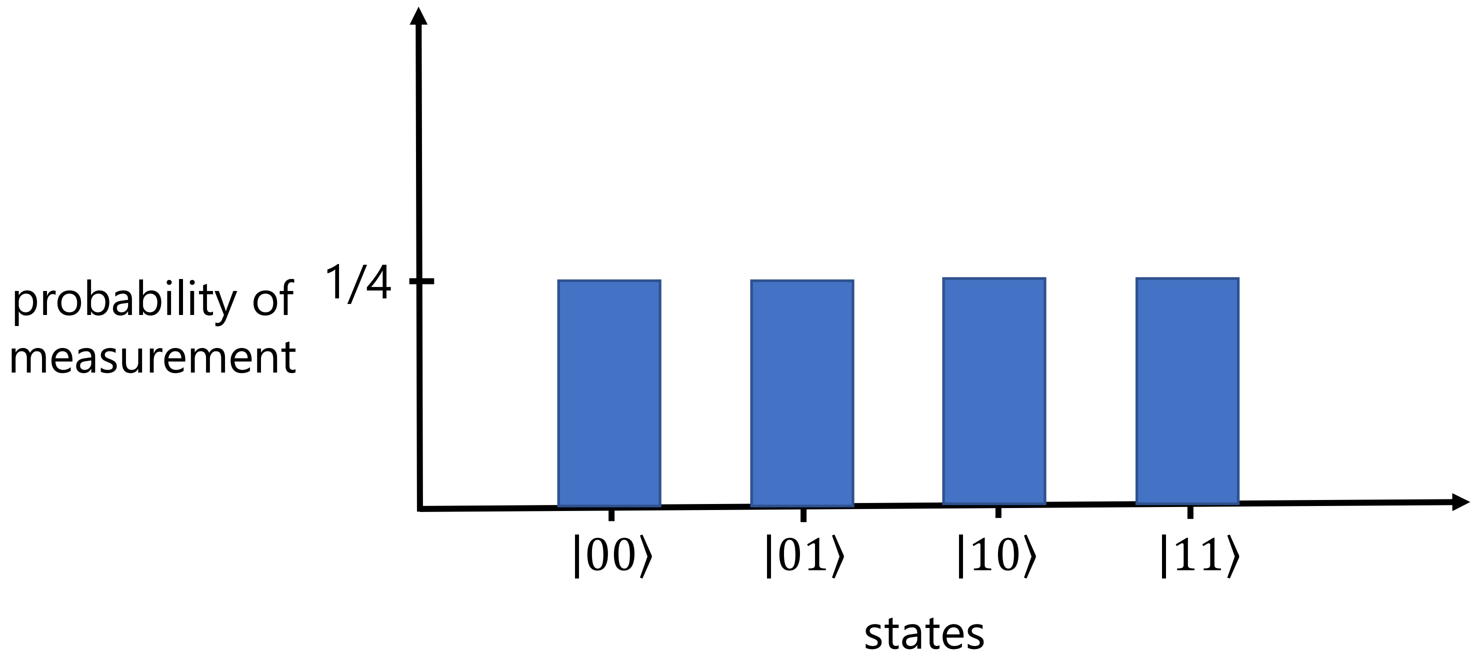
$$|\alpha|^2 = |\beta|^2 = |\gamma|^2 = |\delta|^2 = \frac{1}{4}$$

© 2020 The Coding School

MEASUREMENT PROBABILITY FOR 2-QUBIT STATE

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$|\alpha|^2 = |\beta|^2 = |\gamma|^2 = |\delta|^2 = \frac{1}{4}$$



© 2020 The Coding School

Canvas attendance quiz passcode:

A LOADED TWO QUBIT STATE

2-qubit state: $|\psi\rangle$

qubit 2 / qubit 1	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	1/2	0
$\|1\rangle$	0	1/2

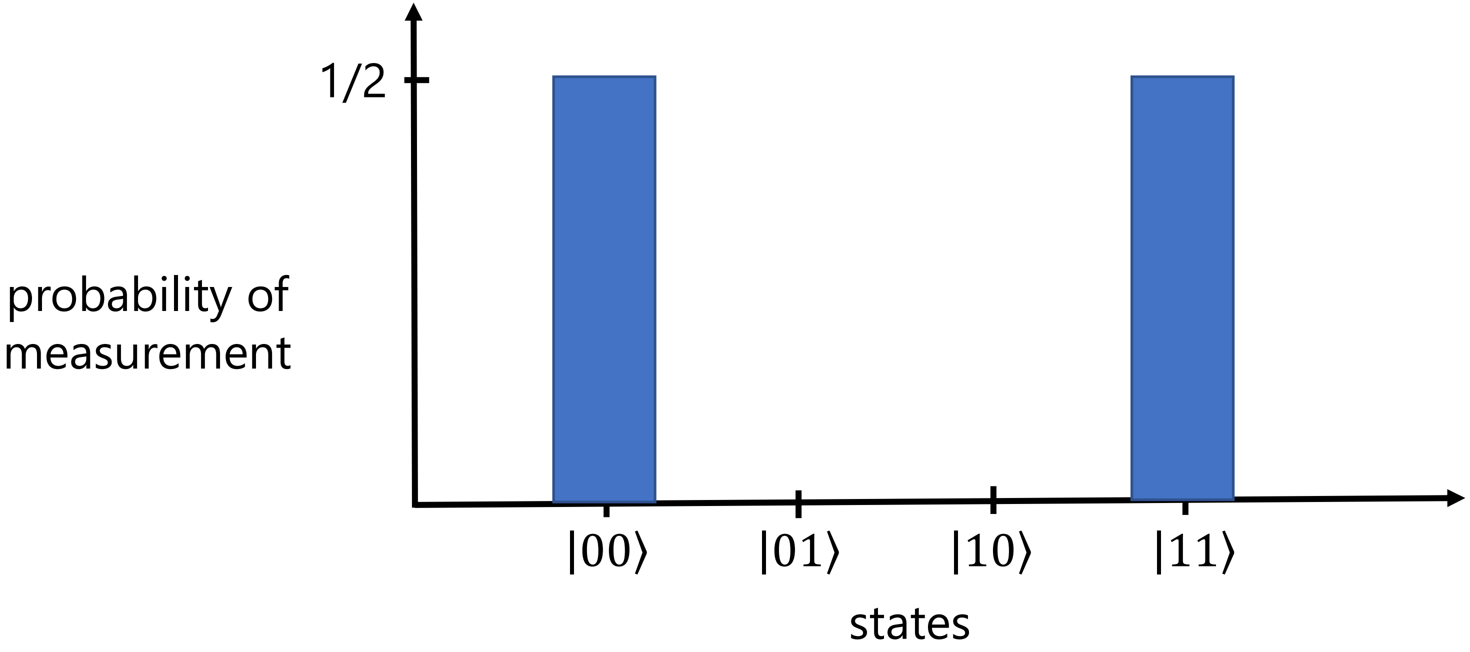
$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Entanglement!

© 2020 The Coding School

THE 2-QUBIT ENTANGLED STATE

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$



© 2020 The Coding School

Canvas attendance quiz passcode:

EXTRA PROBLEMS

© 2020 The Coding School

MORE INNER PRODUCTS

Inner product examples:

Let $\vec{w} = |0\rangle$ and $\vec{v} = |1\rangle$. We want to find $\langle \vec{w}, \vec{v} \rangle$

$$\begin{aligned}\langle \vec{w}, \vec{v} \rangle &= \vec{w}^{\dagger} \vec{v} \\ &= |0\rangle^{\dagger} |1\rangle = \langle 0|1\rangle \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\end{aligned}$$

Let $\vec{w} = |1\rangle$ and $\vec{v} = |0\rangle$. We want to find $\langle \vec{w}, \vec{v} \rangle$

$$\begin{aligned}\langle \vec{w}, \vec{v} \rangle &= \vec{w}^{\dagger} \vec{v} \\ &= |1\rangle^{\dagger} |0\rangle = \langle 1|0\rangle \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\end{aligned}$$

© 2020 The Coding School

Canvas attendance quiz passcode: