INTRO TO QUANTUM COMPUTING
LECTURE #14

The Qubit & Bloch Sphere

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ANNOUNCEMENTS

TODAY'S LECTURE

➔ Use the Bloch Sphere to conceptualize the state of a qubit

➔ View quantum gates as actions on the Bloch sphere

What is a qubit?

- Building block of quantum computers
- A two-level system
- Can be in a superposition of two values

Qubit Reviewed

Superposition: a qubit can be $|0\rangle$ and $|1\rangle$ at the same time!

This is how we show it: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|0\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}$$

$$|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$|1\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$$

$\alpha, \beta \in \mathbb{C}$ ↳ complex

$$|\alpha|^2 + |\beta|^2 = 1$$

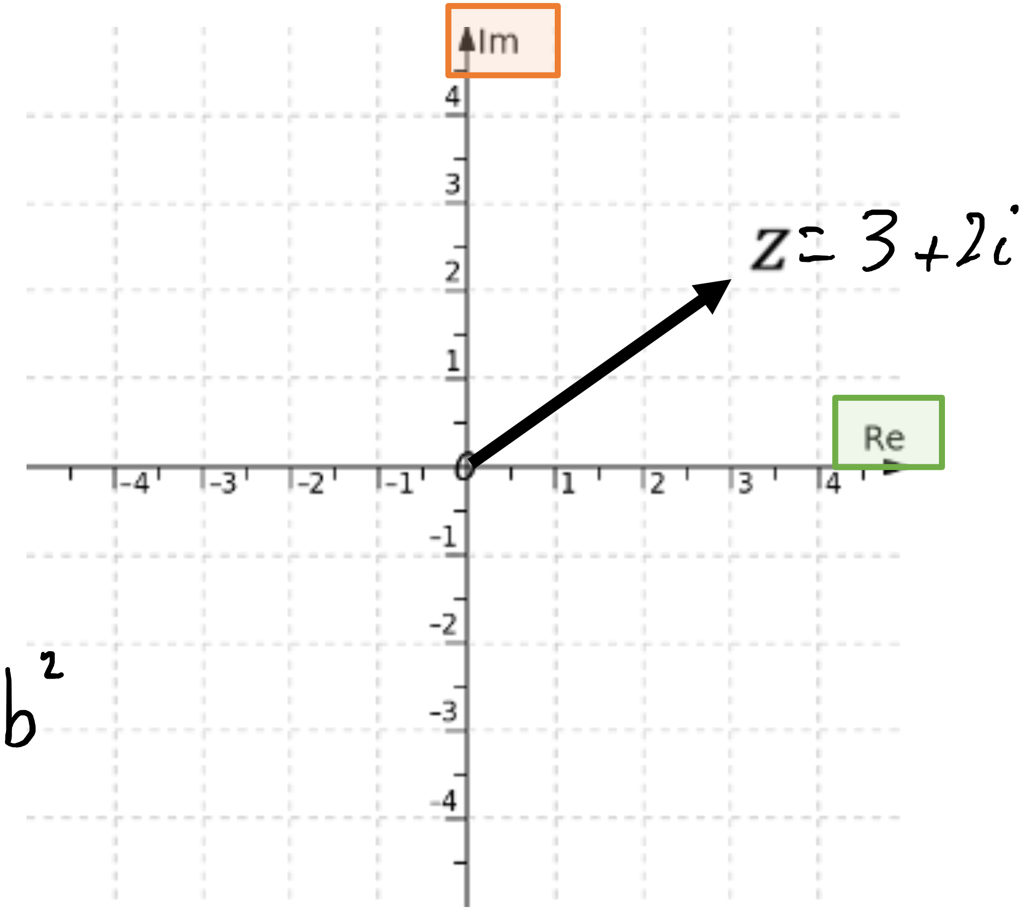
Complex Numbers Review

A complex number consists of both a ***real*** and ***imaginary*** component.

$$z = \underbrace{a}_{\text{real}} + i\,\underbrace{b}_{\text{imaginary}}$$

complex conjugate

$$z^{*} = a - i\,b$$

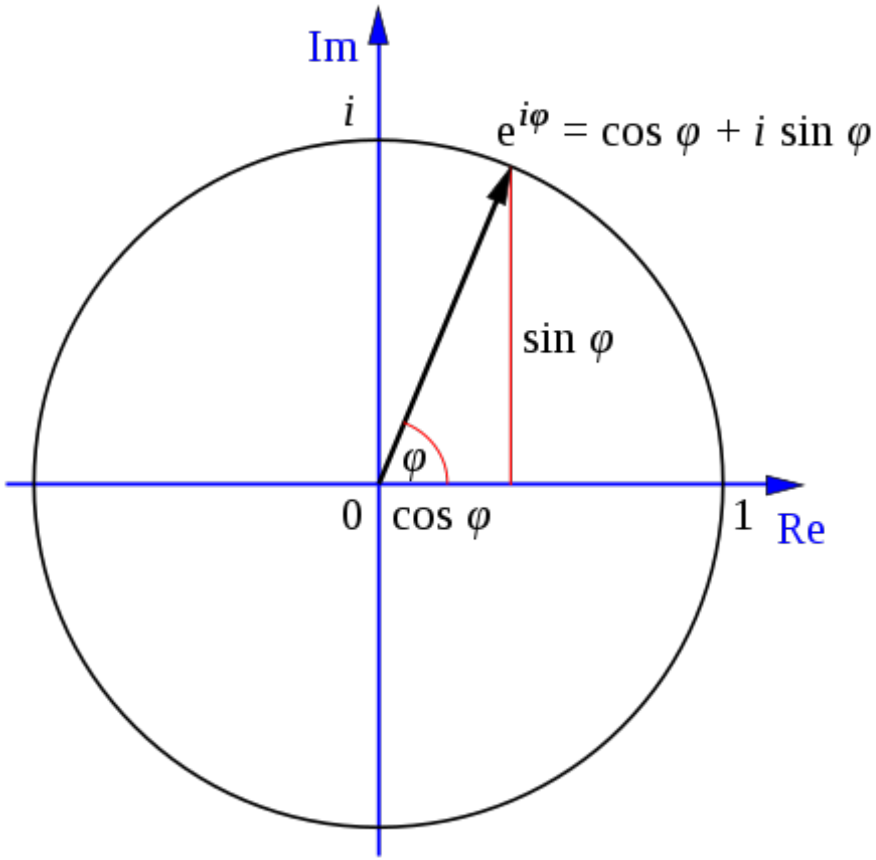


$$|z|^2 = z \cdot z^{*} = z^{*} \cdot z$$

$$(a+ib)(a-ib) = a^2 - i\cancel{ab} + i\cancel{ab} - (i)^2 b^2$$

$$= a^2 + b^2$$

EULER'S FORMULA & COMPLEX EXPONENTIALS



Source: Wikipedia (CC)

Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Polar representation of complex numbers!

$$z = x + iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

(vector radius)

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

(vector angle)

What is phase?

$$e^{i\varphi}$$

$\varphi \equiv \phi$ (phi)

$$e^{i\phi} = \underbrace{\cos(\phi) + i\sin(\phi)}$$

$$|e^{i\phi}| = \sqrt{\cos^2(\phi) + \sin^2(\phi)} = 1$$

$$\phi = 0 \longrightarrow e^{i\phi} = e^0 = \cos(0) + i\sin(0) = 1$$

$$\phi = \pi/2 \longrightarrow e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

$$\phi = \pi \longrightarrow e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

In quantum mechanics global phase has **NO _physical significance_** and **NO _effect_** therefore can be ignored!!

$$e^{i\varphi}|\psi\rangle \rightarrow |\psi\rangle$$

$$-|0\rangle \rightarrow |0\rangle$$

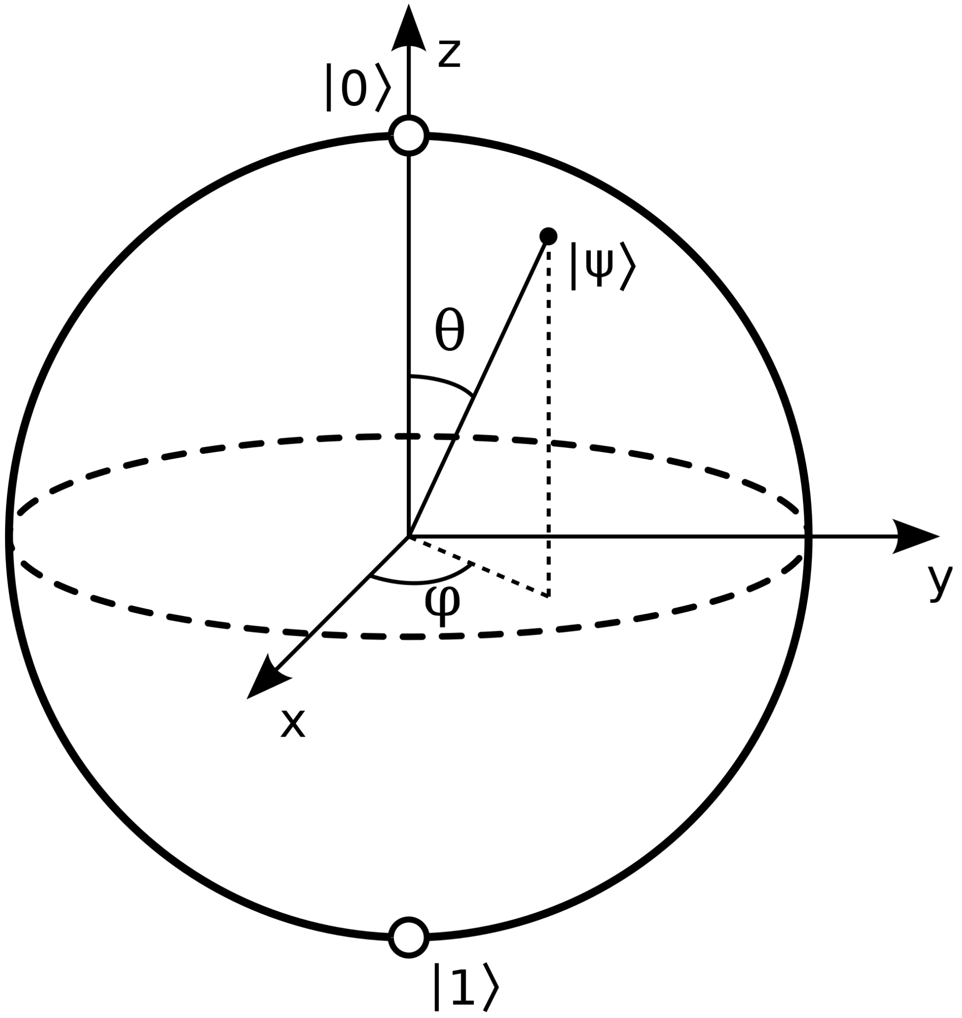
$$\frac{|0\rangle - |1\rangle}{\sqrt{2}} \neq \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Bloch Sphere

The ***state*** of a qubit can be represented as a point on the Bloch Sphere

unit sphere

$r=1$



Qubit

Back to our qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$ are complex

$\alpha = |\alpha| e^{i\phi_\alpha}$

$\beta = |\beta| e^{i\phi_\beta}$

$$|\psi\rangle = |\alpha| e^{i\phi_\alpha} |0\rangle + |\beta| e^{i\phi_\beta} |1\rangle$$

$$= \underbrace{\cancel{e^{i\phi_\alpha}}}_{\text{global phase}} \left[|\alpha| |0\rangle + |\beta| \underbrace{e^{i(\phi_\beta - \phi_\alpha)}}_{\text{relative phase}} |1\rangle \right]$$

$$\phi \equiv \phi_\beta - \phi_\alpha$$

$$= |\alpha| |0\rangle + |\beta| e^{i\phi} |1\rangle$$

$|\alpha|^2 + |\beta|^2 = 1$

$|\alpha| = \cos(\theta/2)$

$|\beta| = \sin(\theta/2)$

$$|\psi\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2) e^{i\phi} |1\rangle$$

$\cos^2(\theta/2) + \sin^2(\theta/2) = 1$

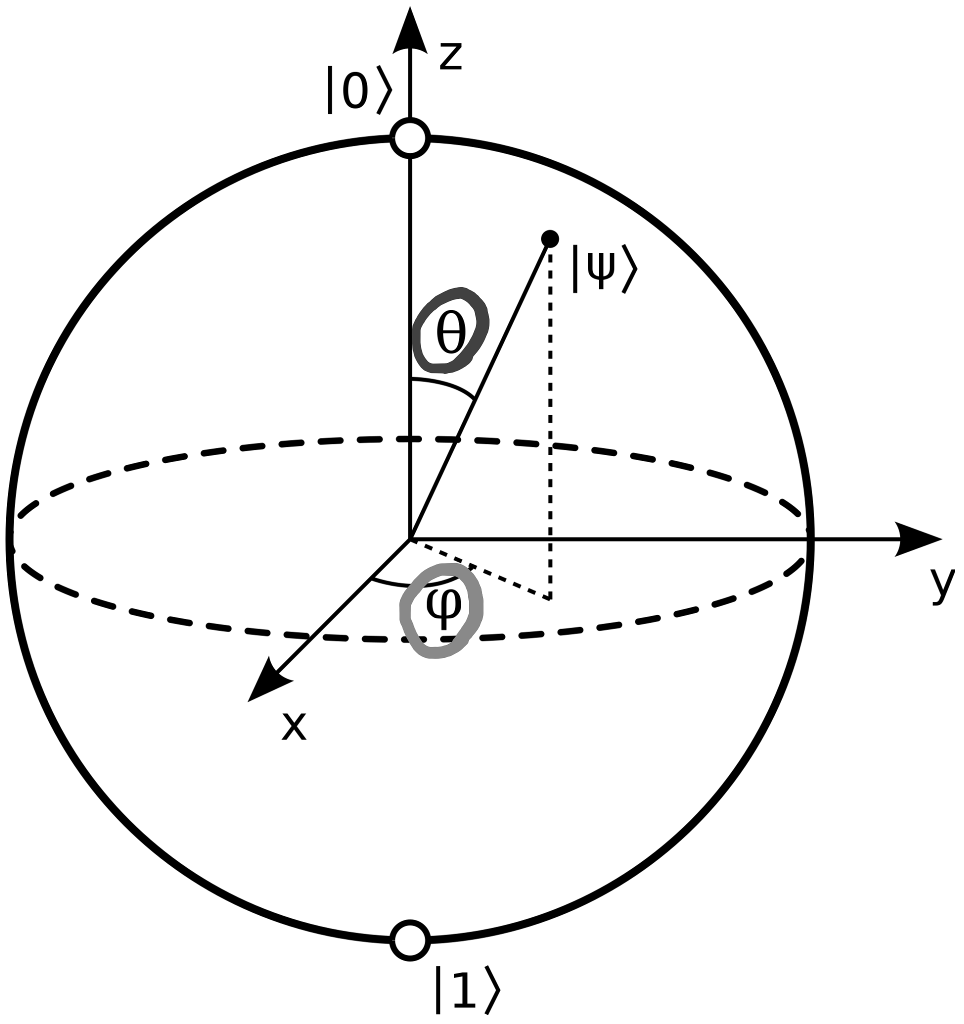
Bloch Sphere

$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle$

North pole: $|0\rangle$

South pole: $|1\rangle$

Classical bits are only on the North pole or the South pole



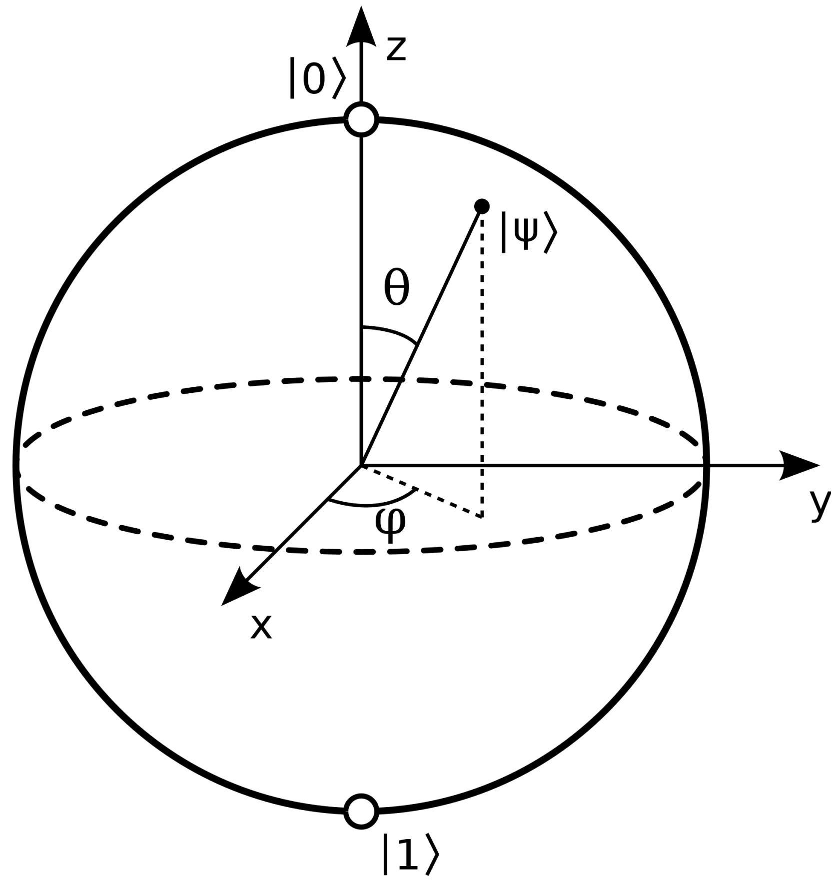
Bloch Sphere

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle$$

What is the state of the qubit based on the position on the Bloch sphere?

$\theta=0$: $|\psi\rangle = \overset{1}{\cancel{\cos(0)}}\,|0\rangle + \overset{0}{\cancel{\sin(0)}}\, e^{i\phi}\,|1\rangle = |0\rangle$

$\theta=\pi$: $|\psi\rangle = \overset{0}{\cancel{\cos(\pi/2)}}\,|0\rangle + \overset{1}{\cancel{\sin(\pi/2)}}\, e^{i\phi} = \underbrace{\cancel{e^{i\phi}}}_{\text{global phase}}|1\rangle$



Bloch Sphere

$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle$

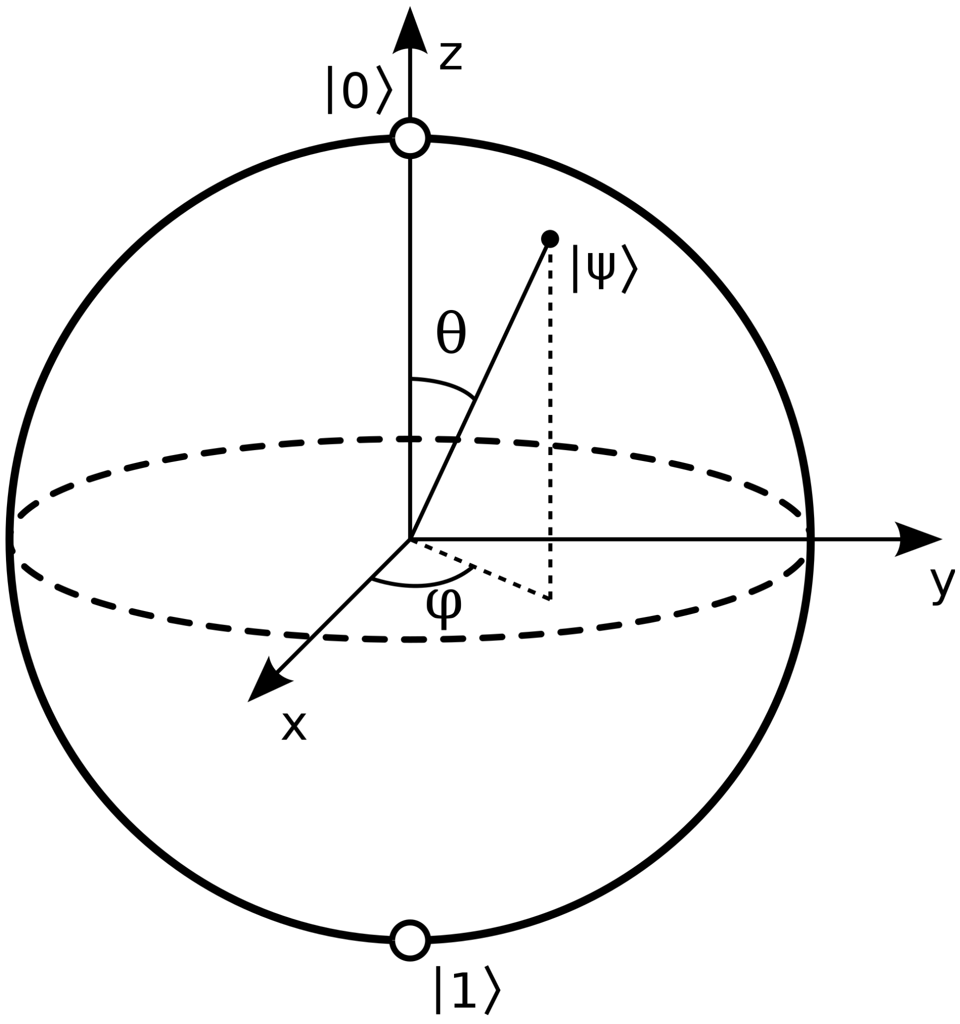
More examples:

$\theta = \pi/2$, $\varphi = 0$: $|\psi\rangle = \cos(\pi/4)|0\rangle + \sin(\pi/4)e^{i0}|1\rangle$

$= 1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$\theta = \pi/2$, $\varphi = \pi$: $|\psi\rangle = \cos(\pi/4)|0\rangle + \sin(\pi/4)\overbrace{e^{i\pi}}^{-1}|1\rangle$

$= 1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$



10 MIN BREAK!

$SU(2) \longrightarrow SO(3)$

2d complex 3D real

Quantum gates

- Maps one the quantum state to another quantum state
- Moves a point on the Bloch Sphere to another point
- We can represent them with matrices

Important Quantum Gates

Pauli gates:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ Pauli-X operator (X)

$$\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$ Pauli-Y operator (Y)

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli-Z operator (Z)

Important Quantum Gates

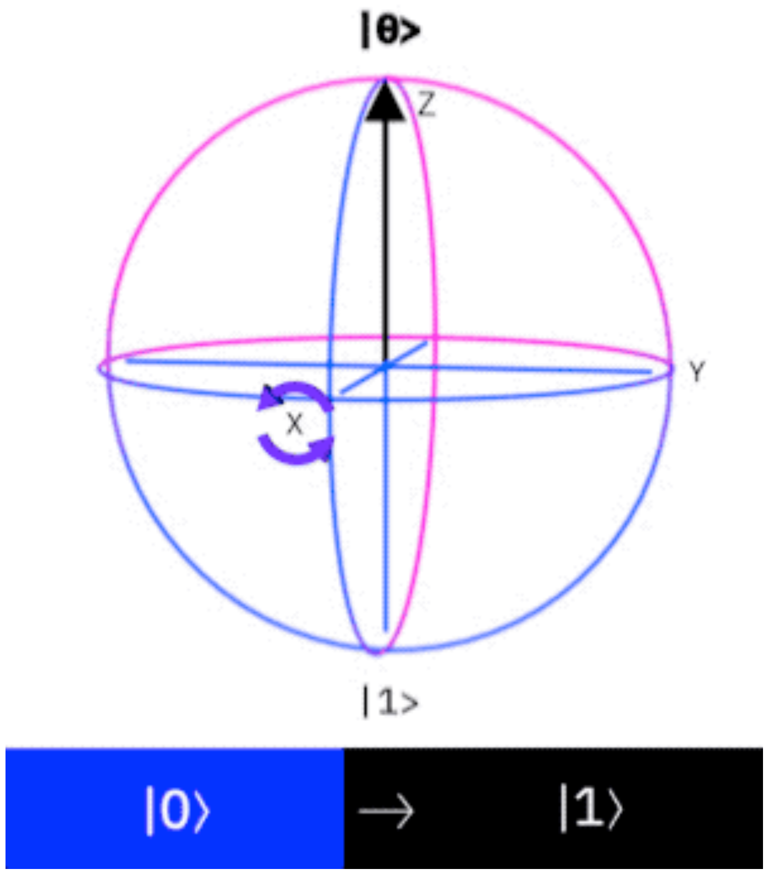
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Pauli-X operator (X)

$$\sigma_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$\sigma_x |1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

(quantum bit-flip)



Rotates the state around the ***X*** axis by 180 degrees

Important Quantum Gates

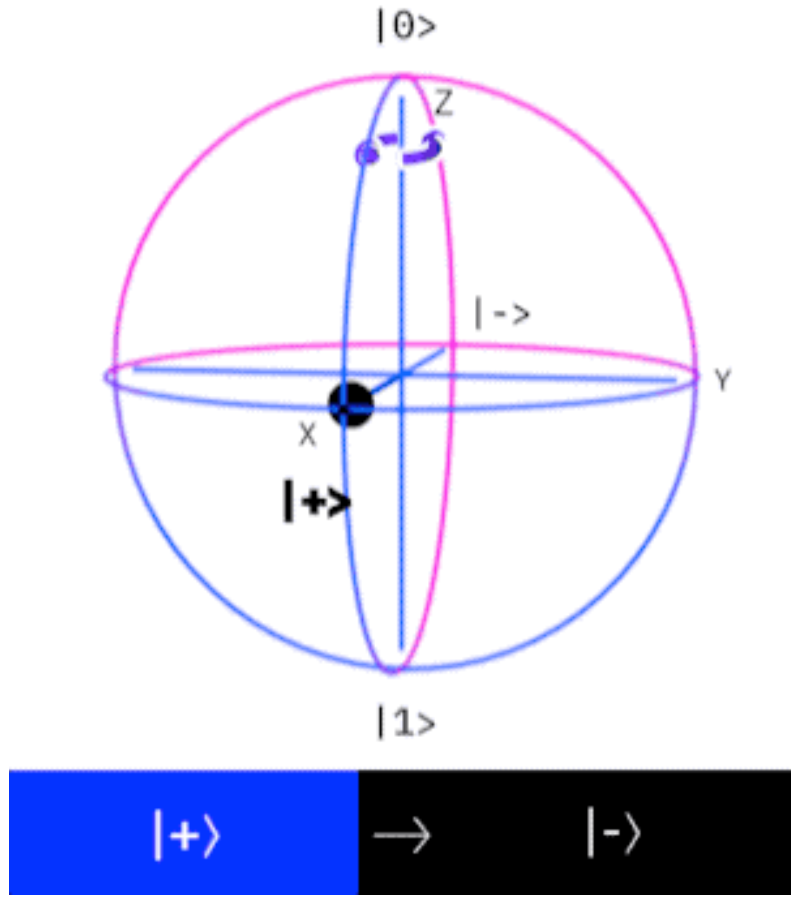
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli-Z operator (Z)

$$\sigma_z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$\sigma_z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

(phase operator)



Rotates the state around the ***Z*** axis by 180 degrees

Important Quantum Gates

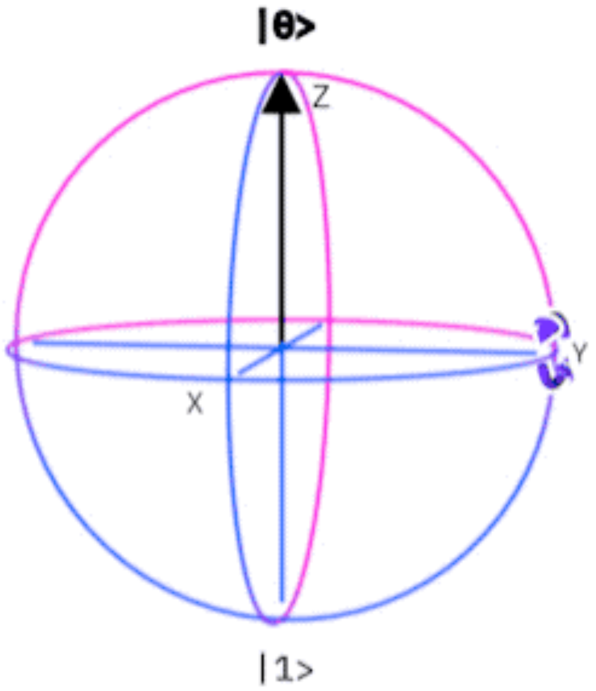
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$ Pauli-Y operator (Y)

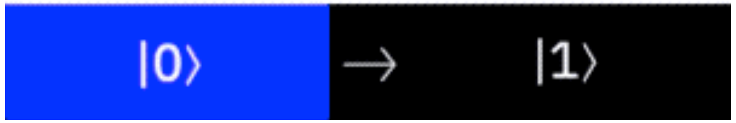
$$\sigma_y |0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle$$

$$\sigma_y |1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

(combination of bit-flip and phase-flip)

$$\sigma_y = i\sigma_x \sigma_z$$





Rotates the state around the ***Y*** axis by 180 degrees

Rotations on the Bloch Sphere

We're not just limited by 180 degree rotations

For example we can rotate around the x axis by 90 degrees

$$\underbrace{R}_{\text{rotation}}{}_{\underbrace{x}_{\text{axis}}}\underbrace{(\pi/2)}_{\text{angle}}$$

Rotations on the Bloch Sphere

We can rotate by an angle θ around x or y or z axis:

Hermitian

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z}$$

Rotations on the Bloch Sphere

Let's take rotation by an angle θ around the x axis as an example:

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x} = \cos\frac{\theta}{2}\, I - i\sin\frac{\theta}{2}\,\sigma_x$$

Matrix representation?

$$R_x(\theta) = \cos(\theta/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin(\theta/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta/2) & 0 \\ 0 & \cos(\theta/2) \end{pmatrix} + \begin{pmatrix} 0 & -i\sin(\theta/2) \\ -i\sin(\theta/2) & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

Rotations on the Bloch Sphere

Let's take rotation by an angle θ around the x axis as an example:

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z} = \cos\frac{\theta}{2}\, I - i\sin\frac{\theta}{2}\,\sigma_z$$

Matrix representation?

$$R_z(\theta) = \cos(\theta/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin(\theta/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta/2) - i\sin(\theta/2) & 0 \\ 0 & \cos(\theta/2) - i\sin(\theta/2) \end{pmatrix}$$

Rotations on the Bloch Sphere

Examples:

$$R_x(\pi)\,|0\rangle = \left(\cos(\pi/2)\, I - i \sin(\pi/2)\,\sigma_x\right)|0\rangle$$

$\downarrow$

X π-pulse

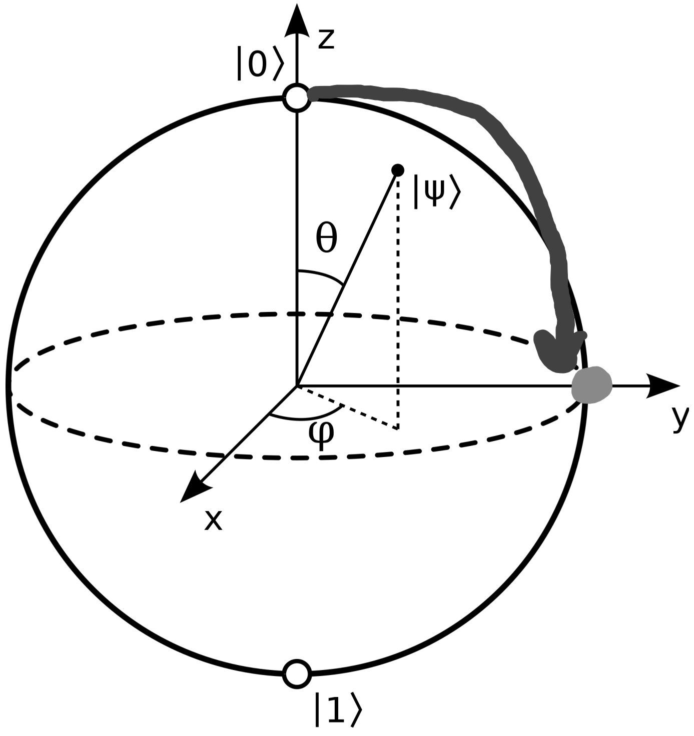
$$= \cos(\pi/2)\cdot I\,|0\rangle - i \sin(\pi/2)\,\sigma_x|0\rangle$$

$$= -i\,\sigma_x|0\rangle = \cancel{-i}\,|1\rangle$$

$$R_x(\pi/2)\,|0\rangle = \left(\cos(\pi/4)\cdot I - i \sin(\pi/4)\,\sigma_x\right)|0\rangle$$

$$= \cos(\pi/4)\cdot I\,|0\rangle - i \sin(\pi/4)\,\sigma_x|0\rangle$$

$$= 1/\sqrt{2}\,|0\rangle - i\frac{1}{\sqrt{2}}|1\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$



Rotations on the Bloch Sphere

Examples:

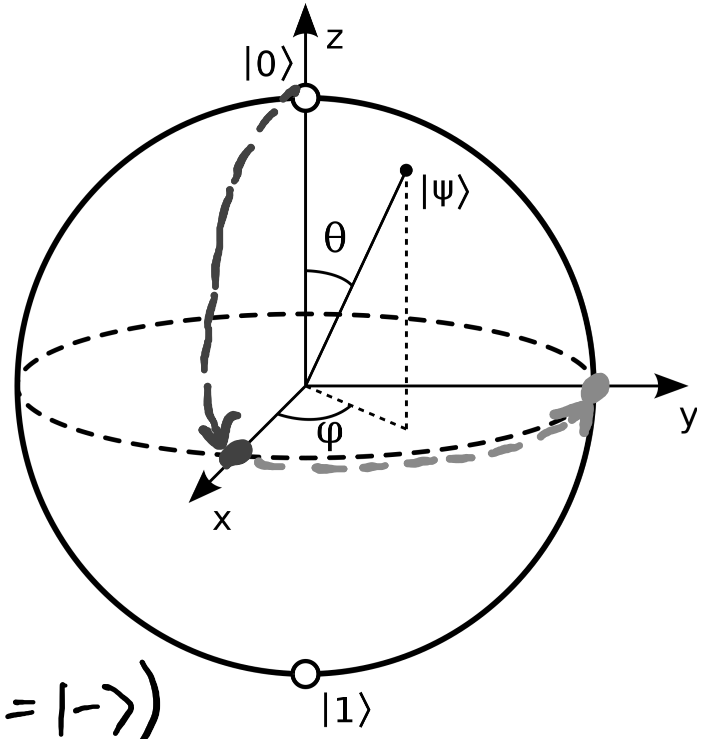
$$R_y(\pi/2)|0\rangle = \cos(\pi/4)\cdot I\,|0\rangle - i\sin(\pi/4)\,\sigma_y\,|0\rangle$$

$$= 1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \equiv |+\rangle$$

$$R_z(\pi/2)|+\rangle = \cos(\pi/4)\cdot I\,|+\rangle - i\sin(\pi/4)\,\sigma_z\,|+\rangle$$

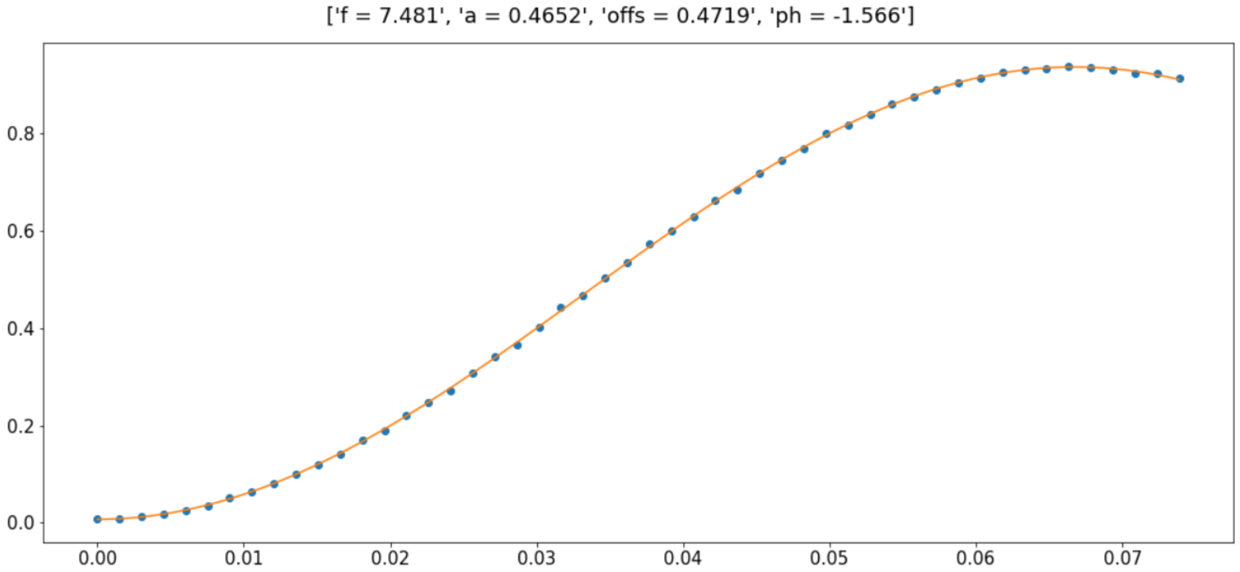
$$= 1/\sqrt{2}\,|+\rangle - \frac{i}{\sqrt{2}}|-\rangle \qquad (\sigma_z|+\rangle = |-\rangle)$$

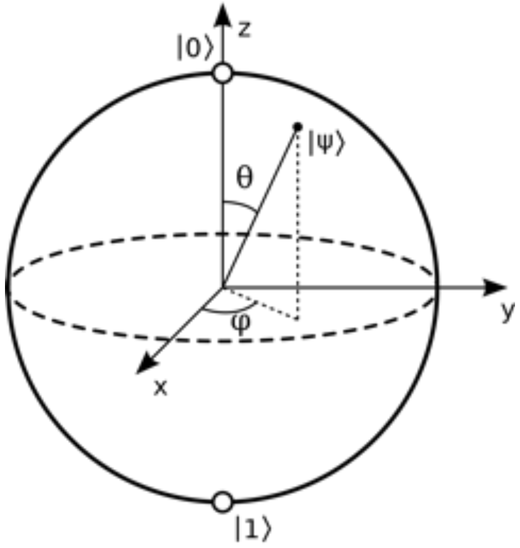
$$= 1/\sqrt{2}\;1/\sqrt{2}\left[|0\rangle+|1\rangle - i|0\rangle + i|1\rangle\right] = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$



Experimental Implementations

- All quantum gates are engineered using rotations on the Bloch sphere
- In our quantum hardware we can control each transition with some level of accuracy
- Example: Rabi oscillation:





Quantum Universality

Any quantum computation operation can be made by using a combination of:

{CNOT, H, S, T}

Quantum Universality

What is the S gate?

$$S = R_z(\pi/2) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$S^2 = Z$$

What is the T gate?

$$T = R_z(\pi/4) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$T^2 = S$$

$$T^4 = Z$$