Problem 5: Optional Challenge Problem

The Orbit Equation

This is an optional problem, and is quite mathematically intensive. We will explore a real world application of coordinate system transformation.

In many fields of physics, we are often interested in describing the motion of objects in space under the influence of gravity. For example, we may want to describe the motion of a planet around a star. This motion is described by the “orbit equation”.

a) First take a moment to think about which coordinate system (Cartesian or polar) might be better to describe this type of system. Explain the reasoning behind your choice.

The orbit equation in Cartesian coordinates can be written as:

$$x = [(1 + \frac{y^2}{x^2})^{\frac{1}{2}} + \varepsilon]^{-1} \tag{2.1}$$

This equation is quite cumbersome, it is long and x appears on both sides of the equations so it is not immediately apparent how x and y depend on each other. As a reminder, we can convert between Cartesian and polar coordinates with the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

b) Using the relationships between Cartesian and polar coordinates and trigonometric identities, show that the orbit equation (Eq(2.1)) can be written in polar coordinates as:

$$r = \frac{1}{1 + \varepsilon\cos\theta}$$

Notice how in this form of the equation the coordinate variables r and θ are separated, appearing only on one side of the equals sign. This allows us to much more easily infer how they relate to each other. Also, the presence of the $\cos\theta$ directly hints at periodicity in this equation, which makes sense for orbits. Try to plot the orbit equation, and see how the parameter ε, which is called the “orbit eccentricity”, changes the shape of an orbit.

Problem 5 Solution

a) While both coordinate systems can be used to describe orbital motion, it is more intuitive to use polar coordinates. Problems that involve circular symmetry or circular motion are often best described with polar coordinates. In this case, the circular symmetry comes from the gravitational force, which depends only on how far the object is from the source of gravity, and is the same strength in all directions.

b) Our starting equation:

$$x = [(1 + \frac{y^2}{x^2})^{\frac{1}{2}} + \varepsilon]^{-1}$$

Substitute the following:

$$x = r\cos\theta \qquad\qquad \frac{y^2}{x^2} = \tan^2\theta$$

$$r\cos\theta = [(1 + \tan^2\theta)^{\frac{1}{2}} + \varepsilon]^{-1}$$

Use identity: $1 + \tan^2\theta = \sec^2\theta$

$$\begin{aligned} r\cos\theta &= [(\sec^2\theta)^{\frac{1}{2}} + \varepsilon]^{-1} \\ r\cos\theta &= [\sec\theta + \varepsilon]^{-1} \\ r\cos\theta &= \frac{1}{\sec\theta + \varepsilon} \\ r &= \frac{1}{\cos\theta(\sec\theta + \varepsilon)} \\ r &= \frac{1}{1 + \varepsilon\cos\theta} \end{aligned}$$