Problem 1: Optional Challenge Problem

Bloch Sphere

So far we've talked about complex numbers and vectors. It turns out that in quantum mechanics, quantum states are described by vectors with complex components. The state of a qubit in a quantum computer is given by a two dimensional vector of the form

$$\vec{v} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

These qubit states live on the surface of the **Bloch Sphere**, where θ is the latitudinal angle (ranging from 0 to π) and ϕ is the azimuth angle (ranging from 0 to 2π).

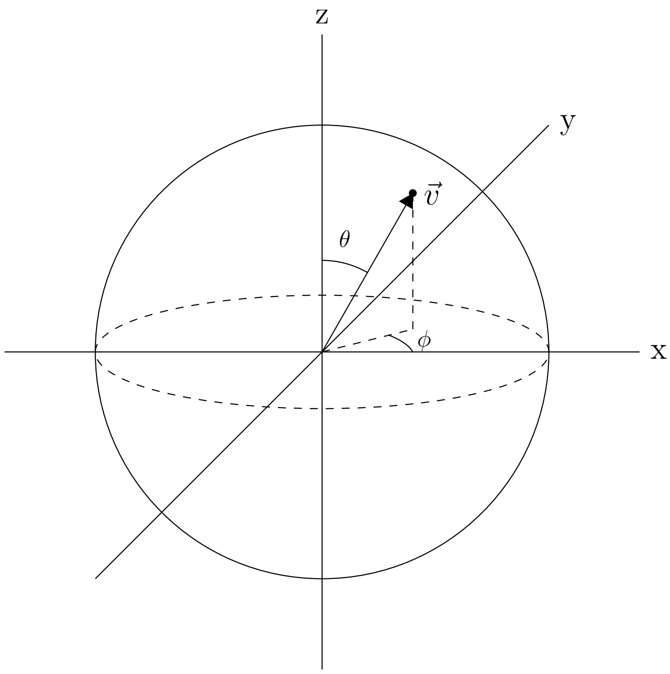


Figure 1: The Bloch sphere showing a general Bloch vector with coordinates described by θ and ϕ.

Understanding the Bloch sphere is at the heart of manipulating quantum states in quantum computing, so let's take some time to practice with Bloch vectors.

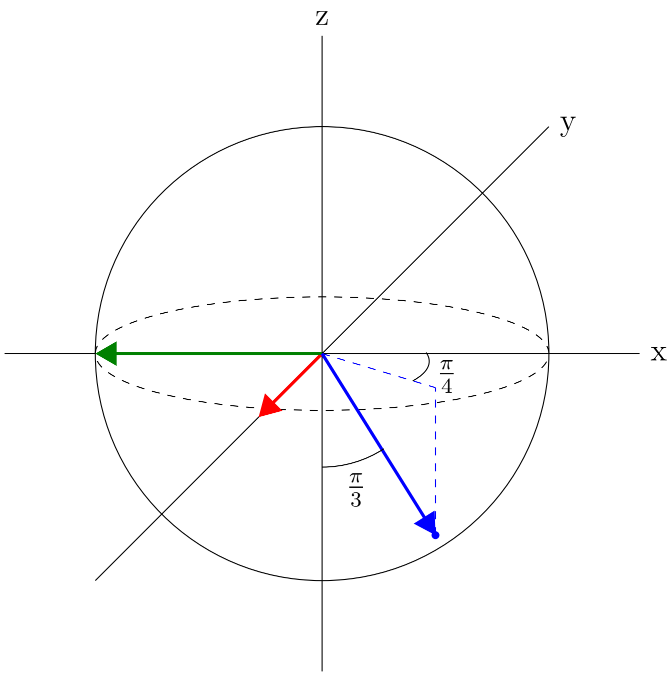
a) Where on the Bloch sphere do these vectors point?

i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

iii) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$

b) Write the vector form of the red, green and blue vectors on the following Bloch sphere.



Problem 2: Optional Challenge Problem

Note: This problem is very difficult! If you attempt this problem, we highly encourage working together with your classmates to solve it. Good luck!

Complex Impedance

Because Euler's formula links complex numbers so closely to the periodic functions sin and cos, we often use complex numbers to describe periodic processes. For example, this is done when analyzing voltages and currents in a circuit that uses alternating current (AC) sources.

In any electric circuit, there are three major quantities: Voltage (V), Current (I) and Resistance (R), which are related via *Ohm's law*:

$$\frac{V}{I} = R \tag{1}$$

Which tells us the relationship between an input voltage and a current response in a circuit. This equation works quite well when V, I and R do not change in time. However, certain components in a circuit may have very different responses to an AC source, which Ohm's law cannot easily address. To analyze these time-varying voltages and currents, we must use complex numbers.

An AC source is a voltage source which oscillates in time, and usually takes the form:

$$V(t) = V_0 \cos(\omega t + \phi_V)$$

V_0 the magnitude of the signal, ω is the *driving frequency*, and ϕ_V is the *phase*. We often represent this oscillating voltage using a *complex voltage*:

$$\tilde{V}(t) = V_0 e^{i(\omega t + \phi_V)}$$

Part (a): Show that the complex AC voltage and real AC voltage are related in the following way:

$$V(t) = \mathrm{Re}[\tilde{V}(t)]$$

(Note: $\mathrm{Re}[z]$ denotes the operation of taking the real part of a complex number z. So for $z = a + bi$, $\mathrm{Re}[z] = a$.)

Now suppose we have a circuit to which we apply an oscillating voltage source represented by our complex $\tilde{V}(t)$. In general we may get an oscillating current response in the circuit that can also be represented as a complex number:

$$\tilde{I}(t) = I_0 e^{i(\omega t + \phi_I)}$$

Where again, the real current measured is given by $I(t) = \mathrm{Re}\,\tilde{I}(t)$.

Part (b): Using the complex voltage $\tilde{V}(t)$ and complex current $\tilde{I}(t)$, find the *complex impedance* of a circuit Z, defined as:

$$Z = \frac{\tilde{V}(t)}{\tilde{I}(t)} \tag{2}$$

in terms of variables V_0, I_0, ϕ_V and ϕ_I.

Note how the complex impedance is very similar to Ohm's Law (Eq.1), except we have replaced the real voltage and current with their complex counterparts. Physically, Z represents something very similar to resistance, except it is a complex number, and takes into account how the components of the circuit react to time-varying voltages and currents.

Part (c):
Suppose we have a circuit with complex impedance given by:

$$Z = \frac{1 - \omega^2 LC}{i\omega C}$$

Where L and C are constants. We drive the circuit with a time-varying complex voltage:

$$\tilde{V}(t) = V_0 e^{i\omega t}$$

i) Using Eq.2, find the current response for the circuit $\tilde{I}(t)$ in terms of V_0, ω, L and C.

ii) Write $\tilde{I}(t)$ in the form $\tilde{I}(t) = I_0 e^{i(\omega t + \phi_I)}$. What is I_0 and ϕ_I in terms of V_0, ω, L and C?

iii) If we were to measure the current in this circuit, what would the actual current measured $I(t)$? (Recall that the actual current is the real part of the complex current)

Even more challenge: What happens to the measured current as the driving frequency ω approaches the value $\frac{1}{\sqrt{LC}}$? The behavior of the circuit as ω approaches this value is called *resonance.*

What's the point?
The idea of impedance and resonance is very important to quantum computing, especially in the application of superconducting qubits like the ones implemented by many industry quantum computing research groups. The quantum behavior of these qubits come from the superconducting material which exhibits very unique impedance characteristics at low temperatures. Resonance is also central to qubit implementation, as the superconducting qubit is actually a very special type of resonant circuit!