This is a supplemental sheet to help with derivations in **Week 5 Homework problem 6**, regarding quantum tunneling and geometric distributions.

Part E: In order to show that the geometric distribution is normalized we must verify

$$\sum_{n=1}^{\infty} P(n) = \sum_{n=1}^{\infty} p(1-p)^{n-1} = 1$$

First we look at the singular p term, since there is no n attached to it, we can pull it out of the sum.

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = p\sum_{n=1}^{\infty}(1-p)^{n-1} \tag{1}$$

$\sum_{n=1}^{\infty}(1-p)^{n-1}$ looks almost like the geometric series identity $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$. The differences are that this identity starts from $m=0$ whereas the sum we have starts from $n=1$ and the exponent in the identity is simply m, but in our sum it is $n-1$. They look very similar, but it isn't obvious that they are exactly the same so we should be careful in how we apply the identity.

A neat trick we can do here is to redefine our summation index. We can define

$$k = n-1$$

We can now rewrite our sum in terms of k instead of n. The first obvious place things change is in the exponent, $(1-p)^{n-1} \to (1-p)^k$. The next place to look for changes is in the limits of our sum. When $n=1$, $k=1-1=0$; and as $n\to\infty$, $k\to\infty$ as well. Applying these to our sum, we can rewrite it as:

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = p\sum_{k=0}^{\infty}(1-p)^k$$

We see now that our rewritten form of the sum is exactly identical to the geometric series identity $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$, where $r=1-p$. Applying the identity gives the result:

$$p\sum_{k=0}^{\infty}(1-p)^k = p\left(\frac{1}{1-(1-p)}\right) = \frac{p}{p} = 1$$

Here is a summary of the entire process:

$$\sum_{n=1}^{\infty} P(n) = \sum_{n=1}^{\infty} p(1-p)^{n-1} = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{k=0}^{\infty}(1-p)^k = p\left(\frac{1}{1-(1-p)}\right) = \frac{p}{p} = 1$$

Extra Challenge: This part is really tricky. The common way to solve this problem involves the use of calculus, but it is possible to solve without it. We will go through both solutions, starting with the method that does not use calculus.

Solution Without Calculus

Our starting equation is:

$$< n >= \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n \cdot p(1-p)^{n-1}$$

Again, we first take out the single p term, which leaves us with:

$$\sum_{n=1}^{\infty} nP(n) = p\sum_{n=1}^{\infty} n(1-p)^{n-1} \tag{2}$$

The key to solving this problem will be evaluating the summation $\sum_{n=1}^{\infty} n(1-p)^{n-1}$, so this is what we will focus on. First, we will write out the first few terms of the sum. The $n = 1$ term is $1(1-p)^0 = 1$, the $n = 2$ term is $2(1-p)$, etc..

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = [1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + ...]$$

Here comes the really tricky part. Notice that the sum when written out, is equivalent to this summation of other geometric sums:

$$[1+2(1-p)+3(1-p)^2+4(1-p)^3+...] = [\sum_{k=0}^{\infty}(1-p)^k+\sum_{k=1}^{\infty}(1-p)^k+\sum_{k=2}^{\infty}(1-p)^k+\sum_{k=3}^{\infty}(1-p)^k+...]$$

where the starting k value increases with each sum in the series.

Let's take some time to dissect this equation. First, we can explicitly write out the sum $\sum_{k=0}^{\infty}(1-p)^k$.

$$\sum_{k=0}^{\infty}(1-p)^k = 1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4...$$

If we think instead about $\sum_{k=1}^{\infty}(1-p)^k$, the sum is exactly the same, except we exclude the 1 that originally came from $k = 0$:

$$\sum_{k=1}^{\infty}(1-p)^k = (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4...$$

Doing this for a few of the terms reveals a pattern:

$$\sum_{k=0}^{\infty}(1-p)^k = 1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 ...$$
$$\sum_{k=1}^{\infty}(1-p)^k = (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 ...$$
$$\sum_{k=2}^{\infty}(1-p)^k = (1-p)^2 + (1-p)^3 + (1-p)^4 ...$$
$$\sum_{k=3}^{\infty}(1-p)^k = (1-p)^3 + (1-p)^4 ...$$

If we add these we only get 1 showing up once, since it is only present in $\sum_{k=0}^{\infty}(1-p)^k$. We only get $(1-p)$ twice, since it shows up in $\sum_{k=0}^{\infty}(1-p)^k$ and $\sum_{k=1}^{\infty}(1-p)^k$, but none of the others. This is how in the end we get

$$\sum_{k=0}^{\infty}(1-p)^k + \sum_{k=1}^{\infty}(1-p)^k + \sum_{k=2}^{\infty}(1-p)^k + \sum_{k=3}^{\infty}(1-p)^k + ... = 1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + ...$$

This is where we are in the analysis so far

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \sum_{k=0}^{\infty}(1-p)^k + \sum_{k=1}^{\infty}(1-p)^k + \sum_{k=2}^{\infty}(1-p)^k + \sum_{k=3}^{\infty}(1-p)^k + ... \tag{3}$$

We would like to use the geometric series equation $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$ to evaluate each term in the sum, but we must contend with the starting k value being different in each term. Let's analyze this type of sum and see how we can connect it to the geometric series. Consider a general term in the sum:

$$\sum_{k=a}^{\infty}(1-p)^k$$

where we start the sum at a. We can separate the sum in the following way:

$$\sum_{k=a}^{\infty}(1-p)^k = \sum_{k=a}^{\infty}(1-p)^a(1-p)^{k-a}$$

The $(1-p)^a$ term does not depend on the summation index k, so we can pull it out the sum:

$$\sum_{k=a}^{\infty}(1-p)^a(1-p)^{k-a} = (1-p)^a \sum_{k=a}^{\infty}(1-p)^{k-a}$$

The summation now looks very similar to what we dealt with in part (e) in Eq 1, where we had the special case of $a = 1$. So we will employ the same strategy to solve it; define

$$m = k - a$$

We can immediately replace the exponent with m, so $(1-p)^{k-a} \to (1-p)^m$. We then adjust the limits of the sum and get the result:

$$(1-p)^a \sum_{k=a}^{\infty} (1-p)^{k-a} = (1-p)^a \sum_{m=0}^{\infty} (1-p)^m$$

which we use the geometric series equation to evaluate.

$$\sum_{k=a}^{\infty} (1-p)^k = (1-p)^a \left(\frac{1}{1-(1-p)} \right) = \frac{(1-p)^a}{p} \tag{4}$$

We can use this result for each term in our sum from Eq 3, by evaluating for $a=0$, $a=1$, etc.

$$\begin{aligned}
\sum_{n=1}^{\infty} n(1-p)^{n-1} &= \sum_{k=0}^{\infty} (1-p)^k + \sum_{k=1}^{\infty} (1-p)^k + \sum_{k=2}^{\infty} (1-p)^k + \sum_{k=3}^{\infty} (1-p)^k + ... \\
&= \frac{1}{p} + \frac{(1-p)}{p} + \frac{(1-p)^2}{p} + \frac{(1-p)^3}{p} + ... \\
&= \frac{1}{p}[1 + (1-p) + (1-p)^2 + (1-p)^3 + ...]
\end{aligned}$$

From here we notice that our end result is actually a geometric series which we can evaluate

$$\begin{aligned}
\frac{1}{p}[1 + (1-p) + (1-p)^2 + (1-p)^3 + ...] &= \frac{1}{p} \sum_{m=0}^{\infty} (1-p)^m \\
&= \frac{1}{p} \left(\frac{1}{1-(1-p)} \right) \\
&= \frac{1}{p^2}
\end{aligned}$$

Finally we arrive at the result:

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{1}{p^2} \tag{5}$$

We can circle all the way back to Eq 2 to get our final result:

$$\begin{aligned}
< n > = \sum_{n=1}^{\infty} nP(n) = p \sum_{n=1}^{\infty} n(1-p)^{n-1} \\
< n > = p \left(\frac{1}{p^2} \right) \\
< n > = \frac{1}{p}
\end{aligned}$$

Solution Using Calculus

Again, the expression we want to analyze is

$$< n >= \sum_{n=1}^{\infty} nP(n) = p \sum_{n=1}^{\infty} n(1-p)^{n-1}$$

Here we will show how to arrive at the result shown in Eq 5 using calculus. The key to this method is the fact that the derivative operation is linear. This means that for two functions $f(x)$ and $g(x)$

$$\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)]$$

This idea can be applied to an entire sum of derivatives, and the derivative operator can be pulled out entirely. If we have N functions, each labeled by index k, $f_k(x)$, the following identity holds

$$\sum_{k=1}^{N} \frac{\mathrm{d}}{\mathrm{d}x} f_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{k=1}^{N} f_k(x)\right] \tag{6}$$

What this equation is effectively saying is that we can either take the derivative before doing the sum, or do the entire sum then take a derivative, the answer is the same.

Now back to the sum we are interested in, $\sum\limits_{n=1}^{\infty} n(1-p)^{n-1}$. We can write $n(1-p)^{n-1}$ using a derivative:

$$n(1-p)^{n-1} = -\frac{\mathrm{d}}{\mathrm{d}p}[(1-p)^n]$$

Plugging this directly into our sum gives

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \sum_{n=1}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}p}[(1-p)^n]$$

From Eq 6, we know we can pull the derivative to the front

$$\sum_{n=1}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}p}[(1-p)^n] = -\frac{\mathrm{d}}{\mathrm{d}p}\left[\sum_{n=1}^{\infty}(1-p)^n\right] \tag{7}$$

We can make the sum inside identical to a geometric sum using the equation:

$$\begin{aligned}\sum_{n=1}^{\infty}(1-p)^n &= -1 + \sum_{n=0}^{\infty}(1-p)^n \\ &= -1 + \frac{1}{p} = \frac{1-p}{p}\end{aligned}$$

Or equivalently using the identity we derived in Eq. 4 for $a = 1$. Using this identity we can simplify Eq 7:

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}p}\left[\sum_{n=1}^{\infty}(1-p)^n\right] &= -\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1-p}{p}\right] \\ &= \frac{1}{p^2}\end{aligned}$$

From here we arrive at the desired result:

$$< n >= p \sum_{n=1}^{\infty} n(1-p)^{n-1} = p\left(\frac{1}{p^2}\right)$$

$$< n > = \frac{1}{p}$$

Using derivatives in this way is a very powerful method to analyze sums in discrete probability theory.