ADDITIONAL PRACTICE 4

VECTORS AND MATRICES

This worksheet is meant to provide additional practice problems for the major concepts from lecture 4 and those in homework problems. In particular, it focuses on properties and operations with vectors and introduces matrices. This worksheet is not graded, but should help students get a solid foundation in the mathematics we will use throughout the course. The solutions to the additional practice problems can be found at the end of worksheet.

Problem 1: Inner Products

Compute the following inner products:

a) $< \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 16 \end{pmatrix} >$

b) $< \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} >$

c) $< \begin{pmatrix} -4 \\ -5 \end{pmatrix}, \begin{pmatrix} 6 \\ -3 \end{pmatrix} >$

d) $< \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} >$

e) $< \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ \frac{3}{2} \\ 3 \end{pmatrix} >$

f) $< \begin{pmatrix} \pi \\ 4 \end{pmatrix}, \begin{pmatrix} 12 \\ -7 \end{pmatrix} >$

g) $< \begin{pmatrix} i \\ 10 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} >$

h) $< \begin{pmatrix} i \\ -1 \\ -i \end{pmatrix}, \begin{pmatrix} i \\ 2 \\ i \end{pmatrix} >$

Problem 2: Vector Orthogonality

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $<\vec{a}, \vec{b}>= 0$.

State whether the following vectors are orthogonal:

a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

b) $\begin{pmatrix} 9 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

d) $\begin{pmatrix} 12 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

e) $\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

f) $\begin{pmatrix} 1 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

g) $\begin{pmatrix} \frac{3}{4} \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

h) $\begin{pmatrix} 6 \\ i \end{pmatrix}, \begin{pmatrix} \frac{1}{6} \\ i \end{pmatrix}$

i) Given the vectors: $\vec{a} = \begin{pmatrix} 1 \\ 12 \\ -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

   i) Verify that they orthogonal to each other.

   ii) Find x, y in a third vector $\vec{v} = \begin{pmatrix} x \\ 1 \\ y \end{pmatrix}$ such that $\vec{v}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

j) Given the vectors: $\vec{c} = \begin{pmatrix} 12 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{d} = \begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}$

   i) Verify that they orthogonal to each other.

   ii) Find x, y in a third vector $\vec{w} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ such that $\vec{w}$ is orthogonal to both $\vec{c}$ and $\vec{d}$.

Problem 3: Linear Combinations

Consider the vectors:

$$\vec{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

A linear combination of these vectors corresponds to doing the operation:

$$x\vec{a} + y\vec{b}$$

for any constants x and y.

Example: $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} -2 \\ -2 \end{pmatrix} = x\vec{a} + y\vec{b}$$
$$\begin{pmatrix} -2 \\ -2 \end{pmatrix} = x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2x \\ x \end{pmatrix} + \begin{pmatrix} -3y \\ 2y \end{pmatrix}$$

Which gives the system:

$$\begin{aligned} 2x + 3y &= -2 \\ 3 + 2y &= -2 \end{aligned}$$

Solving this gives: $x = 2, y = -2$. So $\begin{pmatrix} -2 \\ -2 \end{pmatrix} = 2\vec{a} - 2\vec{b}$

What are the linear combinations of $\vec{a}$ and $\vec{b}$ that form the following vectors:

a) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 12 \\ 7 \end{pmatrix}$

e) $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$

f) $\begin{pmatrix} 48 \\ 30 \end{pmatrix}$

g) $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$

h) $\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

i) $\begin{pmatrix} 18 \\ 11 \end{pmatrix}$

j) $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

k) $\begin{pmatrix} 2i \\ i \end{pmatrix}$

l) $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$

Problem 4: Matrix Operations

Evaluate the following expressions that involve matrices:

a) $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

d) $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

e) $\begin{pmatrix} 2 & -1 \\ i & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

f) $\begin{pmatrix} -1 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -12 & 11 \end{pmatrix}$

g) $\begin{pmatrix} 14 & -3 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 7 & -8 \\ 5 & 2 \end{pmatrix}$

h) $\begin{pmatrix} -i & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 12 & 0 \\ -1 & i \end{pmatrix}$

Solutions to Problem 1

a) 22

b) 0

c) -9

d) 3

e) 15

f) 12π -28

g) 3i

h) -2

Solutions to Problem 2

a) Yes

b) Yes

c) No

d) No

e) Yes

f) No

g) Yes

h) Yes

i) i) $<\vec{a},\vec{b}>= \begin{pmatrix}1 & 12 & -2\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = 2+0+-2=0$. Since the inner product of $\vec{a}$ and $\vec{b}$ is 0, then they must be orthogonal.

   ii) $x = \frac{-12}{5}$ and $y = \frac{24}{5}$

j) i) $<\vec{c},\vec{d}>= \begin{pmatrix}12 & 1 & 3\end{pmatrix}\begin{pmatrix}-1\\6\\2\end{pmatrix} = -12+6+6=0$. Since the inner product of $\vec{c}$ and $\vec{d}$ is 0, then they must be orthogonal.

   ii) $x = \frac{-16}{73}$ and $y = \frac{-27}{73}$

Solutions to Problem 3

a) $x = -1; y = 1$

b) $x = 0; y = 0$

c) $x = -5; y = 3$

d) $x = 3; y = 2$

e) $x = -2; y = 4$

f) $x = 6; y = 12$

g) $x = -1; y = -1$

h) $x = 0; y = 1/2$

i) $x = 3; y = 4$

j) $x = 2; y = 1$

k) $x = i; y = 0$

l) $x = -1; y = -2$

Solutions to Problem 4

a) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

d) $\begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$

e) $\begin{pmatrix} 1 \\ i+3 \end{pmatrix}$

f) $\begin{pmatrix} -50 & 43 \\ -66 & 69 \end{pmatrix}$

g) $\begin{pmatrix} 83 & -118 \\ 30 & 12 \end{pmatrix}$

h) $\begin{pmatrix} -12i & 0 \\ 10 & 2i \end{pmatrix}$