mathematics for quantum

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Dirac notation

It's also known as bra-ket notation

- It allows us to abstract away parts of the complicated underlying math for quantum mechanics!
- It uses the angle brackets "⟨" and "⟩" and a vertical bar "|", to construct bras and kets.

~A ket looks like $|v\rangle$ ~A bra looks like $\langle f|$

ket

- ket ($|\ \rangle$): can be represented with a column vector

Example: Bit to Qubit $0 \to |0\rangle$, $1 \to |1\rangle$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

column vectors

QUANTUM SUPERPOSITION

- Quantum object can be in two states at once

SUPERPOSITION: a qubit can be $|0\rangle$ and $|1\rangle$ at the same time

~This is how we show it. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix} + \begin{pmatrix}0\\\beta\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

Bra

- Bra ($\langle\ |$): can be represented with a row vector

-It is the complex-conjugate of the ket

$\langle 0| = (1\ \ 0)$

$\langle 1| = (0\ \ 1)$

Inner Product

- We use the inner product to find the overlap between two quantum states

Braket (bra + ket) $= \langle\psi|\varphi\rangle$

Example: $\langle 0|0\rangle = (1\ \ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1 \times 1 + 0 \times 0 = 1$

Inner product of superpositions: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Example: $\langle 0|\psi\rangle = \langle 0|\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha\langle 0|0\rangle + \beta\langle 0|1\rangle = \alpha + \beta\cdot 0 = \alpha$

TWO DEFINITIONS

- Two states $|\psi\rangle$ and $|\varphi\rangle$ are "orthogonal" if $\langle\psi|\varphi\rangle = 0$ ~perpendicular
- State $|\psi\rangle$ is "normal" if: $\langle\psi|\psi\rangle = 1$ ~ $|0\rangle$ and $|1\rangle$ are normal

MEASUREMENT

- Collapses the quantum state of the quibit to either 0 or 1

~collapses the quantum state of the quibit $|\psi\rangle$ to either $|0\rangle$ or $|1\rangle$

Probability of measuring $|0\rangle$: $|\alpha|^2 \rightarrow P(0) = |\langle 0|\psi\rangle|^2 = |\langle 0|(\alpha|0\rangle + \beta|1\rangle)|^2$

Probability of measuring $|1\rangle$: $|\beta|^2 \rightarrow P(1) = |\langle 1|\psi\rangle|^2 = |\langle 1|(\alpha|0\rangle + \beta|1\rangle)|^2$

QUANTUM OPERATION

Transforms a quantum state to another

With matrices, we can represent them

Example: quantum gates

─[X]─ ─[Z]─ ─[H]─

Important quantum operators

Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Pauli-X operator}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \text{Pauli-Y operator}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Pauli-Z operator}$$

Properties:

1. Linearility

$$\hat{A}(\alpha|0\rangle + \beta|1\rangle) = \alpha(\hat{A}|0\rangle) + \beta(\hat{A}|1\rangle)$$

2. Can be composited

$$\hat{A}(\hat{B}|\psi\rangle) = (\hat{A}\hat{B})|\psi\rangle$$

3. Orden Matters $\hat{A}\cdot\hat{B} \neq \hat{B}\cdot\hat{A}$

Conjugate Traspose

$$\vec{v}^{\dagger} = (\vec{v}^{T})^{*} = (\vec{v}^{*})^{T}$$

$$\sim (\hat{A}|\psi\rangle)^{\dagger} = \langle\psi|\hat{A}^{\dagger}$$

Scanned with CamScanner

Hermitian operators

- All observable operators are Hermitian.

  Ex: position, momentum, energy

- Hermitian: $A = A^\dagger$ operator is equal to its own conjugate traspose

  Ex:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x = \sigma_x^\dagger \Rightarrow \text{Hermitian}$$

Unitary operators

- All reversible quantum operations are unitary, → all quantum gates are unitary

  Ex: Time evolution

- Unitary: $A \cdot A^\dagger = A^\dagger \cdot A = 1$

  Ex:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$S \cdot S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$S^\dagger \cdot S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = I$$

∴ S is unitary