INTRO TO QUANTUM COMPUTING
LECTURE #8

MATHEMATICS FOR QUANTUM

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TODAY'S LECTURE

a) Dirac notation

b) Inner product

c) Quantum operations

   - Hermitian operations

   - Unitary operations

Dirac Notation

Or bra-ket notation allows us to abstract away parts of the complicated underlying math for quantum mechanics!

Bit to Qubit

Bit	Qubit
0	$\|0\rangle$
1	$\|1\rangle$

$|\cdot\rangle$ is a ket and it indicates that we're talking about a quantum state.

example: QxQ $\mapsto$ $|\ \text{QxQ}\ \rangle$

Ket

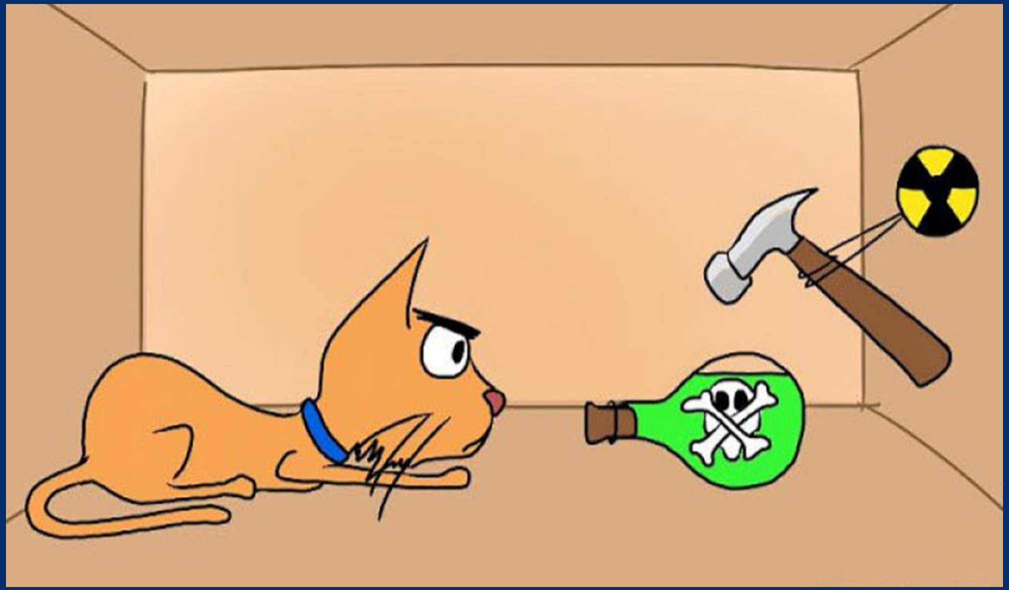
Ket ($|\ \rangle$): can be represented with a ***<u>column vector</u>***!

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{|1\rangle} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quantum Superposition

Quantum object can be in two states at once!



Superposition

Superposition: a qubit can be $|0\rangle$ and $|1\rangle$ at the same time!

This is how we show it: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

psi

$$|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix} + \begin{pmatrix}0\\\beta\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

Practice

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

Bra

Bra (⟨ |): can be represented with a ***row vector***!

the bra is the complex-conjugate of the ket!

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Practice

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \longrightarrow \langle\psi| = \frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}}\end{pmatrix} \longrightarrow \langle\psi| = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}^* = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|$$

Inner Product

We use the inner product to find the overlap between two quantum states

braket (bra+ket): $\langle\psi|\varphi\rangle$

Inner Product

- $\langle 0 | 0 \rangle$: $\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \times 1 + 0 \times 0 = 1$
- $\langle 1 | 0 \rangle$: $\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \times 0 + 0 \times 1 = 0$
- $\langle 0 | 1 \rangle$: $\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \times 0 + 0 \times 1 = 0$
- $\langle 1 | 1 \rangle$: $\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \times 0 + 1 \times 1 = 1$

Inner product of superpositions

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$x(y+z) = x \cdot y + x \cdot z$$

$$\langle 0|\psi\rangle = \langle 0|\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha\langle 0|0\rangle + \beta\langle 0|1\rangle$$

$$= \alpha + \beta \cdot 0 = \alpha$$

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

$$|\phi\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$

$$\langle\psi|\phi\rangle = \begin{pmatrix} \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \tfrac{1}{\sqrt{2}} \\ -\tfrac{1}{\sqrt{2}} \end{pmatrix} = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$\hookrightarrow \left(\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|\right) \cdot \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$\tfrac{1}{2}\overset{1}{\langle 0|0\rangle} + \tfrac{1}{2}\overset{0}{\langle 1|0\rangle} - \tfrac{1}{2}\overset{0}{\langle 0|1\rangle} - \tfrac{1}{2}\overset{1}{\langle 1|1\rangle}$$

$$= \tfrac{1}{2} + \cancel{\tfrac{1}{2} \times 0} - \cancel{\tfrac{1}{2} \times 0} - \tfrac{1}{2} = 0$$

Two definitions

Two states $|\psi\rangle$ and $|\varphi\rangle$ are ***"orthogonal"*** if: $\langle\psi|\varphi\rangle=0$ (orthogonal=perpendicular)

$|0\rangle$ and $|1\rangle$ are orthogonal $\langle 0|1\rangle = 0$

State $|\psi\rangle$ is ***"normal"*** if: $\langle\psi|\psi\rangle=1$

$|0\rangle$ and $|1\rangle$ are normal

$\langle 0|0\rangle = 1$

$\langle 1|1\rangle = 1$

$$|\psi\rangle = \tfrac{1}{\sqrt{3}}|0\rangle + \sqrt{\tfrac{2}{3}}|1\rangle$$

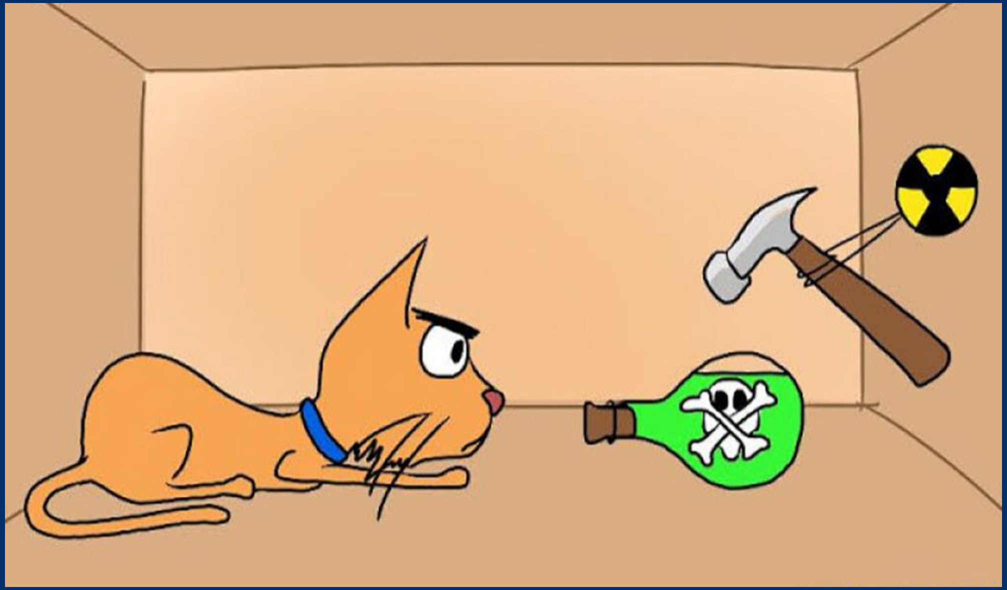
$$\langle\psi|\psi\rangle = \left(\tfrac{1}{\sqrt{3}}\langle 0| + \sqrt{\tfrac{2}{3}}\langle 1|\right)\left(\tfrac{1}{\sqrt{3}}|0\rangle + \sqrt{\tfrac{2}{3}}|1\rangle\right)$$

$$= \tfrac{1}{3}\underbrace{\langle 0|0\rangle}_{1} + \tfrac{\sqrt{2}}{3}\underbrace{\langle 0|1\rangle}_{0} + \tfrac{\sqrt{2}}{3}\underbrace{\langle 1|0\rangle}_{0} + \tfrac{2}{3}\underbrace{\langle 1|1\rangle}_{1}$$

$$= \tfrac{1}{3} + \tfrac{2}{3} = 1$$

Measurement

collapses the quantum state of the qubit to either 0 or 1



Measurement

Qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

measurement: collapses the quantum state of the qubit $|\psi\rangle$ to either $|0\rangle$ or $|1\rangle$

probability of measuring $|0\rangle$: $|\alpha|^2$

$P(0) = |\langle 0|\psi\rangle|^2 = |\langle 0|(\alpha|0\rangle + \beta|1\rangle)|^2$

$|\alpha|^2$

probability of measuring $|1\rangle$: $|\beta|^2$

$P(1) = |\langle 1|\psi\rangle|^2 = |\langle 1|(\alpha|0\rangle + \beta|1\rangle)|^2$

$= |\beta|^2$

$|\alpha|^2 + |\beta|^2 = 1$

10 MIN BREAK!

Quantum states:

Inputs and outputs of the quantum computer

Quantum operations:

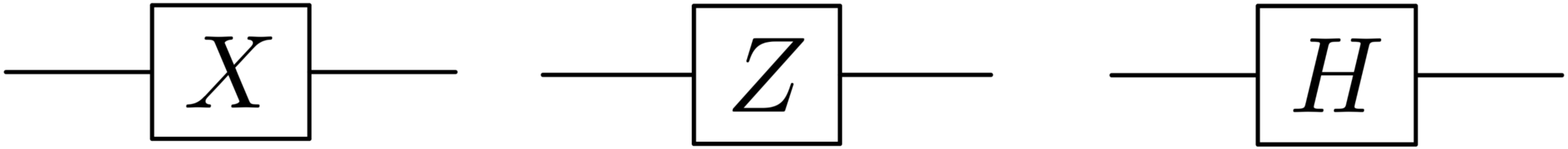
Perform computation in a quantum computer

Quantum Operation

Transforms a quantum state to another

We can represent them as ***matrices***

example: quantum gates!



Important Quantum Operators

Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ Pauli-X operator (X)

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$ Pauli-Y operator (Y)

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli-Z operator (Z)

Practice

Pauli operators:

$$\sigma_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$\sigma_x |1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$\sigma_z |0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$\sigma_z |1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

Quantum Operation

Quantum operation properties:

1. Linearity $\hat{A}(\alpha|0\rangle + \beta|1\rangle) = \alpha(\hat{A}|0\rangle) + \beta(\hat{A}|1\rangle)$

$$\sigma_x\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) = \tfrac{1}{\sqrt{2}}(\sigma_x|0\rangle) + \tfrac{1}{\sqrt{2}}(\sigma_x|1\rangle) = \tfrac{1}{\sqrt{2}}|1\rangle + \tfrac{1}{\sqrt{2}}|0\rangle$$

$$\sigma_z\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) = \tfrac{1}{\sqrt{2}}(\sigma_z|0\rangle) + \tfrac{1}{\sqrt{2}}(\sigma_z|1\rangle) = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$

Quantum operation properties:

2. Can be composited $\hat{A}(\hat{B}|\psi\rangle) = (\hat{A}\hat{B})|\psi\rangle$

$$\sigma_z \cdot (\sigma_x |0\rangle) = \sigma_z (|1\rangle) = -|1\rangle$$

$$(\sigma_z \cdot \sigma_x)|0\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

$-1 \times |1\rangle$

$1 \times |1\rangle$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Quantum Operation

Quantum operation properties:

normal algebra: $x \cdot y = y \cdot x$

3. Order matters $\hat{A} \cdot \hat{B} \neq \hat{B} \cdot \hat{A}$

$$\sigma_x \cdot (\sigma_z |0\rangle) = \sigma_x |0\rangle = |1\rangle$$

$$\neq$$

$$\sigma_z \cdot (\sigma_x |0\rangle) = \sigma_z |1\rangle = -1 \times |1\rangle$$

Conjugate Transpose

Conjugate Transpose:

$$\vec{v}^{\dagger} = (\vec{v}^{T})^{*} = (\vec{v}^{*})^{T}$$

$$\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \hat{A}^{\dagger} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{*} = \begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix}$$

$$(\hat{A}\hat{B})^{\dagger} = \hat{B}^{\dagger} \cdot \hat{A}^{\dagger}$$ (switches order)

$$(\hat{A}|\psi\rangle)^{\dagger} = \langle\psi|\hat{A}^{\dagger}$$

Hermitian Operators

All observable operators are Hermitian!!

example:

- Position
- Momentum
- Energy

$p = m \cdot \vec{v}$

mass

velocity

Hermitian Operators

All observable operators are Hermitian

Hermitian: $A=A^{\dagger}$ operator is equal to its own conjugate transpose

example:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x^{\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x = \sigma_x^{\dagger} \implies \text{Hermitian}$$

Reversibility

Given the output of a gate, we can determine what the inputs are.

- **Reversible gate:** preserves all the information
- **Non-reversible gate:** loses some information

Unitary Operators

All reversible quantum operations are unitary!

example:

- Time evolution

Unitary Operators

All reversible operations are unitary → all quantum gates are unitary

Unitary: A.A†=A†.A= I

$A^\dagger = A^{-1}$

example:

$S = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$

$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$i \times i = -1$

$S . S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$S^\dagger . S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\Big\} \rightarrow$ S is unitary

ANNOUNCEMENTS