HOMEWORK 9

VECTOR SPACES AND EIGENVALUES

Considera vector space V which contains real 3-component vectors $|a\rangle$, $|b\rangle$ and $|c\rangle$. For **Questions 1-4**, state whether the following would also be contained in V. (Answer True if it is contained in V or False if it is not.)

1. $|w\rangle = 2\,|a\rangle$

   a) True

   b) False

2. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

   a) True

   b) False

3. Inner product: $\langle a|b\rangle$

   a) True

   b) False

4. $|v\rangle = \alpha\,|a\rangle + \beta\,|b\rangle + \gamma\,|c\rangle$, for any real numbers α, β and γ

   a) True

   b) False

5. Given a set of 3 vectors $\{|x\rangle\,, |y\rangle\,, |z\rangle\}$ and coefficients α, β and γ, the vectors are **linearly independent** if:

   a) $\alpha\,|x\rangle + \beta\,|y\rangle + \gamma\,|z\rangle = 0$ has exactly one **non-zero** soluton.

   b) $\alpha\,|x\rangle + \beta\,|y\rangle + \gamma\,|z\rangle = 0$ is only solved by $\alpha = \beta = \gamma = 0$

   c) $\alpha\,|x\rangle + \beta\,|y\rangle + \gamma\,|z\rangle = 0$ has infinitely many solutions

6. For a set of vectors $\{|x\rangle, |y\rangle, |z\rangle\}$, what is meant by **span**$(\{|x\rangle, |y\rangle, |z\rangle\})$?

   a) The set of all vectors that **can** be constructed using linear combinations of $\{|x\rangle, |y\rangle, |z\rangle\}$.
   b) The set of all vectors that **cannot** be constructed using linear combinations of $\{|x\rangle, |y\rangle, |z\rangle\}$.
   c) The set of all vectors that **are** orthogonal to $\{|x\rangle, |y\rangle, |z\rangle\}$.
   d) The set of all vectors that **are not** orthogonal to $\{|x\rangle, |y\rangle, |z\rangle\}$.

7. True or False: The span of a set of vectors is a vector space.

   a) True
   b) False

8. Which of the following statements best describes the link between the **span of a set of vectors** and the **dimension of the vector space** it generates.

   a) The span of **any 3 vectors** is a vector space with dimension of 3.
   b) The span of **3 linearly independent vectors** is a vector space with dimension of 3.
   c) The span of **3-component vectors** is a vector space with dimension of 3.
   d) Only the span of **exactly 3 linearly independent 3-component vectors** can be a vector space with dimension of 3.

For **Questions 9-11**, state whether the following vectors are linearly independent.

9. $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$

   a) Linearly Independent
   b) Linearly Dependent

10. $\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}\}$

   a) Linearly Independent
   b) Linearly Dependent

11. $\{\begin{pmatrix}3\\-4\end{pmatrix}, \begin{pmatrix}7\\1\end{pmatrix}\}$

   a) Linearly Independent
   b) Linearly Dependent

For **Questions 12-14**, state whether the following sets of vectors span $\mathbb{R}^2$. Answer True if the span of the set is $\mathbb{R}^2$ or False if it is not.

12. $\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\}$

    a) True

    b) False

13. $\{\begin{pmatrix} 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\}$

    a) True

    b) False

14. $\{\begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$

    a) True

    b) False

Consider an operator represented by a square matrix A. The eigenvalue equation for A is:

$$A\left|v\right\rangle = \lambda\left|v\right\rangle$$

where λ is the eigenvalue and $\left|v\right\rangle$ is the eigenvector.

Eigenvalues and eigenvectors are like the fingerprint of a matrix. It is possible to uniquely identify a matrix using only its eigenvalues and eigenvectors. In quantum mechanics, the eigenvalues and eigenvectors play an important role in **measurement**. The eigenvectors, called **eigenstates**, gives the possible outcomes of any given measurement.

For **Questions 15-17**, verify that the given matrix and vector follow the eigenvalue equation. Then state the eigenvalue λ.

15. $\begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    a) $\lambda = 1$
    b) $\lambda = 5$
    c) $\lambda = -1$
    d) $\lambda = 4$

16. $\begin{pmatrix} 7 & 0 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
    a) $\lambda = 7$
    b) $\lambda = 10$
    c) $\lambda = 6$
    d) $\lambda = 2$

17. $\begin{pmatrix} 7 & -1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$
    a) $\lambda = 1$
    b) $\lambda = \frac{3}{2}$
    c) $\lambda = -3$
    d) $\lambda = 5$