HOMEWORK 5

DISCRETE PROBABILITY

As we will learn quite soon, quantum states are closely linked to probability distributions that describe the outcome of measurements on the state. In order to have a good intuition for how probability fits into quantum mechanics, it is important to develop an understanding of the laws of probability.

Problem 1: Coins and Dice

Suppose you have a fair 6-sided die ($\frac{1}{6}$ probability of rolling each number) and a fair coin ($\frac{1}{2}$ probability of flipping heads or tails).

Calculate the probability of each of the following events (note (c)-(d) involve using both the die and the coin)

a) Flipping heads twice in a row

b) Rolling 1 twice in a row

c) Flipping tails and rolling a 3

d) Flipping heads and rolling an even number

e) Flipping tails and rolling a 1 or 2

Problem 1 Solutions

a) $\frac{1}{4}$

b) $\frac{1}{36}$

c) $\frac{1}{12}$

d) $\frac{1}{4}$

e) $\frac{1}{6}$

Problem 2: Random Variables

Let X and Y be independent discrete random variables that can be one of the following values: $\{2, 5, 6, 8\}$. However, the probability for X and Y to take each value is different. The following table shows the corresponding probability distributions for X and Y

v	P(X=v)	P(Y=v)
2	0.1	0.4
5	0.2	0.15
6	0.2	0.35
8	0.5	0.1

a) What is the expectation of X?

b) What is the expectation of Y?

c) What is the variance of X?

d) What is the variance of Y?

e) Evaluate the following probabilities:

i) P(X=2 and Y=2)

ii) P(X=5 and Y=6)

iii) P(X=8 and Y=5)

Problem 2 Solutions

a) $E[X] = 6.4$

b) $E[Y] = 4.5$

c) $\text{var}[X] = 3.64$

d) $\text{var}[Y] = 4.55$

e) i) 0.04

ii) 0.07

iii) 0.075

Problem 3: Dirac Notation

Consider the following quantum states:

$$|\alpha\rangle = a\begin{pmatrix}1\\0\end{pmatrix} \qquad |\beta\rangle = b\begin{pmatrix}1\\1\end{pmatrix} \qquad |\gamma\rangle = c\begin{pmatrix}5\\3e^{i\frac{\pi}{4}}\end{pmatrix}$$

a) Write the following in vector form:

i) $\langle\alpha|$

ii) $\langle\beta|$

iii) $\langle\gamma|$

b) One of the most important properties of a quantum state is that it is normalized. This means that for any quantum state $|\psi\rangle$, the following must be satisfied:

$$\langle\psi|\psi\rangle = 1$$

i) Using $\langle\alpha|\alpha\rangle = 1$, determine the value of a.

ii) Using $\langle\beta|\beta\rangle = 1$, determine the value of b.

iii) Using $\langle\gamma|\gamma\rangle = 1$, determine the value of c.

c) Calculate the following inner products:

i) $\langle\alpha|\gamma\rangle$

ii) $\langle\gamma|\beta\rangle$

iii) $\langle\beta|\gamma\rangle$

Problem 3 Solutions

Part (a):

i) $\langle\alpha| = a(1, 0)$

ii) $\langle\beta| = b(1, 1)$

iii) $\langle\gamma| = c(5, 3e^{-i\frac{\pi}{4}})$

Part (b):

i) $a = 1$

ii) $b = \frac{1}{\sqrt{2}}$

iii) $c = \frac{1}{\sqrt{34}}$

Part (c):

i) $\langle\alpha|\gamma\rangle = 5$

ii) $\langle\gamma|\beta\rangle = 5 + 3e^{-i\frac{\pi}{4}}$

iii) $\langle\beta|\gamma\rangle = 5 + 3e^{i\frac{\pi}{4}}$

Problem 4: Optional Problem

Error Mitigation Code

An automated irrigation system uses a complex algorithm that takes in environmental data (temperature, humidity, sunlight, etc.) and uses it to determine whether the system should be off (output bit 0) or on (output bit 1). However, the system is quite faulty, and is prone to errors. 10% of the time it encounters a bit flip error, where the output bit shows 0 when it should be 1 or vice versa.

a) If the algorithm is executed once a day, what is the expected value for the number of errors in the system over the course of one year?

To counter these errors, engineers have implemented a 3 bit error code in the system. The code works by encoding one logical bit using 3 physical bits. This means that a logical 0, written as $\bar{0}$, is encoded as 000. **Any combination of physical bits that contain 2 or more 0's, is still recognized as** $\bar{0}$. Similarly, a logical 1, $\bar{1}$, is encoded as 111, and **any string with two or more 1's is interpreted as** $\bar{1}$. The irrigation system is now only on if it reads $\bar{1}$ and off if it reads $\bar{0}$.

b) For each of the possible combinations of 3 physical bits, state whether they would be interpreted as $\bar{0}$ or $\bar{1}$:

Physical Bits	Logical Bit
000	$\bar{0}$
001	$\bar{0}$
010	
011	
100	
101	
110	$\bar{1}$
111	$\bar{1}$

c) How many simultaneous bit flip errors must occur in the system to cause $\bar{0}$ (initially 000) to be incorrectly interpreted as $\bar{1}$?

d) Given the 10% error rate for the physical bits, what is the probability of encountering an error where $\bar{0}$ is flipped to $\bar{1}$?

Problem 4 Solutions

a) 36.5

b)

Physical Bits	Logical Bit
000	$\bar{0}$
001	$\bar{0}$
010	$\bar{0}$
011	$\bar{1}$
100	$\bar{0}$
101	$\bar{1}$
110	$\bar{1}$
111	$\bar{1}$

c) 2 bit flips, for example $000 \to 011$ would have the effect of $\bar{0} \to \bar{1}$

d) The probability of two simultaneous bit flips is $\frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100} = 1\%$

Problem 5: Optional Challenge Problem

Quantum Tunneling

One of the strangest phenomena in quantum mechanics is quantum tunneling. Quantum tunneling is also a feature of the wave-like nature of quantum objects. It is the idea that there is sometimes a small but significant probability that quantum particles may overcome energy barriers.

Consider a quantum particle confined to a box that has a very large width L. The particle moves between the walls of the box, but each time it makes contact with the walls, there is a small probability p that the particle will tunnel through the wall. Fig.5.1 shows a sketch of the setup.

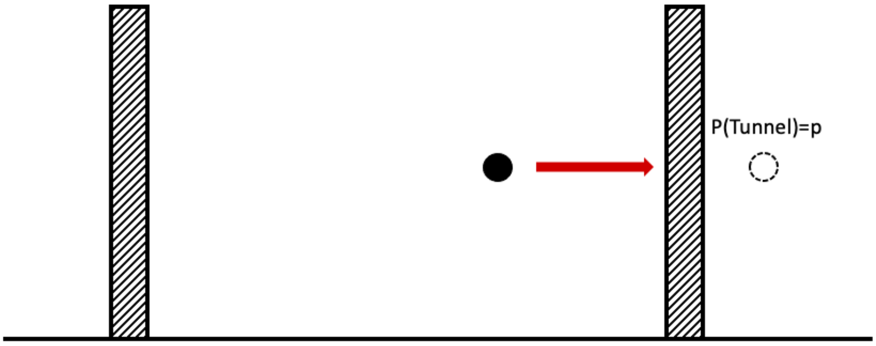

Figure 5.1: Sketch of a particle trapped in a box. The probability of the particle tunneling through the walls of the box is p

a) What is the probability that the particle tunnels through the wall on the 1st collision?

b) What is the probability that the particle does not tunnel through the wall after the 1st collision?

c) What is the probability that the particle tunnels through the wall on the 2nd collision?

d) What is the probability distribution $P(n)$, that describes the probability of the particle tunneling through the wall on the nth collision? (Note: This probability distribution is known as a *geometric distribution*.)

e) Show that this probability distribution is normalized by verifying that:

$$\sum_{n=1}^{\infty} P(n) = 1$$

The following identity for a geometric series may be useful: $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$

The expected number of collisions before the particle escapes $< n >$ is given by the expected value of our probability distribution:

$$< n >= \sum_{n=1}^{\infty} nP(n)$$

It can be shown that for this type of distribution $< n >$ takes the following form:

$$< n >= \frac{1}{p} \tag{5.1}$$

f) If there is a 0.1% probability of the particle tunneling in each collision, what will be the average number of collisions before it escapes?

Extra Challenge: Try to derive Eq 5.1. **Warning**: this is very difficult to do. We encourage students who are interested in the derivation to try on their own, then use the provided walk-through of the derivation if necessary.

Problem 5 Solutions

Part A: The probability to go through the wall is p.

Part B: Since there is probability p to go through the wall, the probability that the particle does not tunnel through is $1 - p$

Part C: In order for the particle to tunnel through after exactly 2 collisions, it must not tunnel through the wall on the first collision, then tunnel through on the second collision: $P(2) = p(1 - p)$

Part D: Extending our thinking from part (c), for the particle to pass through on the nth collision, it must have failed to tunnel through in the previous $n - 1$ collisions:

$$P(n) = p(1 - p)^{n-1}$$

This is the geometric distribution.

Part E: In order to show that the geometric distribution is normalized we must verify

$$\sum_{n=1}^{\infty} P(n) = \sum_{n=1}^{\infty} p(1 - p)^{n-1} = 1$$

First we look at the singular p term, since there is no n attached to it, we can pull it out of the sum.

$$\sum_{n=1}^{\infty} p(1 - p)^{n-1} = p \sum_{n=1}^{\infty} (1 - p)^{n-1} \tag{5.2}$$

$\sum\limits_{n=1}^{\infty} (1 - p)^{n-1}$ looks almost like the geometric series identity $\sum\limits_{m=0}^{\infty} r^m = \frac{1}{1-r}$. The differences are that this identity starts from $m = 0$ whereas the sum we have starts from $n = 1$ and the exponent in the identity is simply m, but in our sum it is $n - 1$. They look very similar, but it isn't obvious that they are exactly the same so we should be careful in how we apply the identity.

A neat trick we can do here is to redefine our summation index. We can define

$$k = n - 1$$

We can now rewrite our sum in terms of k instead of n. The first obvious place things change is in the exponent, $(1 - p)^{n-1} \to (1 - p)^k$. The next place to look for changes is in the limits of our sum. When $n = 1$, $k = 1 - 1 = 0$; and as $n \to \infty$, $k \to \infty$ as well. Applying these to our sum, we can rewrite it as:

$$\sum_{n=1}^{\infty} p(1 - p)^{n-1} = p \sum_{k=0}^{\infty} (1 - p)^k$$

We see now that our rewritten form of the sum is exactly identical to the geometric series identity $\sum\limits_{m=0}^{\infty} r^m = \frac{1}{1-r}$, where $r = 1 - p$. Applying the identity gives the result:

$$p \sum_{k=0}^{\infty} (1 - p)^k = p \left(\frac{1}{1 - (1 - p)} \right) = \frac{p}{p} = 1$$

Here is a summary of the entire process:

$$\sum_{n=1}^{\infty} P(n) = \sum_{n=1}^{\infty} p(1-p)^{n-1} = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{k=0}^{\infty}(1-p)^{k} = p\left(\frac{1}{1-(1-p)}\right) = \frac{p}{p} = 1$$

Part F: Using Eq 5.1 we can quickly see for $p = 0.1\% = \frac{1}{1000}$ that $< n >= 1000$ collisions.

Extra Challenge: This part is really tricky. The common way to solve this problem involves the use of calculus, but it is possible to solve without it. We will go through both solutions, starting with the method that does not use calculus.

Solution Without Calculus

Our starting equation is:

$$< n >= \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n \cdot p(1-p)^{n-1}$$

Again, we first take out the single p term, which leaves us with:

$$\sum_{n=1}^{\infty} nP(n) = p\sum_{n=1}^{\infty} n(1-p)^{n-1} \tag{5.3}$$

The key to solving this problem will be evaluating the summation $\sum_{n=1}^{\infty} n(1-p)^{n-1}$, so this is what we will focus on. First, we will write out the first few terms of the sum. The $n = 1$ term is $1(1-p)^0 = 1$, the $n = 2$ term is $2(1-p)$, etc..

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = [1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + ...]$$

Here comes the really tricky part. Notice that the sum when written out, is equivalent to this summation of other geometric sums:

$$[1+2(1-p)+3(1-p)^2+4(1-p)^3+...] = [\sum_{k=0}^{\infty}(1-p)^k+\sum_{k=1}^{\infty}(1-p)^k+\sum_{k=2}^{\infty}(1-p)^k+\sum_{k=3}^{\infty}(1-p)^k+...]$$

where the starting k value increases with each sum in the series.

Let's take some time to dissect this equation. First, we can explicitly write out the sum $\sum_{k=0}^{\infty}(1-p)^k$.

$$\sum_{k=0}^{\infty}(1-p)^k = 1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4...$$

If we think instead about $\sum_{k=1}^{\infty}(1-p)^k$, the sum is exactly the same, except we exclude the 1 that originally came from $k = 0$:

$$\sum_{k=1}^{\infty}(1-p)^k = (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4...$$

Doing this for a few of the terms reveals a pattern:

$$\sum_{k=0}^{\infty}(1-p)^k = 1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 ...$$
$$\sum_{k=1}^{\infty}(1-p)^k = (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 ...$$
$$\sum_{k=2}^{\infty}(1-p)^k = (1-p)^2 + (1-p)^3 + (1-p)^4 ...$$
$$\sum_{k=3}^{\infty}(1-p)^k = (1-p)^3 + (1-p)^4 ...$$

If we add these we only get 1 showing up once, since it is only present in $\sum_{k=0}^{\infty}(1-p)^k$. We only get $(1-p)$ twice, since it shows up in $\sum_{k=0}^{\infty}(1-p)^k$ and $\sum_{k=1}^{\infty}(1-p)^k$, but none of the others. This is how in the end we get

$$\sum_{k=0}^{\infty}(1-p)^k + \sum_{k=1}^{\infty}(1-p)^k + \sum_{k=2}^{\infty}(1-p)^k + \sum_{k=3}^{\infty}(1-p)^k + ... = 1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + ...$$

This is where we are in the analysis so far

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \sum_{k=0}^{\infty}(1-p)^k + \sum_{k=1}^{\infty}(1-p)^k + \sum_{k=2}^{\infty}(1-p)^k + \sum_{k=3}^{\infty}(1-p)^k + ... \tag{5.4}$$

We would like to use the geometric series equation $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$ to evaluate each term in the sum, but we must contend with the starting k value being different in each term. Let's analyze this type of sum and see how we can connect it to the geometric series. Consider a general term in the sum:

$$\sum_{k=a}^{\infty}(1-p)^k$$

where we start the sum at a. We can separate the sum in the following way:

$$\sum_{k=a}^{\infty}(1-p)^k = \sum_{k=a}^{\infty}(1-p)^a(1-p)^{k-a}$$

The $(1-p)^a$ term does not depend on the summation index k, so we can pull it out the sum:

$$\sum_{k=a}^{\infty}(1-p)^a(1-p)^{k-a} = (1-p)^a \sum_{k=a}^{\infty}(1-p)^{k-a}$$

The summation now looks very similar to what we dealt with in part (e) in Eq 5.2, where we had the special case of $a = 1$. So we will employ the same strategy to solve it; define

$$m = k - a$$

We can immediately replace the exponent with m, so $(1-p)^{k-a} \rightarrow (1-p)^m$. We then adjust the limits of the sum and get the result:

$$(1-p)^a \sum_{k=a}^{\infty}(1-p)^{k-a} = (1-p)^a \sum_{m=0}^{\infty}(1-p)^m$$

which we use the geometric series equation to evaluate.

$$\sum_{k=a}^{\infty}(1-p)^k = (1-p)^a \left(\frac{1}{1-(1-p)}\right) = \frac{(1-p)^a}{p} \tag{5.5}$$

We can use this result for each term in our sum from Eq 5.4, by evaluating for $a = 0$, $a = 1$, etc.

$$\begin{aligned}
\sum_{n=1}^{\infty} n(1-p)^{n-1} &= \sum_{k=0}^{\infty}(1-p)^k + \sum_{k=1}^{\infty}(1-p)^k + \sum_{k=2}^{\infty}(1-p)^k + \sum_{k=3}^{\infty}(1-p)^k + ... \\
&= \frac{1}{p} + \frac{(1-p)}{p} + \frac{(1-p)^2}{p} + \frac{(1-p)^3}{p} + ... \\
&= \frac{1}{p}[1 + (1-p) + (1-p)^2 + (1-p)^3 + ...]
\end{aligned}$$

From here we notice that our end result is actually a geometric series which we can evaluate

$$\begin{aligned}
\frac{1}{p}[1 + (1-p) + (1-p)^2 + (1-p)^3 + ...] &= \frac{1}{p}\sum_{m=0}^{\infty}(1-p)^m \\
&= \frac{1}{p}\left(\frac{1}{1-(1-p)}\right) \\
&= \frac{1}{p^2}
\end{aligned}$$

Finally we arrive at the result:

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{1}{p^2} \tag{5.6}$$

We can circle all the way back to Eq 5.3 to get our final result:

$$\begin{aligned}
< n > = \sum_{n=1}^{\infty} nP(n) &= p\sum_{n=1}^{\infty} n(1-p)^{n-1} \\
< n > &= p\left(\frac{1}{p^2}\right) \\
< n > &= \frac{1}{p}
\end{aligned}$$

Solution Using Calculus

Again, the expression we want to analyze is

$$< n >= \sum_{n=1}^{\infty} nP(n) = p \sum_{n=1}^{\infty} n(1-p)^{n-1}$$

Here we will show how to arrive at the result shown in Eq 5.6 using calculus. The key to this method is the fact that the derivative operation is linear. This means that for two functions $f(x)$ and $g(x)$

$$\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)]$$

This idea can be applied to an entire sum of derivatives, and the derivative operator can be pulled out entirely. If we have N functions, each labeled by index k, $f_k(x)$, the following identity holds

$$\sum_{k=1}^{N} \frac{\mathrm{d}}{\mathrm{d}x} f_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{k=1}^{N} f_k(x)\right] \tag{5.7}$$

What this equation is effectively saying is that we can either take the derivative before doing the sum, or do the entire sum then take a derivative, the answer is the same.

Now back to the sum we are interested in, $\sum_{n=1}^{\infty} n(1-p)^{n-1}$. We can write $n(1-p)^{n-1}$ using a derivative:

$$n(1-p)^{n-1} = -\frac{\mathrm{d}}{\mathrm{d}p}[(1-p)^n]$$

Plugging this directly into our sum gives

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \sum_{n=1}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}p}[(1-p)^n]$$

From Eq 5.7, we know we can pull the derivative to the front

$$\sum_{n=1}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}p}[(1-p)^n] = -\frac{\mathrm{d}}{\mathrm{d}p}\left[\sum_{n=1}^{\infty}(1-p)^n\right] \tag{5.8}$$

We can make the sum inside identical to a geometric sum using the equation:

$$\begin{aligned}\sum_{n=1}^{\infty}(1-p)^n &= -1 + \sum_{n=0}^{\infty}(1-p)^n \\ &= -1 + \frac{1}{p} = \frac{1-p}{p}\end{aligned}$$

Or equivalently using the identity we derived in Eq. 5.5 for $a = 1$. Using this identity we can simplify Eq 5.8:

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}p}\left[\sum_{n=1}^{\infty}(1-p)^n\right] &= -\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1-p}{p}\right] \\ &= \frac{1}{p^2}\end{aligned}$$

From here we arrive at the desired result:

$$< n >= p \sum_{n=1}^{\infty} n(1-p)^{n-1} = p \left(\frac{1}{p^2} \right)$$

$$< n > = \frac{1}{p}$$

Using derivatives in this way is a very powerful method to analyze sums in discrete probability theory.