ADDITIONAL PRACTICE 3

VECTORS AND COMPLEX NUMBERS

This worksheet is meant to provide additional practice problems for the major concepts from lecture 3 and those in homework problems. It is not graded, but should help students get a solid foundation in the mathematics we will use throughout the course. The solutions to the additional practice problems can be found at the end of worksheet.

Problem 1: Vector Operations

Calculate the resulting vector for each of the following operations.

a) $\vec{a} = 3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

b) $\vec{b} = -\frac{1}{2} \cdot \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

c) $\vec{c} = \begin{pmatrix} 6 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

d) $\vec{d} = -1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

e) $\vec{e} = 3 \cdot \begin{pmatrix} 5 \\ 4 \end{pmatrix} - 2 * \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

f) $\vec{f} = -3 \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

g) $\vec{g} = -1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} - 3 \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

h) $\vec{h} = -2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i) $\vec{i} = -\frac{3}{4} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{7}{2} \cdot \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Problem 2: Vector Magnitude and Direction

Calculate the magnitude and direction of the following vectors. The direction of the vector should be written as an angle in radians.

a) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

b) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 2 \\ -7 \end{pmatrix}$

e) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

f) $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$

g) $\begin{pmatrix} -9 \\ -9 \end{pmatrix}$

h) $\begin{pmatrix} (\frac{1}{3})^2 \\ 13 \end{pmatrix}$

i) $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$

j) $\begin{pmatrix} \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} \end{pmatrix}$

Problem 3: Trajectories

a) A plane is mid-flight, and is supposed to be flying directly east from Chicago to Boston. However, there is a north-south wind blowing at 50 mph, so the plane's actual and expected path differ. While the expected velocity of the plane is 600 mph moving east, both the speed and direction of the plane are altered by the wind. What is the actual velocity of the plane when wind is taken into account? Give your answer as a vector with magnitude (in mph) and direction (angle θ in radians).

b) A frog is taking a leisurely swim down a stream traveling north to south. Unfortunately, the stream has a direct east to west current. If the frog is swimming 3 cm/s, and the stream's current is 2 cm/s, what is the actual velocity and direction of the frog? Give your answer as a vector with magnitude (in cm/s) and direction (angle θ in radians). If he is 20 cm from the shore, how long will it take until the current pushes him out of the river?

Problem 4: Adding and Multiplying Complex Numbers

a) Add the following complex numbers:

i) $(3+i)+(-1+i)$

ii) $(2-i)+(-2+i)$

iii) $(15+2i)+(-6-11i)$

iv) $(24+12i)+-3\cdot(1+i)$

v) $(15+2i)+(-6-11i)+(13+4i)$

vi) $\frac{1}{2}\cdot(2-2i)+(\frac{i^2}{3})$

vii) $(15+2i)+(-6-11i)+(13+4i)$

viii) $(-1+3i)+(\frac{1}{2}-\frac{7i}{2})$

ix) $(3+i)+(2-12i)$

x) $(16+7i)-(-3-31i)$

b) Multiply the following complex numbers:

i) $(1+i)\cdot(1-i)$

ii) $(1+i)\cdot(1+i)$

iii) $(2+3i)^3$

iv) $(-1+i)\cdot(1+i)$

v) $(12+2i)\cdot(-1-4i)$

vi) $(1+4i)\cdot(3-2i)$

vii) $i\cdot i\cdot i$

viii) $i\cdot i\cdot i\cdot i$

ix) i^5

x) i^{16}

Problem 5: Modulus of Complex Numbers

Recall that the modulus of a complex number α is defined as:

$$|\alpha| = \sqrt{\alpha \cdot \bar{\alpha}}$$

where $\bar{\alpha}$ is the complex conjugate of α.

a) What is the complex conjugate of the following complex numbers?

i) i

ii) 3

iii) $(1 + 2i)$

iv) $(-1 + 2i)$

v) $(-1 - 2i)$

vi) $(3 + 3i) + (1 + i)$

vii) $(1 + 4i^2)$

viii) $(\frac{1}{2} + \frac{i}{2})$

ix) $(3 + i) \cdot (2 + i)$

x) $(2 + 6i)$

b) Find the modulus of the following complex numbers (Hint: simplify first):

i) i

ii) 3

iii) $(4 + 2i)$

iv) $(2 - 6i)$

v) $(-12 - 5i)$

vi) $(5 - 12i)$

vii) $(3 + 4i)$

viii) $(6 + 5i) - (3 + 2i)$

ix) $(2 + 2i) \cdot (-1 - 4i)$

x) $(-11 + 3i)$

Problem 6: Polar Form of Complex Numbers

Complex numbers can also be represented in polar form using Euler's formula. Euler's formula for a complex number is:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

It is related to our standard form of complex numbers $(a + bi)$ in the following way:

$$r = |a + bi| = \sqrt{a^2 + b^2} \qquad \theta = \arctan\left(\frac{b}{a}\right)$$

a) Express the following complex numbers in polar form.

i) 0

ii) $3 + i$

iii) $\sqrt{3} + 2i$

iv) $2 - 2i$

v) $-13 + 13i$

vi) $6i$

vii) $-2 - i$

b) What is the complex conjugate of the following complex numbers?

i) $e^{i\pi}$

ii) $3e^{-i\frac{\pi}{6}}$

iii) $-e^{-i}$

iv) $1 + e^{-i\frac{\pi}{4}}$

v) $1 - e^{-i\frac{\pi}{4}}$

vi) $-6e^{i\frac{\pi}{2}}$

vii) $3 + 3e^{i\frac{3\pi}{4}}$

c) What is the modulus of the following complex numbers?

i) $\alpha = e^{i\pi}$

ii) $\beta = 3e^{i\frac{\pi}{2}}$

iii) $\gamma = 2e^{-i\pi}$

iv) $\delta = e^{i\frac{\pi}{3}}$

v) $\epsilon = 3e^{i\frac{3\pi}{4}}$

vi) $\zeta = e^{i\pi} \cdot e^{i\pi}$

vii) $\eta = 2e^{-i\frac{\pi}{3}} \cdot e^{-i\frac{\pi}{3}}$

Solutions to Problem 1

a) $\vec{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

b) $\vec{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

c) $\vec{c} = \begin{pmatrix} 10 \\ -6 \end{pmatrix}$

d) $\vec{d} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

e) $\vec{e} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$

f) $\vec{f} = \begin{pmatrix} -6 \\ -7 \end{pmatrix}$

g) $\vec{g} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

h) $\vec{h} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

i) $\vec{i} = \begin{pmatrix} \frac{61}{4} \\ \frac{81}{4} \end{pmatrix}$

Solutions to Problem 2

a) Magnitude: 5 ; Direction: 2.21

b) Magnitude: 0 ; Direction: 0

c) Magnitude: $\sqrt{2}$; Direction: $\frac{\pi}{4}$

d) Magnitude: $\sqrt{53}$; Direction: 4.99

e) Magnitude: $\sqrt{2}$; Direction: $\frac{3\pi}{4}$

f) Magnitude: $\sqrt{26}$; Direction: 2.94

g) Magnitude: $9\sqrt{2}$; Direction: $-\frac{3\pi}{4}$

h) Magnitude: $\frac{37\sqrt{10}}{9}$; Direction: 1.562

i) Magnitude: 1 ; Direction: 0.9553

j) Magnitude: 3 ; Direction: $\frac{-\pi}{4}$

Solutions to Problem 3

a) Velocity: 602.08. mph, Direction: -0.083 rad

b) Velocity: 3.606 cm/s, Direction: -2.158 rad, Time: 10 seconds

Solutions to Problem 4

a)

i) $2 + 2i$

ii) 0

iii) $9 - 9i$

iv) $21 + 9i$

v) $22 - 5i$

vi) $-i + \frac{2}{3}$

vii) $-\frac{1}{2} - \frac{1}{2}i$

viii) $5 - 11i$

ix) $19 + 38i$

b)

i) 2

ii) $2i$

iii) $-46 + 9i$

iv) -2

v) $-4 - 50i$

vi) $11 + 10i$

vii) $-i$

viii) 1

ix) i

x) 1

Solutions to Problem 5

a)

i) $-i$

ii) 3

iii) $(1-2i)$

iv) $(-1-2i)$

v) $(-1+2i)$

vi) $(4-4i)$

vii) (-4)

viii) $(\frac{1}{2}-\frac{i}{2})$

ix) $(5-5i)$

x) $(2-6i)$

b)

i) 1

ii) 3

iii) $2\sqrt{5}$

iv) $2\sqrt{6}$

v) 13

vi) 13

vii) 5

viii) $3\sqrt{2}$

ix) $2\sqrt{34}$

x) $\sqrt{130}$

Solutions to Problem 6

a)

i) 0

ii) $\sqrt{10}e^{0.32i}$

iii) $\sqrt{7}e^{0.86i}$

iv) $2\sqrt{2}e^{-i\frac{\pi}{4}}$

v) $13\sqrt{2}e^{i\frac{3\pi}{4}}$

vi) $6e^{i\frac{\pi}{2}}$

vii) $\sqrt{5}e^{3.61i}$

b)

i) $e^{-i\pi}$

ii) $3e^{i\frac{\pi}{6}}$

iii) $-e^{i}$

iv) $1 + e^{i\frac{\pi}{4}}$

v) $1 - e^{i\frac{\pi}{4}}$

vi) $-6e^{-i\frac{\pi}{2}}$

vii) $3 + 3e^{-i\frac{3\pi}{4}}$

c)

i) 1

ii) 3

iii) 2

iv) 1

v) 3

vi) 1

vii) 2

Note: For any complex number in the form $re^{i\phi}$, the magnitude is simply r. This one reason why the polar form of complex numbers is so useful!