Bounds on The 1-Visibility Localization Number on Trees

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Outline of Presentation

In this presentation, we will discuss the following:

- The Localization Game [1]
- One-Proximity Game
- 1-Visibility Localization [3]
- Bounds on ζ_1 on trees

Key Terminology

Definitions:

- **Tree:** A connected graph with no cycles, denoted as T.
- **Component:** A maximal connected subgraph.
- **Forest:** A collection of disjoint trees.
- $\Delta(T)$**:** The maximum vertex degree found in graph T.

New Ideas:

- **Centroid Vertex:** A vertex v that divides a tree T into components T_i of a forest $T - v$, sizes of these components at most $\frac{n}{2}$.
- **Subtree:** A subgraph of a graph that is a tree.

The 1-Visibility Localization Game

Cop's Move

- The robber occupies a vertex and may move to a neighboring vertex or remain in place.
- The cops' move is the placement of cops on a set of vertices, playing off the graph.

Cop Probes

- Each cop sends out a cop probe d_i during the round.
- If a cop is on the robber's vertex, $d_i = 0$.
- If a cop is adjacent to the robber, $d_i = 1$. Otherwise, $d_i = *$.

Game Outcome

- The cops win if they can determine a unique candidate for the robber's location after a finite number of rounds.
- If the robber evades capture, the robber wins.

One-Visibility Localization Game (Cont'd)

Graph Parameter: $\zeta_1(G)$

- For a graph G, the one-visibility localization number $\zeta_1(G)$ is the least positive integer k for which k cops have a winning strategy in the one-visibility Localization game.

One-Proximity Game

- The one-proximity game is played similarly to the 1-visibility Localization game, except cops are considered to have "captured" the robber if the robber is observed to be at most distance 1 away from at least one cop.
- For instance, suppose a cop probes 1 at a vertex that has multiple adjacent vertices. It suffices to say that the cop has captured the robber if they observe the robber at a distance of 1 away.
    - This paper uses 1-proximity game $\text{prox}_1(G)$ (which is introduced by Bosek et al. [4] as the Blind Localization Game)

Overview

Lemma 16:

For any tree T, $\text{prox}_1(T) \leq \zeta_1(T) \leq \text{prox}_1(T) + 1$.
Furthermore, if $\text{prox}_1(T) \geq \Delta(T)$, then $\zeta_1(T) = \text{prox}_1(T)$.

Lemma 17:

Every tree has at least one centroid vertex.

These lemmas are useful for proving the theorems below.

Theorem 18:

If T is a tree of order $n \geq 2$, then $\zeta_1(T) \leq \lceil \log_2 n \rceil$.

Theorem 19:

For a tree T of depth d, we have that $\zeta_1(T) \leq \lfloor \frac{d}{4} \rfloor + 2$.

Lemma 17

Let T be a tree and v a vertex in T. We call v a *centroid vertex* of T if $T - v$ gives components T_i such that no component contains more than $\frac{n}{2}$ vertices.

Lemma 17:

Every tree has at least one centroid vertex.

Theorem 18:

Theorem

If T is a tree of order $n \geq 2$, then $\zeta_1(T) \leq \lceil \log_2 n \rceil$.

We prove this theorem by strong induction.

Theorem 18: Proof by Strong Induction

Base Case: $n = 2$

- Let T be a tree on $n = 2$ vertices. The tree is unique and must be of size $n - 1 = 1$, and $\log_2(n) = \log_2(2) = 1$ cops suffice in capturing the robber in the 1-Visiblity Localization game.
- The base case holds.

Theorem 18: Proof by Strong Induction (Cont'd)

Inductive Hypothesis: $\zeta_1(T_i) \leq \log_2(n')$ for $T_i \subset T$

- Assume $\zeta_1(T_i) \leq \log_2(n')$.
- Let T be a tree on n vertices, and x a *Centroid Vertex* of T. Place cop c_1 on x.
- Consider the resulting "forest" $T - x$ with order of each component T_i at most $\frac{n}{2}$ by *Lemma 17*. We say "forest" to mean a cop c_1 probing x at every round.
- Suppose $|T_i| = n'$. We have $n' \leq \frac{n}{2} < n$ by choice of x.
- We have $\zeta_1(T_i) \leq \log_2(n')$ by the Inductive Hypothesis.
- So, in component T_i, we need at most $\log_2(n') \leq \log_2(\frac{n}{2})$ cops to find the robber.

Theorem 18: Proof by Strong Induction (Cont'd)

Inductive Step: $\zeta_1(T) \leq \log_2(n)$ for T on n Vertices

- Consider again T. Now the robber cannot travel between components in $T - x$ without capture by cop c_1 stationed at vertex x.
- Since each component T_i contains at most $\frac{n}{2}$ vertices, we require at most $\lceil \log_2(n/2) \rceil = \log_2(n) - 1$ cops to win on T_i.
- We want to show that $\zeta_1(T) \leq \lceil \log_2(n) \rceil$.
- $T - v = \bigcup T_i$ is a disjoint union of "forest" $T - v$.
- Then $\zeta_1(T - v) = \max(\zeta_1(T_i)) \leq \log_2(\frac{n}{2}) = \log_2(n) - 1$
- Recall that we have an additional cop c_1 on vertex x. Then $\zeta_1(T) \leq \log_2(n) - 1 + 1 = \log_2(n)$, as wanted. ■

Example: T_4^2

Consider the family of complete k-ary trees of depth d, T_d^k.
Suppose $k = 2$ and $d = 4$. For T_4^2, there exists one centroid V_1.

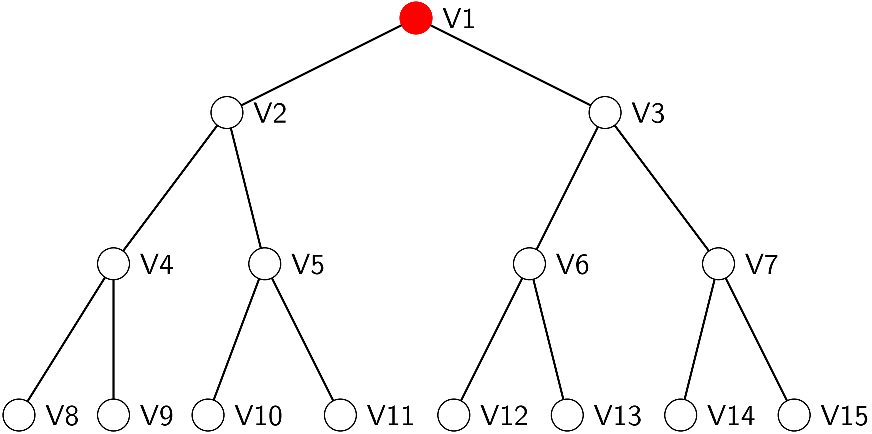

Figure: Complete Binary Tree of Depth 4 with Red Centroid Vertex

Example (Cont'd): Components of $T_4^2 - V1$

$T_4^2 - V1$ has two components.

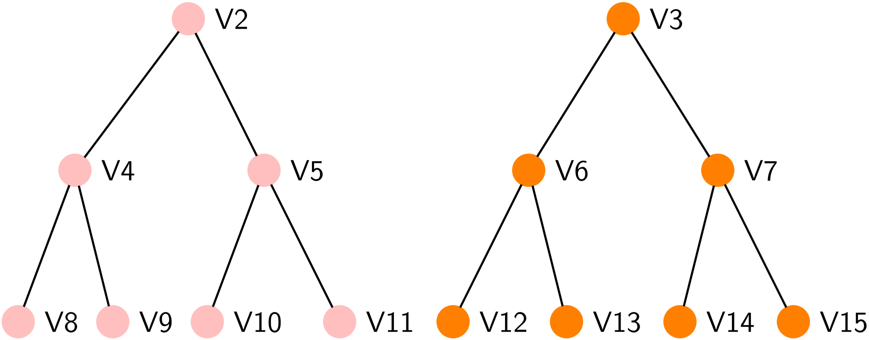

Figure: $T_4^2 - V1$ Complete Binary Tree of Depth 4 Minus Centroid Vertex

Example (Cont'd): Components of T_4^2

$T_4^2 - \{V1, V2\}$ and $T_4^2 - \{V1, V3\}$ each have two components.

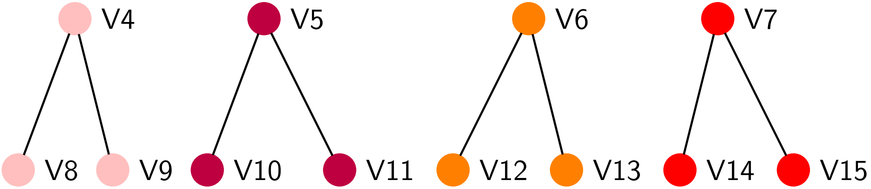

Figure: Subtrees of T_4^2

Fixing a centroid vertex for each of these components gives the singleton graphs, which each require 1 cop to capture the robber. Recursively add a cop at each midway vertex until we have the original tree.

Lemma 16

For any tree T, $\mathrm{prox}_1(T) \leq \zeta_1(T) \leq \mathrm{prox}_1(T) + 1$.
Furthermore, if $\mathrm{prox}_1(T) \geq \Delta(T)$, then $\zeta_1(T) = \mathrm{prox}_1(T)$.

Main Takeaways

- If we win the 1-proximity game, we need at most one more cop to win the 1-visibility Localization game.
- If we can the one-proximity game with $\Delta(T)$-many cops, then we can also win the 1-visibility Localization game.

Proof Sketch

- We play the one-visibility Localization game using $\mathrm{prox}_1(T) = m$ cops which suffice in the one-proximity game.
- We place spare cop at arbitrarily chosen root r.
- We use winning prox_1 strategy to approximate the robber's location, and use the additional cop to reduce the robber territory.

Theorem 19

Theorem

For a tree T of depth d, we have that $\zeta_1(T) \leq \lfloor \frac{d}{4} \rfloor + 2$.

Proof Sketch / Main Idea

1. Play the one-proximity game.
2. The paths of the tree are either "clean" or "infected".
3. Clear paths sequentially based on an ordering of the leaves.
4. Fix a cop at the split point of the tree, to ensure the robber cannot reinfect previously infected paths with a lower index in each round.
5. Use two cop moves to clear each path.

Theorem 19

Notation

Let $\ell_1, \ell_2, \ldots, \ell_p$ denote an ordering of the leaves obtained by performing a depth-first search on T.
Let $P_i = u_1 u_2 \ldots u_q = \ell_i$ denote the path from the root to leaf ℓ_i.

Base Case: Clear path P_1

Clear P_1. Then, on the first round, $T - P_1$ is composed of infected trees.

Inductive Hypothesis: Clear path P_i

The only infected descendants of v_i are in P_i. Each subtree in $T - P_i$ is either cleared or infected.

Theorem 19

Inductive Step: Clear paths P_{i+1}

- Suppose path P_{i+1} is $q \leq d$ long.
- Place a cop on vertices u_{4j} of P_i, for $0 \leq j \leq \lfloor \frac{q}{4} \rfloor$, using at most $\lfloor \frac{q}{4} \rfloor + 1$ cops.
- Each cop u_{4j} clears a ball of radius 1 around u_{4j}.
- If the robber was on P_{i+1} and not captured, then they must have been on u_{4j} for some $0 \leq j \leq \lfloor \frac{q-2}{4} \rfloor$.
- Otherwise, if $q \equiv 3 \pmod 4$, the robber must be on $u_q = \ell_i$
- If robber is not on P_i, it must be on an infected subtree and not P_i or a cleared tree. Repeat the process to complete the inductive step. ■

Example: Spider Graph

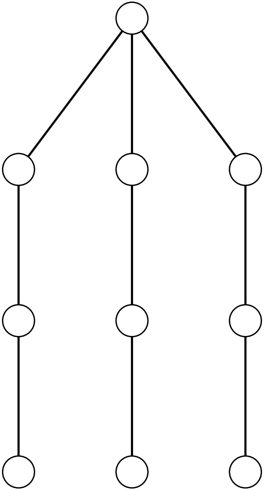

Figure: Spider of depth 4

Bounds on T_d^k Using Theorem 18 & 19

Table: Example where bound in Theorem 18 is better than Theorem 19.

T_d^k	Depth d	Order n	1st Bound	2nd Bound
T_2^3	2	4	1	2
T_3^3	3	13	1	4
T_4^3	4	40	2	6
T_5^3	5	121	2	7
T_6^3	6	364	2	9

Summary

We discussed two bounds on ζ_1 for trees

- If T is a tree of order $n \geq 2$, then $\zeta_1(T) \leq \lceil \log_2 n \rceil$.
- For a tree T of depth d, we have that $\zeta_1(T) \leq \lfloor \frac{d}{4} \rfloor + 2$.
- There is one additional bound (Theorem 20) in the paper
- Each bound works well for certain kinds of trees

Discussion / Conclusion

- This paper also introduces ζ_1 on cartesian grids, etc.
- ζ_k bounds on trees discussed in new paper [2]

References

- Anthony Bonato, *An invitation to pursuit-evasion games and graph theory*, Student Mathematical Library, vol. 97, American Mathematical Society, Providence, RI, 2022.
- Anthony Bonato, Trent G. Marbach, John Marcoux, and JD Nir, *The k-visibility localization game*, arXiv preprint arXiv:2311.01582 (2023).
- Anthony Bonato, Trent G. Marbach, Michael Molnar, and JD Nir, *The one-visibility localization game*, Theoret. Comput. Sci. **978** (2023), 114–186.
- Bartłomiej Bosek, Przemysław Gordinowicz, Jarosław Grytczuk, Nicolas Nisse, Joanna Sokół, and Małgorzata Śleszyńska-Nowak, *Localization game on geometric and planar graphs*, Discrete Appl. Math. **251** (2018), 30–39.