Assignment 5

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Question 1

Use an energy argument to show that the eigenvalues for the problem

$$-(x^2y')' = \lambda y$$

$$1 < x < e, y(1) = 0, y(e) = 0$$

are strictly positive.

Solution:

We use an energy argument which reduces to the process of multiplying the given ODE by y and integrating with respect to x from a to b. We start with

$$-(x^2y')'dx = \lambda y dx$$

Multiply LHS and RHS by y

$$-\int_1^e y(x^2y')'dx = \int_1^e \lambda y^2 dx$$

We want to show that

$$\lambda = -\frac{\int_1^e y(x^2y')'dx}{\int_1^e y^2 dx} > 0$$

We have the denominator $\int_1^e y^2 dx > 0$ since $y^2 > 0$ for all $y \in \mathbb{R}$.
Apply integration by parts using $u = y$ and $dv = \frac{d}{dx}x^2y'$ which gives $du = y'$ and $v = x^2y'$. So we have

$$uv\Big|_a^b - \int_a^b v du = y(x^2y')\Big|_1^e - \int_1^e x^2(y')^2 dx$$

Given our boundary conditions $y(1) = 0$ and $y(e) = 0$, we have

$$y(x^2y')\Big|_1^e = y(e)(e^2y'(e)) - y(1)(e^2y'(1)) = 0 - 0 = 0$$

and

$$\int_1^e x^2(y')^2 dx > 0$$

since $x^2 > 0$ and $(y')^2 > 0$. We have $(y')^2 > 0$ since if $y' = 0$ then $y = c$ but we have $y(1) = y(e) = 0$ which means $y = 0$ which is the trivial solution, and so y cannot be an eigenfunction. Thus, it cannot be that $y' = 0$.
So we must have that $y'^2 > 0$.
In that case, $\int_1^e (y')^2 dx > 0$ and thus $-\int_1^e (y')^2 dx < 0$. So the entire numerator is negative, as wanted.
Since a negative numerator and a positive denominator give a negative value of λ, we have that $\lambda > 0$, as wanted.

Question 2

Use the eigenfunction expansion method to solve

$$u_{xx} + u_{yy} = 0, 0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2}$$

$$u_x(0, y) = 0 = u(\frac{\pi}{2}, y), y > 0$$

$$u(x, \frac{\pi}{2}) = \cos(x) + \cos(3x), u_x(x, 0) = 0$$

Solution:

We look for separable solutions of the form $u(x, y) = X(x)Y(y)$ and substitute into the given PDE to get

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Multiply by $\frac{1}{X(x)Y(y)}$ and rearrange to get

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda$$

We then have

$$u_x(0, y) = X'(0)Y(y) = 0$$
$$u(\frac{\pi}{2}, y) = X(\frac{\pi}{2})Y(y) = 0$$

which implies $X'(0) = 0$ and $X(\frac{\pi}{2}) = 0$ since $Y(y) \neq 0$.

This gives

$$X''(x) + \lambda X(x) = 0, X'(0) = 0, X(\frac{\pi}{2}) = 0$$

so

$$X_n(x) = \cos(nx)$$

since $X_n'(x) = -n\sin(nx) = -n\sin(0) = 0$ at $x = 0$, and $\lambda_n = n^2, n = 1, 2, ...$

We have $X'(x)Y(0) = 0$ which implies that $Y(0) = 0$ since $X'(x) \neq 0$.

Plugging in $\lambda_n = n^2$ gives

$$Y''(y) - n^2Y(y) = 0, Y(0) = 0$$

so $m^2 = n^2 \rightarrow m = \pm n$

So

$$Y(y) = c_1e^{ny} + c_2e^{-ny}$$

From lecture notes, we have $Y(y) = \sinh(ny)$.

Thus,

$$u_n(x, y) = X_n(x)Y_n(y) = \cos(nx) \cdot \sinh(ny), n = 1, 2, ...$$

So $u(x, y) = \sum_{n=1}^{\infty} A_n \cos(nx)\sinh(ny)$

To find A_n, recall that we can impose $u(x, \frac{\pi}{2}) = X(x)Y(\frac{\pi}{2}) = \cos(x) + \cos(3x)$

So

$$u(x, \frac{\pi}{2}) = \sum_{n=1}^{\infty} A_n \cos(nx)\sinh(n\frac{\pi}{2}) = \cos(x) + \cos(3x)$$

$$A_n \cos(nx)\sinh(n\frac{\pi}{2}) = \cos(x) + \cos(3x)$$

$$A_n \cos(nx)\sinh(n\frac{\pi}{2}) = \cos(x) + \cos(3x)$$

$$A_n = \frac{\cos(x) + \cos(3x)}{\cos(nx)\sinh(n\frac{\pi}{2})}$$

Therefore, by superposition, $u(x, y) = \sum_{n=1}^{\infty} \frac{\cos(x)+\cos(3x)}{\cos(nx)\sinh(n\frac{\pi}{2})} \cos(nx)\sinh(ny)$

Question 3

Using Laplace transform, solve partial differential equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0$$

Boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = 0$$

Initial conditions:

$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = -\sin(\pi x)$$

Solution:

The given partial differential equation (PDE) is:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0$$

subject to boundary conditions:

$$u(0,t) = 0$$

$$u(1,t) = 0$$

and initial conditions:

$$u(x,0) = \sin(\pi x)$$

$$u_t(x,0) = -\sin(\pi x)$$

To solve this PDE using the Laplace transform, we apply the Laplace transform to both sides of Equation (1) with respect to the time variable t. Let $\mathcal{L}\{u(x,t)\}$ denote the Laplace transform of $u(x,t)$ with respect to t. The Laplace transforms of the second-order partial derivatives with respect to time and space are given by:

$$\mathcal{L}\{u_{tt}\} = s^2 \mathcal{L}\{u_t\} - su(x,0) - u_t(x,0)$$

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{xx}\}$$

Substituting these into Equation (1) and using the initial conditions, we obtain:

$$s^2 \mathcal{L}\{u\} - su(x,0) - u_t(x,0) = c^2 \mathcal{L}\{u_{xx}\}$$

Let $\mathcal{L}\{u_t\}$ be denoted as $U_t(x,s)$ and $\mathcal{L}\{u_{xx}\}$ as $U_{xx}(x,s)$. Also, let $\mathcal{L}\{u(x,t)\}$ be $U(x,s)$. The Laplace transforms of the initial conditions are given by:

$$\mathcal{L}\{u(x,0)\} = U(x,0)$$

$$\mathcal{L}\{u_t(x,0)\} = U_t(x,0)$$

Substituting these into Equation (4), as well as the initial conditions, we get the transformed equation to solve for $U(x,s)$.

$$s^2 \mathcal{L}\{u\} - ssin(\pi x) + sin(\pi x) = c^2 \mathcal{L}\{u_{xx}\}$$

Re-write and we have

$$sin(\pi x)(1-s) = \mathcal{L}\{u_{xx}\} - \frac{s^2}{c^2}\mathcal{L}\{u\}$$

We want $u(x,s) = U^{CF}(x,s) + U^{PI}(x,s)$.
$U^{CF}(x,s)$ must satisfy $U_{xx} - \frac{s^2}{c^2}U = 0$
Let $U = e^{mx}$. Then $m^2 - \frac{s^2}{c^2} = 0 \rightarrow m = \pm\frac{s}{c}$

So,

$$U^{CF} = A(s)e^{\frac{s}{c}x} + B(s)e^{-\frac{s}{c}x}$$

To find U^{PI}, we use the method of undetermined coefficients to get $U^{PI} = c_1 + c_2\sin(\pi x) + c_3\cos(\pi x)$. Then

$$U_x^{PI} = c_2\pi\cos(\pi x) - c_3\pi\sin(\pi x)$$

$$U_{xx}^{PI} = -c_2\pi^2\sin(\pi x) - c_3\pi^2\cos(\pi x)$$

Then

$$-c_2\pi^2\sin(\pi x) - c_3\pi^2\cos(\pi x) - \frac{s^2}{c^2}(c_1 + c_2\sin(\pi x) + c_3\cos(\pi x)) = sin(\pi x)(1-s)$$

Factorizing common terms, we get the following system of equations:

$$1. \quad -c_2\pi^2 - \frac{s^2}{c^2}c_2 = 1 - s \tag{1}$$

$$2. \quad -c_3\pi^2 - \frac{s^2}{c^2}c_3 = 0 \tag{2}$$

$$3. \quad -\frac{s^2}{c^2}c_1 = 0 \tag{3}$$

$$\tag{4}$$

This gives $c_1 = c_3 = 0$ and

$$c_2 = \frac{1-s}{-\pi^2 - \frac{s^2}{c^2}}$$

This gives $A(s) = B(s) = 0$ from looking at U^{CF} and U^{PI}.
We then have $U(x,s) = \frac{1-s}{-\pi^2-\frac{s^2}{c^2}}\sin(\pi x)$ since $c_1 = c_3 = 0$
Taking the inverse Laplace transform of this gives

$$\begin{aligned}
u(x,t) &= \mathcal{L}^{-1}\left\{\frac{1-s}{-\pi^2 - \frac{s^2}{c^2}}\sin(\pi x)\right\} \\
&= \sin(\pi x)\mathcal{L}^{-1}\left\{\frac{1-s}{-\pi^2 - \frac{s^2}{c^2}}\right\} \\
&= -\sin(\pi x)\mathcal{L}^{-1}\left\{\frac{1}{-\pi^2 - \frac{s^2}{c^2}} - \frac{s}{-\pi^2 - \frac{s^2}{c^2}}\right\} \\
&= -\sin(\pi x)\mathcal{L}^{-1}\left\{\frac{1}{\pi^2 + \frac{s^2}{c^2}} - \frac{s}{\pi^2 + \frac{s^2}{c^2}}\right\} \\
&= -c^2\sin(\pi x)\mathcal{L}^{-1}\left\{\frac{1}{c^2\pi^2 + s^2}\frac{\pi c}{\pi c}\right\} - c^2\sin(\pi x)\mathcal{L}^{-1}\left\{\frac{s}{c^2\pi^2 + s^2}\right\} \\
&= -\frac{c}{\pi}\sin(\pi x)\mathcal{L}^{-1}\left\{\frac{\pi c}{c^2\pi^2 + s^2}\right\} - c^2\sin(\pi x)\mathcal{L}^{-1}\left\{\frac{s}{c^2\pi^2 + s^2}\right\} \\
&= -\frac{c}{\pi}\sin(\pi x)\sin(\pi cx) - c^2\sin(\pi cx)\cos(\pi cx)
\end{aligned}$$