Assignment 4

Mariam Walaa

November 14, 2023

Question 1

Consider the differential equation

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + (1+\epsilon)y = 0,$$

where $y(0) = 0$ and $\frac{dy}{dt}(0) = 1$. Assume $\epsilon > 0$.

Use the **method of multiple scales** to find the leading-order approximation to the solution.

Solution:

To find the leading-order approximation using the method of multiple scales, we assume that the solution to the given differential equation can be expressed as a power series in a small parameter ϵ:

$$Y(t,\tau) = Y_0(t,\tau) + \epsilon Y_1(t,\tau) + \epsilon^2 Y_2(t,\tau) + \dots$$

with $\tau = \epsilon t$. Now, substitute this expression into the differential equation and collect terms of the same order in ϵ. We'll start with the first two terms and assume that $Y_0(t)$ varies slowly compared to $Y_1(t)$. Assume the solution:

$$Y(t,\tau) = Y_0(t,\tau) + \epsilon Y_1(t,\tau)$$

Now, differentiate with respect to time:

$$\frac{dY}{dt} = \frac{dY_0}{dt} + \epsilon(\frac{dY_0}{dt} + \frac{dY_1}{dt})$$

$$\frac{d^2Y}{dt^2} = \frac{d^2Y_0}{dt^2} + \epsilon(\frac{d^2Y_0}{dtd\tau} + \frac{d^2Y_0}{d\tau dt} + \frac{d^2Y_1}{dt^2})$$

Now, substitute these expressions into the differential equation:

$$\left(\frac{d^2Y_0}{dt^2} + \epsilon(\frac{d^2Y_0}{dtd\tau} + \frac{d^2Y_0}{d\tau dt} + \frac{d^2Y_1}{dt^2})\right) + \epsilon\left(\frac{dY_0}{dt} + \epsilon(\frac{dY_0}{dt} + \frac{dY_1}{dt})\right) + (1+\epsilon)(Y_0 + \epsilon Y_1) = 0,$$

Now, collect terms at each order in ϵ to get:

$$O(1): \frac{d^2Y_0}{dt^2} + Y_0 = 0$$

$$O(\epsilon): \frac{d^2Y_0}{d\tau dt} + \frac{d^2Y_0}{dtd\tau} + \frac{d^2Y_1}{dt^2} + \frac{dY_0}{dt} + Y_0 + Y_1 = 0$$

Now, solve the leading-order equation:

$$\frac{d^2Y_0}{dt^2} + Y_0 = 0$$

We assume a solution of the form

$$Y_0(t,\tau) = Y_0^{CF} + Y_0^{PI} = A(\tau)cos(t) + B(\tau)\sin(t)$$

which gives

$$Y_0(t,\tau) = A(\tau)\cos(t) + B(\tau)\sin(t)$$

$$\frac{dY_0}{dt} = -A(\tau)\sin(t) + B(\tau)\cos(t)$$

$$\frac{d^2Y_0}{d\tau dt} = -A'(\tau)\sin(t) + B'(\tau)\cos(t) = \frac{d^2Y_0}{dtd\tau}$$

Now, move on to the next order, and plug in the above, to get:

$$\frac{d^2Y_0}{d\tau dt} + \frac{d^2Y_0}{dtd\tau} + \frac{d^2Y_1}{dt^2} + \frac{dY_0}{dt} + Y_1 + Y_0 = 0 \leftrightarrow \frac{d^2Y_1}{dt^2} + Y_1 = -\frac{d^2Y_0}{d\tau dt} - \frac{d^2Y_0}{dtd\tau} - \frac{dY_0}{dt} - Y_0$$

$$\to \frac{d^2Y_1}{dt^2} + Y_1 = 2A'(\tau)\sin(t) - 2B'(\tau)\cos(t) + A(\tau)\sin(T) - B(\tau)\cos(t) - A(\tau)\cos(t) - B(\tau)\sin(t)$$

$$\to \frac{d^2Y_1}{dt^2} + Y_1 = \sin(t)\left(2A'(\tau) + A(\tau)\right) - \cos(t)\left(2B'(\tau) + B(\tau)\right) - A(\tau)\cos(t) - B(\tau)\sin(t)$$

So we want

$$2A'(\tau) + A(\tau) = 0$$

$$2B'(\tau) + B(\tau) = 0$$

Which is

$$2A'(\tau) = -A(\tau)$$

$$2B'(\tau) = -B(\tau)$$

This gives

$$A(\tau) = ae^{-\frac{1}{2}\tau}$$

$$B(\tau) = be^{-\frac{1}{2}\tau}$$

It follows that

$$y = Y_0(t,\tau) = ae^{-\frac{1}{2}\tau}\cos(t) + be^{-\frac{1}{2}\tau}\sin(t)$$

Using the given initial conditions, we have

$$y(0) = 0 = ae^{-\frac{1}{2}\tau}\cos(0) + be^{-\frac{1}{2}\tau}\sin(0)$$

$$\to 0 = ae^{-\frac{1}{2}\tau} \to a = 0$$

$$y'(0) = 1 = -ae^{-\frac{1}{2}\tau}sin(0) + be^{-\frac{1}{2}\tau}cos(0)$$

$$\to 1 = be^{-\frac{1}{2}\tau} \to b = 1$$

Finally, we have

$$y = e^{-\frac{1}{2}\tau}sin(t)$$

Question 2(A)

Consider the differential equation $-y'' = \lambda y$ with the boundary conditions $y(0) = 0$ and $y'(2\pi) = 0$. Find the eigenvalues and eigenfunctions corresponding to this differential equation.

Solution:

Assume a solution of the form $y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. Suppose $\lambda > 0$. Then, the first and second derivatives are given by:

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$y''(x) = -A\lambda\cos(kx) - B\lambda\sin(kx)$$

Now, impose the boundary conditions to get:

$$y(0) = A\cos(\sqrt{\lambda}0) + B\sin(\sqrt{\lambda}0) = A\cos(\sqrt{\lambda}0) = 0 \to A = 0$$

$$y'(2\pi) = A\cos(\sqrt{\lambda}2\pi) + B\sin(\sqrt{\lambda}2\pi) = A\cos(\sqrt{\lambda}2\pi) = 0$$

From the second equation, we have $A\cos(\sqrt{\lambda}2\pi) = 0$. Since $A \neq 0$, we must have

$$\sqrt{\lambda}2\pi = \frac{\pi}{2} + n\pi, n = 0, 1, ...$$

.

This gives

$$\lambda = \left(\frac{2n+1}{4}\right)^2$$

with the corresponding eigenfunctions

$$y(x) = \sin\left(\frac{2n+1}{4}x\right)$$

Now, suppose $\lambda = 0$. If $\lambda = 0$, then $r = 0, 0$. A repeated root of zero leads to $y = A + Bx$. Impose boundary conditions to get $y(0) = 0 \to A = 0$, and $y'(2\pi) = 0 \to B = 0$. This implies that $y = 0$, which is a trivial solution. Therefore, there are no eigenvalues and eigenfunctions in this case.

Now, finally suppose $\lambda < 0$. If $\lambda < 0$, then $r = \pm\sqrt{-\lambda}$, so $r = \pm\sqrt{\lambda}$. For two real roots, the solution to the DE is $y = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Impose boundary conditions to get

$$y(0) = 0 \to A + B = 0 \to B = -A$$

$$y'(2\pi) = A\sqrt{\lambda}e^{\sqrt{\lambda}2\pi} - B\sqrt{\lambda}e^{-\sqrt{\lambda}2\pi}$$

$$A\sqrt{\lambda}(e^{\sqrt{\lambda}2\pi} + e^{-\sqrt{\lambda}2\pi}) = 0 \to A = 0$$

This gives $B = -A = 0 \to y = 0$, which is the trivial solution, so there are no eigenvalues and eigenfunctions in this case.

In summary,

$$\lambda = \lambda_n = \left(\frac{2n+1}{4}\right)^2, n = 0, 1, 2, ...$$

$$y = y_n = \sin\left(\frac{2n+1}{4}x\right)$$

Question 2(B)

Show that the eigenfunctions

$$y = y_n = \sin\left(\frac{2n+1}{4}x\right)$$

in (a) are orthogonal.

Solution:

To show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal, we need to prove the following integral:

$$\int_0^{2\pi} \sin\left(\frac{2n+1}{4}x\right)\sin\left(\frac{2m+1}{4}x\right)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

Let's proceed with the proof:
Case $n = m$:

$$\begin{aligned}
&\int_0^{2\pi} \sin\left(\frac{2n+1}{4}x\right)\sin\left(\frac{2n+1}{4}x\right)\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (1 - \cos\left(\frac{2(2n+1)}{4}x\right))\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (1 - \cos\left(\frac{2n+1}{2}x\right))\,dx \\
&= \frac{1}{2}\int_0^{2\pi} dx - \frac{1}{2}\int_0^{2\pi} \cos\left(\frac{2n+1}{2}x\right)\,dx \\
&= \frac{1}{2}[2\pi - 0] - \frac{1}{2}\frac{2}{2n+1}[\sin\left(\frac{2n+1}{2}2\pi\right) - \sin(0)] \\
&= \frac{1}{2}[2\pi - 0] - \frac{1}{2}\frac{2}{2n+1}[\sin\left((2n+1)\pi\right)] \\
&= \frac{1}{2}[2\pi - 0] \\
&= \frac{2\pi}{2} \\
&= \pi.
\end{aligned}$$

Case $n \neq m$:

$$\begin{aligned}
&\int_0^{2\pi} \sin\left(\frac{2n+1}{4}x\right)\sin\left(\frac{2m+1}{4}x\right)\,dx \\
&= \frac{1}{2}\int_0^{2\pi} \cos\left(\frac{(2n+1)-(2m+1)}{4}x\right) - \cos\left(\frac{(2n+1)+(2m+1)}{4}x\right)\,dx \\
&= \frac{1}{2}\int_0^{2\pi} \cos\left(\frac{(2n-2m)}{4}x\right) - \cos\left(\frac{(2n+2m+2)}{4}x\right)\,dx \\
&= \frac{1}{2}\int_0^{2\pi} \cos\left(\frac{2(n-m)}{4}x\right) - \cos\left(\frac{(n+m+1)}{2}x\right)\,dx.
\end{aligned}$$

Now, let $k = \frac{n-m}{2}$ and $l = \frac{n+m+1}{2}$, so the integral becomes:

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{2\pi} \cos(kx) - \cos(lx)\,dx \\
&= \frac{1}{2}\left[\frac{\sin(kx)}{k} - \frac{\sin(lx)}{l}\right]_0^{2\pi} \\
&= \frac{1}{2}\left(\frac{\sin(k\cdot 2\pi)}{k} - \frac{\sin(l\cdot 2\pi)}{l} - \left(\frac{\sin(0)}{k} - \frac{\sin(0)}{l}\right)\right) \\
&= \frac{1}{2}\left(\frac{\sin(0)}{k} - \frac{\sin(0)}{l}\right) \\
&= 0 \quad (\text{since } \sin(0) = 0 \text{ for any angle}).
\end{aligned}
$$

Therefore, the integral is zero when $n \neq m$, and the eigenfunctions are orthogonal.

Question 2(C)

Using the above eigenfunctions

$$y = y_n = \sin\left(\frac{2n+1}{4}x\right)$$

from (a), find the expansion of $f(x) = x$ on the interval $0 < x < 2\pi$.

Solution:

To find the expansion of $f(x) = x$ using the given eigenfunctions $\sin\left(\frac{2n+1}{4}x\right)$, we need to compute the expansion coefficients c_n:

$$c_n = \frac{(f, y_n)}{||y_n||^2} = \frac{2}{2\pi}\int_0^{2\pi} x\sin\left(\frac{2n+1}{4}x\right)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin\left(\frac{2n+1}{4}x\right)\,dx$$

We compute c_n now. First, we have

$$\begin{aligned}
||y_n||^2 &= ||\sin(\frac{2n+1}{4}x)||^2 \\
&= \left[\sqrt{\left(\sin\left(\frac{2n+1}{4}x\right), \sin\left(\frac{2n+1}{4}x\right)\right)}\right]^2 \\
&= \int_0^{2\pi} \sin^2\left(\frac{2n+1}{4}x\right) dx \text{ using } \sin^2(x) = 1-\cos^2(x) \\
&= \frac{1}{2}\int_0^{2\pi} 1-\cos\left(\frac{2(2n+1)}{4}x\right) dx \\
&= \frac{x}{2} - \frac{1\cdot 4}{2\cdot(2n+1)}\sin\left(\frac{2n+1}{2}x\right) \\
&= \frac{(2\pi-0)}{2} - \frac{2}{(2n+1)}\sin\left(\frac{2n+1}{4}x\right) \\
&= \pi
\end{aligned}$$

To calculate (f, y_n), we do integration by parts using $u = x$ and $dv = \sin\left(\frac{(2n+1)x}{4}\right) dx$. This gives $du = dx$ and $v = -\cos\left(\frac{2n+1}{4}x\right)\frac{4}{2n+1}$

$$\begin{aligned}
(f, y_n) &= \int_0^{2\pi} x\sin\left(\frac{2n+1}{4}x\right) dx \\
&= \left(-\frac{4}{2n+1}x\cos\left(\frac{(2n+1)x}{4}\right)\Bigg|_0^{2\pi} + \frac{4}{2n+1}\int_0^{2\pi}\cos\left(\frac{(2n+1)x}{4}\right) dx\right) \\
&= \left(-\frac{4}{2n+1}2\pi\cos\left(\frac{(2n+1)2\pi}{4}\right) - \left(-\frac{4}{2n+1}0\cos\left(\frac{(2n+1)0}{4}\right)\right) + \frac{4}{2n+1}\int_0^{2\pi}\cos\left(\frac{(2n+1)x}{4}\right) dx\right) \\
&= \left(\left(-\frac{4}{2n+1}2\pi\cos\left(\frac{(2n+1)2\pi}{4}\right)\right) + \frac{4}{2n+1}\int_0^{2\pi}\cos\left(\frac{(2n+1)x}{4}\right) dx\right) \\
&= \frac{-8\pi}{2n+1}\cos\left(\frac{(2n+1)\pi}{2}\right) + (\frac{4}{2n+1})^2\sin\left(\frac{2n+1}{2}\pi\right) \\
&= \frac{-8\pi(2n+1)\cos(\frac{2n+1}{2}\pi) + 16\sin(\frac{2n+1}{2}\pi)}{(2n+1)^2}
\end{aligned}$$

Finally, we have

$$c_n = \frac{\left(x, \sin(\frac{2n+1}{4}x)\right)}{||\sin(\frac{2n+1}{4}x)||^2} = \frac{1}{\pi}\frac{-8\pi(2n+1)\cos(\frac{2n+1}{2}\pi) + 16\sin(\frac{2n+1}{2}\pi)}{(2n+1)^2}$$

The expansion of $f(x)$ is then given by:

$$f(x) \approx \sum_{n=0}^{\infty} c_n \sin\left(\frac{2n+1}{4}x\right)$$

$$f(x) \approx \sum_{n=0}^{\infty} \frac{1}{\pi} \frac{-8\pi(2n+1)\cos(\frac{2n+1}{2}\pi) + 16\sin(\frac{2n+1}{2}\pi)}{(2n+1)^2} \sin\left(\frac{2n+1}{4}x\right)$$