Assignment 6

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Question 1

Use Laplace transforms to solve

$$u_{tt} = u_{xx} - g, \quad x > 0, \quad t > 0, \tag{1}$$

subject to the boundary condition:

$$u(0,t) = 0, \tag{2}$$

and initial conditions:

$$u(x,0) = 0, \tag{3}$$
$$u_t(x,0) = 0. \tag{4}$$

Assume that u is bounded as x goes to infinity, and that g is a positive constant.

Solution:

Taking the Laplace transform of both sides of (1) with respect to t, and utilizing the initial conditions, we obtain:

$$s^2U(x,s) - su(x,0) - u_t(x,0) = U_{xx}(x,s) - \frac{g}{s}, \tag{5}$$

where $U(x,s)$ is the Laplace transform of $u(x,t)$.
Rearranging terms and substituting (5), we have the ODE in x:

$$U_{xx}(x,s) - s^2U(x,s) = \frac{g}{s}. \tag{6}$$

Solving this gives $U = U_{CF} + U_{PI}$ where $U_{CF} = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x}$.
Since u is bounded as x goes to infinity, we must have $A(s) = 0$ since the $e^{\sqrt{s}x}$ term would make u unbounded.
For U_{PI}, we use method of undetermined coefficents and guess $U_{PI} = A$, where A is a constant. This is because $\frac{g}{s}$ can be treated as a constant.
Then $U_{PI(x)} = U_{PI(xx)} = 0$, so $s^2A = 0 - \frac{g}{s}$ which implies $A = -\frac{g}{s^3}$, so

$$U = U_{CF} + U_{PI} = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} - \frac{g}{s^3}$$

Applying the boundary condition, we have $u(0,t) = 0 = A(s) + B(s) - \frac{g}{s^3}$ which implies $B(s) = \frac{g}{s^3}$.
Finally, this gives

$$u(x,t) = \frac{g}{s^3}e^{-\sqrt{s}x} - \frac{g}{s^3}$$

Question 2

Use a Fourier transform to solve the partial differential equation (PDE):

$$u_t = u_x + u_t, \quad -\infty < x < +\infty, \quad t > 0$$

subject to the initial condition:

$$u(x, 0) = 0$$

Solution:

Applying the Fourier transform to both sides of the PDE and utilizing the initial condition, we can obtain the transformed equation.

$$\mathcal{F}\{u(x,t) : x \to \zeta\} = \hat{u}(\zeta, t)$$
$$\mathcal{F}\{u_t(x,t) : x \to \zeta\} = \hat{u}_t(\zeta, t)$$
$$\mathcal{F}\{u_x(x,t) : x \to \zeta\} = -i\zeta\hat{u}(\zeta, t)$$
$$\mathcal{F}\{t : x \to \zeta\} = t \cdot \mathcal{F}\{1 : x \to \zeta\} = 2\pi t\delta(\zeta)$$
$$\mathcal{F}\{u(x,0) : x \to \zeta\} = \mathcal{F}\{0\} = 0$$

This gives the new transformed PDE with transformed initial condition as follows

$$\mathcal{F}\{u_t\} = \mathcal{F}\{u_x\} + \mathcal{F}\{t\},$$

$$\hat{u}_t(\zeta, t) = -i\zeta\hat{u}(\zeta, t) + 2\pi t\delta(\zeta)$$

with initial condition

$$\hat{u}(\zeta, 0) = 0$$

Simplify and re-write

$$\hat{u}_t = -i\zeta\hat{u} + 2\pi t\delta(\zeta)$$

which is a first order non-homogenous linear ODE. Consider the homogenous equation

$$\hat{u}_t = -i\zeta\hat{u}$$

with general solution

$$\hat{u} = A(\zeta)e^{-i\zeta t}$$

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We can apply the inverse Fourier transform to get

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\zeta, t)e^{-i\zeta x} d\zeta$$

$$= \int_{-\infty}^{\infty} A(\zeta)e^{-i\zeta t}e^{-i\zeta x} d\zeta$$

$$= \int_{-\infty}^{\infty} A(\zeta)e^{-i\zeta(t+x)} d\zeta$$

Considering the non-homogenous equation again

$$\hat{u}_t = -i\zeta\hat{u} + 2\pi t\delta(\zeta)$$

We use the method of undetermined coefficients to guess a particular solution $\hat{u}_{PS} = B(\zeta)t$.
Plugging this particular solution back in, we have

$$B(\zeta) = -i\zeta B(\zeta)t + 2\pi t\delta(\zeta)$$

Solving for $B(\zeta)$, we have

$$B(\zeta)(1+i\zeta t) = 2\pi t\delta(\zeta)$$

$$B(\zeta) = \frac{2\pi t\delta(\zeta)}{1+i\zeta t}$$

Then

$$\hat{u}_{PS} = B(\zeta)t = \frac{2\pi t\delta(\zeta)}{1+i\zeta t}t$$

So we have

$$\begin{aligned} u_{PS} &= \int_{-\infty}^{\infty} \hat{u}_{PS} e^{-i\zeta x} d\zeta \\ &= \int_{-\infty}^{\infty} \frac{2\pi t\delta(\zeta)}{1+i\zeta t} t e^{-i\zeta x} d\zeta \end{aligned}$$

Finally,

$$u(x,t) = u_{GI} + u_{PS} = \int_{-\infty}^{\infty} A(\zeta) e^{-i\zeta(t+x)} d\zeta + \int_{-\infty}^{\infty} \frac{2\pi t\delta(\zeta)}{1+i\zeta t} t e^{-i\zeta x} d\zeta$$

Question 3

Use a Fourier transform to solve the partial differential equation (PDE):

$$u_t = Du_{xx} + u_x, \quad -\infty < x < +\infty, \quad t > 0, \tag{7}$$

subject to the initial condition:

$$u(x, 0) = \delta(x - 1), \tag{8}$$

where $\delta(x - 1)$ is the Dirac delta function centered at $x = 1$, and D is a positive constant.

Solution:

Applying the Fourier transform to both sides of the PDE and utilizing the initial condition, we can obtain the transformed equation.

$$\mathcal{F}\{u(x,t) : x \to \zeta\} = \hat{u}(\zeta, t)$$
$$\mathcal{F}\{u_t(x,t) : x \to \zeta\} = \hat{u}_t(\zeta, t)$$
$$\mathcal{F}\{u_{xx}(x,t) : x \to \zeta\} = -\zeta^2\hat{u}(\zeta, t)$$
$$\mathcal{F}\{u_x(x,t) : x \to \zeta\} = -i\zeta\hat{u}(\zeta, t)$$
$$\mathcal{F}\{u(x,0) : x \to \zeta\} = \mathcal{F}\{\delta(x-1)\} = \hat{\delta}(\zeta - 1)$$

This gives the new transformed PDE with transformed initial condition as follows

$$\mathcal{F}\{u_t\} = D\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_x\}, \tag{9}$$

$$\hat{u}_t(\zeta, t) = -D\zeta^2\hat{u}(\zeta, t) - i\zeta\hat{u}(\zeta, t) \tag{10}$$

with initial condition

$$\hat{u}(\zeta, 0) = \hat{\delta}(\zeta - 1)$$

Simplifying, we have

$$\hat{u}_t = -D\zeta^2\hat{u} - i\zeta\hat{u} \tag{11}$$

$$\hat{u}_t = -\hat{u}(D\zeta^2 - i\zeta) \tag{12}$$

$$\frac{\hat{u}_t}{\hat{u}} = -(D\zeta^2 - i\zeta) \tag{13}$$

Integrating both sides gives

$$ln|\hat{u}| = -(D\zeta^2 - i\zeta)t + ln|A(\zeta)| \tag{14}$$

Applying the exponential on both sides to get

$$\hat{u} = A(\zeta)e^{-(D\zeta^2 - i\zeta)t} \tag{15}$$

The initial condition gives

$$\hat{u}(\zeta, 0) = A(\zeta)e^{-(D\zeta^2 - i\zeta)0} = A(\zeta) = \hat{\delta}(\zeta - 1)$$

So

$$\hat{u}(\zeta, t) = \hat{\delta}(\zeta - 1)e^{-(D\zeta^2 + i\zeta)t}$$

Applying the inverse Fourier transform to the above, we have

$$\hat{u} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\delta}(\zeta - 1)e^{-(D\zeta^2 + \zeta i)t}e^{-i\zeta x}d\zeta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\delta(s-1)e^{\zeta s}ds\right)e^{-(D\zeta^2+\zeta i)t}e^{-i\zeta x}d\zeta$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\delta(s-1)e^{\zeta s}ds\right)\int_{-\infty}^{\infty}e^{-(D\zeta^2 t-\zeta it-\zeta x)}d\zeta$$

Using Euler's formula and symmetry of cosines, we have

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\delta(s-1)ds\cdot 2\int_{0}^{\infty}e^{-D\zeta^2 t}\cos(\zeta(s-t-x))d\zeta$$

Also, $\int_0^\infty e^{-ay^2}cosby = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}}$ with $a=t$ and $b=s-t-x$ gives

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\zeta(s-1)\cdot\frac{1}{2}\sqrt{\frac{\pi}{t}}e^{\frac{-(s-t-x)^2}{4t}}ds$$

Finally, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\delta(s-1)e^{\frac{-(s-t-x)^2}{4t}}ds$$