Tutorial 4: Basic Solutions, Simplex Method, Graphical Interpretation

0.1 Administrative

- In the first 15 minutes of this tutorial, we will be going through the basic feasible solutions algorithm, which will be preparation for your quiz. In the remaining time until your quiz, we will go through an introduction to the simplex method.
- We will start the quiz at 6:35 PM. You will receive your quiz in Quercus Inbox. You will have 25 minutes to complete the quiz and 10 minutes to submit the quiz. If you are having trouble submitting to Quercus, please email your quiz to me at mariam.walaa@mail.utoronto.ca or through Quercus Inbox.
- **Next week**: Quiz 4, More on Simplex Method
- **Your tentative midterm date:** Saturday, March 5. No quiz on March 3.
- **My office hours:** Fridays @ 6:30PM on Zoom
- **Textbook:** Elementary Linear Programming with Applications (Kolman, Beck)

0.2 Learning Objectives

In this tutorial, you should expect to achieve the following:

- Learn the Basic Feasible Solutions algorithm.
- Learn the simplex algorithm.
- Understand the difference between the two algorithms.
- Understand simplex method terminology and flowchart.

0.3 Recall

By now, you have learned and should be able to recall definitions for the following:

- Forms of Linear Programming Problems (LPP)
- Matrix notation
- Feasible regions
- Feasible solutions and optimal solutions
- Convex sets and combinations
- Graphical solutions to LPPs

0.4 Finding all basic feasible solutions to a LPP

Reading: Section 1.5 Basic Solutions

To apply the basic solutions algorithm, we typically require s decision variables and m constraints in a LPP where $s \geq m$, and we therefore have some linearly dependent columns that we need to get rid of in order to solve the system of linear equations. To do this, we follow the basic feasible solution algorithm.

Basic Feasible Solution Algorithm

1. Set the LPP to canonical form.
2. Find m linearly independent column vectors in the matrix A. Call this matrix A'.
3. For the remaining columns which are linearly dependent to the columns in matrix A', set the corresponding values in the vector x to 0.
4. Set this matrix A' multiplied by x' to the corresponding values of b and solve the system of linear equations. What follows is a *basic* solution to your LPP. (**Note:** It is not necessarily a basic *feasible* solution.)
   - Repeat steps 2 - 4 for every set of m linearly independent column vectors.
5. For each basic solution, select those that satisfy all the constraints of the LPP. You now have a list of the basic feasible solutions.
6. For each basic feasible solution, compute the objective function z and choose the basic feasible solution x resulting in the maximum or minimum value depending on your LPP.

0.5 Basic Solutions Terminology

- **Basic solution:** A basic solution is obtained when solving the system of linear equations pertaining to a linearly independent set of columns of A. It includes exactly $s - m$ zero's.
- **Basic feasible solution:** A basic feasible solution is a basic solution that is feasible for the LPP. Specifically, $x_i \geq 0$ for all i. A basic feasible solution is equivalent to an extreme point of the LPP, and there are finitely many basic feasible solutions in an LPP.
- **Non-basic variable:** Non-basic variables are the variables x_i that we set to 0. There are exactly $s - m$ variables x_i that we set to 0.
- **Basic variable:** Basic variables are the variables x_i that we do *not* set to zero, however, they can take the value 0. There are exactly m such variables.

0.6 Graphical Interpretation of Simplex Method

Reading: Section 2.1 The Simplex Method for Standard Problems

The idea behind the simplex method is that we improve on the BFS algorithm by reducing the number of extreme points we check. To better understand the intuition behind the simplex method, consider a graphical interpretation of the algorithm that moves from extreme point to extreme point until an optimal solution is found.

Suppose we have a LPP defined by the feasible region below for the following objective function

$$\text{maximize } z = x + y$$

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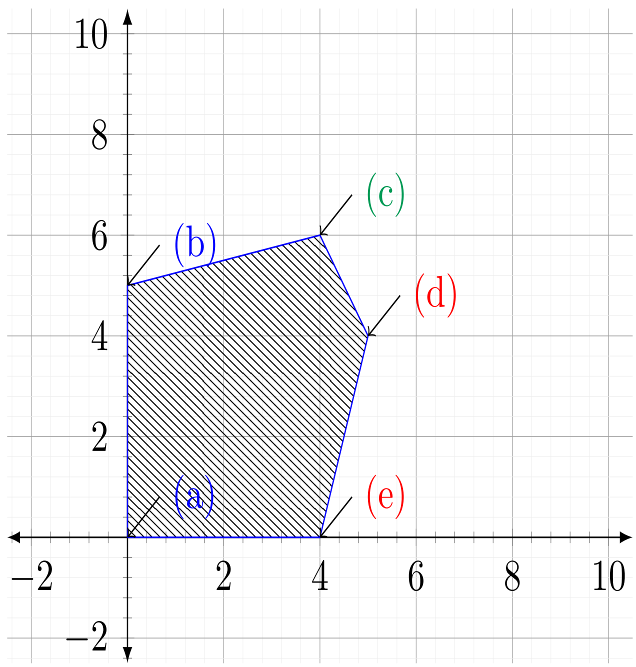

Steps by the simplex algorithm:

- **(a):** This is the point (0,0) where the algorithm starts
- **(b):** This is the next point (0,5) based on the simplex method
- **(c):** This is the terminal point (4,6) at which optimal value is found
- **(d), (e):** These points are skipped by the simplex algorithm

Thinking Questions:

- Note that the simplex algorithm starts at the origin $(0, 0)$. Does the simplex algorithm start at the origin for every LPP?
- How do we know that (c) and (d) do not contain values that are more optimal?

0.7 Simplex Method Algorithm

Reading: Section 2.1 The Simplex Method for Standard Problems

To use the simplex method, the simplex algorithm has two initial requirements:

1. Starting with the standard form and converting to canonical form
   - **Why?** This gives us a starting BFS due to the slack variables added when converting to canonical form. The Simplex algorithm requires a starting point to determine what the next best optimal value is.
2. Having all b_i's non-negative
   - **Why?** The b_i's give us the initial BFS. Having the b_i's satisfy the non-negative constraint allows the initial BFS to be feasible.

Once we have the correct form that satisfies the above requirements, we may convert the LPP to its corresponding initial Simplex Tableau using the matrix notation c, x, A, b, and z.

0.8 Simplex Tableau

Reading: Section 2.1 The Simplex Method for Standard Problems

The following is the general representation of a canonical form LPP in a simplex tableau.

$$\left[\begin{array}{ccc|c} A & I & z & b \\ \hline -c^T & 0 & 1 & 0 \end{array}\right] \tag{1}$$

Why do we use a tableau and not a matrix?
The LPP is converted from matrix notation to a tableau in order to aid in checking for optimality as we perform row operations.

$$\begin{array}{c} \\ u \\ v \\ \\ \end{array} \left[\begin{array}{ccccc|c} x & y & u & v & z & \\ \hline 2 & 2 & 1 & 0 & 0 & 3 \\ 2 & 2 & 0 & 1 & 0 & 3 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \tag{2}$$

0.9 Simplex Method Algorithm

Reading: Section 2.1 The Simplex Method for Standard Problems

The simplex algorithm moves from BFS to BFS by modifying basic and non-basic variables through row operations as follows:

1. A starting BFS is acquired by setting non-slack (non-basic) variables to zero, and slack variables (basic variables) to the values b_i. The initial tableau corresponds to the initial BFS.
2. The next tableau displays the next (adjacent) BFS acquired by selecting a pivotal column and pivotal row, performing row operations, and therefore increasing the objective value z.

At each step of the simplex algorithm, observe the tableau and ask the following:

- What is the BFS for the current tableau?
- What is the value of z for the current tableau?

Question: When do you stop?

- By looking at the current tableau, there is an indicator for whether you are currently at the optimal value of z: Are all the entries in the objective row non-negative? Particularly, are all entries corresponding to basic variables zero, and all entries corresponding to non-basic variables positive? If so, you have reached the optimal value at the current BFS and may now terminate the algorithm.
- Another indicator that you may terminate the algorithm: All entries in the pivotal column are either zero or negative. In this case, the given LPP has no finite optimal solution, and you may terminate the algorithm.
- However, if the solution is not yet optimal, you will see negative entries in the columns corresponding to non-basic variables. In that case, the next simplex tableau will enable you to move to an adjacent extreme point in a way that increases the objective function value.

Another question: Why does the simplex tableau result in the optimal value?

- Note that every line in the simplex tableau has an algebraic representation.
- You can write the algebraic representation to further understand why these manipulations lead us to find the optimal value.

0.10 Simplex Algorithm Flowchart

Reading: Section 2.1 The Simplex Method for Standard Problems

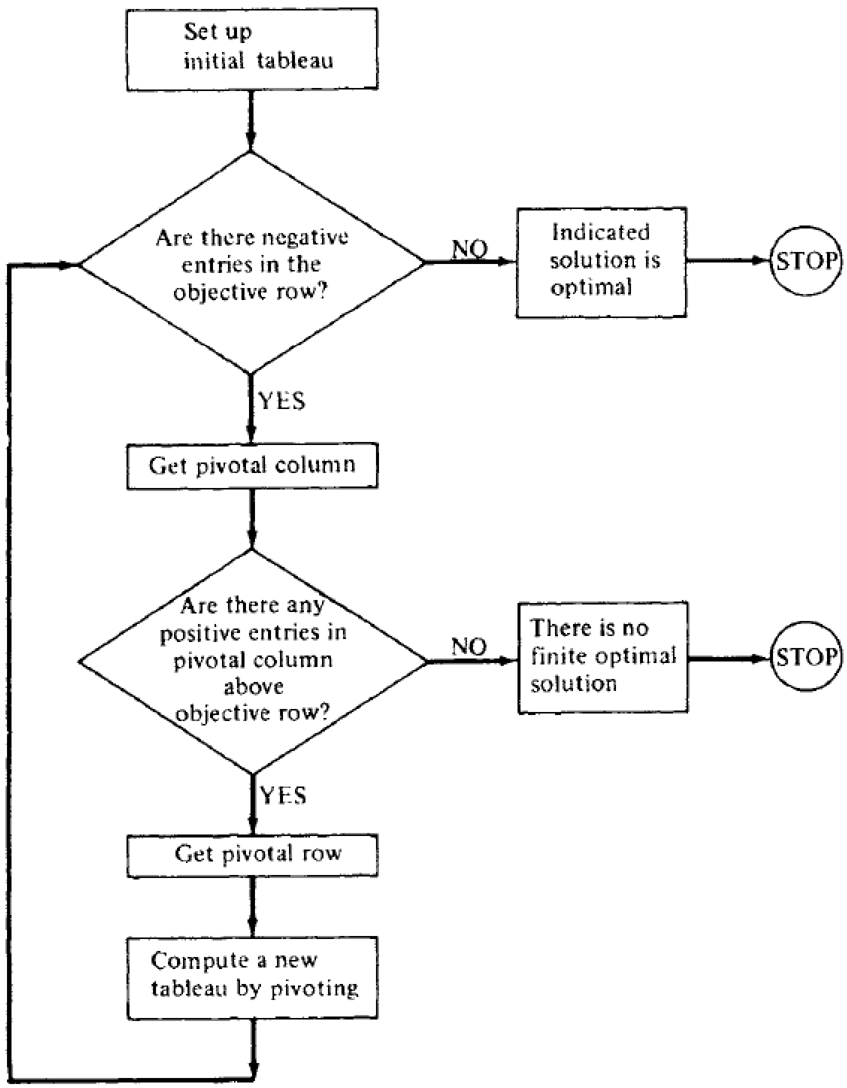

FIGURE 2.1 Flowchart for simplex algorithm (standard form, $\mathbf{b} \geq \mathbf{0}$).