Tutorial 2: Terminology, Fundamental Theorem, Matrix Notation, Graphical Solutions, Quiz

0.1 Administrative

- In the first 10 minutes of this tutorial, we will be going through a review of how to convert between forms of linear programming problems, and how to set up a linear programming problem. This will be preparation for your quiz. In the remaining 15 minutes of this tutorial, we will go through assignment 2 and related topics.
- We will start our quiz at 6:40 PM. You will have **25 minutes** to complete the quiz. It will end at 7:05 PM. You will have until 7:20 PM to submit. If you finish early, feel free to submit and leave early. If you are having trouble submitting to Quercus, please email your quiz to me at mariam.walaa@mail.utoronto.ca or through Quercus Inbox.
- **Next week**: Quiz 2, Extreme Point Theorem, Convexity, Basic Solutions
- **Your tentative midterm date:** Saturday, March 5.
- Any questions about assignment 1 solutions?

0.2 Learning Objectives

- Understand the fundamental theorem of linear programming.
- Understand fundamental terminology including feasible, optimal, slack variable, convex.
- Solve a 2D linear programming problem graphically.
- Convert a linear programming problem to its matrix equivalent.

0.3 Definitions & Terminology

- Feasible region: The feasible region is the set of all feasible solutions to a linear programming problem. All points in the feasible region satisfy the linear programming problem.

- Feasible solution: A feasible solution is a single point in the feasible region. A feasible solution satisfies the linear programming problem but is not necessarily an optimal solution.

- Optimal solution: An optimal solution is a feasible solution that is also a maximum solution or a minimum solution. A feasible solution that is not a maximum or minimum of a feasible region is not an optimal solution.

- Slack variable: When we want to convert from $\leq$ to $=$ or $\geq$ to $=$, we use a **slack variable** u to balance the inequality. In the first case, we need to add some non-negative number u to make an equality. In the second case, we need to subtract some non-negative number v to make an equality.

  Note: Separate slack variables need to be set for separate inequalities.

- Convex set: A subset K of $\mathbb{R}^n$ is called a **convex set** if for any x_1 and x_2 in K,

  $$x = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1$$

  is also in K. In other words, any point on this line segment x is also in K.

  A convex set is **bounded** if it can be enclosed in a rectangle $\{x \in \mathbb{R}^n | a_i \leq x_i \leq b_i\}$.

0.4 Fundamental Theorem of Linear Programming

Let f be a linear function.
Let U be a non-empty region in R^2.
Let U be defined by linear inequalities, including its boundaries.

- **If U is bounded:**
  - f has a maximum and a minimum on U.
  - The maximum and minimum must occur at corner points of U.
- **If U is unbounded and f has a maximum or minimum:**
  - The maximum or minimum must occur at a corner point of U.

Note: If x is a corner point of U, x cannot also be an interior point of U. It is possible that a corner point x of a line segment $\lambda_1 x + (1 - \lambda_1)x$, $\lambda_1 \in [0, 1]$ is an interior point of another line segment $\lambda_2 x + (1 - \lambda_2)x$, $\lambda_2 \in [0, 1]$.

0.5 Matrix Notation

A canonical linear programming problem is defined in **matrix notation** as follows.

$$\begin{aligned} \max \quad & z = c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

A standard linear programming problem is defined in **matrix notation** as follows.

$$\begin{aligned} \max \quad & z = c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Recall: The components of linear programming models.

1. Decision variables or unknowns: This would be the x_i's in the vector x.
2. Parameters: This would be the coefficients a_{ij} in the matrix A, and the c_i's in the vector c.

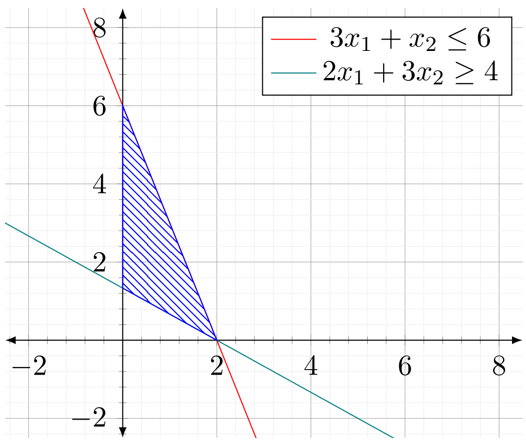
   Note: Think of the "parameters" as the known values which you use to **describe** the model.
3. Constraints: This would be the constants b_i in the vector b.
4. Objective function: This would be the function z. z is just a constant.

Note: Slack variables must be added to the notation when converting from canonical form to standard form. This will increase the size of the matrices and vectors.

0.6 Graphical Solutions

We can sometimes solve some linear programming problems in 2D or 3D via a graphical solution. Below is an example.

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 3x_1 + x_2 \leq 6 \\ & 2x_1 + 3x_2 \geq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$



Note that we only plot the constraints! The objective function is not plotted. We sometimes plot the objective function for certain feasible solutions and we call that the **level set**.
View the solution to this particular problem here: Online Optimizer

0.7 Convexity Proofs

Next week!