Tutorial 3: Convexity, Graphical Solution, Basic Solutions

0.1 Administrative

- In the first 15 minutes of this tutorial, we will be going through graphical solutions and convexity, which will be preparation for your quiz. In the remaining time until your quiz, we will go through the basic feasible solutions algorithm.
- We will start the quiz at 6:35 PM. You will receive your quiz in Quercus Inbox. You will have 25 minutes to complete the quiz and 10 minutes to submit the quiz. If you are having trouble submitting to Quercus, please email your quiz to me at mariam.walaa@mail.utoronto.ca or through Quercus Inbox.
- **Next week**: Quiz 3, Basic Solutions, Simplex Method
- **Your tentative midterm date:** Saturday, March 5.

0.2 Learning Objectives

In this tutorial, you should expect to achieve the following:

- Learn how to solve LPPs using graphical solution.
- Understand convexity terminology, including convex sets and convex combinations.
- Learn the Basic Feasible Solutions algorithm.

0.3 Recall

By now, you have learned and should be able to recall definitions for the following:

- Forms of LPPs
- Matrix notation
- Feasible regions
- Feasible solutions
- Optimal solutions
- Graphical solutions
- Convex sets

0.4 Matrix Notation

A linear programming problem is defined in **matrix notation** as follows.
Find a vector $x \in \mathbb{R}^n$ that will

$$\begin{aligned} \max \quad & z = c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Note: Writing $Ax \leq b$ means each entry of Ax is less than or equal to the corresponding entry in b. Similarly, writing $x \geq 0$ means each entry of x is greater than or equal to 0.

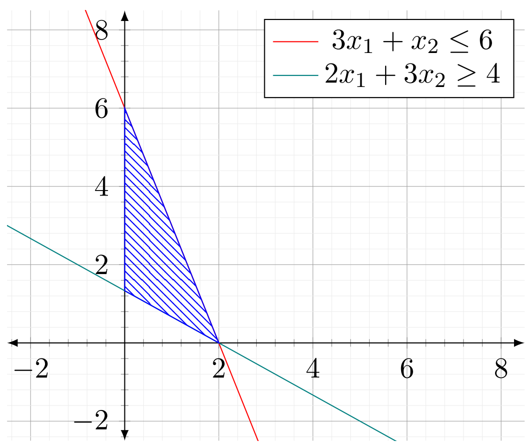
Exercise: Check that the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a feasible solution to the following linear programming problem

$$\begin{aligned} \max \quad & z = \begin{bmatrix} 120 & 100 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 8 \\ 15 \end{bmatrix} \\ & \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \end{aligned}$$

0.5 Graphical Solutions

We can sometimes solve some linear programming problems in 2D or 3D via a graphical solution. Below is an example.

$$
\begin{aligned}
\text{maximize} \quad & 3x_1 + 2x_2 \\
\text{subject to} \quad & 3x_1 + x_2 \leq 6 \\
& 2x_1 + 3x_2 \geq 4 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$



Note that we only plot the constraints! The objective function is not plotted. We sometimes plot the objective function for certain feasible solutions and we call that the **level set**.
View the solution to this particular problem here: Online Optimizer

0.6 Closed Half-Spaces & Convexity

Reading: Section 1.3 Geometry of Linear Programming Problems

Definition. *Closed Half-Space.* A closed half-space is a set of points $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ that satisfy the constraint $a^T x \leq b_i$ where $a^T = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$.

- The shaded areas in Figure 1.4(a) and Figure 1.4(b) are both **closed half-spaces**.
- By Theorem 1.1, a closed half-space is a convex set.
- By Theorem 1.3, the intersection of a finite set of convex sets is a convex set.
- Therefore, the region in Figure 1.4(c) is a convex set.

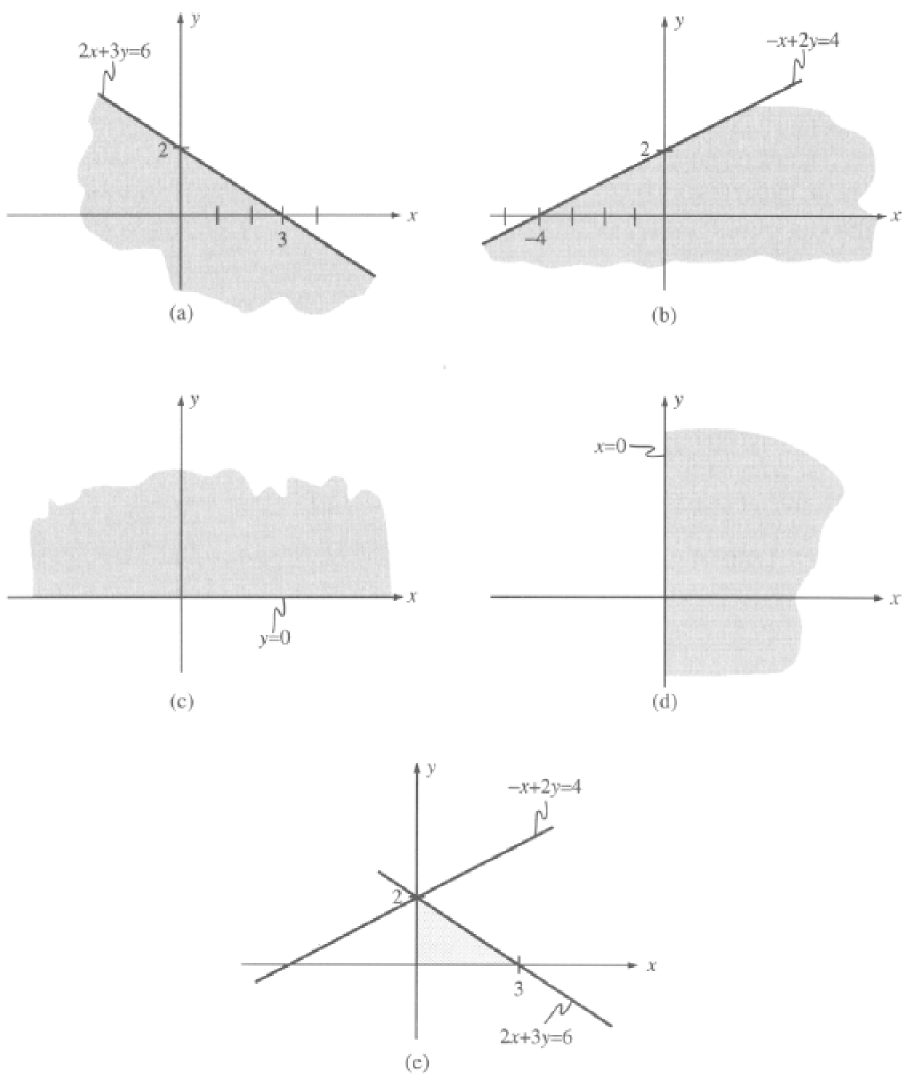


FIGURE 1.4 Set of all feasible solutions (two dimensions).

0.7 How to find all basic solutions to an LPP

Reading: Section 1.5 Basic Solutions

We typically have s decision variables and m constraints in a LPP where $s \geq m$. Typically $s \geq m$ and we therefore have some linearly dependent columns that we need to get rid of in order to solve the system of linear equations. To do this, we follow the basic feasible solution algorithm.

Basic Feasible Solution Algorithm

1. Set the LPP to canonical form.
2. Find m linearly independent column vectors in the matrix A. Call this matrix A'.
3. For the remaining columns which are linearly dependent to the columns in matrix A', set the corresponding values in the vector x to 0.
4. Set this matrix A' multiplied by x' to the corresponding values of b and solve the system of linear equations. What follows is a *basic* solution to your LPP. (**Note:** It is not necessarily a basic *feasible* solution.)
   - Repeat steps 2 - 4 for every set of m linearly independent column vectors.
5. For each basic solution, select those that satisfy all the constraints of the LPP. You now have a list of the basic feasible solutions.
6. For each basic feasible solution, compute the objective function z and choose the basic feasible solution x resulting in the maximum or minimum value depending on your LPP.

0.8 Understanding Basic Solutions Terminology

- **Basic solution:** A basic solution is obtained when solving the system of linear equations pertaining to a linearly independent set of columns of A.
- **Basic feasible solution:** A basic feasible solution is a basic solution that is feasible for the LPP. A basic feasible solution is equivalent to an extreme point of the LPP, and there are finitely many basic feasible solutions.
- **Non-basic variable:** These are the $s - m$ variables x_i that we set to 0.
- **Basic variable:** These are the m variables x_i that we do NOT set to 0.