MATB61 Midterm Preparation - Summary of Theorems

Refer to Textbook Chapters:

- 1 Introduction to Linear Programming
- 2 The Simplex Method
- 3 Further Topics in Linear Programming

0.1 Summary of Important Theorems and Definitions in Chapter 1

Definition 1.1 *A vector* $x \in \mathbb{R}^n$ *satisfying the constraints of a linear programming problem is called a* ***feasible solution*** *to the problem.*

Definition 1.2 *A feasible solution that maximizes or minimizes the objective function of a linear programming problem is called an* ***optimal solution****.*

Definition 1.3 *A subset* S *of* $\mathbb{R}^n$ *is called convex if for any two distinct points* x_1 *and* x_2 *in* S*, the line segment joining* x_1 *and* x_2 *lies in* S*. That is,* S *is convex if, whenever* x_1 *and* $x_2 \in S$*, so is* $x = \lambda x_1 + (1 - \lambda)x_2$ *for* $0 \leq \lambda \leq 1$*.*

Definition 1.4 *A point* $x \in \mathbb{R}^n$ *is a convex combination of the points* $x_1, x_2, ..., x_r$ *in* R^n *if for some real numbers* $c_1, c_2, ..., c_r$ *which satisfy* $\sum_{i=1}^{r} c_i = 1$ *and* $c_i \geq 0$*,* $1 \leq i \leq r$*, we have* $x = \sum_{i=1}^{r} c_i x_i$*.*

Theorem 1.1 *A closed half-space is a convex set.*

Theorem 1.2 *A hyper-plane is a convex set.*

Theorem 1.3 *The intersection of a finite collection of convex sets is convex.*

Theorem 1.4 *Let* A *be an* $m \times n$ *matrix, and let* b *be a vector in* $\mathbb{R}^m$*. The set of solutions to the system of linear equations* $Ax = b$*, if non-empty, is a convex set.*

Theorem 1.5 *The set of all convex combinations of a finite set of points in* $\mathbb{R}^n$ *is a convex set.*

Definition 1.5 *A point* u *in a convex set* S *is called an extreme point of* S *if it is not an interior point of any line segment in* S*. That is,* u *is an extreme point of* S *if there are no distinct points* x_1 *and* x_2 *in* S *such that* $u = \lambda x_1 + (1 - \lambda)x_2, 0 < \lambda < 1$*.*

Theorem 1.6 *Let* S *be a convex set in* $\mathbb{R}^n$*. A point* u *in* S *is an extreme point of* S *if and only if* u *is not a convex combination of other points of* S*.*

Theorem 1.7 *Let* S *be the set of feasible solutions to a general linear programming problem.*

1. *If* S *is non-empty and bounded, then an optimal solution to the problem exists and occurs at an extreme point.*
2. *If* S *is non-empty and not bounded and if an optimal solution to the problem exists, then an optimal solution occurs at an extreme point.*

3. If an optimal solution to the problem does not exist, then either S is empty or S is unbounded.

Theorem 1.8 *Suppose that the last m columns of A, which we denote by $A'_1, A'_2, ..., A'_m$, are linearly independent and suppose that $x'_1A'_1 + x'_2A'_2 + ... + x'_mA'_m = b$, where $x'_i \geq 0$ for $i = 1, 2, ..., m$. Then the point $x = (0, 0, ..., 0, x'_1, x'_2, ..., x'_m)$ is an extreme point of S.*

Theorem 1.9 *If $x = (x_1, x_2, ..., x_s)$ is an extreme point of S, then the columns of A that correspond to positive x_j form a linearly independent set of vectors in R^m.*

Theorem 1.10 *At most m components of any extreme points of S can be positive. The rest must be zero.*

Theorem 1.11 *Consider the linear programming problem*

$$\max \quad z = c^\top x \tag{1}$$

$$\begin{aligned} \textit{subject to} \quad Ax &= b \\ x &\geq 0 \end{aligned} \tag{2}$$

For the linear programming problem determined by (1) and (2), every basic feasible solution is an extreme point, and, conversely, every extreme point is a basic feasible solution.

Theorem 1.12 *The problem determined by (1) and (2) has a finite number of basic feasible solutions.*

Theorem 1.13 *Every extreme point of S yields an extreme point of S' when slack variables are added. Conversely, every extreme point of S', when truncated, yields an extreme point of S.*

Theorem 1.14 *The convex set S of all feasible solutions to a linear programming problem in standard form has a finite number of extreme points.*

Definition 1.6 *A basic feasible solution to the linear programming problem given by (1) and (2) is a basic solution that is also feasible.*

1.2 Summary of Important Theorems and Definitions in Chapter 23

Definition 2.1 *Two distinct extreme points in S' are said to be adjacent if as basic feasible solutions they have all but one basic variable in common.*

2.3 Summary of Important Theorems and Definitions in Chapter 3

Theorem 3.1 *Given a primal problem as follows*

$$\begin{aligned} \max \quad & z = c^\top x \\ \textit{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

The dual of its dual is again the primal problem.

Theorem 3.2 *The linear programming problem in canonical form given by*

$$\begin{aligned} \max \quad & z = c^\top x \\ \textit{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

has for its dual the linear programming problem

$$\begin{aligned} \max \quad & z' = b^\top w \\ \textit{subject to} \quad & A^T w \geq c \\ & w \ \textit{unrestricted} \end{aligned}$$