Tutorial 1: Intro, Linear Programs, Forms, Applications

0.1 Admin

- The first quiz will take place next week for 20-30 minutes in tutorial.
- It will be on assignment 1, lecture materials, and readings.
- You will need to submit your quiz as a PDF on Quercus and complete it with cameras on.
- In this tutorial, we'll be going through an introductory overview of linear programming as well as some of the 15 assignment 1 questions.
- **Next week**: Quiz 1, Assignment 2, Graphical solutions, Matrix notation, Convexity

0.2 Learning Objectives

- Understand the components of a linear programming problem.
- Understand the types of linear programming forms.
- Learn how to solve some applications of linear programming.
- Learn how to convert between different forms of linear programming models.
- Understand the fundamental theorem of linear programming.

0.3 Components of a Linear Program

What is a linear program? "Linear programming finds the least expensive way to meet given needs with available resources."

1. Decision variables or unknowns: The variables that we want to solve for. Ideally, we want these to correspond to an optimal allocation of the scarce resources presented by the model, such as how much to invest in each type of financial security in a portfolio. Using math notation, this would be the x_i's.

2. Parameters: The inputs that are known by us, which are typically estimates or approximations in a real-life scenario. For example, this could be the cost of purchasing ingredients needed to produce the final goods. Using math notation, this would be the coefficients a_{ij}.

3. Constraints: The conditions that limit which values our variables can take. For example, there may be a limit to how much we can spend on the cost of purchasing ingredients. Using math notation, this would be the constants b_i.

4. Objective function: The function that we want to maximize or minimize. This is typically something like maximizing profit or minimizing cost. Using math notation, this would be the function z.

0.4 Forms of Linear Programs

1. **General**: Can be a minimization or maximization problem.

$$\begin{aligned} \text{minimize or maximize} \quad & c_1x_1 + c_2x_2 + ... + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n (\leq)(\geq)(=) b_1 \\ & a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n (\leq)(\geq)(=) b_2 \\ & ... \\ & a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n (\leq)(\geq)(=) b_m \end{aligned}$$

2. **Canonical**: Must be a maximization problem. All decision variables must be non-negative. All constraints need to be equalities.

$$\begin{aligned} \text{maximize} \quad & c_1x_1 + c_2x_2 + ... + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ & ... \\ & a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \\ & x_i \geq 0 \qquad \forall i = 1, 2, ..., n \end{aligned}$$

3. **Standard**: Must be a maximization problem. All decision variables must be non-negative. All constraints need to be inequalities.

$$\begin{aligned} \text{maximize} \quad & c_1x_1 + c_2x_2 + ... + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2 \\ & ... \\ & a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m \\ & x_i \geq 0 \qquad \forall i = 1, 2, ..., n \end{aligned}$$

0.5 Converting Between Different Forms of Linear Programs

- Convert minimization problem to a maximization problem.
  - Multiply the objective function by -1 to turn it into a maximization problem.
- Handle all inequalities and/or equalities.
  - **General to Standard:** Convert all $\geq$ and $=$ to $\leq$.
  - **General to Canonical:** Convert all $\geq$ and $\leq$ to $=$.
  - **Canonical to Standard:** Convert all $=$ to $\leq$.
  - **Standard to Canonical:** Convert all $\leq$ to $=$.
- Handle all variables x_i with no non-negative constraint.
  - x_i is a real number: Think of a negative constraint x_i as being the difference of two non-negative numbers x_i^+ and x_i^-. Then substitute $x_i = x_i^+ - x_i^-$ in your definition of the linear program model.
  - $x_i \leq 0$: Take $x_i' = -x_i$. Then you have a non-negative variable.
  - Other scenarios need to be handled depending on the case.

0.6 Assignment 1 Problems

1. **Identify the LP components**: A pesticide blends two existing pesticides PEST and BUG. Each kilogram of PEST requires 30g of carbaryl and 40g of Malathion, while each kilogram of BUG requires 40g of carbaryl and 20g of Malathion. The final blend must consist of at least 120g of carbaryl and at most 80g of Malathion. If each kilogram of PEST costs \$3 and each kilogram of BUG costs \$2.50, how many kilograms of each should be used to minimize the cost?

   (a) **What are the x_i's? (Decision variables or unknowns)**:

   Amounts of PEST and BUG. We need to determine how much of each is needed.

   (b) **What are the a_{ij}'s? (Parameters)**:

   We know the cost of PEST and BUG, and the required amounts of carbaryl and Malathion for each of PEST and BUG are also known.

   (c) **What are the b_i's? (Constraints)**:

   We know the final blend needs at least 120g of carbaryl and at most 80g of Malathion.

   (d) **What is z? (Objective function)**:

   We want to minimize the cost of each of PEST and BUG.

2. **Fill in the blanks**: A doctor treats a disease by perscribing two brand-name pills, Palium and Timade. Palium costs \$0.40/pill and Timade costs \$0.30/pill. Each contains SND plus an activator. Each dosage requires at least 10mg of SND per day. Activators are limited to a maximum of 2mg per day. Each contain 0.5mg of activator per pill. Palium contains 4mg of SND and Timade contains 2mg of SND per day. How many of each pill per day should the doctor prescribe to minimize costs, provide enough SND, and not exceed the maximum limit of activator?

$$\begin{aligned} \text{minimize} \quad & \text{Objective function} : \_\_x_1 + \_\_x_2 \\ \text{subject to} \quad & \text{Constraint1} : \_\_x_1 + \_\_x_2 \le \_\_ \\ & \text{Constraint2} : \_\_x_1 + \_\_x_2 \ge \_\_ \\ & x_i \ge 0 \qquad \forall i = 1, 2 \end{aligned}$$

3. **Book binding**: A book publisher plans to bind a bestseller in 3 different bindings: paperback, book club, and library. Each book goes through a sewing and gluing process. The time required for each process is given in the table below.

	Paperback	*Book club*	*Library*
Sewing (min)	2	2	3
Gluing (min)	4	6	10

Suppose the sewing process is available 7hr per day and the gluing process is available 10hr per day. Assume profits are \$0.50 for paperback, \$0.80 for book club, and \$1.20 for library.

How many books should be manufactured in each binding to maximize profit?

Make sure to get your units of time right.

4. **Air pollution problem**: An airshed's main source of air pollution comes from a cement-manufacturing plant. This plant produces 2.5 million barrels of cement and emits 2lb of dust per barrel of cement. The plant plans to install precipitators A and B for each production process to reduce emissions. The goal is to reduce emissions by at least 84%. Precipitator A reduces emissions by 1.5lb of dust per barrel and costs \$0.14/barrel to operate, while Precipitator B reduces emissions by 1.8 lb of dust/barrel and would cost \$0.18/barrel to operate. How many barrels should be produced using each precipitator to minimize the cost of controls and meet the EPA requirements?

   Set the constraint for EPA requirements carefully. The goal is to reduce by at least 84%.

   - x is the number of barrels produced with Precipitator A.
   - y is the number of barrels produced with Precipitator B.

$$1.5x + 1.8y \geq 84\% \cdot (2x + 2y)$$

   The total amount of emissions reduced by x and y barrels produced using the new precipitators needs to be at least 84% of the emissions that would have been produced by that many barrels.

5. **Investment problem**: A trust fund wants to invest up to \$200,000 in 3 types of financial securities: a 9%-dividend stock, a 4%-dividend stock, and a 5%-interest bond. The total amount invested in stocks must be at most half of the total amount invested. Moreover, the amount invested in the 9%-dividend stock must not exceed \$40K, and the amount invested in the bond must be at least \$70K. How much should be invested in each security to maximize return?

$$x_1 + x_2 \leq \frac{1}{2} \cdot (x_1 + x_2 + x_3)$$

6. **Mixing problem**: A food store packages 3 snack foods: Chewy, Crunchy, and Nutty, by mixing sunflower seeds, raisins, and peanuts. The specifications for each mixture are given in the table below.

Mixture	*Sunflower seeds*	*Raisins*	*Peanuts*	*Selling price per kilogram ($)*
Chewy		At least 60%	At most 20%	2.00
Crunchy	At least 60%			1.60
Nutty	At most 20%		At least 60%	1.20

Consider each of the above statements as a single constraint.

For example: The Chewy Mixture requires at least 60% raisins. Let x_{11}, x_{12}, x_{13} be the proportions of sunflower seeds, raisins, and peanuts in Chewy respectively. Then we must have

$$x_{12} \geq 60\% \cdot (x_{11} + x_{12} + x_{13})$$

Repeat the same process for the remaining constraints.

7. **Convert general to standard**:

$$\begin{aligned}
\text{maximize} \quad & 4x_1 + 2x_2 + x_3 \\
\text{subject to} \quad & -x_1 + 3x_2 - x_3 \geq 1 \\
& 5x_1 + 3x_3 = 5 \\
& x_1 + x_2 + x_3 \leq 1 \\
& x_1 \geq -1 \\
& x_2 \leq 2 \\
& x_3 \geq 0
\end{aligned}$$

(a) Make decision variables non-negative.

$$\begin{aligned}
x_1 &\geq -1 \\
x_1 + 1 &\geq 0 \\
x_1' := x_1 + 1 &\geq 0
\end{aligned}$$

Repeat the process to get $x_2' := 2 - x_2 \geq 0$.

(b) Change constraints to inequalities.

$$\begin{aligned}
\text{maximize} \quad & 4x_1' - 2x_2' + x_3 \\
\text{subject to} \quad & x_1' + 3x_2' + x_3 \leq 6 \\
& 5x_1' + 3x_3 \leq 10 \\
& -5x_1' - 3x_3 \leq -10 \\
& x_1' - x_2' + x_3 \leq 0 \\
& x_1' \geq 0 \\
& x_2' \geq 0 \\
& x_3 \geq 0
\end{aligned}$$

Plug in the values of x_1 and x_2 and simplify to get the final form.

Make sure you finish with all non-negative decision variables at the end.

8. **Convert general to canonical**:

$$\begin{aligned} \text{minimize} \quad & x_1 - 12x_2 + x_3 \\ \text{subject to} \quad & 5x_1 - x_2 - 2x_3 = 10 \\ & 2x_1 + x_2 - 10x_3 \geq -30 \\ & x_2 \leq 0 \\ & 1 \leq x_3 \leq 4 \end{aligned}$$

Always start with fixing x_i's to be non-negative.

Here, take $x_1 = x_1^+ - x_1^-$, $x_2' := -x_2$, and $x_3' := x_3 - 1$. Then $x_2 = -x_2'$ and $x_3 = x_3' + 1$.

Now all decision variables are non-negative, and you can plug them in (Exercise).

0.7 Fundamental Theorem of Linear Programming

Let f be a linear function.
Let U be a non-empty region in R^2 such that it is defined by linear inequalities.

- **If U is bounded:**
  - f has a maximum and a minimum on U.
  - These values occur at corner points of U.
- **If U is unbounded and if f has a maximum or minimum:**
  - It occurs at a corner point of U.