Tutorial 5: Special Cases in the Simplex Method

0.1 Administrative

- **This week**: Reading week. No tutorial or office hours. Study well!
- **Next week:** In-person tutorial and midterm review. No quiz.
- **Your midterm date:** Saturday, March 5.
- **My office hours:** Fridays @ 6:30PM on Zoom

0.2 Recall

By now, you have learned and should be able to recall definitions for the following:

- Forms of Linear Programming Problems (LPP)
- Matrix notation
- Feasible regions
- Feasible solutions and optimal solutions
- Convex sets and combinations
- Graphical solutions to LPPs
- Basic feasible solution algorithm

0.3 Learning Objectives

In this tutorial, you should expect to achieve the following:

- Learn about degeneracy and cycling cases when using the Simplex Method.
- Learn how to avoid degeneracy and cycling by using Bland's Rule.
- Learn the Two-Phase and Big M algorithms for the Simplex Method.

0.4 Degeneracy and Cycling

Reading: Section 2.2 Degeneracy and Cycling

We have learned the simplex method which is a more efficient way to search for the optimal value in the finite set of extreme points (or BFS). We now know how to analyze the simplex tableau to determine the current BFS as well as the current optimal value, determine whether the current tableau is a terminal tableau, and whether the objective function is bounded or unbounded.

In addition, we learned how to translate the results of the simplex tableau back to the original LP problem. We also learned that, given a 2-dimensional (or even 3-dimensional) LP problem, we now have multiple ways of solving the problem whether through the graphical method, BFS algorithm, or simplex method.

More specifically, in learning the simplex method, we learned about the importance of the θ-ratio in selecting the departing variable.

We now want to investigate more specific cases:

- What happens if there is a negative θ-ratio in the pivotal column?
- What happens if there are multiple rows with the same θ-ratio?

Definition. *Degeneracy.* A basic feasible solution x that satisfies an LP problem is a degenerate solution if any of its m basic variables equal to zero.

Definition. *Cycling.* An LP problem that has a degenerate solution and results in an iterative loop of the simplex method is cycling.

This degeneracy is a direct result of the case where multiple rows contain the same θ-ratio. In other words, we have multiple candidates for the departing variable, and both choices should lead to the optimal solution. In few cases, this degeneracy may lead to cycling, where the simplex method does not terminate with a particular optimal solution. Therefore, the Bland's Rule has been developed to avoid cycling by terminating the simplex method algorithm.

0.5 Bland's Rule

Reading: Section 2.2 Degeneracy and Cycling

The Bland's Rule works as follows.

Bland's Rule for Degeneracy & Cycling

1. **Select the pivotal column** from the objective row by choosing the column with the smallest subscript among those that correspond to a negative entry in the objective row. (Note: You are NOT choosing the most negative entry).
2. **Select the pivotal row** from the pivotal column by choosing the row with the smallest subscript among those that have equal θ-ratios. (Note: If there are no equal θ-ratios, the Bland's Rule is unnecessary)

Note that the rule prevents the simplex method from cycling by terminating the algorithm, but does not result in an optimal solution.

Thinking Question: Why is the above rule successful in terminating the algorithm in the case of degeneracy?

0.6 Two-Phase Method

Reading: Section 2.3 Artificial Variables

So far, in all LP problems we have worked with, the LP problems have been first converted to standard form and then canonical form, where we then have non-negative b_i's as a result, and therefore the simplex method requirements had been satisfied. However, there are cases when the b_i's may be negative, and it is not possible to convert these to non-negative b_i's. (Recall that we need b_i's to be non-negative in order to have an initial BFS).

More specifically, we are unsuccessful in getting a non-negative b_i in both of the following cases:

- If $a_{1i}x_1 + a_{2i}x_2 + ... + a_{ni}x_n \geq b_i$ where $b_i \geq 0$, we get $b_i \leq 0$ by converting to standard form
- If we add a slack variable by converting to canonical form, we get a negative coefficient on the slack variable in the process of flipping the signs

We therefore require a method that satisfies this more general case.

Two-Phase Method (Generalized Simplex Algorithm)

1. Convert the LP problem to its canonical form.
2. For constraints where no appropriate slack variable exists (i.e., a slack variable with a coefficient of +1), add an artificial variable y_i to the ith constraint.
3. Define a new objective function that minimizes the sum of all artificial variables.
   - **Why?** We aim to have a solution where all artificial variables are set to zero, so that we apply this solution to our original LP problem as an initial BFS.
4. Write the objective function in terms of the variables x_i and convert to a maximization problem. This is now the auxiliary LP problem.
5. Write the corresponding simplex tableau for the auxiliary LP problem.
6. Follow the simplex algorithm to find the optimal solution. Multiple cases may arise:
   - Case 1: The optimal value is negative, which means that at least one artificial variable appears in the BFS. Why?
   - Case 2: The optimal value is zero, which means we can proceed with this optimal solution as the initial BFS in the original LP problem.
7. If the second case arises, proceed with the optimal solution as follows:
   (a) Remove columns corresponding to the artificial variables in the tableau.
   (b) Apply the steps of the simplex method for auxiliary LP problem on the objective row for the original LP problem.
   (c) Replace the objective row for the auxiliary LP problem with the transformed objective row for the original LP problem.

0.7 Big M Method

Reading: Section 2.3 Artificial Variables

The Big M method is an alternative method to solve the problem of having no initial BFS due to negative b_i's. This method works by adding a penalty term M (where M is a very large positive quantity) to each non-zero artificial variable in order to ensure the optimal solution cannot be one that uses an artificial variable. The algorithm works as follows.

Two-Phase Method (Generalized Simplex Algorithm)

1. Convert the LP problem to its canonical form.
2. For constraints where no appropriate slack variable exists (i.e., a slack variable with a coefficient of +1), add an artificial variable y_i to the ith constraint.
   - **Note:** The first two steps are the same as in the Two-Phase method.
3. Define the objective function for the original LP problem, with an additional negative penalty term as follows: $-(My_1 + My_2 + ...My_m)$ where each m corresponds to a single constraint. Recall that $y_i = 0$ when no artificial variable is used in the solution, and $y_i \neq 0$ when an artificial variable is used for some value y_i.
4. Re-express the objective function in terms of the x_i's and coefficients M.
5. Solve with the simplex method.