MTH141 Quiz 6 Solution

TA & Solution Author: Mariam Walaa

October 24, 2023

Question

Solve the following homogeneous system using Gauss-Jordan elimination, write its solution as a linear combination, and find its solution space N_A and $\dim(N_A)$:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 3x_1 - 3x_2 + 2x_3 &= 0 \\ -x_1 - 11x_2 + 6x_3 &= 0 \end{aligned}$$

Solution

To solve the given homogeneous system of linear equations using Gauss-Jordan elimination, we first write the augmented matrix and then perform row operations to reduce it to its row-echelon form (REF) and further to the reduced row-echelon form (RREF).

The system of equations is as follows:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 3x_1 - 3x_2 + 2x_3 &= 0 \\ -x_1 - 11x_2 + 6x_3 &= 0 \end{aligned}$$

Now, let's form the augmented matrix $[A|0]$, where A represents the coefficients of the variables, and 0 represents the right-hand side of the equations: (2 marks for setting up augmented matrix)

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 3 & -3 & 2 & | & 0 \\ -1 & -11 & 6 & | & 0 \end{bmatrix}$$

Step 1: Row 2 = Row 2 $-$ 3 $\cdot$ Row 1

Step 2: Row 3 = Row 3 + Row 1

New augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -9 & 5 & | & 0 \\ 0 & -9 & 5 & | & 0 \end{bmatrix}$$

Step 3: Row 2 = Row 2 $-$ Row 3

New augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & -9 & 5 & | & 0 \end{bmatrix}$$

Now, we have the row-echelon form (REF) of the matrix. Let's continue with the Gauss-Jordan elimination to reduce it to the reduced row-echelon form (RREF).

Step 4: Row 3 = Row $3/(-9)$

New augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{5}{9} & | & 0 \end{bmatrix}$$

Step 5: Row 1 = Row 1 + Row 3

New augmented matrix:

$$\begin{bmatrix} 1 & 2 & -\frac{14}{9} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{5}{9} & | & 0 \end{bmatrix}$$

Step 6: Row 1 = Row $1 - 2 \cdot$ Row 2

Final augmented matrix: (3 marks for correct final matrix i.e. correct row operations)

$$\begin{bmatrix} 1 & 0 & \frac{1}{9} & | & 0 \\ 0 & 1 & -\frac{5}{9} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Now, the matrix is in reduced row-echelon form (RREF). From this, we can extract the solution to the system:

1. $x_1 + 2x_2 - \dfrac{14}{9}x_3 = 0$
2. $x_3 = 0$
3. $\dfrac{1}{9}x_2 - \dfrac{5}{9}x_3 = 0$

Now, express x_1, x_2, and x_3 in terms of a parameter (let's call it t): (1 mark for identifying solution form)

$$x_1 = \frac{-1}{9}t$$
$$x_2 = \frac{5}{9}t$$
$$x_3 = t$$

So, the solution to the homogeneous system is above.

This solution can also be written as a linear combination: (2 marks for identifying solution as linear combination)

$$\mathbf{x} = t\left(\frac{-1}{9}, \frac{5}{9}, t\right)$$

The solution space $N_A = \{t\mathbf{x} : t \in \mathbb{R}\}$ is the set of all possible solutions to the homogeneous system (1 mark).

Its dimension ($\dim(N_A)$) is 1, as there is one parameter (t) that defines the solution space (1 mark).