MTH141 Quiz 2 Solution

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Question

Find the area of the parallelogram defined by two vectors $\mathbf{a} = (0, 2, -1)$ and $\mathbf{b} = (-5, 1, 1)$.

Solution

We can find the area of the parallelogram using either of the following formulas:

$$\begin{aligned}
&\text{Method 1:} \quad A(\mathbf{a}, \mathbf{b}) = \sqrt{G(\mathbf{a}, \mathbf{b})} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \\
&\text{Method 2:} \quad A(\mathbf{a}, \mathbf{b}) = |\mathbf{a}||\mathbf{b} - \text{proj}_{\mathbf{a}}(\mathbf{b})|
\end{aligned}$$

First, let's calculate the values needed for these formulas:

$$\begin{aligned}
|\mathbf{a}|^2 &= (0^2 + 2^2 + (-1)^2) = 5 \\
|\mathbf{b}|^2 &= (-5^2 + 1^2 + 1^2) = 27 \\
\mathbf{a} \cdot \mathbf{b} &= (0 \cdot (-5) + 2 \cdot 1 + (-1) \cdot 1) = 1
\end{aligned}$$

Next, we'll calculate $\text{proj}_{\mathbf{a}}(\mathbf{b})$:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \left(\frac{1}{5}\right)(0, 2, -1) = \left(0, \frac{2}{5}, -\frac{1}{5}\right)$$

Now, we can use either of the methods to find the area:
Method 1:

$$\begin{aligned}
A(\mathbf{a}, \mathbf{b}) &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= \sqrt{(5)(27) - (1^2)} \\
&= \sqrt{135 - 1} \\
&= \sqrt{134}
\end{aligned}$$

Using method 1, the area is then $\sqrt{134}$.
Method 2:

$$\begin{aligned}
A(\mathbf{a}, \mathbf{b}) &= |\mathbf{a}||\mathbf{b} - \text{proj}_{\mathbf{a}}(\mathbf{b})| \\
&= \sqrt{5} \cdot ((-5, 1, 1) - (0, \frac{2}{5}, -\frac{1}{5}) \\
&= ... \\
&= \sqrt{134}
\end{aligned}$$

So, the area of the parallelogram defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ using method 2 is also $\sqrt{134}$ (as expected).