MTH140 Lab 5

TA: Mariam Walaa

October 10-12, 2023

Administrative Items

Structure of Today's Lab

- Please provide feedback on our labs: https://tinyurl.com/MTH140Section17
- Your feedback will count for attendance.
- As usual, 1hr 20 minutes Collaborative Problem-Solving
- Quiz 2 (20-minute) will take place in last 30 minutes of lab
- Slides to labs 3-5 in **Lab Teaching Sessions** page on D2L.

Learning Objectives

This lab will help you achieve the following learning outcomes:

- Find the equation (slope and y-intercept) of a tangent line of a function (Q1)
- Find the derivative function using the definition of derivative (Q2)
- Use the graph of a function to sketch its derivative function (Q3)
- Find the derivative using differentiation rules (Q4)
- Use a given graph to calculate its derivative at various points (Q5)
- Calculate the velocity, acceleration, and time intervals where the function slows down or speeds up (Q6, Q7, Q8)

Recommended Textbook Sections

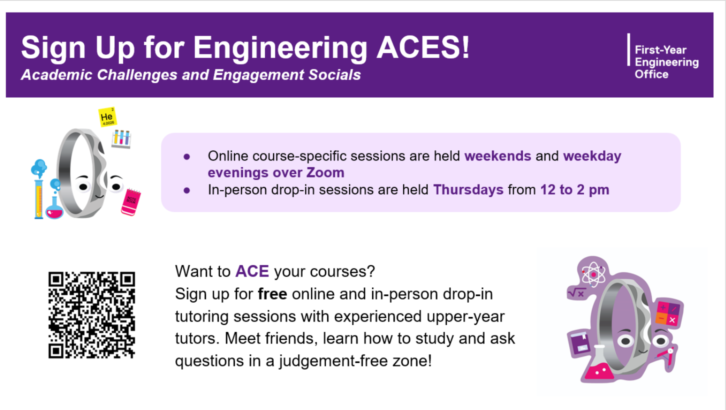
Please read the following sections from the textbook:

- 3.1 Defining the Derivative
- 3.2 The Derivative of a Function
- 3.3 Differentiation Rules
- 3.4 Derivatives as Rates of Change

These sections will provide you with valuable background information and concepts required to solve this week's problem set.

Announcements

Midterm Test Information: 120 minutes: Friday, Oct. 20, at 6:30 pm, In-person, Paper, and Pen.



In addition to ACES, the **Math and Computer Science Support centre** provides 24/7 drop-in support.

Exercise 1a(i): Find Slope of Tangent Line to a Curve

Find the slope of the tangent line to the curve $f(x) = \frac{x}{5} + 6$ at $x = -1$ using the limit formula. **Solution:** We can find the slope of the tangent line by computing the derivative of $f(x)$ and then evaluating it at $x = -1$. One way to compute the derivative is to use the derivative limit formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Exercise 1a(i): Find Slope of Tangent Line to Curve

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(-1+h)}{5} + 6 - \left(\frac{-1}{5} + 6\right)}{h} \\
&= \lim_{h\to 0} \frac{(-1+h+30) - (-1+30)}{5h} \\
&= \lim_{h\to 0} \frac{h}{5h} \\
&= \lim_{h\to 0} \frac{1}{5} \\
&= \frac{1}{5}
\end{aligned}
$$

So, the slope of the tangent line at $x = -1$ is $m = f'(-1) = \frac{1}{5}$.

Exercise 1a(ii): Find Y-Intercept of Tangent Line to Curve

Find the equation of the tangent line to the curve $f(x) = \frac{x}{5} + 6$ at $x = -1$. **Solution:** We've already found that the slope of the tangent line at $x = -1$ is $m = \frac{1}{5}$. To find the equation of the tangent line, we can use the point-slope form of a line to get the **y-intercept**:

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line. In this case, (x_1, y_1) is $(-1, f(-1))$.

$$y - f(-1) = \frac{1}{5}(x - (-1))$$

Now, substitute $f(-1)$ and simplify to get the equation of the tangent line.

Exercise 1a(ii): Find Y-Intercept of Tangent Line to Curve

$$y - f(-1) = \frac{1}{5}(x+1)$$

Now, let's simplify further:

$$y - \left(\frac{-1}{5} + 6\right) = \frac{1}{5}(x+1)$$

$$y + \frac{1}{5} - 6 = \frac{1}{5}(x+1)$$

$$y - \frac{29}{5} = \frac{1}{5}(x+1)$$

This is the equation of the tangent line to $f(x)$ at $x = -1$, unsimplified. Simplifying, we get $y = \frac{1}{5}x + 6$, with $b = 6$.

Problem 1b(i): Find Slope of Tangent Line to Curve

Find the slope of the tangent line to the curve $f(x) = 1 - x - x^2$ at $x = 0$ using the limit formula. **Solution:** We can find the slope of the tangent line by computing the derivative of $f(x)$ and then evaluating it at $x = 0$:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(a) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h\to 0} \frac{1 - (0+h) - (0+h)^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{1 - h - h^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{-h - h^2}{h} \\
&= \lim_{h\to 0} (-1 - h) \\
&= -1
\end{aligned}
$$

Problem 1b(i): Find Slope of Tangent Line to Curve

Find the equation of the tangent line to the curve $f(x) = 1 - x - x^2$ at $x = 0$. **Solution:** We've already found that the slope of the tangent line at $x = 0$ is $m = -1$. To find the equation of the tangent line, we can use the point-slope form of a line:

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line. In this case, (x_1, y_1) is $(0, f(0))$.

$$y - f(0) = -1(x - 0)$$

Now, substitute $f(0)$ and simplify to get the equation of the tangent line.

Problem 1b(i): Find Slope of Tangent Line to Curve

Some more computation to get the final equation.

$$y - f(0) = -1(x - 0)$$

$$y - (1 - 0 - 0^2) = -1(x)$$

$$y - 1 = -x$$

$$y = -x + 1$$

This is the equation of the tangent line to $f(x)$ at $x = 0$, with y-intercept $b = 1$.

Exercise 2: Definition of Derivative

Find the derivative $f'(x)$ of the function $f(x) = x + \frac{1}{x}$ using the definition of the derivative. **Solution:** To find $f'(x)$ using the definition of the derivative, we'll start with the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

First, let's compute $f(x)$ and $f(x+h)$:

$$f(x) = x + \frac{1}{x}$$

$$f(x+h) = x + h + \frac{1}{x+h}$$

Exercise 2: Definition of Derivative (Continued)

Now, we can substitute this and $f(x)$ into the derivative definition and simplify.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\left(x + h + \frac{1}{x+h}\right) - \left(x + \frac{1}{x}\right)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x + h + \frac{1}{x+h} - x - \frac{1}{x}}{h}$$

Exercise 2: Definition of Derivative (Continued)

Now, we can simplify further and factor out common terms.

$$f'(x) = \lim_{h\to 0} \frac{x + h + \frac{1}{x+h} - x - \frac{1}{x}}{h}$$

Simplify.

$$f'(x) = \lim_{h\to 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h}$$

Pull out the first h term into its own fraction (applying sum limit law).

$$f'(x) = \lim_{h\to 0} \frac{h}{h} + \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Next, let's find a common denominator in the numerator.

$$f'(x) = \lim_{h\to 0} 1 + \frac{\frac{x-x-h}{x(x+h)}}{h}$$

Exercise 2: Definition of Derivative (Continued)

Now, we can simplify the numerator further.

$$f'(x) = \lim_{h \to 0} 1 + \frac{\frac{x-x-h}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h \to 0} 1 + \frac{\frac{-h}{x(x+h)}}{h}$$

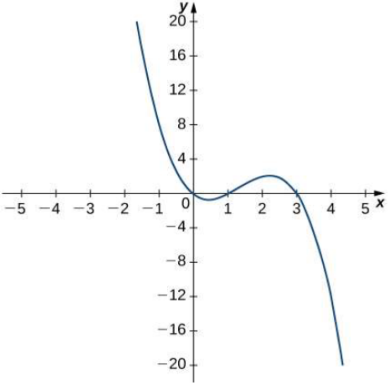
$$f'(x) = \lim_{h \to 0} 1 - \frac{1}{x(x+h)}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

Notice that hx and $(x+h)$ cancel out in the numerator and denominator.
So, the derivative $f'(x)$ of the function $f(x) = x + \frac{1}{x}$ is $f'(x) = 1 - \frac{1}{x^2}$.

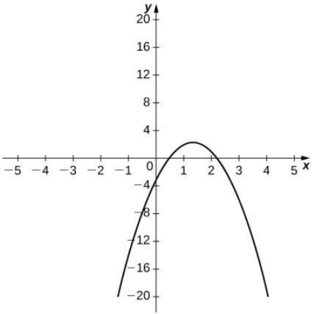
Exercise 3: Sketch Derivative

3. **[(v1) 3.2 Problem 64]** For the following exercise, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.



Solution:

Solution:



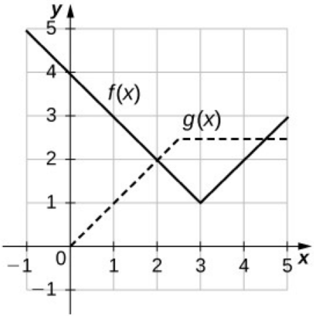
Additional Notes: The original function has the form of a cubic polynomial, hence we expect the derivative to be in the form of a quadratic. Make sure the inflection points and max/min points line up.

Exercise 5: Use Graph to find Derivative

5. **[(v1) 3.3 Problem 130, 132]** For the following exercises, use the figure below to find the derivatives $h'(1)$, $h'(3)$, and $h'(4)$ if they exist.

   (a) $h(x) = f(x) + g(x)$

   (b) $h(x) = \frac{f(x)}{g(x)}$



Exercise 5: Use Graph to find Derivative

Solution:

Solution:

(a) *Using the linearity of differentials:* $h'(x) = \left(f(x) + g(x)\right)' = f'(x) + g'(x)$

(i) *Observing the graph, at 1, the slope of* $f(x)$ *is* -1, *and the slope of g(x) is* 1. *These are then the derivatives of the functions at 1.*
Then, $h'(1) = f'(1) + g'(1) = -1 + 1 = 0$

(ii) *By observation,* $g'(3)$ *DNE, thus* $h'(3) = f'(3) + g'(3)$ *DNE.*

(iii) $f'(4) = 1$, $g'(4) = 0$, *Thus* $h'(4) = 1$

(b) *By the quotient rule:* $h'(x) = \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$

(i) *We have* $f(1) = 3, g(1) = 1$.

$$h'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{\left(g(1)\right)^2} = \frac{(-1)(1) - (3)(1)}{(1)^2} = -5$$

(ii) *As with (a) (ii), DNE since* $g'(3)$ *DNE*

(iii) $f(4) = 2$, $g(4) = 2.5$

$$h'(4) = \frac{(1)(2.5) - (2)(0)}{(2.5)^2} = \frac{1}{2.5} = \frac{2}{5}$$

Exercise 6: Application - Velocity & Acceleration

Question: We are given a function that represents position s of an object at time t: $s(t) = \frac{t}{1+t^2}$. Find the *velocity* and *acceleration*, and determine *time intervals* where function slows down or speeds up.

Exercise 6: Application - Velocity & Acceleration

Solution: **Step 1: Find the Velocity Function** We know that velocity $v(t)$ equals the derivative of position $s(t)$ with respect to time t.

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

Differentiating, we get:

$$v(t) = \frac{1-t^2}{(1+t^2)^2}$$

Exercise 6: Application - Velocity & Acceleration

Step 2: Find the Acceleration Function Acceleration $a(t)$ equals the derivative of velocity $v(t)$ with respect to time t.

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{(1+t^2)^2}\right)$$

Differentiating, we get:

$$a(t) = \frac{-2t(1+t^2)(3-t^2)}{(1+t^2)^4}$$

Now, we have the velocity and acceleration functions:

$$v(t) = \frac{1-t^2}{(1+t^2)^2}$$

$$a(t) = \frac{-2t(1+t^2)(3-t^2)}{(1+t^2)^4}$$

Exercise 6: Application - Velocity & Acceleration

Step 3: Determine Speeding Up and Slowing Down Intervals To find where the object is speeding up or slowing down, we need to consider the sign of acceleration.

- If $a(t) > 0$, the object is **speeding up**.
- If $a(t) < 0$, the object is **slowing down**.

Exercise 6: Application - Velocity & Acceleration

Slowing Down Interval: Notice that the denominator of $a(t)$ is necessarily positive, so we focus on the numerator. For $-2t(1+t^2)(3-t^2) > 0$, we have $t < -\sqrt{3}$ or $t > \sqrt{3}$. The object is slowing down for $t < -\sqrt{3}$ and $t > \sqrt{3}$.

Speeding Up Interval: For $-2t(1+t^2)(3-t^2) < 0$, we have $-\sqrt{3} < t < \sqrt{3}$. The object is speeding up for $-\sqrt{3} < t < \sqrt{3}$.

Exercise 6: Application - Velocity & Acceleration

Summary In summary, we found:

- Velocity function: $v(t) = \frac{1-t^2}{(1+t^2)^2}$
- Acceleration function: $a(t) = \frac{-2t(1+t^2)(3-t^2)}{(1+t^2)^4}$
- The object is **slowing down** for $t < -\sqrt{3}$ and $t > \sqrt{3}$.
- The object is **speeding up** for $-\sqrt{3} < t < \sqrt{3}$.

Exercise 7: Application

7. **[(v1) 3.4 Problem 154]** A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

   (a) Determine how long it takes for the ball to hit the ground.

   (b) Determine the velocity of the ball when it hits the ground.

Exercise 7: Application

Solution:

Solution:

(a) Find first POSITIVE zero.

$$s(t) = 0$$

$$\Rightarrow -16t_{ground}^2 - 8t_{ground} + 64 = 0$$

$$\Rightarrow t_{ground} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(-16)(64)}}{2*(-1\ \ )} = -\frac{1}{4} \pm \frac{\sqrt{65}}{4} \qquad \textit{(Quadratic Formula)}$$

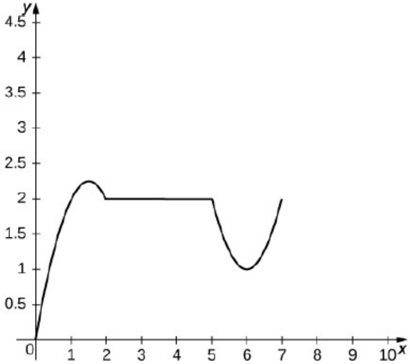
Take the positive root: $t_{ground} = -\frac{1}{4} + \frac{\sqrt{65}}{4} = \frac{1}{4}(\sqrt{65} - 1)\ s$

(b) Find $v(t_{ground})$.

$$v(t) = s'(t) = -32t - 8$$

Exercise 8: Application

8. **[(v1) 3.4 Problem 159]** **Optional - If time permits** The following graph shows the position $y = s(t)$ of an object moving along a straight line.



(a) Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.

(b) Sketch the graph of the velocity function.

(c) Use the graph of the velocity function to determine the time intervals when the

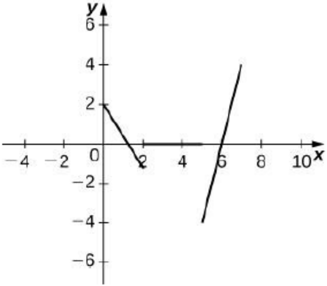
Exercise 8: Application

Solution:

(a) Velocity is positive on $(0, 1.5) \cup (6,7)$, *negative on* $(1.5, 2) \cup (5,6)$, *zero on* $(2,5)$.

Note: the singlet intervals [1.5,1.5] and [6,6] are also technically zero, and the velocity is undefined at x=2,5.

(b)



(c) Using the graph from (b), acceleration is positive on $(5,7)$, *negative on* $(0,2)$, *zero on* $(2, 5)$