MTH140 Lab Progress Check 2 (Section 17) Solution

TA & Solution Author: Mariam Walaa

October 19, 2023

Question 1

(4 Marks Total)

Given that

$$\lim_{x \to -7} \frac{h(x)}{x^2} = 1$$

1. Find $\lim_{x \to -7} h(x)$.
2. Find $\lim_{x \to -7} \frac{h(x)}{x}$.

Solution to Question 1:

Let's solve each part separately:

1. (2 Marks) To find $\lim_{x \to -5} h(x)$, we can use the limit properties. We have:

$$\begin{aligned}
\lim_{x \to -7} \frac{h(x)}{x^2} &= 1 \\
\lim_{x \to -7} h(x) \cdot \lim_{x \to -7} \frac{1}{x^2} &= 1 \\
\lim_{x \to -7} h(x) \cdot \frac{1}{(-7)^2} &= 1 \\
\lim_{x \to -7} h(x) \cdot \frac{1}{49} &= 1 \\
\lim_{x \to -7} h(x) &= 49 \cdot 1 \\
\lim_{x \to -7} h(x) &= 49
\end{aligned}$$

   Therefore, $\lim_{x \to -7} h(x) = 49$.

2. (2 Marks) To find $\lim_{x \to -7} \frac{h(x)}{x}$, we can again use the limit properties:

$$\begin{aligned}
\lim_{x \to -7} \frac{h(x)}{x^2} &= 1 \\
\lim_{x \to -7} \frac{h(x)}{x} \cdot \lim_{x \to -7} \frac{1}{x} &= 1 \\
\left(\lim_{x \to -7} \frac{h(x)}{x}\right) \cdot \left(\lim_{x \to -7} \frac{1}{x}\right) &= 1 \\
\lim_{x \to -7} \frac{h(x)}{x} \cdot \left(\lim_{x \to -7} \frac{1}{x}\right) &= 1 \\
\lim_{x \to -7} \frac{h(x)}{x} \cdot -\frac{1}{7} &= 1 \\
\lim_{x \to -7} \frac{h(x)}{x} &= -7
\end{aligned}$$

Question 2

(4 Marks Total)

Solve the equation $\sin(3\theta) + 1 = 0$ for $0 < \theta < 2\pi$

Solution to Question 2:

To solve the equation $\sin(3\theta) + 1 = 0$ for $0 < \theta < 2\pi$, follow these steps:

1. Bring the $+1$ over to the right hand side.

$$\sin(3\theta) = -1$$

2. Take the inverse sine (arcsin) of both sides:

$$\arcsin(\sin(3\theta)) = 3\theta = \arcsin(-1)$$

3. Find the values of $\arcsin(-1)$ within the given interval:

$$\arcsin(-1) = \frac{3\pi}{2}$$

4. Divide by 3 from both sides to isolate θ:

$$\theta = \frac{3\pi}{2 \cdot 3} = \frac{\pi}{2}$$

5. Since sine is a periodic function with a period of 2π, you can find other solutions by adding multiples of 2π to the initial solution:

$$\theta_1 = \frac{\pi}{2}$$

To find solutions within $0 < \theta < 2\pi$, add 2π repeatedly to θ_1:

$$\theta_2 = \theta_1 + 2\pi = \frac{\pi}{2} + 2\pi$$

$$\theta_3 = \theta_1 + 4\pi = \frac{\pi}{2} + 4\pi$$

Continue adding 2π until you reach values within the interval $0 < \theta < 2\pi$.

So, the solutions for $0 < \theta < 2\pi$ are:

$$\theta_1 = \frac{\pi}{2}, \quad \theta_2 = \frac{\pi}{2} + 2\pi, \quad \theta_3 = \frac{\pi}{2} + 4\pi, \ldots$$

You can simplify these further if needed.

Question 3

(3 Marks: 2 for method, 1 for accuracy)

Consider the functions $p(x) = x^3 + 1$ and q is a one-to-one function with the following values:

$$q(0) = 5$$
$$q(1) = 0$$
$$q(3) = -1$$

We want to find $(p \circ q)^{-1}(0)$.

Solution to Question 3:

Method 1:

Let $c = (p \circ q)^{-1}(0)$.

Then

$$(p \circ q)^{-1}(0) \rightarrow (p \circ q)(c) = 0$$

Recall that a function is invertible if and only if there exists its inverse.

As given, we know $p(x)$ has an inverse $p^{-1}(x)$, since it is invertible.

Moreover, we are given that $q(x)$ is one-to-one. We can assume its also onto, so $q(x)$ also has an inverse $q^{-1}(x)$,

This allows us to say $(p \circ q)^{-1}(p \circ q) = q^{-1}(p^{-1}(p(q(x)))) = x$, since $q^{-1}(q(x)) = x$ and $p^{-1}(p(x)) = x$.

Then we have

$$(p \circ q)(0) = p(q(c)) = 0$$

But we know that q is only -1 if $c = 3$. I.e.,

$$q(c) = -1 \rightarrow c = 3$$

Therefore, it must be that $(p \circ q)^{-1}(0) = 3$.

Method 2:
We are given that p is bijective by its definition, so it is one-to-one and onto.

Recall:

- A function $f : X \rightarrow Y$ is said to be *one-to-one* or *injective* if for every $x_1, x_2 \in X$: if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- A function $f : X \rightarrow Y$ is said to be *onto* or *surjective* if for every $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$.

Assume that q is also one-to-one and onto (a "weak" assumption).
Then

$$(p \circ q)^{-1}(x) = q^{-1} \circ p^{-1} = q^{-1}(p^{-1}(x))$$

$$(p \circ q)^{-1}(0) = q^{-1} \circ p^{-1} = q^{-1}(p^{-1}(0)) = q^{-1}(-1) = 3$$