MTH140 Lab 3

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Administrative Items

Structure of Today's Lab

- Collaborative Problem Solving & Take-up: As usual, 1hr 20 minutes.
- Lab Progress Check: Last 30 minutes of lab.
- You only have 20 minutes to write the quiz.
- Question will be displayed on the projector.
- No calculators or electronic devices during the quiz.
- Please use your own paper for calculations.

Learning Objectives

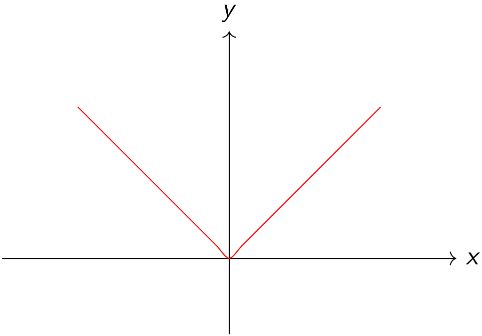
This lab's problem set will help you achieve the following learning outcomes:

- Use the horizontal line test to determine if a function is one-to-one.
- Find inverse functions for given functions, and consider domain restrictions when finding inverse functions.
- Sketch the graph of an inverse function based on the original function's graph.
- Use function composition to determine if two functions are inverses of each other.
- Convert logarithmic equations into exponential form.
- Use logarithmic and exponential identities.
- Sketch the graph of a given function.
- Determine the domain, range, and horizontal asymptote of a function.

Exercise 1: Horizontal Line Test

Problem: Use the horizontal line test to determine if **(a)** the absolute value function and **(b)** the logarithmic function are one-to-one.

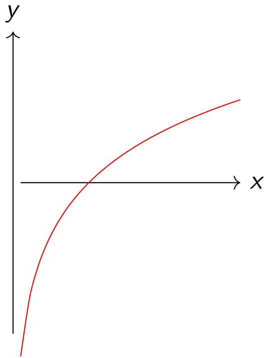
- **(a) Absolute Value Function:**
  - Function: $f(x) = |x|$
  - Graph:



  - Conclusion: The absolute value function is not one-to-one because horizontal lines intersect it more than once.

Exercise 1: Horizontal Line Test (contd.)

- **(b) Logarithmic Function:**
  - Function: $g(x) = \ln(x)$
  - Graph:



  - Conclusion: The logarithmic function is one-to-one because no horizontal line intersects it more than once.

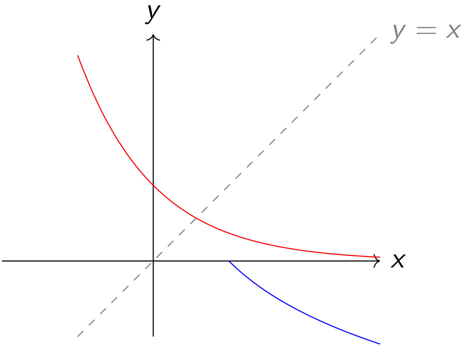
Exercise 2: Finding Inverse Functions

Problem: Find the inverse functions of the following functions:

- **(a):** Find the Inverse of $f(x) = (x-1)^2$ with $x \leq 1$
  - Step 1: Write $y = (x-1)^2$
  - Step 2: Swap x and y: $x = (y-1)^2$
  - Step 3: Solve for y: $\sqrt{(y-1)^2} = y - 1 = \pm\sqrt{x}$. We must choose the negative square root here.
  - Step 4: Isolate y: $y = 1 - \sqrt{x}$
  - Step 5: Determine the Domain: $x \leq 1$
  - Step 6: Write the Inverse Function: $f^{-1}(x) = 1 - \sqrt{x}$, where $x \leq 1$. Note that this could also be correctly written as $f^{-1}(y) = 1 - \sqrt{y}$
- **(b):** Find the Inverse of $g(x) = \frac{1}{x+2}$
  - Step 1: Write $y = \frac{1}{x+2}$
  - Step 2: Swap x and y: $x = \frac{1}{y+2}$
  - Step 3: Solve for y: $y + 2 = \frac{1}{x}$
  - Step 4: Isolate y: $y = \frac{1}{x} - 2$
  - Step 5: Determine the Domain: $x \neq 0$
  - Step 6: Write the Inverse Function: $g^{-1}(x) = \frac{1}{x} - 2$, where $x \neq 0$

Exercise 3: Sketching Inverse Functions

Problem: Use the graph of $y = e^{-x}$ to sketch its inverse function.
Solution:



Exercise 3: Sketching Inverse Functions (contd.)

1. Given Function: $y = e^{-x}$
2. Swap x and y: $x = e^{-y}$
3. Solve for y:

$$\begin{aligned} x &= e^{-y} \\ \ln(x) &= \ln(e^{-y}) \\ \ln(x) &= -y \\ y &= -\ln(x) \end{aligned}$$

4. Inverse Function: $y^{-1} = -\ln(x)$

The graph of the inverse function is a reflection of $y = e^{-x}$ over the line $y = x$.

Exercise 4: Function Composition

Problem: Use function composition to determine if the functions $f(x) = 8x + 3$ and $g(x) = \frac{x-3}{8}$ are inverses.

1. **(a)** $f(x) = 8x + 3$ **and** $g(x) = \frac{x-3}{8}$
   - Find $f(g(x))$:
   $$f(g(x)) = 8\left(\frac{x-3}{8}\right) + 3 = x$$
   - Conclusion: $f(x)$ and $g(x)$ are inverses.
2. **(b)** $f(x) = 2x + 1$ **and** $g(x) = \frac{x-1}{2}$
   - Find $f(g(x))$:
   $$f(g(x)) = 2\left(\frac{x-1}{2}\right) + 1 = x$$
   - Conclusion: $f(x)$ and $g(x)$ are inverses.
3. **(c)** $f(x) = x^2$ **and** $g(x) = \sqrt{x}$
   - Find $f(g(x))$:
   $$f(g(x)) = (\sqrt{x})^2 = x$$
   - Conclusion: $f(x)$ and $g(x)$ are inverses.

- In conclusion, all four pairs of functions are inverses of each other.

Exercise 5: Logarithmic Equations

Problem: Write the logarithmic equation $\log_3(81) = 4$ in its exponential form.
Solution: $3^4 = 81$

Exercise 5: Logarithmic Equations (contd.)

(b) Rewrite the logarithmic equation in its exponential form:
$\ln(1/e^3) = -3$.
Solution (b): To rewrite the equation in exponential form, use the definition of the natural logarithm (ln):

$$\ln\left(\frac{1}{e^3}\right) = -3$$

This equation means that e raised to the power of -3 is equal to $\frac{1}{e^3}$.
Therefore,

$$e^{-3} = \frac{1}{e^3}$$

Exercise 6: Sketching a Function

Problem: Sketch the graph of $f(x) = 5^{x+1} + 2$ and find its domain, range, and horizontal asymptote.
Solution:

- Domain: $(-\infty, \infty)$
- Range: $(2, \infty)$
- Horizontal Asymptote: $y = 2$

Exercise 7: Logarithmic to Exponential Expressions

Problem: Rewrite the expression in terms of exponentials and simplify
(a) $cosh(2x) - sinh(2x)$.
Solution (a): We can rewrite the expression using exponential functions:

$$\cosh(2x) - \sinh(2x) = \frac{e^{2x} + e^{-2x}}{2} - \frac{e^{2x} - e^{-2x}}{2}$$

Simplify the expression:

$$\frac{e^{2x} + e^{-2x}}{2} - \frac{e^{2x} - e^{-2x}}{2} = e^{-2x}$$

Exercise 7: Logarithmic to Exponential Expressions (contd.)

(b) $\ln(\cosh(x) + \sinh(x)) + \ln(\cosh(x) - \sinh(x))$
Solution: We can rewrite the expression using exponential functions:

$$\ln(\cosh(x) + \sinh(x)) + \ln(\cosh(x) - \sinh(x))$$

Simplify the expression after plugging in exponential identities:

$$= \ln\left(e^{x}\right) + \ln\left(e^{-x}\right)$$
$$= x - x = 0$$

Exercise 8: Exponential and Logarithmic Identities

(a) Simplify the expression: $\log(x^4 \cdot y)$.
Solution (a): To simplify this expression, we can use the logarithmic identity $\log_a(b \cdot c) = \log_a(b) + \log_a(c)$:

$$\log(x^4 \cdot y) = \log(x^4) + \log(y)$$

Next, we can use the power rule for logarithms, $\log_a(b^n) = n \cdot \log_a(b)$, to simplify further:

$$4\log(x) + \log(y)$$

So, $\log(x^4 \cdot y)$ simplifies to $4\log(x) + \log(y)$.

Exercise 8: Exponential and Logarithmic Identities (contd)

(b) Simplify the expression: $\log_4\left(\frac{(xy)^{1/3}}{64}\right)$.

Solution (b): To simplify this expression, use the properties of logarithms. First, let's rewrite the fraction as a division inside the logarithm:

$$\log_4\left(\frac{(xy)^{1/3}}{64}\right) = \log_4\left(\frac{(xy)^{1/3}}{4^3}\right)$$

Now, we can use the logarithmic identity $\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$:

$$\log_4\left(\frac{(xy)^{1/3}}{4^3}\right) = \log_4((xy)^{1/3}) - \log_4(4^3)$$

Next, we can use the power rule for logarithms:

$$\frac{1}{3}\log_4(xy) - 3\log_4(4)$$

Since $\log_4(4)$ simplifies to 1, we have:

1

Exercise 8: Exponential and Logarithmic Identities (contd)

(c) Simplify the expression: $\ln\left(\frac{6}{\sqrt{e^3}}\right)$.

Solution (c): To simplify this expression, we can use the properties of logarithms. First, let's rewrite the fraction as a division inside the natural logarithm:

$$\ln\left(\frac{6}{\sqrt{e^3}}\right) = \ln\left(\frac{6}{e^{3/2}}\right)$$

Now, we can use the logarithmic identity $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$:

$$\ln\left(\frac{6}{e^{3/2}}\right) = \ln(6) - \ln(e^{3/2})$$

Next, we know that $\ln(e^x) = x$, so $\ln(e^{3/2}) = 3/2$:

$$\ln(6) - \frac{3}{2}$$

So, $\ln\left(\frac{6}{\sqrt{e^3}}\right)$ simplifies to $\ln(6) - \frac{3}{2}$.