Lab Session 2 → Interpolation

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Group II → 1D Interpolation

1) The goal is to obtain the vector c in which $E(c)$, equation 2 is minimized. We have to calculate the gradient of E in order to find c, when the gradient is equal to zero, which means that E is minimum.

$$E(c) = \frac{1}{M}\sum_{i=1}^{M}\left(f_i - \hat{f}(T_i, c)\right)^2 \quad (2)$$

where

$$\hat{f}(t_i, c) = \Phi^T(T_i)\, c \quad (3)$$

So we have,

$$E(c) = \frac{1}{M}\sum_{i=1}^{M}\left(f_i - \Phi^T(T_i)\, c\right)^2 \quad (4)$$

transforming equation 4 using matrix notation, we get:

$$E = \frac{1}{M}\left(\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_M \end{bmatrix} - \begin{bmatrix} \Phi_1(T_1) & \Phi_2(T_1) & \cdots & \Phi_N(T_1) \\ \Phi_1(T_2) & \Phi_2(T_2) & \cdots & \Phi_N(T_2) \\ \vdots & \vdots & & \vdots \\ \Phi_1(T_M) & \Phi_2(T_M) & \cdots & \Phi_N(T_M) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}\right)^2$$

Following:

$$E(c) = \frac{1}{M} \times (F - \Theta c)^2 = \frac{1}{M} \times (F - \Theta c)^T \times (F - \Theta c) =$$

$$= \frac{1}{M} \times (F^T F - F^T \Theta c - c^T \Theta^T F + c^T \Theta^T \Theta c) =$$

$$= \frac{1}{M} \times (F^T F - 2F^T \Theta c + c^T \Theta^T \Theta c)$$

Properties:

$$(AB)^T = B^T A^T$$

$$A^T A B = B^T A^T A$$

$$\frac{\partial}{\partial u}(Au) = A^T$$

$$\frac{\partial}{\partial u}(u^T u) = 2u$$

Deriving:

$$\nabla_c E(c) = \frac{1}{M} \times \left(\cancel{\frac{\partial F^T F}{\partial c}}^{\,0} - \frac{\partial (2F^T \Theta c)}{\partial c} + \frac{\partial (c^T \Theta^T \Theta c)}{\partial c}\right) =$$

$$= \frac{1}{M}(-2\Theta^T F + 2\Theta^T \Theta c)$$

$$\nabla_c E(c) = 0$$

$$\frac{1}{M} \times (-2\Theta^T F + 2\Theta^T \Theta c) = 0 \iff \Theta^T \Theta c = \Theta^T F \Rightarrow \boxed{c^* = (\Theta^T \Theta)^{-1} \Theta^T F}$$

5) Find a matrix that represents $[c_1 - c_0 \quad c_2 - c_1 \cdots c_{N-1} - c_{N-2}]^T$
this vector is equivalent to $\begin{bmatrix} c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} - \begin{bmatrix} c_0 \\ \vdots \\ c_{N-2} \end{bmatrix} = xc - yc$

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where,

$$x = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \end{bmatrix}}_{(N-1)\times N} \quad \text{and} \quad y = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix}}_{(N-1)\times N}$$

$$A = x - y = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & & 0 \\ 0 & & \ddots & \ddots & \\ & & 0 & -1 & 1 \end{bmatrix}$$

$$E(c) = (f - \Theta c)^T (f - \Theta c) + \alpha (Ac)^T (Ac)$$

obtained in group II)

$$\nabla_c E(c) \Rightarrow \boxed{-2\Theta^T (f - \Theta c)} + 2\alpha A^T * (Ac) = 0$$

$$(\Theta^T \Theta + \alpha A^T A) c = \Theta^T f$$

$$\Rightarrow \boxed{c^* = (\Theta^T \Theta + \alpha A^T A)^{-1} \Theta^T f}$$

8) considering the Energy function $E(c)$ given by equation (16):

$$E(c) = \sum_{i=1}^{M} (f_i - \Phi^T(t_i)c)^2 + \alpha \sum_{k=1}^{N-1} |c_k - c_{k-1}|$$

this can be reformulated by considering that $|a-b| = \frac{|a-b||a-b|}{|a-b|} = \frac{(a-b)^2}{|a-b|}$

thus, $E(c) = \sum_{i=1}^{M} (f_i - \Phi^T(t_i)c)^2 + \alpha \sum_{k=1}^{N-1} \left(\sqrt{B_k}\,(c_k - c_{k-1})\right)^2$

with $B_k = \frac{1}{|c_k - c_{k-1}|}$ and $B = \begin{bmatrix} \sqrt{B_1} & 0 & \cdots & 0 \\ 0 & \sqrt{B_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \sqrt{B_{N-1}} \end{bmatrix}$

Taking the matrice notation

$$E(c) = (F - \Theta c)^T (F - \Theta c) + \alpha (BAc)^T (BAc)$$

$$\nabla_c E(c) = -2\Theta^T F + 2(\Theta^T \Theta c) + 2\alpha (A^T B^T BAc)$$

$$\nabla_c E(c) = 0$$

$$\Rightarrow \boxed{c^* = (\alpha A^T \beta A + \Theta^T \Theta)^{-1} \Theta^T F}$$

Being $\beta = B^T B = B^2 = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & B_{N-1} \end{bmatrix}$

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