Signal and Systems in Bioengineering

Master on Biomedical Engineering

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Interpolation

Lab Session 2

João Miguel Sanches
Bioengineering Department (DBE)
Instituto Superior Técnico / University of Lisbon

I. Introduction

In this work, discrete representation of continuous functions will be addressed. Let us consider a function $f(t)$, $f : R \rightarrow R$ from the space $C(0,1)$ of continuous functions in the interval $I = [0,1]$ and let $\phi_k(x)$ be a set of basis functions spanning a sub-space $V \subset C(a,b)$, where $C(a,b)$ is the space of continuous functions in the interval $[a,b]$, such that,

$$f(t) = g(t) + e(t) = \Phi^T(t)c + e(t) \tag{1}$$

where $\Phi(t) = [\phi_1(t), \phi_2(t), ..., \phi_m(t)]^T$ is a column vector and $g(t) \in V$.

In this work it will be assumed that the basis functions $\phi_k(t)$ are obtained from the mother function, $\phi(t)$ after translation and scaling operations in the domain axis, that is, $\phi_k(t) = \phi(t/\Delta - k)$ where $\Delta = 1/(N-1)$ e $0 \leq k < N$.

Let $F = \{f_i\}$ be a set of M observations of $f(t)$ at non-evenly spaced locations $T = \{t_i\}$ with $0 \leq t_i \leq 1$. The goal of this work is solving the following problem:

Problema: Given M observations of $f(t)$, $F = \{f_1, f_2, ...f_M\}^T$, at non evenly spaced locations $T = \{t_1, t_2, ..., t_M\}^T$, where $f_i = f(t_i)$, compute the N coefficients $\mathbf{c} = [c_1, c_2, ..., c_N]^T$ that minimize the following energy function

$$E(c) = \frac{1}{M}\sum_{i=1}^{M}\left(f_i - \hat{f}(t_i, \mathbf{c})\right)^2 \tag{2}$$

where

$$\hat{f}(t_i, \mathbf{c}) = \Phi^T(t_i)\mathbf{c} \tag{3}$$

II. 1D Interpolation

1) Derive the expression to obtain the vector $\mathbf{c}$ from F, T and $\Phi(t) = [\phi_1(t), \phi_2(t), ..., \phi_m(t)]^T$ that minimize (2). Suggestion: Use the following matrix to re-write equation (2) using matrix notation,

$$\Theta = \begin{pmatrix} \phi_1(t_1) & \phi_2(t_1) & ... & \phi_N(t_1) \\ \phi_1(t_2) & \phi_2(t_2) & ... & \phi_N(t_2) \\ ... & ... & ... & ... \\ \phi_1(t_M) & \phi_2(t_M) & ... & \phi_N(t_M) \end{pmatrix} = \begin{pmatrix} \Phi^T(t_1) \\ \Phi^T(t_2) \\ ... \\ \Phi^T(t_M) \end{pmatrix} \tag{4}$$

2) Sample the function

$$f(t) = sin(2\pi t)e^{-20(t-0.5)^2} \tag{5}$$

at $M = 10$ random locations uniformly distributed in the interval $I = [0,1]$. Store the values of the function, F, and the corresponding positions, T. Display grafically the function $f(t)$ in the interval $[0,1]$ with 500 points (to simulate a continuous representaion) and overlap in this graph the samples (T, F), obtained before.

3) Based on the samples obtained in the previous item and on the expression obtained in 1) compute the vector **c**, with dimension $N = 10$, which allows to estimate the function $f(t)$ from (T, F), according with,

$$f(t) \approx \hat{f}(t) = \Phi^T(t)c. \tag{6}$$

Use the following basis functions,

$$\phi_k(t) = sinc(t/\Delta - k) = \sin(\pi(t/\Delta - k))/\pi(t/\Delta - k). \tag{7}$$

Display in the same graph the original function $f(t)$ and the estimated one $\hat{f}(t)$ at the same high density time points. Compute the *signal to noise ratio* in dB, according with

$$SNR = 10\log_{10}\left[\frac{\|f\|^2}{\|f-g\|^2}\right] \tag{8}$$

em que $\|x\| = \frac{1}{L}\sqrt{\sum_{i=1}^{L} x_i^2}$, com $L = 500$, $t_i = (i-1)/(L-1)$ e $1 \leq i \leq L$.

4) Using again $M = 10$ random samples of $f(t)$ compute the vector **c** with dimension $N = 50$. Observe and comment the result. Increase the number of samples for $M = 100$ and observe the result. Comment.

5) Consider (2) and introduce an additional term as follows

$$E(\mathbf{c}) = \sum_{i=1}^{M}(f_i - \Phi^T(t_i)\mathbf{c})^2 + \alpha\sum_{k=1}^{N-1}(c_k - c_{k-1})^2 \tag{9}$$

Derive the expression to obtain the vector **c** that minimize (9).
Suggestion: express the additional term in matrix notation. Find a matrix such that $\Phi\mathbf{c} = [c_0-0, c_1-c_0, c_2-c_1, ..., c_{N-1}-c_{N-2}]^T$).

6) Recompute 4) with the new expression and comment the differences (choose the value of the parameter α that leads to better results).

7) Explain and comment the role of the additional term.
8) Propose a method to minimize the following energy function

$$E(\mathbf{c}) = \sum_{i=1}^{M}(f_i - \Phi^T(t_i)\mathbf{c})^2 + \alpha\sum_{k=1}^{N-1}|c_k - c_{k-1}| \tag{10}$$

III. 2D Interpolation

This section illustrates the **aliasing** phenomenon in 2D. The sampling process will be simulated by using the **down sampling** operation applied to the discrete signals. The main goal of this work is to observe the artifacts due the spectral overlapping and, simultaneously, to show the effectiveness of the **anti-aliasing** filter to eliminate them. Interpolation will also be considered in order to increase the sampling rate and therefore the image dimensions.

A. Sampling

1) Load the image (lena) by using the command *imread()* to the variable x and show it in a graphical window (use *subplot(311); imagesc(x)*).

2) Reduce the image dimensions ten times (1/10) using the command ">>y=x(1:10:end, 1:10:end)" and visualize the new image, side-by-side with the original one using the command subplot(312).
Comment the observation.

3) Repeat the previous item by firstly low pass filtering the original image. Use the following commands:

```
>>B=fir1(100,0.1);
>>xLP=conv2(B,B',x,'same');
```

These commands generate the coefficients B of a 100 order low-pass FIR filter used to filter the image, first along the columns and next along the lines. Observe the filtered image and the reduced one.
Comment both images and compare the reduced versions, without and with the anti-aliasing filter.

B. *Reconstruction*

1) Split the original image x in 10×10 pixel blocks. In each block put all pixels equal to zero but the first one at the upper left one. Observe the resulting image, called here $xDirac$, and comment it. Can you observe the original details?

2) Build a new image from the previous one by putting all pixels equal to the upper left corner one in each block. This is what is usually called ”pixeling” effect. Comment the result, referring specifically, what is the interpolation method involved.

3) Filter the image $xDirac$ from III.B.1) with a low-pass filter by using Q, Hamming windows with several dimensions. Use the following sequence of commands,

```
>>P=10;
>>Q=5*P; h=fir1(Q,1/P,hamming(Q+1));
>>xSinc=conv2(h,h',xDirac,'same');
```

   Visualize and comment the results. Compare them with the results obtained in III.B.2).

4) Make an ”ideal” reconstruction from the image obtained in III.B.1). Explain the procedure.

5) Compare the results obtained in III-B.2-3).