Circuit Theory and Electronics Fundamentals

Integrated Master in Aerospace Engineering, Técnico, University of Lisbon

Lab2: RC Circuit Analysis

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Contents

1 Introduction

The objective of this laboratory assignment is to study a circuit with four elementary meshes and eight nodes. This circuit contains an independent voltage source v_s, a voltage source dependent from a current V_d, a current source dependent from a voltage I_b and a capacitor C. Besides this, it has seven resistors, R_1, R_2, R_3, R_4, R_5, R_6 and R_7. The circuit can be seen in Figure 1.
The main goal of our analysis is to compute v_6, beginning with its natural solution and then discovering the forced solution. We also analysed how v_6, v_s and v_c vary with frequency.
In Section 2, a theoretical analysis of the circuit is presented, using Octave and node method. In Section 9, the circuit is analysed by simulation using ngspice. The conclusions of this study are outlined in Section 10. In addiction, the results are compared to the theoretical results obtained in Section 2.

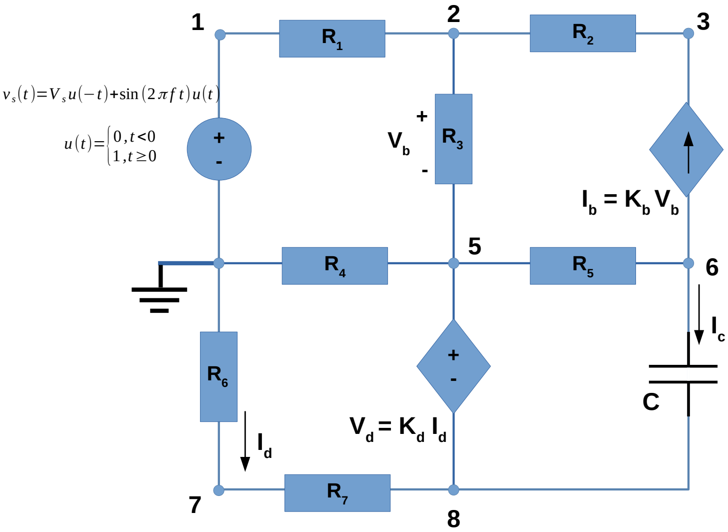

Figure 1: Circuit analysed.

2 Theoretical Analysis

In this section, the circuit shown in Figure 1 is analysed theoretically.
At first, we are going to use the nodal method to compute voltages in all nodes and currents in all branches for t < 0. After that, we will determine the equivalent resistance as seen from the capacitor terminals and the time constant, as well. Then, using the equivalent resistance and the time constant discovered before, we are able to compute the natural solution of v_6(t) in the interval [0,20]ms and plot the result obtained. We will also compute the forced solution of v_6(t) in the same interval, using phasors. After, we are going to compute de final solution of v_6 and plot v_6(t) and v_s(t) in the interval of [-5,20]ms. At last, we can plot v_s(f), v_c(f) and v_6(f) for frequency range 0,1Hz to 1MHz. To compute the values mentioned above, we use the following values (Resistances in Ohm, voltages in Volts, capacity in F, K_b in S and K_d in Ohms):

Table 1: Given values by Python.

R_1	1.00332071e+03
R_2	2.04460853e+03
R_3	3.08291730e+03
R_4	4.16061679e+03
R_5	3.04022345e+03
R_6	2.06711403e+03
R_7	1.03302701e+03
V_s	5.13988034e+00
C	1.02475824e-06
K_b	7.05445350e-03
K_d	8.16113798e+03

3 Nodal method for t < 0

For t < 0, since u(t) = 0 and u(-t) = 1, vs = Vs. Also, we are considering that the voltage source has been turned on a long time ago for t < 0,therefore, all values are constant. The current flowing through the capacitor can be given by:

$$I_c = C\frac{dV_c}{dt} \tag{1}$$

We can conclude that v_c doesn't vary in time, so I_c is 0 for t < 0. There are seven unknown node voltages (V_1(t<0), V_2(t<0), V_3(t<0), V_5(t<0), V_6(t<0), V_7(t<0) and V_8(t<0)), so we need seven equations to compute these values. In this analysis, we consider currents diverging from the node as positive values and currents converging as negative values and we used KCL, first on the nodes not connected to voltage sources and, after that, we have written additional equations for nodes related by voltage sources.
Starting with node 2, for this node, we considered all currents diverging, so we have this equation:

$$\frac{V_2(t<0)-V_1(t<0)}{R_1}+\frac{V_2(t<0)-V_5(t<0)}{R_3}+\frac{V_2(t<0)-V_3(t<0)}{R_2}=0 \tag{2}$$

In node 3, I_b is converging and the current flowing through R_2 is diverging. I_b equals to K_bV_b. V_B is equal to V_2 - V_5, due to the direction represented in the circuit for R_3 voltage:

$$\frac{V_3(t<0)-V_2(t<0)}{R_2}-K_b(V_2(t<0)-V_5(t<0))=0 \tag{3}$$

Moving on to node 6, in which all currents are diverging and I_c is 0, having this equation:

$$\frac{V_6(t<0)-V_5(t<0)}{R_5}+K_b(V_2(t<0)-V_5(t<0))=0 \tag{4}$$

On the other side, in node 7, I_d is converging and the current flowing through R_7 is diverging:

$$\frac{V_7(t<0)-V_8(t<0)}{R_7}+\frac{V_7(t<0)}{R_6}=0 \tag{5}$$

The next equation establish the relation between node 1 and GND voltages:

$$V_1 = V_s \tag{6}$$

The relation between nodes 5 and 8 is showed in the next equation:

$$V_5(t<0)-V_8(t<0)=\frac{-K_dV_7(t<0)}{R_6} \tag{7}$$

Since there is still an equation missing, we considered a super node containing v_s branch and all currents diverging:

$$\frac{V_1(t<0)-V_2(t<0)}{R_1}-\frac{V_5(t<0)}{R_4}-\frac{V_7(t<0)}{R_6}=0 \tag{8}$$

All these equations can be transformed in a matricial system as it is showed here:

$$\begin{bmatrix} -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & -\frac{1}{R_2}-K_b & \frac{1}{R_2} & K_b & 0 & 0 & 0 \\ 0 & K_b & 0 & -K_b-\frac{1}{R_5} & \frac{1}{R_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 0 & -\frac{1}{R_4} & 0 & -\frac{1}{R_6} & 0 \end{bmatrix} \begin{bmatrix} V_1(t<0) \\ V_2(t<0) \\ V_3(t<0) \\ V_5(t<0) \\ V_6(t<0) \\ V_7(t<0) \\ V_8(t<0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}$$

Table 2: Node voltages for t < 0.

Nodes Voltages	Values obtained (Volts)
V_1(t<0)	5.13988034e+00
V_2(t<0)	4.92714703e+00
V_3(t<0)	4.47273618e+00
V_5(t<0)	4.95865171e+00
V_6(t<0)	5.63433632e+00
V_7(t<0)	-2.02531215e+00
V_8(t<0)	-3.03744893e+00

Solving the matricial system using Ocatve and the values given by Python, we have obtained the values showed above, in Table 2.
Having discovered nodes voltages, we need now to obtain the currents flowing in each resistance, I_b and currents in V_s and V_d branches, with the following equations:

$$I_b = K_b(V_2(t<0) - V_5(t<0)) \tag{9}$$

$$R_1[i] = \frac{(V_1(t<0) - V_2(t<0))}{R_1} \tag{10}$$

$$R_2[i] = \frac{(V_2(t<0) - V_3(t<0))}{R_2} \tag{11}$$

$$R_3[i] = \frac{(V_2(t<0) - V_5(t<0))}{R_3} \tag{12}$$

$$R_4[i] = \frac{-V_5(t<0)}{R_4} \tag{13}$$

$$R_5[i] = \frac{(V_5(t<0) - V_6(t<0))}{R_5} \tag{14}$$

$$R_6[i] = \frac{-V_7(t<0)}{R_6} \tag{15}$$

$$R_7[i] = \frac{V_7(t<0) - V_8(t<0)}{R_7} \tag{16}$$

$$v_s[i] = -R_1[i] \tag{17}$$

$$V_d[i] = R_3[i] + R_4[i] - R_5[i] \tag{18}$$

Table 3: Branch currents for t < 0.

Branch currents	Values obtained (Ampers)
I_b	-2.22248338e-04
$R_1[i]$	2.12029224e-04
$R_2[i]$	2.22248338e-04
$R_3[i]$	-1.02191147e-05
$R_4[i]$	-1.19180688e-03
$R_5[i]$	-2.22248338e-04
$R_6[i]$	9.79777657e-04
$R_7[i]$	9.79777657e-04
$v_s[i]$	-2.12029224e-04
$V_d[i]$	-9.79777657e-04

4 Equivalent resistance and time constant

In this section, we analyse the circuit for t = 0 and, consequently, v_s equals 0, as well as V_1. To compute the equivalent resistance as seen from the capacitor terminals and the time constant, we replace the capacitor with a voltage source:

$$V_x = V_6(t<0) - V_8(t<0), \tag{19}$$

where V_6(t<0) and V_8(t<0) are the values computed in Section 3. In a circuit with dependent sources, as the one in analysis, we can't turn off all sources to compute the equivalent resistance as seen from one component. Having this said, We need to discover the equivalent current, flowing through the capacitor, which we named I_x, in this case, and the equivalent voltage, V_x, which we know already from equation 19). Using this procesure, we also ensure the continuity of the capacitor voltage.
Hence, we now have six voltages to compute (V_2(t=0), V_3(t=0), V_5(t=0), V_6(t=0), V_7(t=0) and V_8(t=0)), needing, thus, six equations, some similar to the ones from Section 3. In node 2, 3 and 7 and in the supernode equation, the only change is that now V_1(t=0) is 0:

$$\frac{V_2(t=0)}{R_1} + \frac{V_2(t=0) - V_5(t=0)}{R_3} + \frac{V_2(t=0) - V_3(t=0)}{R_2} = 0 \tag{20}$$

$$\frac{V_3(t=0) - V_2(t=0)}{R_2} - K_b(V_2(t=0) - V_5(t=0)) = 0 \tag{21}$$

$$\frac{V_7(t=0) - V_8(t=0)}{R_7} + \frac{V_7(t=0)}{R_6} = 0 \tag{22}$$

$$-\frac{V_2(t=0)}{R_1} - \frac{V_5(t=0)}{R_4} - \frac{V_7(t=0)}{R_6} = 0 \tag{23}$$

Moving to the new equation, that relates voltages of nodes 6 and 8, which are now connected by a voltage source V_x:

$$V_6(t=0) - V_8(t=0) = V_x \tag{24}$$

These equations can be transformed in the following matricial system:

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 & 0 \\ -\frac{1}{R_2} - K_b & \frac{1}{R_2} & K_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \\ -\frac{1}{R_1} & 0 & -\frac{1}{R_4} & 0 & -\frac{1}{R_6} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_2(t=0) \\ V_3(t=0) \\ V_5(t=0) \\ V_6(t=0) \\ V_7(t=0) \\ V_8(t=0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_x \end{bmatrix}$$

Table 4: Node voltages for t = 0.

Nodes Voltages	Values obtained (Volts)
V_2(t=0)	0.00000000e+00
V_3(t=0)	0.00000000e+00
V_5(t=0)	0.00000000e+00
V_6(t=0)	8.67178526e+00
V_7(t=0)	0.00000000e+00
V_8(t=0)	0.00000000e+00

Having solved this system in Octave, we have obtained the following voltages for each node: Having discovered nodes voltages, we need now to obtain the currents flowing in each resistance, I_b and currents in V_s and V_d branches, with the following equations:

$$I_b = K_b(V_2(t=0) - V_5(t=0)) \tag{25}$$

$$R_1[i] = -\frac{V_2(t=0)}{R_1} \tag{26}$$

$$R_2[i] = \frac{(V_2(t=0) - V_3(t=0))}{R_2} \tag{27}$$

$$R_3[i] = \frac{(V_2(t=0) - V_5(t=0))}{R_3} \tag{28}$$

$$R_4[i] = \frac{-V_5(t=0)}{R_4} \tag{29}$$

$$R_5[i] = \frac{(V_5(t=0) - V_6(t=0))}{R_5} \tag{30}$$

$$R_6[i] = \frac{-V_7(t=0)}{R_6} \tag{31}$$

$$R_7[i] = \frac{V_7(t=0) - V_8(t=0)}{R_7} \tag{32}$$

$$v_s[i] = -R_1[i] \tag{33}$$

$$V_d[i] = R_3[i] + R_4[i] - R_5[i] \tag{34}$$

Table 5: Branch currents for t = 0.

Branch currents	Values obtained (Ampers)
I_b	0.00000000e+00
$R_1[i]$	0.00000000e+00
$R_2[i]$	0.00000000e+00
$R_3[i]$	0.00000000e+00
$R_4[i]$	0.00000000e+00
$R_5[i]$	-2.85235128e-03
$R_6[i]$	0.00000000e+00
$R_7[i]$	0.00000000e+00
$v_s[i]$	0.00000000e+00
$V_d[i]$	2.85235128e-03

After this, we are able to compute I_x, since this current is given by, using KCL in node 6:

$$I_x = -\frac{(V_6(t=0) - V_5(t=0))}{R_5} - K_b(V_2(t=0) - V_5(t=0)) \tag{35}$$

At last, we calculated R_{eq}, in Ohms, and τ, in seconds, using the following equations:

$$R_{eq} = |\frac{V_x}{I_x}| \tag{36}$$

$$\tau = R_{eq}C \tag{37}$$

The results obtained are showed in Table 6:

Table 6: Equivalent resistance and time constant

V_X	8.67178526e+00
I_x	-2.85235128e-03
R_{eq}	3.04022345e+03
τ	3.11549404e-03

5 Natural solution v_6n(t)

In this section, the main goal is to compute the natural solution of v_{6n}(t). For that, we used the general solution given by:

$$v_{6n}(t) = V_6(+\infty) + (V_6(t=0) - V_6(+\infty))e^{\frac{-t}{\tau}} \tag{38}$$

In this circuit, we can conclude that the capacitor begins charged and, as time goes by, it discharges. Also, in t = 0, V_8 equals 0. For both reasons, $v_6(+\infty)$ equals 0, meaning that v_{6n}(t) can be computed with the following equation:

$$v_{6n}(t) = V_6(t=0)e^{\frac{-t}{\tau}} \tag{39}$$

It can also be written, using V_x and V_8(t=0):

$$v_{6n}(t) = (V_x + V_8(t=0))e^{\frac{-t}{\tau}} \tag{40}$$

Since V_8 equals 0, as mentioned before, we can simply write:

$$v_{6n}(t) = V_x e^{\frac{-t}{\tau}} \tag{41}$$

To finish this section, we plot this function in the interval of [0,20]ms.

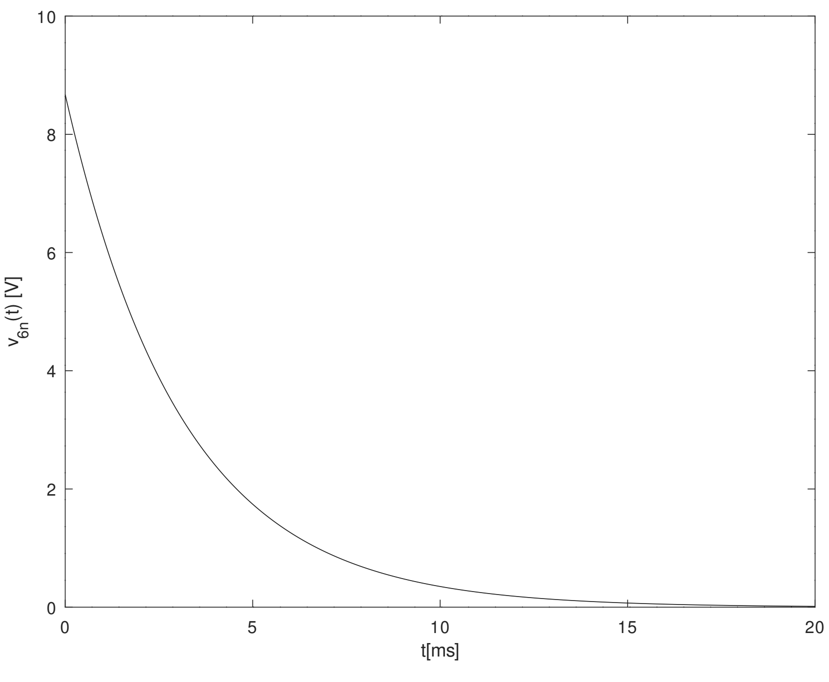

Figure 2: Natural Solution $v_{6n}(t)$ in [0,20]ms.

6 Forced solution $v_6 f$(t)

In this section, the main goal is to compute the forced solution of v_{6f}(t). First, we have to compute the complex amplitudes of voltages in each node, using the node method, in the same way we did previously in Section 3, but considering, this time, the capacitor and the currents flowing through it, using its impedance. Hence, the only equation that is different from those written in Section 3 is the one referring to node 6. In the capacitor, we have:

$$Z_c = \frac{1}{iwc} \tag{42}$$

$$v_c = I_c Z_c \tag{43}$$

Therefore, in node 6, considering all currents diverging and v_c = V_6(t > 0) - V_8(t > 0), KCL can be given by the following equation:

$$\frac{V_6(t>0) - V_5(t>0)}{R_5} + K_b(V_2(t>0) - V_5(t>0)) + \frac{V_6(t>0) - V_8(t>0)}{Z_c} = 0 \tag{44}$$

This method gives us enough equations to build a matricial system to solve, with seven voltages to compute and seven equations:

$$\begin{bmatrix} -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & -\frac{1}{R_2}-K_b & \frac{1}{R_2} & K_b & 0 & 0 & 0 \\ 0 & K_b & 0 & -K_b-\frac{1}{R_5} & \frac{1}{R_5}+\frac{1}{Z_c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \\ \frac{1}{R_1} & -\frac{1}{R_1} & 0 & -\frac{1}{R_4} & 0 & -\frac{1}{R_6} & 0 \end{bmatrix} \begin{bmatrix} V_1(t>0) \\ V_2(t>0) \\ V_3(t>0) \\ V_5(t>0) \\ V_6(t>0) \\ V_7(t>0) \\ V_8(t>0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}$$

Solving the given system, we obtain the complex amplitude voltages in all nodes, including node 6. The results are expressed in the following table:

Table 7: Complex Amplitude Voltages In All Nodes (Volts).

V_1(t > 0)	1.00000000e+00 $e^{0.00000000e+00i}$
V_2(t > 0)	9.58611233e-01 $e^{-6.66173098e-18i}$
V_3(t > 0)	8.70202395e-01 $e^{-9.85626946e-16i}$
V_5(t > 0)	9.64740692e-01 $e^{5.45596884e-17i}$
V_6(t > 0)	5.92831416e-01 $e^{1.45518915e-01i}$
V_7(t > 0)	3.94038774e-01 $e^{5.45596884e-17i}$
V_8(t > 0)	5.90957129e-01 $e^{5.45596884e-17i}$

To obtain the forced solution of v_6, the following expression can be used:

$$V_{6f}(t) = V_6(t>0) sin(wt) \tag{45}$$

7 Total solution v_6(t)

To obtain the total solution for the voltage of node 6 and of the voltage source v_s, in the given time range, [-5,20ms] we need to consider three cases for t<0, t=0 and t>0. In the first case, v_s equals V_s and v_6 is the voltage we obtained in Section 3. In the moment t = 0, v_s also equals V_s, since sin(0) = 0, and v_6 is the value obtained in Section 4. The third and last case, v_s equals $sin(2\pi * f * t)$, as we can see by the expression in Figure 1, but to compute V_6 we have to consider the natural and forced solution:

$$V_6(t) = V_{6f}(t) + V_{6n}(t) \tag{46}$$

Finally, these three separate conditions can be summed up in the following graph, where v_s and v_6 are plotted:

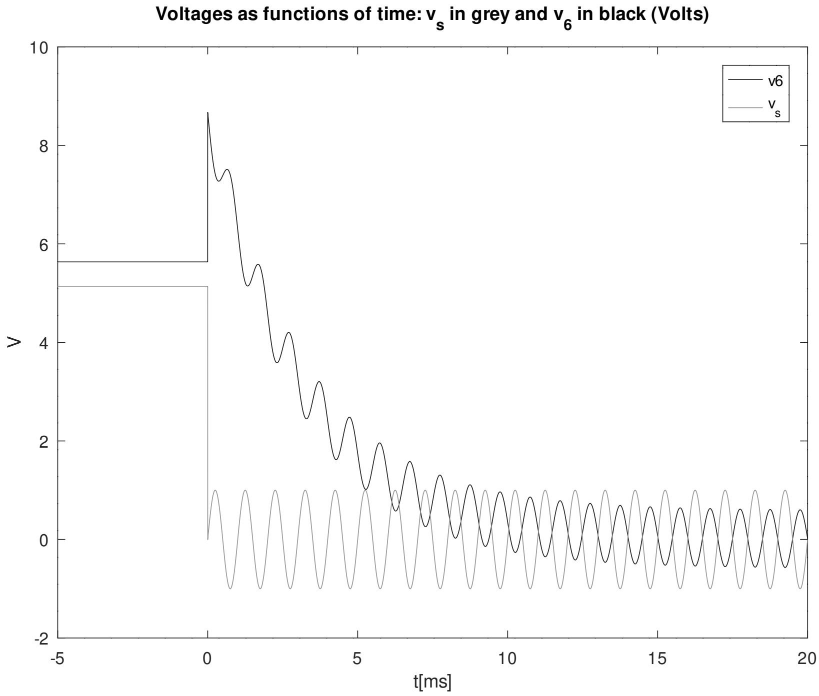

Figure 3: $v_s(t)$ and Total Solution $v_6(t)$ in [-5,20]ms.

8 $v_s f$(f), $v_c f$(f), $v_6 f$(f)

9 Simulation Analysis

9.1 Operating point analysis

This section covers the circuit simulation in ngspice. However, for this simulation, a new node 8 and an auxiliary independent voltage source - V_8 - with a voltage of 0V had to be created. Node 8 is located between the resistance R_6 and node 3, while the terminals of V_8 are connected to nodes 3 and 8. This happens because, to create a current dependent voltage source, ngspice needs the current value of a voltage source. As the current I_c doesn't flow through any voltage source, one had to be created.
The values obtained, by ngspice, for currents flowing in each resistance (Ampers) and nodes voltages (Volts) are showed in the following table:

Name	Value [A or V]
@c1[i]	0.000000e+00
@gb[i]	-2.22248e-04
@r1[i]	2.120292e-04
@r2[i]	2.222483e-04
@r3[i]	-1.02191e-05
@r4[i]	-1.19181e-03
@r5[i]	-2.22248e-04
@r6[i]	9.797777e-04
@r7[i]	9.797777e-04
v(1)	5.139880e+00
v(2)	4.927147e+00
v(3)	4.472736e+00
v(5)	4.958652e+00
v(6)	5.634336e+00
v(7)	-2.02531e+00
v(8)	-3.03745e+00
v(9)	0.000000e+00

Table 8: Operating point. A variable preceded by @ is of type *current* and expressed in Ampere; other variables are of type *voltage* and expressed in Volt. (g in "gib" refers to the Ngspice notation of a current source controlled by a voltage).

Name	Value [A or V]
@gb[i]	0.000000e+00
@r1[i]	0.000000e+00
@r2[i]	0.000000e+00
@r3[i]	0.000000e+00
@r4[i]	0.000000e+00
@r5[i]	-2.85235e-03
@r6[i]	0.000000e+00
@r7[i]	0.000000e+00
v(1)	0.000000e+00
v(2)	0.000000e+00
v(3)	0.000000e+00
v(5)	0.000000e+00
v(6)	8.671785e+00
v(7)	0.000000e+00
v(8)	0.000000e+00
v(9)	0.000000e+00

Table 9: Operating point. A variable preceded by @ is of type *current* and expressed in Ampere; other variables are of type *voltage* and expressed in Volt. (g in "gib" refers to the Ngspice notation of a current source controlled by a voltage).

10 Conclusion

Name	Value [A or V]
@c1[i]	0.000000e+00
@gb[i]	-2.22248e-04
@r1[i]	2.120292e-04
@r2[i]	2.222483e-04
@r3[i]	-1.02191e-05
@r4[i]	-1.19181e-03
@r5[i]	-2.22248e-04
@r6[i]	9.797777e-04
@r7[i]	9.797777e-04
v(1)	5.139880e+00
v(2)	4.927147e+00
v(3)	4.472736e+00
v(5)	4.958652e+00
v(6)	5.634336e+00
v(7)	-2.02531e+00
v(8)	-3.03745e+00
v(9)	0.000000e+00

Name	Value [A or V]
I_c	0
I_b	-2.22248338e-04
$R_1[i]$	2.12029224e-04
$R_2[i]$	2.22248338e-04
$R_3[i]$	-1.02191147e-05
$R_4[i]$	-1.19180688e-03
$R_5[i]$	-2.22248338e-04
$R_6[i]$	9.79777657e-04
$R_7[i]$	9.79777657e-04
V_1	5.13988034e+00
V_2	4.92714703e+00
V_3	4.47273618e+00
V_5	4.95865171e+00
V_6	5.63433632e+00
V_7	-2.02531215e+00
V_8	-3.03744893e+00

Table 10: Simulation (right) and Theoretical(left) Node Voltages and Branch currents t < 0.

Name	Value [A or V]
@gb[i]	0.000000e+00
@r1[i]	0.000000e+00
@r2[i]	0.000000e+00
@r3[i]	0.000000e+00
@r4[i]	0.000000e+00
@r5[i]	-2.85235e-03
@r6[i]	0.000000e+00
@r7[i]	0.000000e+00
v(1)	0.000000e+00
v(2)	0.000000e+00
v(3)	0.000000e+00
v(5)	0.000000e+00
v(6)	8.671785e+00
v(7)	0.000000e+00
v(8)	0.000000e+00
v(9)	0.000000e+00

Name	Value [A or V]
I_b	0.00000000e+00
$R_1[i]$	0.00000000e+00
$R_2[i]$	0.00000000e+00
$R_3[i]$	0.00000000e+00
$R_4[i]$	0.00000000e+00
$R_5[i]$	-2.85235128e-03
$R_6[i]$	0.00000000e+00
$R_7[i]$	0.00000000e+00
V_1(t=0)	0
V_2(t=0)	0.00000000e+00
V_3(t=0)	0.00000000e+00
V_5(t=0)	0.00000000e+00
V_6(t=0)	8.67178526e+00
V_7(t=0)	0.00000000e+00
V_8(t=0)	0.00000000e+00

Table 11: Simulation (right) and Theoretical(left) Node Voltages and Branch currents t = 0.