Decomposing labelled proof theory for intuitionistic modal logic

Sonia Marin*
IT-Universitetet i København
Denmark

Marianela Morales
Universidad Nacional de Córdoba
Argentina

Lutz Straßburger
Inria Saclay & LIX
France

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Labelled deduction has been proposed by Gabbay [7] in the 80s as a unifying framework throughout proof theory in order to provide proof systems for a wide range of logics. For modal logics it can take for example the form of labelled natural deduction and labelled sequent systems, as used by Simpson [3], Viganò [4] and Negri [2].

These formalisms make explicit use not only of labels, but also of relational atoms referring to the accessibility relation of a Kripke model. In this short note we propose a system that represents both the *accessibility relation* (for modal logics) and the *preorder relation* (for intuitionistic logic), using the full power of the bi-relational semantics for intuitionistic modal logics, and developing fully the idea of [1].

A *bi-relational frame* [5, 6] $\mathcal{B}$ is a triple $\langle W, R, \leq \rangle$ of a non-empty set of worlds W equipped with an accessibility relation R and a preorder $\leq$, satisfying:

(F_1) For all worlds x, y, z, if xRy and $y \leq z$, there exists a u such that $x \leq u$ and uRz.

(F_2) For all worlds x, y, z, if xRy and $x \leq z$, there exists a u such that $y \leq u$ and zRu.

Reflecting this definition, we define our two-sided intuitionistic labelled sequents to be of the form $\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}$ where $\mathcal{B}$ denotes a set of relational atoms xRy and preorder atoms $x \leq y$, and $\mathcal{L}$ and $\mathcal{R}$ are multi-sets of labelled formulas $x\colon A$ (for x and y taken from the set of labels and A an intuitionistic modal formula).

Furthermore, our system has to incorporate the two semantic conditions into deductive rules as follows:

$$\mathsf{F}_1 \frac{\mathcal{B}, xRy, y \leq z, x \leq u, uRz, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

$$\mathsf{F}_2 \frac{\mathcal{B}, xRy, x \leq z, y \leq u, zRu, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

In the intuitionistic setting, the validity of a modal formula has to be defined using both the R and the $\leq$ relation as: $x \Vdash \Box A$ iff for all y and z s.t. $x \leq y$ and yRz, $z \Vdash A$.

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Again, our system reflects exactly this definition in the rules introducing the $\Box$-operator:

$$\Box_{\mathsf{L}} \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L}, x\colon \Box A, z\colon A \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x \leq y, yRz, x\colon \Box A \Rightarrow \mathcal{R}}$$

$$\Box_{\mathsf{R}} \frac{\mathcal{B}, x \leq y, yRz, \mathcal{L} \Rightarrow \mathcal{R}, z\colon A}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon \Box A}\ y, z \text{ fresh}$$

By complementing these rules with the standard labelled rules for intuitionistic modal logic of [3], we get a system that is sound and complete wrt. the birelational semantics.

In [6], Plotkin and Stirling give a correspondence result for intuitionistic modal logic extended with a family of axioms wrt. some classes of bi-relational frames. For example, the frames that validate the axiom $4_\Diamond\colon \Diamond\Diamond A \supset \Diamond A$ are exactly the ones satisfying the condition:

($\blacklozenge_4$) if wRv and vRu, there exists a u' s.t. $u \leq u'$ and wRu'.

Incorporating the preorder symbol into the syntax of our sequents allows us to also obtain a sound and complete proof system for the intuitionistic modal logic extended with axiom $4_\Diamond$, by designing the following rule:

$$\blacklozenge_4 \frac{\mathcal{B}, wRv, vRu, u \leq u', wRu', \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, wRv, vRu, \mathcal{L} \Rightarrow \mathcal{R}}\ u' \text{ fresh}$$

Therefore, we decompose further the formalism of labelled sequents and extend the reach of labelled deduction to the logics studied in [6]. These systems enjoy cut-elimination via usual arguments, the generality of the result is subject of ongoing study.

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