Decomposing labelled proof theory for intuitionistic modal logic

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Abstract

We present a labelled deduction for intuitionistic modal logic that comes with two relation symbols, one for the accessible world relation associated with the Kripke semantics for modal logics, and one for the preorder relation associated with the Kripke semantics for intuitionistic logic. Thus, our labelled system is in close correspondence to the birelational Kripke models.

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1 Introduction

Labelled deduction has been proposed by Gabbay [8] in the 80's as a unifying framework throughout proof theory in order to provide proof systems for a wide range of logics. For modal logics it can take the form of labelled natural deduction and labelled sequent systems as used, for example, by Simpson [3], Vigano [4] and Negri [2]. These formalisms make explicit use not only of labels, but also of relational atoms. To our knowledge, all labelled systems proposed so far had only one relational symbol, [Sonia: Incorrect. We even cite [1] below] either representing the *accessible world relation* R (for modal logics) or the *future relation* $\leq$ (for intuitionistic logic). In this short note we propose a system that unifies both, and thus is able to use the full power of the bi-relational semantics for intuitionistic modal logics.

[Sonia: Removed paragraph. Simpson's system can easily be made multi-conclusion in the style of Maehara without the $\leq$-atoms.] [Lutz: maybe, but getting all the theorems right is not so easy, we had a hard time in the nested variant] [Sonia: I did not mean it like that. I mean that the point of our system is not that it is multi-conclusion (it is that it allows us to give systems for some more logics!). This paragraph is confusing as such because it shifts the attention of the reader. I think that we can actually restrict to single conclusion with no difference.]

For sequent systems for intuitionistic logics there is always a choice to be made: make the system *single conclusion* following Gentzen [?] or *multiple conclusion* following Maehara [?]. In our work we choose the multiple conclusion variant because of the closer correspondence to the semantics. In that respect, our system is closer to [?] than to [3] and [?].

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2 Intuitionistic modal logics

[Sonia: Removed the part on the classical labelled calculus. Don't think we need it in a short version.]

[Sonia: Formulas need to be defined.

The language of intuitionisitic modal logic is obtained from the one of intuitionistic propositional logic by adding the modal connectives $\Box$ and $\Diamond$. Starting with a set $\mathcal{A}$ of atomic propositions denoted a, modal formulas are constructed from the following grammar:

$$A ::= a \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A$$

]

[Marianela: is this enough as an introduction of the axiomatisations to present the theorem 2.5?] [Sonia: no, we would need the formal definition. But the text is good.]

The axiomatisation that is now generally accepted as intuitionistic modal logic denoted by IK was given by Plotkin and Stirling [6] and is equivalent to the one proposed by Fischer-Servi [5], and by Ewald [7] in the case of intuitionistic tense logic. It then was investigated in detail in [3], in which strong arguments are given in favour of this axiomatic definition: it allows for adapting to intuitionistic logic the standard embedding of modal logic into first-order logic, and also provides an extension of the standard Kripke semantics for classical modal logic to the intuitionistic case.

The Kripke semantics for IK was first defined by Fischer-Servi in [5]. This semantics for intuitionistic modal logic combines the Kripke semantics for intuitionistic propositional logic and the one for classical modal logic, using two distinct relations on the set of worlds.

Definition 2.1. A *bi-relational frame* $\mathcal{F}$ is a triple $\langle W, R, \leq \rangle$ of a non-empty set of worlds W equipped with two binary relations R and $\leq$, where R being the modal *accessibility relation* and $\leq$ a preorder (*i.e.* a reflexive and transitive relation), satisfying the following conditions:

(F1) For all worlds u, v, v', if $u\mathrm{R}v$ and $v \leq v'$, there exists a u' such that $u \leq u'$ and $u'\mathrm{R}v'$.
(F2) For all worlds u', u, v, if $u \leq v$, there exists a v' such that $u'\mathrm{R}v'$ and $v \leq v'$.

Definition 2.2. A *bi-relational model* $\mathcal{M}$ is a quadruple $\langle W, R, \leq, V\rangle$ with $\langle W, R, \leq\rangle$ a bi-relational frame and $V\colon W \to 2^{\mathcal{A}}$ a monotone valuation function, that is, a function mapping each world w to the subset of propositional atoms true at w, additionally subject to:

$$w \leq w' \Rightarrow V(w) \subseteq V(w')$$

We write $w \Vdash a$ iff $a \in V(w)$ and we extend this relation to all formulas by induction, following the rules for both intuitionistic and modal Kripke models:
$w \nVdash \bot$ (i.e. it is never the case that $w \Vdash \bot$)
$w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
$w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
$w \Vdash A \supset B$ iff for all w' with $w \leq w'$, if $w' \Vdash A$ then $w' \Vdash B$
$w \Vdash \Box A$ iff for all w' and u with $w \leq w'$ and $w'Ru$, $u \Vdash A$
$w \Vdash \Diamond A$ iff there exists a u such that wRu and $u \Vdash A$.

Definition 2.3. A formula A is *satisfied* in a model $\mathcal{M} = \langle W, R, \leq, V\rangle$, if for all $w \in W$ we have $w \Vdash A$.

Definition 2.4. A formula A is *valid* in a frame $\mathcal{F} = \langle W, R, \leq\rangle$, if for all valuations V, A is satisfied in $\langle W, R, \leq, V\rangle$.

[Sonia: Removed the figure as it has nothing to do with the theorem. Fisher-Servi as well as Plotkin and Sterling only give Hilbert-style axiomatisations for the birelational models, no sequent calculi. Therefore, if we want to present this theorem, the axiomatisations also need to be introduced.]

Theorem 2.5 (Fischer-Servi [5], Plotkin and Stirling [6]). *A formula A is a theorem of* IK *if and only if A is valid in every bi-relational frame.*

3 Capturing intuitionistic modal logics with labels

[Sonia: I moved Simpson's system labIK here as we need to make explicit that we are aware of his approach and we propose something else.]

Simpson [3] followed the lines of Gentzen in a labelled context, namely, he developed a labelled natural deduction framework for modal logics and then converted it into sequent systems with the consequent restriction to one formula on the right-hand side of each sequent. This worked as well in the labelled setting as in the ordinary sequent case: we present Simpson's sequent system labIK (1) where intuitionistic labelled sequents are written $\mathcal{G}, \mathcal{L} \Rightarrow z\colon C$ for some multiset of labelled formulas $\mathcal{L}$, some formula C, some label z and a set of relational atoms $\mathcal{G}$.

[TODO: Some background needs to be added] [Marianela: what about the paragraph that I just added before?]

$$\mathsf{id}\ \frac{}{\mathcal{G}, \mathcal{L}, x\colon a \Rightarrow x\colon a} \qquad \bot_L\ \frac{}{\mathcal{G}, \mathcal{L}, x\colon \bot \Rightarrow z\colon A}$$

$$\wedge_L\ \frac{\mathcal{G}, \mathcal{L}, x\colon A, x\colon B \Rightarrow z\colon C}{\mathcal{G}, \mathcal{L}, x\colon A \wedge B \Rightarrow z\colon C}$$

$$\wedge_R\ \frac{\mathcal{G}, \mathcal{L} \Rightarrow x\colon A \quad \mathcal{G}, \mathcal{L} \Rightarrow x\colon B}{\mathcal{G}, \mathcal{L}, \Rightarrow x\colon A \wedge B}$$

$$\vee_L\ \frac{\mathcal{G}, \mathcal{L}, x\colon A \Rightarrow z\colon C \quad \mathcal{G}, \mathcal{L}, x\colon B \Rightarrow z\colon C}{\mathcal{G}, \mathcal{L}, x\colon A \vee B \Rightarrow z\colon C}$$

$$\vee_{R1}\ \frac{\mathcal{G}, \mathcal{L} \Rightarrow x\colon A}{\mathcal{G}, \mathcal{L} \Rightarrow x\colon A \vee B} \qquad \vee_{R2}\ \frac{\mathcal{G}, \mathcal{L} \Rightarrow x\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow x\colon A \vee B}$$

$$\supset_L\ \frac{\mathcal{G}, \mathcal{L} \Rightarrow x\colon A \quad \mathcal{G}, \mathcal{L}, x\colon B \Rightarrow z\colon C}{\mathcal{G}, \mathcal{L}, x\colon A \supset B \Rightarrow z\colon C}$$

$$\supset_R\ \frac{\mathcal{G}, \mathcal{L}, x\colon A \Rightarrow x\colon B}{\mathcal{G}, \mathcal{L}, x\colon A \Rightarrow x\colon B}$$

$$\Box_L\ \frac{\mathcal{G}, x\mathrm{R}y, \mathcal{L}, x\colon \Box A, y\colon A \Rightarrow z\colon B}{\mathcal{G}, x\mathrm{R}y, \mathcal{L}, x\colon \Box A \Rightarrow z\colon B}$$

$$\Box_R\ \frac{\mathcal{G}, x\mathrm{R}y, \mathcal{L} \Rightarrow y\colon A}{\mathcal{G}, \mathcal{L} \Rightarrow x\colon \Box A}\ y \text{ is fresh}$$

$$\Diamond_L\ \frac{\mathcal{G}, x\mathrm{R}y, \mathcal{L}, y\colon A \Rightarrow z\colon B}{\mathcal{G}, \mathcal{L}, x\colon \Diamond A \Rightarrow z\colon B}\ y \text{ is fresh}$$

$$\Diamond_R\ \frac{\mathcal{G}, x\mathrm{R}y, \mathcal{L} \Rightarrow y\colon A}{\mathcal{G}, x\mathrm{R}y, \mathcal{L} \Rightarrow x\colon \Diamond A}$$

Figure 1. System labIK

Theorem 3.1 (Simpson [3]). *A formula A is provable in the calculus* labIK *if and only if A is valid in every bi-relational frame.*

Echoing the definition of bi-relational structures, another extension of labelled deduction to the intuitionistic setting would be to use two sorts of relational atoms, one for the modal relation R and another one for the intuitionistic relation $\leq$. This is the approach developed by Maffezioli, Naibo and Negri in [1]. The idea is to extend labelled sequents with a preorder relation symbol in order to capture intuitionistic modal logics, that is to define intuitionistic labelled sequents from labelled formulas $x\colon A$, relational atoms $x\mathrm{R}y$, and preorder atoms of the form $x \leq y$, where x, y range over a set of labels and A is an intuitionistic modal formula.

A two-sided intuitionistic labelled sequent would be of the form $\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}$ where $\mathcal{G}$ denotes a set of relational and preorder atoms, and $\mathcal{L}$ and $\mathcal{R}$ are multiset of labelled formulas. We then obtain a proof system lab♡IK (2) for intuitionistic modal logic in this formalism. [Marianela: I can't put the figure 2 before this!!]

$$id^{lab}\ \dfrac{}{\mathcal{G}, \mathcal{L}, x\colon a \Rightarrow x\colon a} \qquad \bot_L^{lab}\ \dfrac{}{\mathcal{G}, \mathcal{L}, x\colon \bot \Rightarrow z\colon A} \qquad \top_R^{lab}\ \dfrac{}{\mathcal{G}, \mathcal{L} \Rightarrow x\colon \top}$$

$$\wedge_L\ \dfrac{\mathcal{G}, \mathcal{L}, x\colon A, x\colon B \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\colon A \wedge B \Rightarrow \mathcal{R}} \qquad \wedge_R\ \dfrac{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \quad \mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \wedge B}$$

$$\vee_L\ \dfrac{\mathcal{G}, \mathcal{L}, x\colon A \Rightarrow \mathcal{R} \quad \mathcal{G}, \mathcal{L}, x\colon B \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\colon A \vee B \Rightarrow \mathcal{R}} \qquad \vee_R\ \dfrac{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A, x\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \vee B}$$

$$\supset_R\ \dfrac{\mathcal{G}, \mathcal{L}, x \leq y, y\colon A \Rightarrow \mathcal{R}, y\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \supset B}\ y \text{ fresh}$$

$$\supset_L\ \dfrac{\mathcal{G}, \mathcal{L}, x \leq y, x\colon A \supset B \Rightarrow \mathcal{R}, y\colon A \quad \mathcal{G}, \mathcal{L}, x \leq y, x\colon A \supset B, y\colon B \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x \leq y, x\colon A \supset B \Rightarrow \mathcal{R}}$$

$$\Box_L\ \dfrac{\mathcal{G}, \mathcal{L}, x \leq y, y\mathrm{R}z, x\colon \Box A, z\colon A \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x \leq y, y\mathrm{R}z, x\colon \Box A \Rightarrow \mathcal{R}} \qquad \Box_R\ \dfrac{\mathcal{G}, \mathcal{L}, x \leq y, y\mathrm{R}z \Rightarrow \mathcal{R}, z\colon A}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon \Box A}\ y, z \text{ fresh}$$

$$\Diamond_L\ \dfrac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, y\colon A \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\colon \Diamond A \Rightarrow \mathcal{R}}\ y \text{ fresh} \qquad \Diamond_R\ \dfrac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y \Rightarrow \mathcal{R}, x\colon \Diamond A, y\colon A}{\mathcal{G}, \mathcal{L}, x\mathrm{R}y \Rightarrow \mathcal{R}, x\colon \Diamond A}$$

$$refl\ \dfrac{\mathcal{G}, x \leq x, \mathcal{L}, \mathcal{R}}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}} \qquad trans\ \dfrac{\mathcal{G}, x \leq y, y \leq z, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{G}, x \leq y, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$F1\ \dfrac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, y \leq z, x \leq u, u\mathrm{R}z \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, y \leq z \Rightarrow \mathcal{R}}\ u \text{ fresh} \qquad F2\ \dfrac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, x \leq z, y \leq u, z\mathrm{R}u \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, x \leq z \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

Figure 2. System lab♡IK

Theorem 3.2. *A formula A is provable in the calculus* lab♡IK *if and only if A is valid in every bi-relational frame.*

[Sonia: Rewrote the last two paragraphs more succinctly. Should we say more?]

On the one hand, we prove directly that each rule from our system is sound wrt. bi-relational structures. On the other hand, we show that lab♡IK is complete wrt. Simpson's labIK, and the theorem then follows from Theorem 3.1. [Marianela: Sonia, what do you think about mention your result of completeness with cut elimination?]

Proof of $4_\Box : \Box A \supset \Box\Box A$

$$\supset_R \dfrac{\Box_R \dfrac{\mathsf{F}_1 \dfrac{\text{trans} \dfrac{\blacksquare_4 \dfrac{\Box \dfrac{\text{id} \dfrac{}{x \leq w, w \leq w', w'Rv, v \leq v', v'Ru, w' \leq t, tRv', w \leq t, tRu, w : \Box A, u : A \Rightarrow u : A}}{x \leq w, w \leq w', w'Rv, v \leq v', v'Ru, w' \leq t, tRv', w \leq t, \boldsymbol{tRu}, w : \Box A \Rightarrow u : A}}{x \leq w, w \leq w', w'Rv, v \leq v', \boldsymbol{v'Ru}, w' \leq t, \boldsymbol{tRv'}, w \leq t, w : \Box A \Rightarrow u : A}}{x \leq w, w \leq w', w'Rv, v \leq v', v'Ru, \boldsymbol{w' \leq t}, \boldsymbol{tRv'}, w : \Box A \Rightarrow u : A}}{x \leq w, w \leq w', \boldsymbol{w'Rv}, \boldsymbol{v \leq v'}, v'Ru, w : \Box A \Rightarrow u : A}}{x \leq w, w : \Box A \Rightarrow w : \Box\Box A}}{x : \Box A \supset \Box\Box A}$$

Proof of $4_\Diamond : \Diamond\Diamond A \supset \Diamond A$:

$$\supset_R \dfrac{\Diamond_L \dfrac{\blacklozenge_4 \dfrac{\Diamond_R \dfrac{\text{id} \dfrac{}{x \leq w, wRv, vRu, u \leq u', wRu' u : A \Rightarrow w : \Diamond A, u' : A}}{x \leq w, wRv, vRu, u \leq u', wRu' u : A \Rightarrow w : \Diamond A}}{x \leq w, wRv, vRu, u : A \Rightarrow w : \Diamond A}}{x \leq w, w : \Diamond\Diamond A \Rightarrow w : \Diamond A}}{x : \Diamond\Diamond A \supset \Diamond A}$$

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