Proposal: Decomposing labelled proof theory for intuitionistic modal logic

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The reference work on proof theory for intuitionistic modal logics is Simpson's PhD thesis [6].

1 Classical modal logic

1.1 Syntax

The language of classical modal logic is obtained from the one of classical propositional logic by adding the modal connectives $\Box$ and $\Diamond$, standing for example for *necessity* and *possibility*. Starting with a set $\mathcal{A}$ of atomic propositions denoted a and their duals $\bar{a}$, modal formulas are constructed from the following grammar:

$$A ::= \quad a \mid \bar{a} \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid \Box A \mid \Diamond A$$

In a classical setting, we always assume that formulas are in negation normal form, that is, negation is restricted to atoms. When we write $\neg A$ in this case, we mean the result of computing the de Morgan dual of connectives and atomic propositions within A, i.e. $\neg\neg A \equiv A$, $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and $\neg\Box A \equiv \Diamond\neg A$, where $\equiv$ denotes syntactic equality. Implication can be defined from this set of connectives by $A \supset B := \neg A \vee B$. $\top$ and $\bot$ are the usual *units* of the binary connectives $\wedge$ and $\vee$ respectively.

The classical modal logic K is then obtained from classical propositional logic by adding:

- the *necessitation* rule: if A is a theorem of K then $\Box A$ is too; and
- the axiom of *distributivity* $\mathsf{k} := \Box(A \supset B) \supset (\Box A \supset \Box B)$.

1.2 Semantics

Kripke gave the first systematic treatment of *possible-worlds semantics* (therefore also known as Kripke semantics). One starts with a graph: a *frame* $\mathcal{F}$ is a pair $\langle W, R\rangle$ of a non-empty set W of *possible worlds* and a binary relation $R \subseteq W \times W$, called the *accessibility relation*. Then one adds a mechanism to evaluate formulas: a *model* $\mathfrak{M}$ is a frame together with a *valuation* function $V \colon W \to 2^{\mathcal{A}}$, which assigns to each world w a subset of propositional variables that are "true" in w. The *truth* of a modal formula at a world w in a relational structure is the smallest relation $\Vdash$ satisfying:

$$\begin{array}{lll}
w \Vdash a & \text{iff} & a \in V(w) \\
w \Vdash \bar{a} & \text{iff} & a \notin V(w) \\
w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B \\
w \Vdash \Box A & \text{iff} & \text{for all } v \in W \text{ such that } (w,v) \in R \text{ one has } v \Vdash A \\
w \Vdash \Diamond A & \text{iff} & \text{there exists } v \in W \text{ such that } (w,v) \in R \text{ and } v \Vdash A
\end{array}$$

$$\text{id}\ \frac{}{\mathcal{G} \Rightarrow \mathcal{R}, x : a, x : \bar{a}} \qquad \top\ \frac{}{\mathcal{G} \Rightarrow \mathcal{R}, x : \top}$$

$$\wedge\ \frac{\mathcal{G} \Rightarrow \mathcal{R}, x : A \quad \mathcal{G} \Rightarrow \mathcal{R}, x : B}{\mathcal{G} \Rightarrow \mathcal{R}, x : A \wedge B} \qquad \vee\ \frac{\mathcal{G} \Rightarrow \mathcal{R}, x : A, x : B}{\mathcal{G} \Rightarrow \mathcal{R}, x : A \vee B}$$

$$\Box\ \frac{\mathcal{G}, xRy \Rightarrow \mathcal{R}, y : A}{\mathcal{G} \Rightarrow \mathcal{R}, x : \Box A}\ y \text{ is fresh} \qquad \Diamond\ \frac{\mathcal{G}, xRy \Rightarrow \mathcal{R}, x : \Diamond A, y : A}{\mathcal{G}, xRy \Rightarrow \mathcal{R}, x : \Diamond A}$$

Figure 1: System labK

We say that a formula A is *satisfied in a model* $\mathfrak{M} = \langle W, R, V \rangle$, denoted by $\mathfrak{M} \models A$, if for every $w \in W$, $w \Vdash A$. We say that a formula A is *valid in a frame* $\mathcal{F} = \langle W, R \rangle$, denoted by $\mathcal{F} \models A$, if for every valuation V, $\langle W, R, V \rangle \models A$.

Theorem 1 (Kripke [1])**.** *A formula A is a theorem of* K *if and only if A is valid in every frame.*

1.3 Labelled proof theory

Labelled sequents are formed from by *labelled formulas* of the form $x : A$ and *relational atoms* of the form xRy , where x, y range over a set of variables (called *labels*) and A is a modal formula. A (one-sided) labelled sequent is then of the form $\mathcal{G} \Rightarrow \mathcal{R}$ where $\mathcal{G}$ denotes a set of relational atoms and $\mathcal{R}$ a multiset of labelled formulas. A simple proof system (shown on Figure 1) for classical modal logic K can be obtained in this formalism.

Theorem 2 (Negri [3])**.** *A formula A is provable in the calculus* labK *if and only if A is valid in every frames.*

2 Intuitionistic modal logic

2.1 Syntax

In the intuitionistic case, we work with a different set of connectives. Starting with a set $\mathcal{A}$ of atomic propositions still denoted a, formulas are constructed from the following grammar:

$$A ::= a \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A$$

When we write $\neg A$ in this case, we mean $A \supset \bot$.

Obtaining the intuitionistic variant of K is more involved than the classical variant. Lacking De Morgan duality, there are several variants of k that are classically but not intuitionistically equivalent. Five axioms have been considered as primitives in the literature. The intuitionistic modal logic IK is obtained from ordinary intuitionistic propositional logic by adding:

- the *necessitation rule*: $\Box A$ is a theorem if A is a theorem; and
- the following five variants of the k axiom.

k_1: $\Box(A \supset B) \supset (\Box A \supset \Box B)$ $\quad$ k_3: $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$ $\quad$ k_5: $\Diamond\bot \supset \bot$

k_2: $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ $\quad$ k_4: $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$

2.2 Semantics

The Kripke semantics for IK combines the Kripke semantics for intuitionistic propositional logic and the one for classical modal logic, using two distinct relations on the set of worlds. So, a *bi-relational frame* $\mathcal{F}$ is a triple $\langle W, \leq, R\rangle$ of a non-empty set of worlds W with two binary relations: $R \subseteq W \times W$ and $\leq$ a pre-order on W (*i.e.* a reflexive and transitive relation) satisfying the conditions:

(F1) For all worlds u, v, v', if uRv and $v \leq v'$, there exists a u' such that $u \leq u'$ and $u'Rv'$:

$$\begin{tikzcd}
u' \arrow[r, dotted, "R"'] & v' \\
u \arrow[u, dotted, "\leq"] \arrow[r, "R"'] & v \arrow[u, "\leq"']
\end{tikzcd}$$

(F2) For all worlds u', u, v, if $u \leq u'$ and uRv, there exists a v' such that $u'Rv'$ and $v \leq v'$:

$$\begin{tikzcd}
u' \arrow[r, dotted, "R"] & v' \\
u \arrow[u, "\leq"] \arrow[r, "R"] & v \arrow[u, dotted, "\leq"']
\end{tikzcd}$$

A *bi-relational model* $\mathfrak{M}$ is a quadruple $\langle W, \leq, R, V\rangle$ with $\langle W, \leq, R\rangle$ a frame and V a monotone *valuation* function $V \colon W \to 2^{\mathcal{A}}$ which is a function that maps each world w to the subset of propositional atoms that are true in w, subject to:

$$w \leq w' \Rightarrow V(w) \subseteq V(w')$$

As in the classical case, we write $w \Vdash a$ if $a \in V(w)$ and we extend this relation to all formulas by induction, following the rules for both intuitionistic and modal Kripke models:

$$\begin{array}{lll}
w \nVdash \bot & \text{iff} & \\
w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B \\
w \Vdash A \supset B & \text{iff} & \text{for all } w' \text{ with } w \leq w', \text{ if } w' \Vdash A \text{ then also } w' \Vdash B \\
w \Vdash \Box A & \text{iff} & \text{for all } w' \text{ and } u \text{ with } w \leq w' \text{ and } w'Ru, \text{ we have } u \Vdash A \\
w \Vdash \Diamond A & \text{iff} & \text{there is a } u \in W \text{ such that } wRu \text{ and } u \Vdash A
\end{array}$$

We write $w \nVdash A$ if it is not the case that $w \Vdash A$, but contrarily to the classical case, we do not have $w \Vdash \neg A$ iff $w \nVdash A$ (since $\neg A$ is defined as $A \supset \bot$).

We say that a formula A *is satisfied in a model* $\mathfrak{M} = \langle W, R, \leq, V\rangle$, if for all $w \in W$ we have $w \Vdash A$. A formula A *is valid in a frame* $\langle W, R, \leq\rangle$, if for all valuations V, A is satisfied in $\langle W, R, \leq, V\rangle$

Theorem 3 (Fischer-Servi [5], Plotkin and Stirling [4])**.** *A formula A is a theorem IK if and only if A is valid in every bi-relational frame.*

2.3 Labelled proof theory

Echoing to the definition of bi-relational structures, the straightforward extension of labelled deduction to the intuitionistic setting would be to use two sorts of relational atoms, one for the modal relation R and another one for the intuitionistic relation $\leq$. This is the approach developed by Maffezioli, Naibo and Negri in [2]. To our knowledge this has not yet been investigated much further, but could be a fruitful perspective.

The idea is to extend labelled sequents with a preorder relation symbol in order to capture intuitionistic modal logics; that is, to define intuitionistic labelled sequents from labelled formulas $x : A$, relational atoms xRy, and *preorder atoms* of the form $x \leq y$, where x, y range over a set of labels and A is an intuitionistic modal formula. A *two-sided intuitionistic labelled sequent* would be of the form $\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}$ where $\mathcal{B}$ denotes a set of relational and preorder atoms, and $\mathcal{L}$ and $\mathcal{R}$ are multisets of labelled formulas. We then would want to obtain a proof system $\mathsf{lab}\heartsuit\mathsf{IK}$ for intuitionistic modal logic IK in this formalism and prove the following as a new theorem.

Question 4. *A formula A is provable in the calculus $\mathsf{lab}\heartsuit\mathsf{IK}$ if and only if A is valid in every bi-relational frame.*

2.4 Answer

$$\mathsf{id}\ \dfrac{}{\mathcal{B}, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, y : a} \qquad \bot_L\ \dfrac{}{\mathcal{B}, \mathcal{L}, x : \bot \Rightarrow \mathcal{R}} \qquad \top_R\ \dfrac{}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : \top}$$

$$\wedge_L\ \dfrac{\mathcal{B}, \mathcal{L}, , x : A, x : B \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x : A \wedge B \Rightarrow \mathcal{R}} \qquad \wedge_R\ \dfrac{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : A \quad \mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : A \wedge B}$$

$$\vee_L\ \dfrac{\mathcal{B}, \mathcal{L}, x : A \Rightarrow \mathcal{R} \quad \mathcal{B}, \mathcal{L}, x : B \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x : A \vee B \Rightarrow \mathcal{R}} \qquad \vee_R\ \dfrac{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : A, x : B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : A \vee B}$$

$$\supset_L\ \dfrac{\mathcal{B}, x \leq y, \mathcal{L}, x : A \supset B \ \Rightarrow \mathcal{R}, y : A \quad \mathcal{B}, x \leq y, \mathcal{L}, y : B \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}}$$

$$\supset_R\ \dfrac{\mathcal{B}, x \leq x', \mathcal{L}, x' : A \Rightarrow \mathcal{R}, x' : B}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B}\ x' \text{ fresh}$$

$$\Box_L\ \dfrac{\mathcal{B}, x \leq u, uRv, \mathcal{L}, x : \Box A, v : A \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq u, uRv, \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}} \qquad \Box_R\ \dfrac{\mathcal{B}, x \leq x', x'Ry', \mathcal{L} \Rightarrow \mathcal{R}, y' : A}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A}\ x', y' \text{ fresh}$$

$$\Diamond_L\ \dfrac{\mathcal{B}, xRy', \mathcal{L}, y' : A \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}}\ y' \text{ fresh} \qquad \Diamond_R\ \dfrac{\mathcal{B}, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A, y : A}{\mathcal{B}, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A}$$

....

$$\mathsf{refl}\ \dfrac{\mathcal{B}, x \leq x, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}} \qquad \mathsf{trans}\ \dfrac{\mathcal{B}, x \leq y, y \leq z, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$\mathsf{F}_1\ \dfrac{\mathcal{B}, xRy, y \leq z, x \leq u, uRz, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}\ u \text{ fresh} \qquad \mathsf{F}_2\ \dfrac{\mathcal{B}, xRy, y \leq z, x \leq u, uRz, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{B}, xRy, x \leq u, \mathcal{L} \Rightarrow \mathcal{R}}\ z \text{ fresh}$$

Figure 2: System $\mathsf{lab}\heartsuit\mathsf{IK}$

Conjecture 5. *For any formula A, the following are equivalent.*

1. *A is a theorem of IK*
2. *A is provable in $\mathsf{lab}\heartsuit\mathsf{IK} + \mathsf{cut}$ where cut is* $\quad \mathsf{cut}\ \dfrac{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, z : C \quad \mathcal{B}_2, \mathcal{L}, z : C \Rightarrow \mathcal{R}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}}$
3. *A is provable in $\mathsf{lab}\heartsuit\mathsf{IK}$*
4. *A is valid in every birelational frames*

Proof of 2 → 3. By induction on number of cuts + (rank,height) of the left-most top-most cut.

Commutative cases:

$$
{\scriptstyle \mathsf{cut}}\ \dfrac{{\scriptstyle \supset_L}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, z : C, y : A} \quad \dfrac{\mathcal{D}_2}{\mathcal{B}_1, x \leq y, \mathcal{L}, y : B \Rightarrow \mathcal{R}, z : C}}{\mathcal{B}_1, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, z : C} \quad \dfrac{\mathcal{D}_3}{\mathcal{B}_2, \mathcal{L}, x : A \supset B, z : C \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}} \leadsto
$$

$$
{\scriptstyle \supset_L}\ \dfrac{{\scriptstyle \mathsf{cut}}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, z : C, y : A} \quad \dfrac{\mathcal{D}_3^{\mathsf{w}}}{\mathcal{B}_2, \mathcal{L}, x : A \supset B, z : C \Rightarrow \mathcal{R}, y : A}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, y : A} \quad {\scriptstyle \mathsf{cut}}\ \dfrac{\dfrac{\mathcal{D}_2}{\mathcal{B}_1, x \leq y, \mathcal{L}, y : B \Rightarrow \mathcal{R}, z : C} \quad \dfrac{\mathcal{D}_3[v/y]^{\supset^\bullet_L}}{\mathcal{B}_2, \mathcal{L}, y : B, z : C \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, y : B \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}}
$$

We need to make sure that y does not appear in $\mathcal{D}_3$, before applying Lemma 8. If it does we rewrite it with a fresh variable v first.

$$
{\scriptstyle \mathsf{cut}}\ \dfrac{{\scriptstyle \supset_R}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq x', \mathcal{L}, x' : A \Rightarrow \mathcal{R}, x' : B, z : C}}{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B, z : C}\ x' \text{ fresh} \quad \dfrac{\mathcal{D}_2}{\mathcal{B}_2, \mathcal{L}, z : C \Rightarrow \mathcal{R}, x : A \supset B}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B}
$$

$$
\leadsto \quad {\scriptstyle \supset_R}\ \dfrac{{\scriptstyle \mathsf{cut}}\ \dfrac{\dfrac{\mathcal{D}_1[x''/x']}{\mathcal{B}_1, x \leq x'', \mathcal{L}, x'' : A \Rightarrow \mathcal{R}, x'' : B, z : C} \quad \dfrac{\mathcal{D}_2^{\supset^\bullet_R}}{\mathcal{B}_2, x \leq x'', \mathcal{L}, z : C, x'' : A \Rightarrow \mathcal{R}, x'' : B}}{\mathcal{B}_1, \mathcal{B}_2, x \leq x'' \mathcal{L}, x'' : A \Rightarrow \mathcal{R}, x'' : B}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B}\ x'' \text{ fresh (also in } \mathcal{D}_2)
$$

$$
{\scriptstyle \mathsf{cut}}\ \dfrac{{\scriptstyle \Box_L}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq u, uRv, \mathcal{L}, x : \Box A, v : A \Rightarrow \mathcal{R}, z : C}}{\mathcal{B}_1, x \leq u, uRv, \mathcal{L}, x : \Box \Rightarrow \mathcal{R}, z : C} \quad \dfrac{\mathcal{D}_2}{\mathcal{B}_2, \mathcal{L}, x : \Box A, z : C \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}}
$$

$$
\leadsto \quad {\scriptstyle \Box_L}\ \dfrac{{\scriptstyle \mathsf{cut}}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq u, uRv, \mathcal{L}, x : \Box A, v : A \Rightarrow \mathcal{R}, z : C} \quad \dfrac{\mathcal{D}_2^{\mathsf{w}}}{\mathcal{B}_2, \mathcal{L}, x : \Box A, v : A, z : C \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L}, x : \Box A, v : A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}}
$$

$$
{\scriptstyle \mathsf{cut}}\ \dfrac{{\scriptstyle \Box_R}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq x', x'Ry' \mathcal{L} \Rightarrow \mathcal{R}, y' : A, z : C}}{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A, z : C}\ x', y' \text{ fresh} \quad \dfrac{\mathcal{D}_2}{\mathcal{B}_2, \mathcal{L}, z : C \Rightarrow \mathcal{R}, x : \Box A}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A}
$$

$$
\leadsto \quad {\scriptstyle \Box_R}\ \dfrac{{\scriptstyle \mathsf{cut}}\ \dfrac{\dfrac{\mathcal{D}_1}{\mathcal{B}_1, x \leq u, uRv, \mathcal{L} \Rightarrow \mathcal{R}, v : A, z : C} \quad \dfrac{\mathcal{D}_2^{\Box^\bullet_R}}{\mathcal{B}_1, x \leq u, uRv, \mathcal{L}, z : C \Rightarrow \mathcal{R}, v : A}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L} \Rightarrow \mathcal{R}, v : A}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A}\ u, v \text{ fresh (also in } \mathcal{D}_2)
$$

$$\text{cut}\ \dfrac{\Diamond_L\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, xRy', \mathcal{L}, y' : A \Rightarrow \mathcal{R}, z : C}}{\mathcal{B}_1, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}, z : C}\ y' \text{ fresh} \qquad \overset{\mathcal{D}_2}{\mathcal{B}_2, \mathcal{L}, x : \Diamond A, z : C \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}} \rightsquigarrow \Diamond_L\ \dfrac{\text{cut}\ \dfrac{\overset{\mathcal{D}_1[y''/y']}{\mathcal{B}_1, xRy'', \mathcal{L}, y'' : A \Rightarrow \mathcal{R}, z : C} \quad \overset{\mathcal{D}_2^{\Diamond_L^{\bullet}}}{\mathcal{B}_2, \mathcal{L}, y'' : A, z : C \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, xRy'', \mathcal{L}, y'' : A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}}\ y'' \text{ fresh (also in } \mathcal{D}_2)$$

$$\text{cut}\ \dfrac{\Diamond_R\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A, y : A, z : C}}{\mathcal{B}_1, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A, z : C} \qquad \overset{\mathcal{D}_2}{\mathcal{B}_2, \mathcal{L}, z : C \Rightarrow \mathcal{R}, x : \Diamond A}}{\mathcal{B}_1, \mathcal{B}_2, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A} \rightsquigarrow \Diamond_R\ \dfrac{\text{cut}\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A, y : A, z : C} \quad \overset{\mathcal{D}_2^{\mathsf{w}}}{\mathcal{B}_2, \mathcal{L}, z : C \Rightarrow \mathcal{R}, x : \Diamond A, y : A}}{\mathcal{B}_1, \mathcal{B}_2, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A, y : A}}{\mathcal{B}_1, \mathcal{B}_2, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A}$$

Key cases:

$$\text{cut}\ \dfrac{\text{id}\ \dfrac{}{\mathcal{B}_1, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, y : a} \qquad \overset{\mathcal{D}_2}{\mathcal{B}_2, \mathcal{L}, x : a, y : a \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}} \rightsquigarrow \text{mon}_L\ \dfrac{\overset{\mathcal{D}_2^{\mathsf{w}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, x : a, y : a \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}}$$

$$\text{cut}\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, x : a, y : a} \qquad \text{id}\ \dfrac{}{\mathcal{B}_2, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, y : a}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, y : a} \rightsquigarrow \text{mon}_R\ \dfrac{\overset{\mathcal{D}_1^{\mathsf{w}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, x : a, y : a}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, y : a}$$

$$\text{cut}\ \dfrac{\supset_R\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, x \leq x', \mathcal{L}, x' : A \Rightarrow \mathcal{R}, x' : B}}{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B} \qquad \supset_L\ \dfrac{\overset{\mathcal{D}_2}{\mathcal{B}_2, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, y : A} \quad \overset{\mathcal{D}_3}{\mathcal{B}_3, x \leq y, \mathcal{L}, y : B \Rightarrow \mathcal{R}}}{\mathcal{B}_2, \mathcal{B}_3, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}} \rightsquigarrow$$

$$\text{cut}\ \dfrac{\text{cut}\ \dfrac{\supset_R\ \dfrac{\overset{\mathcal{D}_1^{\mathsf{w}}}{\mathcal{B}_1, x \leq x', \mathcal{L}, x' : A \Rightarrow \mathcal{R}, x' : B, y : A}}{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B, y : A} \qquad \overset{\mathcal{D}_2}{\mathcal{B}_2, x \leq y, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, y : A}}{\mathcal{B}_1, \mathcal{B}_2, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, y : A} \qquad \text{cut}\ \dfrac{\overset{\mathcal{D}_1[y/x']}{\mathcal{B}_1, x \leq y, \mathcal{L}, y : A \Rightarrow \mathcal{R}, y : B} \quad \overset{\mathcal{D}_3}{\mathcal{B}_3, x \leq y, \mathcal{L}, y : B \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_3, x \leq y, \mathcal{L}, y : A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$\text{cut}\ \dfrac{\Box_R\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, x \leq x', x'Ry', \mathcal{L} \Rightarrow \mathcal{R}, y' : A}}{\mathcal{B}_1, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A} \qquad \Box_L\ \dfrac{\overset{\mathcal{D}_2}{\mathcal{B}_2, x \leq u, uRv, \mathcal{L}, x : \Box A, v : A \Rightarrow \mathcal{R}}}{\mathcal{B}_2, x \leq u, uRv, \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L} \Rightarrow \mathcal{R}} \rightsquigarrow$$

$$\text{cut}\ \dfrac{\overset{\mathcal{D}_1[u/x', v/y']}{\mathcal{B}_1, x \leq u, uRv, \mathcal{L} \Rightarrow \mathcal{R}, v : A} \qquad \text{cut}\ \dfrac{\Box_R\ \dfrac{\overset{\mathcal{D}_1^{\mathsf{w}}}{\mathcal{B}_1, x \leq x', x'Ry', \mathcal{L}, v : A \Rightarrow \mathcal{R}, x : \Box A, y' : A}}{\mathcal{B}_1, \mathcal{L}, v : A \Rightarrow \mathcal{R}, x : \Box A} \quad \overset{\mathcal{D}_2}{\mathcal{B}_2, x \leq u, uRv, \mathcal{L}, x : \Box A, v : A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L}, v : A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, x \leq u, uRv, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$\text{cut}\ \dfrac{\Diamond_R\ \dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A, y : A}}{\mathcal{B}_1, xRy, \mathcal{L} \Rightarrow \mathcal{R}, x : \Diamond A} \qquad \Diamond_L\ \dfrac{\overset{\mathcal{D}_2}{\mathcal{B}_2, xRy', \mathcal{L}, y' : A \Rightarrow \mathcal{R}}}{\mathcal{B}_2, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}}\ y' \text{ is fresh}}{\mathcal{B}_1, \mathcal{B}_2, xRy, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$
\mathsf{cut}\,\dfrac{\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\mathcal{B}_1, xRy, \mathcal{L} \Rightarrow \mathcal{R}, y : A, x : \Diamond A} \qquad \Diamond_L\,\dfrac{\overset{\mathcal{D}_2^{\mathsf{w}}}{\mathcal{B}_2, xRy', \mathcal{L}, y' : A \Rightarrow \mathcal{R}, y : A}}{\mathcal{B}_2, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}, y : A}\; y' \text{ is fresh}}{\mathcal{B}_1, \mathcal{B}_2, xRy, \mathcal{L} \Rightarrow \mathcal{R}, y : A} \qquad \overset{\mathcal{D}_2[y/y']}{\mathcal{B}_2, xRy, \mathcal{L}, y : A \Rightarrow \mathcal{R}}}{\mathcal{B}_1, \mathcal{B}_2, xRy, \mathcal{L} \Rightarrow \mathcal{R}}
$$

☺

Lemma 6. *The following rules are admissible in* lab♡IK.

$$
\mathsf{mon}_L\,\dfrac{\mathcal{B}, x \leq y, \mathcal{L}, x : a, y : a \Rightarrow \mathcal{R}}{\mathcal{B}, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}} \qquad \mathsf{mon}_R\,\dfrac{\mathcal{B}, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, x : a, y : a}{\mathcal{B}, x \leq y, \mathcal{L} \Rightarrow \mathcal{R}, y : a}
$$

Proof.

$$
\mathsf{mon}_L\,\dfrac{\mathsf{r}\,\dfrac{\mathcal{B}', x \leq y, \mathcal{L}', x : a, y : a \Rightarrow \mathcal{R}', x : a}{\mathcal{B}, x \leq y, \mathcal{L}, x : a, y : a \Rightarrow \mathcal{R}, x : a}}{\mathcal{B}, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, x : a} \quad \rightsquigarrow \quad \mathsf{r}\,\dfrac{\mathsf{mon}_L\,\dfrac{\mathcal{B}', x \leq y, \mathcal{L}', x : a, y : a \Rightarrow \mathcal{R}', x : a}{\mathcal{B}', x \leq y, \mathcal{L}', x : a \Rightarrow \mathcal{R}', x : a}}{\mathcal{B}, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, x : a}
$$

$$
\mathsf{mon}_L\,\dfrac{\mathsf{id}\,\dfrac{}{\mathcal{B}, x \leq y, x : a, y : a \Rightarrow \mathcal{R}, x : a}}{\mathcal{B}, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, x : a} \quad \rightsquigarrow \quad \mathsf{id}\,\dfrac{}{\mathcal{B}, x \leq y, x : a \Rightarrow \mathcal{R}, x : a}
$$

$$
\mathsf{mon}_L\,\dfrac{\mathsf{id}\,\dfrac{}{\mathcal{B}, x \leq y, x : a, y : a \Rightarrow \mathcal{R}, y : a}}{\mathcal{B}, x \leq y, \mathcal{L}, x : a \Rightarrow \mathcal{R}, y : a} \quad \rightsquigarrow \quad \mathsf{id}\,\dfrac{}{\mathcal{B}, x \leq y, x : a \Rightarrow \mathcal{R}, y : a}
$$

☺

Lemma 7.

1. *If there exists a proof* $\overset{\mathcal{D}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : \bot}$ *then there exists a proof* $\overset{\mathcal{D}^{\bot}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}}$

2. *If there exists a proof* $\overset{\mathcal{D}}{\mathcal{B}, \mathcal{L}, x : \top \Rightarrow \mathcal{R}}$ *then there exists a proof* $\overset{\mathcal{D}^{\top}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}}$

3. *If there exists a proof* $\overset{\mathcal{D}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}}$ *then there exists a proof* $\overset{\mathcal{D}^{\mathsf{w}}}{\mathcal{B}, xRy, u \leq v, \mathcal{L}, z : A \Rightarrow \mathcal{R}, w : B}$

Proof. Standard ☺

Lemma 8.

1. *If there exists a proof* $\overset{\mathcal{D}}{\mathcal{B}, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}}$ *then there exists a proof* $\overset{\mathcal{D}^{\supset^{\bullet}_L}}{\mathcal{B}, \mathcal{L}, y : B \Rightarrow \mathcal{R}}$ *of the same (or smaller) height, for any label y that does not appear in $\mathcal{D}$.*

2. *If there exists a proof* $\overset{\mathcal{D}}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : A \supset B}$ *then there exists a proof* $\overset{\mathcal{D}^{\supset^{\bullet}_R}}{\mathcal{B}, x \leq y, \mathcal{L}, y : A \Rightarrow \mathcal{R}, y : B}$ *of the same (or smaller) height, for any label y that does not appear in $\mathcal{D}$.*

3. *If there exists a proof* $\dfrac{\mathcal{D}\,\Big\|}{\mathcal{B}, \mathcal{L} \Rightarrow \mathcal{R}, x : \Box A}$ *then there exists a proof* $\dfrac{\mathcal{D}^{\Box^\bullet_R}\,\Big\|}{\mathcal{B}, x \leq u, uRv, \mathcal{L} \Rightarrow \mathcal{R}, v : A}$ *of the same (or smaller) height, for any label u and v that do not appear in $\mathcal{D}$.*

4. *If there exists a proof* $\dfrac{\mathcal{D}\,\Big\|}{\mathcal{B}, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}}$ *then there exists a proof* $\dfrac{\mathcal{D}^{\Diamond^\bullet_L}\,\Big\|}{\mathcal{B}, xRy, \mathcal{L}, y : A \Rightarrow \mathcal{R}}$ *of the same (or smaller) height, for any label y that does not appear in $\mathcal{D}$.*

Proof. In each case, we reason by induction on the height of $\mathcal{D}$.

- $\supset_L$: For a proof of height 1, it is straightforward. For example, if $\mathcal{D} = \mathsf{id}\,\dfrac{}{\mathcal{B}, u \leq v, \mathcal{L}, u : a, x : A \supset B \Rightarrow \mathcal{R}, v : a}$,

  then we take $\mathcal{D}^{\supset^\bullet_L}$ to be $\mathsf{id}\,\dfrac{}{\mathcal{B}, u \leq v, \mathcal{L}, u : a, y : B \Rightarrow \mathcal{R}, v : a}$.

  For a proof $\mathcal{D}$ of height greater than 1 we have two cases, depending on whether the last rule of $\mathcal{D}$ acts on $x : A \supset B$ or only on some part of the context.

  First let us fix a given index y that does not appear in $\mathcal{D}$.

  If we start with a proof

$$\mathcal{D} = \mathsf{r}\,\dfrac{\dfrac{\mathcal{D}'\,\Big\|}{\mathcal{B}', \mathcal{L}', x : A \supset B \Rightarrow \mathcal{R}'}}{\mathcal{B}, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}}$$

  Then by induction hypothesis there exists a proof

$$\dfrac{\mathcal{D}'^{\supset^\bullet_L}\,\Big\|}{\mathcal{B}', \mathcal{L}', y : B \Rightarrow \mathcal{R}'}$$

  of the same (or smaller) height as $\mathcal{D}'$ (as y also does not appear in $\mathcal{D}'$).

  Therefore, we have the proof

$$\mathcal{D}^{\supset^\bullet_L} = \mathsf{r}\,\dfrac{\dfrac{\mathcal{D}'^{\supset^\bullet_L}\,\Big\|}{\mathcal{B}', \mathcal{L}', y : B \Rightarrow \mathcal{R}'}}{\mathcal{B}, \mathcal{L}, y : B \Rightarrow \mathcal{R}}$$

  of the same (or smaller) height as $\mathcal{D}$.

  If we start with a proof

$$\supset_L \dfrac{\dfrac{\mathcal{D}_1\,\Big\|}{\mathcal{B}, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, z : A} \quad \dfrac{\mathcal{D}_2\,\Big\|}{\mathcal{B}, \mathcal{L}, z : B \Rightarrow \mathcal{R}}}{\mathcal{B}, \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}}\ x \leq z \text{ appears in } \mathcal{B}$$

  then we take $\mathcal{D}^{\supset^\bullet_L}$ to be $\dfrac{\mathcal{D}_2[y/z]\,\Big\|}{\mathcal{B}, \mathcal{L}, y : B \Rightarrow \mathcal{R}}$ (as y also does not appear in $\mathcal{D}_2$) and its height is smaller than the one of $\mathcal{D}$.

- $\Diamond_L$:

If we start with a proof

$$\mathcal{D} = \mathsf{r}\,\dfrac{\begin{array}{c}\mathcal{D}' \,\Big\| \\ \mathcal{B}', \mathcal{L}', x : \Diamond A \Rightarrow \mathcal{R}'\end{array}}{\mathcal{B}, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}}$$

Then by induction hypothesis there exists a proof

$$\begin{array}{c}\mathcal{D}'^{\Diamond^{\bullet}_L} \,\Big\| \\ \mathcal{B}', xRy, \mathcal{L}', y : A \Rightarrow \mathcal{R}'\end{array}$$

of the same (or smaller) height as $\mathcal{D}'$.

Therefore, we have the proof

$$\mathcal{D}^{\Diamond^{\bullet}_L} = \mathsf{r}\,\dfrac{\begin{array}{c}\mathcal{D}'^{\Diamond^{\bullet}_L} \,\Big\| \\ \mathcal{B}', xRy, \mathcal{L}', y : A \Rightarrow \mathcal{R}'\end{array}}{\mathcal{B}, xRy, \mathcal{L}, y : A \Rightarrow \mathcal{R}}$$

of the same (or smaller) height as $\mathcal{D}$, for any label y that does not appear in $\mathcal{D}$.

If we start with a proof

$$\Diamond_L\,\dfrac{\begin{array}{c}\mathcal{D}' \,\Big\| \\ \mathcal{B}, xRy', \mathcal{L}, y' : A \Rightarrow \mathcal{R}\end{array}}{\mathcal{B}, \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}}\; y' \text{ fresh}$$

then we take $\mathcal{D}^{\Diamond^{\bullet}_L}$ to be $\begin{array}{c}\mathcal{D}'[y/y'] \,\Big\| \\ \mathcal{B}, xRy, \mathcal{L}, y : A \Rightarrow \mathcal{R}\end{array}$ for any label y that does not appear in $\mathcal{D}$.

- $\supset_R$ and $\Box_R$: Similar. ☺

Lemma 9. *The following rule is admissible in* lab♡IK*:* $\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \Rightarrow \mathcal{R}, y : A}$

Proof. By induction of the size of A.

$$\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : a \Rightarrow \mathcal{R}, y : a} \quad\rightsquigarrow\quad \mathsf{id}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : a \Rightarrow \mathcal{R}, y : a}$$

$$\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \wedge B \Rightarrow \mathcal{R}, y : A \wedge B} \quad\rightsquigarrow$$

$$\wedge_L\,\dfrac{\wedge_R\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A, x : B \Rightarrow \mathcal{R}, y : A} \quad \mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A, x : B \Rightarrow \mathcal{R}, y : B}}{\mathcal{B}, x \leq y; \mathcal{L}, x : A, x : B \Rightarrow \mathcal{R}, y : A \wedge B}}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \wedge B \Rightarrow \mathcal{R}, y : A \wedge B}$$

$$\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \vee B \Rightarrow \mathcal{R}, y : A \vee B} \quad\rightsquigarrow$$

$$\vee_L\,\dfrac{\vee_R\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \Rightarrow \mathcal{R}, y : A}}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \Rightarrow \mathcal{R}, y : A \vee B} \quad \vee_R\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : B \Rightarrow \mathcal{R}, y : B}}{\mathcal{B}, x \leq y; \mathcal{L}, x : B \Rightarrow \mathcal{R}, y : A \vee B}}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \vee B \Rightarrow \mathcal{R}, y : A \vee B}$$

$$\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, y : A \supset B} \quad \rightsquigarrow$$

$$\supset_R \dfrac{trans\,\dfrac{\supset_L \dfrac{refl\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y, y \leq z, x \leq z, z \leq z; \mathcal{L}, x : A \supset B, z : A \Rightarrow \mathcal{R}, z : B, z : A}}{\mathcal{B}, x \leq y, y \leq z, x \leq z; \mathcal{L}, x : A \supset B, z : A \Rightarrow \mathcal{R}, z : B, z : A} \quad refl\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y, y \leq z, x \leq z, z \leq z; \mathcal{L}, z : B, z : A \Rightarrow \mathcal{R}, z : B}}{\mathcal{B}, x \leq y, y \leq z, x \leq z; \mathcal{L}, z : B, z : A \Rightarrow \mathcal{R}, z : B}}{\mathcal{B}, x \leq y, y \leq z, x \leq z; \mathcal{L}, x : A \supset B, z : A \Rightarrow \mathcal{R}, z : B}}{\mathcal{B}, x \leq y, y \leq z; \mathcal{L}, x : A \supset B, z : A \Rightarrow \mathcal{R}, z : B}}{\mathcal{B}, x \leq y; \mathcal{L}, x : A \supset B \Rightarrow \mathcal{R}, y : A \supset B}$$

$$\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}, y : \Box A} \quad \rightsquigarrow$$

$$\Box_R \dfrac{trans\,\dfrac{\Box_L \dfrac{refl\,\dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y, y \leq z, x \leq z, zRw, w \leq w; \mathcal{L}, z : \Box A, w : A \Rightarrow \mathcal{R}, w : A}}{\mathcal{B}, x \leq y, y \leq z, x \leq z, zRw; \mathcal{L}, z : \Box A, w : A \Rightarrow \mathcal{R}, w : A}}{\mathcal{B}, x \leq y, y \leq z, x \leq z, zRw; \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}, w : A}}{\mathcal{B}, x \leq y, y \leq z, zRw; \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}, w : A}}{\mathcal{B}, x \leq y; \mathcal{L}, x : \Box A \Rightarrow \mathcal{R}, y : \Box A}$$

$$\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y; \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}, y : \Diamond A} \quad \rightsquigarrow$$

$$\Diamond_L \dfrac{F_2\,\dfrac{\Diamond_R \dfrac{\mathsf{id_g}\,\dfrac{}{\mathcal{B}, x \leq y, xRz, z \leq u, yRu; \mathcal{L}, z : A \Rightarrow \mathcal{R}, y : \Diamond A, u : A}}{\mathcal{B}, x \leq y, xRz, z \leq u, yRu; \mathcal{L}, z : A \Rightarrow \mathcal{R}, y : \Diamond A}}{\mathcal{B}, x \leq y, xRz; \mathcal{L}, z : A \Rightarrow \mathcal{R}, y : \Diamond A}}{\mathcal{B}, x \leq y; \mathcal{L}, x : \Diamond A \Rightarrow \mathcal{R}, y : \Diamond A}$$

☺

2.5 Comparaison with Simpson's labIK

Proposition 10.

1. *If there is a proof* $\begin{array}{c}\mathcal{D} \, \Vert \\ \Rightarrow z : C\end{array}$ *in* labIK *then there is a proof* $\begin{array}{c}\mathcal{D}_m \, \Vert \\ \Rightarrow z : C\end{array}$ *in* lab♡IK.

2. *If there is a proof* $\begin{array}{c}\mathcal{D} \, \Vert \\ \Rightarrow z : C\end{array}$ *in* lab♡IK *then there is a proof* $\begin{array}{c}\mathcal{D}_s \, \Vert \\ \Rightarrow z : C\end{array}$ *in* labIK.

Proof. 1 by soudness of labIK wrt IK and completeness of lab♡IK wrt IK. 2 by soudness of lab♡IK wrt IK and completeness of labIK wrt IK. ☺

Question 11. *Can we give a direct proof of this result by proof transformation? In which case we might need to generalise the statement to make it suitable for an induction.*

1. *If there is a proof* $\begin{array}{c}\mathcal{D} \, \Vert \\ \mathcal{G}, \mathcal{L} \Rightarrow z : C\end{array}$ *in* labIK *then there is a proof* $\begin{array}{c}\mathcal{D}^{\mathsf{m}} \, \Vert \\ \mathcal{G}, \mathcal{L} \Rightarrow z : C\end{array}$ *in* lab♡IK.

$$\mathsf{id}\ \frac{}{\mathcal{G},\mathcal{L},x:a\Rightarrow x:a}\qquad \bot_L\ \frac{}{\mathcal{G},\mathcal{L},x:\bot\Rightarrow z:A}$$

$$\wedge_L\ \frac{\mathcal{G},\mathcal{L},x:A,x:B\Rightarrow z:C}{\mathcal{G},\mathcal{L},x:A\wedge B\Rightarrow z:C}\qquad \wedge_R\ \frac{\mathcal{G},\mathcal{L}\Rightarrow x:A\quad \mathcal{L}\Rightarrow x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\wedge B}$$

$$\vee_L\ \frac{\mathcal{G},\mathcal{L},x:A\Rightarrow z:C\quad \mathcal{G},\mathcal{L},x:B\Rightarrow z:C}{\mathcal{G},\mathcal{L},x:A\vee B\Rightarrow z:C}\qquad \vee_{R1}\ \frac{\mathcal{G},\mathcal{L}\Rightarrow x:A}{\mathcal{G},\mathcal{L}\Rightarrow x:A\vee B}\qquad \vee_{R2}\ \frac{\mathcal{G},\mathcal{L}\Rightarrow x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\vee B}$$

$$\supset_L\ \frac{\mathcal{G},\mathcal{L}\Rightarrow x:A\quad \mathcal{G},\mathcal{L},x:B\Rightarrow z:C}{\mathcal{G},\mathcal{L},x:A\supset B\Rightarrow z:C}\qquad \supset_R\ \frac{\mathcal{G},\mathcal{L},x:A\Rightarrow x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\supset B}$$

$$\Box_L\ \frac{\mathcal{G},xRy,\mathcal{L},x:\Box A,y:A\Rightarrow z:B}{\mathcal{G},xRy,\mathcal{L},x:\Box A\Rightarrow z:B}\qquad \Box_R\ \frac{\mathcal{G},xRy,\mathcal{L}\Rightarrow y:A}{\mathcal{G},\mathcal{L}\Rightarrow x:\Box A}\ y\text{ is fresh}$$

$$\Diamond_L\ \frac{\mathcal{G},xRy,\mathcal{L},y:A\Rightarrow z:B}{\mathcal{G},\mathcal{L},x:\Diamond A\Rightarrow z:B}\ y\text{ is fresh}\qquad \Diamond_R\ \frac{\mathcal{G},xRy,\mathcal{L}\Rightarrow y:A}{\mathcal{G},xRy,\mathcal{L}\Rightarrow x:\Diamond A}$$

Figure 3: System labIK

2. *If there is a proof* $\begin{array}{c}\mathcal{D}\,\Big\Vert\\ \mathcal{B},\mathcal{L}\Rightarrow\mathcal{R}\end{array}$ *in* lab♡IK *then there is a proof* $\begin{array}{c}\mathcal{D}^{\mathsf{s}}\,\Big\Vert\\ \mathcal{G},\mathcal{L}\Rightarrow z:C\end{array}$ *in* labIK*. TODO: for which $\mathcal{G}$, which z and which C?*

Proof. 1 by case analysis on the last rule in $\mathcal{D}$. Most of the rules in labIK are the same as rules in lab♡IK except for the following:

$$\mathsf{id}\ \frac{}{\mathcal{G},\mathcal{L},x:a\Rightarrow x:a}\quad\rightsquigarrow\quad refl\ \frac{\mathsf{id}\ \frac{}{\mathcal{G},x\leq x,\mathcal{L},x:a\Rightarrow x:a}}{\mathcal{G},\mathcal{L},x:a\Rightarrow x:a}$$

$$\vee_{R1}\ \frac{\overset{\mathcal{D}_1}{\Vert}\ \mathcal{G},\mathcal{L}\Rightarrow x:A}{\mathcal{G},\mathcal{L}\Rightarrow x:A\vee B}\quad\text{or}\quad \vee_{R2}\ \frac{\overset{\mathcal{D}_1}{\Vert}\ \mathcal{G},\mathcal{L}\Rightarrow x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\vee B}\quad\rightsquigarrow\quad \vee_R\ \frac{\overset{\mathcal{D}_1^{\mathsf{mw}}}{\Vert}\ \mathcal{G},\mathcal{L}\Rightarrow x:A,x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\vee B}$$

$$\supset_L\ \frac{\overset{\mathcal{D}_1}{\Vert}\ \mathcal{G},\mathcal{L}\Rightarrow x:A\quad \overset{\mathcal{D}_2}{\Vert}\ \mathcal{G},\mathcal{L},x:B\Rightarrow z:C}{\mathcal{G},\mathcal{L},x:A\supset B\Rightarrow z:C}\quad\rightsquigarrow\quad \supset_L\ \frac{\overset{\mathcal{D}_1^{\mathsf{mw}}}{\Vert}\ \mathcal{G},x\leq x,\mathcal{L},x:A\supset B\Rightarrow x:A\quad \overset{\mathcal{D}_2^{\mathsf{mw}}}{\Vert}\ \mathcal{G},x\leq x,\mathcal{L},x:B\Rightarrow z:C}{refl\ \frac{\mathcal{G},x\leq x,\mathcal{L},x:A\supset B\Rightarrow z:C}{\mathcal{G},\mathcal{L},x:A\supset B\Rightarrow z:C}}$$

$$\supset_R\ \frac{\overset{\mathcal{D}_1}{\Vert}\ \mathcal{G},\mathcal{L},x:A\Rightarrow x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\supset B}\quad\rightsquigarrow\quad \supset_R\ \frac{\overset{\mathcal{D}_1^{\mathsf{mw}}}{\Vert}\ \mathcal{G},x\leq x,\mathcal{L},x:A\Rightarrow x:B}{\mathcal{G},\mathcal{L}\Rightarrow x:A\supset B}$$

$$\Box_L\ \frac{\overset{\mathcal{D}_1}{\Vert}\ \mathcal{G},xRy,\mathcal{L},x:\Box A,y:A\Rightarrow z:B}{\mathcal{G},xRy,\mathcal{L},x:\Box A\Rightarrow z:B}\quad\rightsquigarrow\quad \Box_L\ \frac{\overset{\mathcal{D}_1^{\mathsf{mw}}}{\Vert}\ \mathcal{G},x\leq x,xRy,\mathcal{L},x:\Box A,y:A\Rightarrow z:B}{refl\ \frac{\mathcal{G},x\leq x,xRy,\mathcal{L},x:\Box A\Rightarrow z:B}{\mathcal{G},xRy,\mathcal{L},x:\Box A\Rightarrow z:B}}$$

$$\Box_R\ \frac{\overset{\mathcal{D}_1}{\Vert}\ \mathcal{G},xRy,\mathcal{L}\Rightarrow y:A}{\mathcal{G},\mathcal{L}\Rightarrow x:\Box A}\quad\rightsquigarrow\quad \Box_R\ \frac{\overset{\mathcal{D}_1^{\mathsf{mw}}}{\Vert}\ \mathcal{G},x\leq x,xRy,\mathcal{L}\Rightarrow y:A}{\mathcal{G},\mathcal{L}\Rightarrow x:\Box A}\qquad ☺$$

2.6 Extensions with Scott-Lemmon axioms

Proof of $\mathsf{4}_{\Box}: \Box A \supset \Box\Box A$

$$
\supset_R\,\dfrac{\Box_R\,\dfrac{\mathsf{F}_1\,\dfrac{\mathsf{trans}\,\dfrac{\boxtimes_4\,\dfrac{\Box\,\dfrac{\mathsf{id}\,\dfrac{}{x \le w, w \le w', w'Rv, v \le v', v'Ru, w' \le t, tRv', w \le t, tRu, w : \Box A, u : \mathcal{A} \Rightarrow u : A}}{x \le w, w \le w', w'Rv, v \le v', v'Ru, w' \le t, tRv', w \le t, \boldsymbol{tRu}, w : \Box A \Rightarrow u : A}}{x \le w, w \le w', w'Rv, v \le v', \boldsymbol{v'Ru}, w' \le t, \boldsymbol{tRv'}, w \le t w : \Box A \Rightarrow u : A}}{x \le w, w \le w', w'Rv, v \le v', v'Ru, \boldsymbol{w' \le t}, \boldsymbol{tRv'}, w : \Box A \Rightarrow u : A}}{x \le w, w \le w', \boldsymbol{w'Rv}, \boldsymbol{v \le v'}, v'Ru, w : \Box A \Rightarrow u : A}}{x \le w, w : \Box A \Rightarrow w : \Box\Box A}}{x : \Box A \supset \Box\Box A}
$$

Proof of $\mathsf{4}_{\Diamond}: \Diamond\Diamond A \supset \Diamond A$:

$$
\supset_R\,\dfrac{\Diamond_L\,\dfrac{\diamondplus_4\,\dfrac{\Diamond_R\,\dfrac{\mathsf{id}\,\dfrac{}{x \le w, wRv, vRu, u \le u', wRu' u : A \Rightarrow w : \Diamond A, u' : A}}{x \le w, wRv, vRu, u \le u', wRu' u : A \Rightarrow w : \Diamond A}}{x \le w, wRv, vRu, u : A \Rightarrow w : \Diamond A}}{x \le w, w : \Diamond\Diamond A \Rightarrow w : \Diamond A}}{x : \Diamond\Diamond A \supset \Diamond A}
$$

3 Constructive modal logic(s?)

3.1 Sequent system

$$
\mathsf{id}\,\dfrac{}{\Gamma, A \Rightarrow \Delta, A} \quad \bot_L\,\dfrac{}{\Gamma, \bot \Rightarrow \Delta} \quad \bot_R\,\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot}
$$

$$
\vee_L\,\dfrac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \quad \vee_R\,\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \quad \wedge_L\,\dfrac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \quad \wedge_R\,\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}
$$

$$
\supset_L\,\dfrac{\Gamma, A \supset B \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \quad \supset_R\,\dfrac{\Gamma \Rightarrow A \supset B}{\Gamma \Rightarrow \Delta, A \supset B}
$$

$$
\Diamond_L\,\dfrac{\Gamma_1, A \Rightarrow \Delta_1}{\Box\Gamma_1, \Gamma_2, \Diamond A \Rightarrow \Diamond\Delta_1, \Delta_2} \quad \Box_R\,\dfrac{\Gamma_1 \Rightarrow A}{\Box\Gamma_1, \Gamma_2 \Rightarrow \Delta, \Box A}
$$

3.2 Remarks

We know that k_1 and k_2 are easily derived even if we restrict the succedent to always be a singleton.

$$
\supset_R\,\dfrac{\supset_R\,\dfrac{\Box_R\,\dfrac{\supset_L\,\dfrac{\mathsf{id}\,\dfrac{}{A \supset B, A \Rightarrow A} \quad \mathsf{id}\,\dfrac{}{B, A \Rightarrow B}}{A \supset B, A \Rightarrow B}}{\Box(A \supset B), \Box A \Rightarrow \Box B}}{\Box(A \supset B) \Rightarrow \Box A \supset \Box B}}{\mathsf{k}_1 : \Box(A \supset B) \supset (\Box A \supset \Box B)}
\qquad
\supset_R\,\dfrac{\supset_R\,\dfrac{\Diamond_L\,\dfrac{\supset_L\,\dfrac{\mathsf{id}\,\dfrac{}{A \supset B, A \Rightarrow A} \quad \mathsf{id}\,\dfrac{}{B, A \Rightarrow B}}{A \supset B, A \Rightarrow B}}{\Box(A \supset B), \Diamond A \Rightarrow \Diamond B}}{\Box(A \supset B) \Rightarrow \Diamond A \supset \Diamond B}}{\mathsf{k}_2 : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)}
$$

k_3 becomes derivable as soon as we allow two or more formulas on the right.

$$\supset_R \dfrac{\vee_R \dfrac{\Diamond_L \dfrac{\vee_L \dfrac{\mathsf{id}\,\dfrac{}{A \Rightarrow A, B} \qquad \mathsf{id}\,\dfrac{}{B \Rightarrow A, B}}{A \vee B \Rightarrow A, B}}{\Diamond(A \vee B) \Rightarrow \Diamond A, \Diamond B}}{\Diamond(A \vee B) \Rightarrow \Diamond A \vee \Diamond B}}{\mathsf{k}_3 \colon \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)}$$

k_5 becomes derivable as soon as we allow the righthandside to be empty.

$$\supset_R \dfrac{\bot_R \dfrac{\Diamond_L \dfrac{\bot_L \dfrac{}{\bot \Rightarrow}}{\Diamond\bot \Rightarrow}}{\Diamond\bot \Rightarrow \bot}}{\mathsf{k}_5 \colon \Diamond\bot \supset \bot}$$

k_4 on the other hand, is still underivable.

$$\supset_R \dfrac{\supset_L \dfrac{? \dfrac{???}{\Diamond A \supset \Box B \Rightarrow \Diamond A, \Box(A \supset B)} \qquad \Box_R \dfrac{\supset_R \dfrac{\mathsf{id}\,\dfrac{}{B, A \Rightarrow B}}{B \Rightarrow A \supset B}}{\Box B \Rightarrow \Box(A \supset B)}}{\Diamond A \supset \Box B \Rightarrow \Box(A \supset B)}}{\mathsf{k}_4 \colon (\Diamond A \supset \Box B) \supset \Box(A \supset B)}$$

3.3 Bicolor nested sequent systems

For IK

$$\mathsf{id}\,\dfrac{}{\Gamma_1\{A^\bullet, \llbracket A^\circ, \Gamma_2 \rrbracket\}} \qquad \bot_L \dfrac{}{\Gamma\{\bot^\bullet\}}$$

$$\wedge_L \dfrac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge_R \dfrac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \qquad \vee_L \dfrac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee_R \dfrac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset_{L1} \dfrac{\Gamma_1\{A^\circ\} \quad \Gamma_1\{B^\bullet\}}{\Gamma_1\{A \supset B^\bullet\}} \qquad \supset_{L2} \dfrac{\Gamma_1\{\llbracket A \supset B^\bullet, \Gamma_2 \rrbracket\}}{\Gamma_1\{A \supset B^\bullet, \llbracket \Gamma_2 \rrbracket\}} \qquad \supset_R \dfrac{\Gamma\{\llbracket A^\bullet, B^\circ \rrbracket\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box_L \dfrac{\Gamma_1\{[A^\bullet, \Gamma_2]\}}{\Gamma_1\{\Box A^\bullet, [\Gamma_2]\}} \qquad \Box_R \dfrac{\Gamma\{\llbracket [A^\circ] \rrbracket\}}{\Gamma\{\Box A^\circ\}} \qquad \Diamond_L \dfrac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond_R \dfrac{\Gamma_1\{[A^\circ, \Gamma_2]\}}{\Gamma_1\{\Diamond A^\circ, [\Gamma_2]\}}$$

$$\mathsf{mon}_L \dfrac{\Gamma_1\{\llbracket A^\bullet, \Gamma_2 \rrbracket\}}{\Gamma_1\{A^\bullet, \llbracket \Gamma_2 \rrbracket\}} \qquad \mathsf{mon}_R \dfrac{\Gamma_1\{A^\circ, \llbracket \Gamma_2 \rrbracket\}}{\Gamma_1\{\llbracket A^\circ, \Gamma_2 \rrbracket\}} \qquad \mathsf{F}_1 \dfrac{\Gamma_1\{\llbracket [\Gamma_2] \rrbracket\}}{\Gamma_1\{[\llbracket \Gamma_2 \rrbracket]\}}$$

$$refl_\leq \dfrac{\Gamma_1\{\llbracket \Gamma_2 \rrbracket\}}{\Gamma_1\{\Gamma_2\}} \qquad trans_\leq \dfrac{\Gamma_1\{\llbracket \Gamma_2 \rrbracket, \llbracket \Gamma_3 \rrbracket\}}{\Gamma_1\{\llbracket \Gamma_2, \llbracket \Gamma_3 \rrbracket \rrbracket\}}$$

For $\mathsf{IS4}$ Replace $\Box_L$ and $\Diamond_R$ above by the following:

$$\Box_{L1} \dfrac{\Gamma\{A^\bullet\}}{\Gamma_1\{\Box A^\bullet\}} \qquad \Box_{L2} \dfrac{\Gamma_1\{[\Box A^\bullet, \Gamma_2]\}}{\Gamma_1\{\Box A^\bullet, [\Gamma_2]\}} \qquad \Diamond_{R1} \dfrac{\Gamma\{A^\circ\}}{\Gamma_1\{\Diamond A^\circ\}} \qquad \Diamond_{R2} \dfrac{\Gamma_1\{[\Diamond A^\circ, \Gamma_2]\}}{\Gamma_1\{\Diamond A^\circ, [\Gamma_2]\}}$$

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