Decomposing labelled proof theory for intuitionistic modal logic

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Abstract

We present a labelled deduction for intuitionistic modal logic equipped with two relation symbols, one for the accessibility relation associated with the Kripke semantics for modal logics, and one for the preorder relation associated with the Kripke semantics for intuitionistic logic. As such, it is in close correspondence with the birelational Kripke models for intuitionistic modal logic.

Keywords: Intuitionistic modal logic, Labelled sequents, Proof theory.

1 Introduction

Labelled deduction has been proposed by Gabbay [TODO: add reference] in the 80s as a unifying framework throughout proof theory in order to provide proof systems for a wide range of logics. For modal logics it can take the form of labelled natural deduction and labelled sequent systems as used, for example, by Simpson [3], Vigano [4] and Negri [2]. These formalisms make explicit use not only of labels, but also of relational atoms. To our knowledge, all labelled systems proposed so far had only one relational symbol, [Sonia: Incorrect. We even cite [1] below] either representing the *accessible world relation* R (for modal logics) or the *future relation* $\leq$ (for intuitionistic logic). In this short note we propose a system that unifies both, and thus is able to use the full power of the bi-relational semantics for intuitionistic modal logics.

[Sonia: Removed paragraph. Simpson's system can easily be made multi-conclusion in the style of Maehara without the $\leq$-atoms.]

2 Preliminaries[TODO: give a more informative title]

[Sonia: Removed the part on the classical labelled calculus. Don't think we need it in a short version.]

[Sonia: Formulas need to be defined.

The language of intuitionisitic modal logic is obtained from the one of intuitionistic propositional logic by adding the modal connectives $\Box$ and $\Diamond$. Starting with a set $\mathcal{A}$ of atomic propositions denoted a, modal formulas are constructed from the following grammar:

$$A ::= a \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A$$

]

The Kripke semantics for intuitionistic modal logic combines the Kripke semantics for intuitionistic propositional logic and the one for classical modal logic, using two distinct relations on the set of worlds.

Definition 2.1 A *bi-relational frame* $\mathcal{F}$ is a triple $\langle W, R, \leq \rangle$ of a non-empty set of worlds W equipped with two binary relations R and $\leq$: R being the modal *accessibility relation* and $\leq$ a preorder, satisfying the following conditions:

(F1) For all worlds u, v, v', if $u\mathrm{R}v$ and $v \leq v'$, there exists a u' such that $u \leq u'$ and $u'\mathrm{R}v'$.

(F2) For all worlds u', u, v, if $u \leq v$, there exists a v' such that $u'\mathrm{R}v'$ and $v \leq v'$.

Definition 2.2 A *bi-relational model* $\mathcal{M}$ is a quadruple $\langle W, R, \leq, V \rangle$ with $\langle W, R, \leq \rangle$ a bi-relational frame and $V : W \to 2^{\mathcal{A}}$ a monotone valuation function, that is, a function mapping each world w to the subset of propositional atoms true at w, additionally subject to:

$$w \leq w' \Rightarrow V(w) \subseteq V(w')$$

We write $w \Vdash a$ iff $a \in V(w)$ and we extend this relation to all formulas by induction, following the rules for both intuitionistic and modal Kripke models:

$w \nVdash \bot$
$w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
$w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
$w \Vdash A \supset B$ iff for all w' with $w \leq w'$, if $w' \Vdash A$ then $w' \Vdash B$
$w \Vdash \Box A$ iff for all w' and u with $w \leq w'$ and $w'Ru$, $u \Vdash A$
$w \Vdash \Diamond A$ iff there exists a u such that wRu and $u \Vdash A$.

Definition 2.3 A formula A is *satisfied* in a model $\mathcal{M} = \langle W, R, \leq, V \rangle$, if for all $w \in W$ we have $w \Vdash A$.

Definition 2.4 A formula A is *valid* in a frame $\mathcal{F} = \langle W, R, \leq \rangle$, if for all valuations V, A is satisfied in $\langle W, R, \leq, V \rangle$.

[Sonia: Removed the figure as it has nothing to do with the theorem. Fisher-Servi as well as Plotkin and Sterling only give Hilbert-style axiomatisations for the birelational models, no sequent calculi. Therefore, if we want to present this theorem, the axiomatisations also need to be introduced.]

$$\mathit{id}\,\frac{}{\mathcal{G},\mathcal{L},x\colon a\Rightarrow x\colon a}\qquad \bot_L\,\frac{}{\mathcal{G},\mathcal{L},x\colon \bot\Rightarrow z\colon A}$$

$$\wedge_L\,\frac{\mathcal{G},\mathcal{L},x\colon A,x\colon B\Rightarrow z\colon C}{\mathcal{G},\mathcal{L},x\colon A\wedge B\Rightarrow z\colon C}\qquad \wedge_R\,\frac{\mathcal{G},\mathcal{L}\Rightarrow x\colon A\quad \mathcal{G},\mathcal{L}\Rightarrow x\colon B}{\mathcal{G},\mathcal{L},\Rightarrow x\colon A\wedge B}$$

$$\vee_L\,\frac{\mathcal{G},\mathcal{L},x\colon A\Rightarrow z\colon C\quad \mathcal{G},\mathcal{L},x\colon B\Rightarrow z\colon C}{\mathcal{G},\mathcal{L},x\colon A\vee B\Rightarrow z\colon C}\qquad \vee_{R1}\,\frac{\mathcal{G},\mathcal{L}\Rightarrow x\colon A}{\mathcal{G},\mathcal{L}\Rightarrow x\colon A\vee B}$$

$$\vee_{R2}\,\frac{\mathcal{G},\mathcal{L}\Rightarrow x\colon B}{\mathcal{G},\mathcal{L}\Rightarrow x\colon A\vee B}$$

$$\supset_L\,\frac{\mathcal{G},\mathcal{L}\Rightarrow x\colon A\quad \mathcal{G},\mathcal{L},x\colon B\Rightarrow z\colon C}{\mathcal{G},\mathcal{L},x\colon A\supset B\Rightarrow z\colon C}\qquad \supset_R\,\frac{\mathcal{G},\mathcal{L},x\colon A\Rightarrow x\colon B}{\mathcal{G},\mathcal{L},x\colon A\Rightarrow x\colon B}$$

$$\Box_L\,\frac{\mathcal{G},x\mathrm{R}y,\mathcal{L},x\colon \Box A,y\colon A\Rightarrow z\colon B}{\mathcal{G},x\mathrm{R}y,\mathcal{L},x\colon \Box A\Rightarrow z\colon B}\qquad \Box_R\,\frac{\mathcal{G},x\mathrm{R}y,\mathcal{L}\Rightarrow y\colon A}{\mathcal{G},\mathcal{L}\Rightarrow x\colon \Box A}\;y\text{ is fresh}$$

$$\Diamond_L\,\frac{\mathcal{G},x\mathrm{R}y,\mathcal{L},y\colon A\Rightarrow z\colon B}{\mathcal{G},\mathcal{L},x\colon \Diamond A\Rightarrow z\colon B}\;y\text{ is fresh}\qquad \Diamond_R\,\frac{\mathcal{G},x\mathrm{R}y,\mathcal{L}\Rightarrow y\colon A}{\mathcal{G},x\mathrm{R}y,\mathcal{L}\Rightarrow x\colon \Diamond A}$$

Fig. 1: System labIK

Theorem 2.5 (Fischer-Servi [5], Plotkin and Stirling [6]) *A formula A is a theorem of* IK *if and only if A is valid in every bi-relational frame.*

3 Capturing intuitionistic modal logics with labels

[Sonia: I moved Simpson's system labIK here as we need to make explicit that we are aware of his approach and we propose something else.]

[TODO: Some background needs to be added]

Theorem 3.1 (Simpson [3]) *A formula A is provable in the calculus* labIK *if and only if A is valid in every bi-relational frame.*

Echoing the definition of bi-relational structures, another extension of labelled deduction to the intuitionistic setting would be to use two sorts of relational atoms, one for the modal relation R and another one for the intuitionistic relation $\leq$. This is the approach developed by Maffezioli, Naibo and Negri in [1]. The idea is to extend labelled sequents with a preorder relation symbol in order to capture intuitionistic modal logics, that is to define intuitionistic labelled sequents from labelled formulas $x\colon A$, relational atoms $x\mathrm{R}y$, and preorder atoms of the form $x \leq y$, where x, y range over a set of labels and A is an intuitionistic modal formula.

A two-sided intuitionistic labelled sequent would be of the form $\mathcal{G},\mathcal{L}\Rightarrow\mathcal{R}$ where $\mathcal{G}$ denotes a set of relational and preorder atoms, and $\mathcal{L}$ and $\mathcal{R}$ are multiset of labelled formulas. We then obtain a proof system lab♡IK (2) for intuitionistic modal logic in this formalism.

$$id^{lab}\ \frac{}{\mathcal{G}, \mathcal{L}, x\colon a \Rightarrow x\colon a} \qquad \bot_L^{lab}\ \frac{}{\mathcal{G}, \mathcal{L}, x\colon \bot \Rightarrow z\colon A} \qquad \top_R^{lab}\ \frac{}{\mathcal{G}, \mathcal{L} \Rightarrow x\colon \top}$$

$$\wedge_L\ \frac{\mathcal{G}, \mathcal{L}, x\colon A, x\colon B \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\colon A \wedge B \Rightarrow \mathcal{R}} \qquad \wedge_R\ \frac{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \quad \mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \wedge B}$$

$$\vee_L\ \frac{\mathcal{G}, \mathcal{L}, x\colon A \Rightarrow \mathcal{R} \quad \mathcal{G}, \mathcal{L}, x\colon B \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\colon A \vee B \Rightarrow \mathcal{R}} \qquad \vee_R\ \frac{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A, x\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \vee B}$$

$$\supset_R\ \frac{\mathcal{G}, \mathcal{L}, x \leq y, y\colon A \Rightarrow \mathcal{R}, y\colon B}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon A \supset B}\ y \text{ fresh}$$

$$\supset_L\ \frac{\mathcal{G}, \mathcal{L}, x \leq y, x\colon A \supset B \Rightarrow \mathcal{R}, y\colon A \quad \mathcal{G}, \mathcal{L}, x \leq y, x\colon A \supset B, y\colon B \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x \leq y, x\colon A \supset B \Rightarrow \mathcal{R}}$$

$$\Box_L\ \frac{\mathcal{G}, \mathcal{L}, x \leq y, y\mathrm{R}z, x\colon \Box A, z\colon A \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x \leq y, y\mathrm{R}z, x\colon \Box A \Rightarrow \mathcal{R}}$$

$$\Box_R\ \frac{\mathcal{G}, \mathcal{L}, x \leq y, y\mathrm{R}z \Rightarrow \mathcal{R}, z\colon A}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}, x\colon \Box A}\ y, z \text{ fresh}$$

$$\Diamond_L\ \frac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, y\colon A \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\colon \Diamond A \Rightarrow \mathcal{R}}\ y \text{ fresh} \qquad \Diamond_R\ \frac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y \Rightarrow \mathcal{R}, x\colon \Diamond A, y\colon A}{\mathcal{G}, \mathcal{L}, x\mathrm{R}y \Rightarrow \mathcal{R}, x\colon \Diamond A}$$

$$refl\ \frac{\mathcal{G}, x \leq x, \mathcal{L}, \mathcal{R}}{\mathcal{G}, \mathcal{L} \Rightarrow \mathcal{R}} \qquad trans\ \frac{\mathcal{G}, x \leq y, y \leq z, x \leq z, \mathcal{L} \Rightarrow \mathcal{R}}{\mathcal{G}, x \leq y, y \leq z, \mathcal{L} \Rightarrow \mathcal{R}}$$

$$F1\ \frac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, y \leq z, x \leq u, u\mathrm{R}z \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, y \leq z \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

$$F2\ \frac{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, x \leq z, y \leq u, z\mathrm{R}u \Rightarrow \mathcal{R}}{\mathcal{G}, \mathcal{L}, x\mathrm{R}y, x \leq z \Rightarrow \mathcal{R}}\ u \text{ fresh}$$

Fig. 2: System lab♡IK

Theorem 3.2 *A formula A is provable in the calculus* lab♡IK *if and only if A is valid in every bi-relational frame.*

[Sonia: Rewrote the last two paragraphs more succinctly. Should we say more?]

On the one hand, we prove directly that each rule from our system is sound wrt. bi-relational structures. On the other hand, we show that lab♡IK is complete wrt. Simpson's labIK, and the theorem then follows from Theorem 3.1.

References

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