Introduction to irrational numbers

by

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1. Introduction

My goal here is to give the fundamental nature of irrational numbers. This paper is intended for anyone interested in the topic with a common knowledge of mathematics and calculus. I hope the paper will be useful for anyone reading it. Before we start I have to say something about notation used in this paper. Throughout this paper the notation $f^{(k)}(x)$ will mean the k-th derivative of the function $f(x)$ and rather conventionally, lnx will mean the natural logarithm of x (i.e. the logarithm with base e).

2. The basics

We start by a definition. An irrational number is a number which cannot be expressed as a ratio of two integers. That means, that for an irrational number *c:*

$$c \neq \frac{a}{b}.$$

We will now show the easiest example of an irrational number.

Theorem 1. $\sqrt{2}$ *is an irrational number.*

$$\sqrt{2} \neq \frac{a}{b}.$$

Suppose that $\sqrt{2} = \frac{a}{b}$ where $(a, b) = 1$. Then:

$$2b^2 = a^2.$$

And thus $a = 2m$.

$$2b^2 = 4m^2$$
$$b^2 = 2m^2$$

and so $b = 2n$. But that is a contradiction with our assumption that $(a, b) = 1$. Thus $\sqrt{2}$ is an irrational number.

That was our first irrational number. In fact we can prove also a stronger theorem about n-th roots of numbers, which we will prove when I introduce the reader to the basics of number theory.

Theorem 2 (The Fundamental Theorem of Arithmetic). *Any number n can be expressed uniquely as a product of primes.*

We shall not include a proof of this theorem in this paper. But with that theorem we can find an infinity of other irrationals. Now we will explore some properties of irrational numbers. Suppose some hypothetical irrational number *c*. By Theorem 1, we know that at least one such number exists.

Theorem 3 (Properties of irrationals).

$$Irrational+Rational = Irrational$$
$$Irrational - Rational = Irrational$$
$$Irrational \times Rational = Irrational$$
$$\frac{Irrational}{Rational} = Irrational$$
$$\sqrt[n]{Irrational} = Irrational, n \in N$$

Here the rational number is not equal to zero. Suppose *a, b, d, e* are integers (*c* is irrational). Then we can prove the theorem with proof by contradiction. Firstly consider:

$$c + \frac{a}{b} = \frac{d}{e}$$
$$c = \frac{bd - ae}{be}.$$

Which is a contradiction with the definition of *c*. Thus we have proved *Irrational+Rational = Irrational.* Next case,

$$c - \frac{a}{b} = \frac{d}{e}$$
$$c = \frac{bd + ae}{be}.$$

A contradiction. So *Irrational - Rational = Irrational.*

$$c \times \frac{a}{b} = \frac{d}{e}$$
$$c = \frac{bd}{ae}.$$

A contradiction. So *Irrational×Rational = Irrational.*

$$\frac{bc}{a} = \frac{d}{e}$$
$$c = \frac{ad}{be}.$$

A contradiction. It follows that $\frac{Irrational}{Rational} = Irrational$.

$$\sqrt[n]{c} = \frac{a}{b}$$

$$c = \frac{a^n}{b^n}.$$

Also impossible, so finally $\sqrt[n]{Irrational} = Irrational$
Suppose we have a natural number N, then theorem 4 holds:

Theorem 4. *$\sqrt[m]{N}$ is irrational, unless $N = x^m$.*

Suppose $\sqrt[m]{N} = \frac{a}{b}$, where $(a, b) = 1$, then by manipulating the equation we get $Nb^m = a^m$, and by The Fundamental Theorem of Arithmetic, $b|a^m$ and $p|a^m$ for every prime factor p of b. Thus $p|a$. Therefore $b = 1$. That proves the theorem, because if such number exists it is either an irrational number or an integer of the form x^m.

We have then also proved

Theorem 5. *The set of all irrational numbers is infinite.*

3. Some important concepts

This chapter is necessary to understand the next chapters. We shall introduce some concepts such as Taylor series, infinite series and some theorems about irrationals. Those who understand Taylor series and infinite series can either skip this chapter or dive deeper in the subjects.

From theorem 4 we can conclude following: *If x is a root of an equation*

$$x^m = N,$$

where N is a natural number, then x is either integral or irrational.

Theorem 4 is a special case of a more general theorem.

Theorem 6. *If x is a root of an equation*

$$x^m + c_1 x^{m-1} + \cdots + c_m = 0,$$

with integral coefficients, of which the first is unity, then x is either integral or irrational.

We suppose $c_m \neq 0$ and that the root is $\frac{a}{b}$, where $(a, b) = 1$ and we see that

$$a^m + c_1 b a^{m-1} + \cdots c_m b^m = 0.$$

Hence $b|a^m$ and therefore $b = 1$.

I have borrowed these theorems from An introduction to the theory of numbers by G. H. Hardy and E. M. Wright. I have put these theorems here because they are quite an interesting connection of irrationals and polynomials. Also if we can prove the theorem 4, we will also prove theorem 7.

Theorem 7. *An infinity of irrational numbers is algebraic.*

An algebraic number is a number which is a root of a polynomial with integral coefficients. If it is not a root of such polynomial, we call it *transcendental*.

Now we will move to the infinite series, which will become important in proving irrationality of the number *e.*

Theorem 8. *If* $|r| < 1,$ *then*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

We start by defining S_n as the sum of r^n.

$$S_n = 1 + r + r^2 + \cdots + r^n.$$

We can multiply both sides by *r.*

$$rS_n = r + r^2 + \cdots + r^{n+1}.$$

Now we can subtract rS_n from S_n. And solve for S_n. That will yield the sum of *n* terms.

$$S_n - rS_n = 1 - r^{n+1},$$
$$S_n(1-r) = 1 - r^{n+1},$$
$$S_n = \frac{1-r^{n+1}}{1-r} = \frac{r^{n+1}-1}{r-1}.$$

And as $n \to \infty,$ this sum converges only when $|r| < 1.$ And it converges to

$$\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^{n+1}-1}{r-1} = \frac{1}{1-r}.$$

Now we will move to the Taylor series. Theorem 9 represents the Taylor polynomial of a function of *x.*

Theorem 9. *If* $f(x)$ *is infinitely differentiable real valued function (such that* 0 *is in the domain), then* $f(x)$ *may be expressed as*

$$f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0) + R_{n+1}(x),$$

or more generally (for some a in domain)

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_{n+1}(x).$$

The first one is called Maclaurin polynomial. And in fact, it is possible to express $R_{n+1}(x)$. In case of Maclaurin polynomial,

$$R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(sx),$$

where $0 < s < 1$. This is called Lagrange's remainder.

If we let $n \to \infty$, we get a series, which we call Taylor (or Maclaurin) series. A function such that Taylor series converges in some neighborhood of every point in its domain is called *analytic*. For analytic functions $\lim_{n\to\infty}(R_{n+1}(x)) = 0$.

This will become important later.

4. Irrationality of φ

In this short chapter we shall prove that the golden ratio φ is irrational. The proof is straightforward. Firstly, we need to define φ. We define the golden ratio as a ratio of two quantities *a*, *b*, *b*<*a*, for which holds this equation:

$$\frac{a+b}{a} = \frac{a}{b}.$$

To calculate the golden ratio, we make substitution for *a/b:* $\varphi = \frac{a}{b}$. Then the equation becomes:

$$1 + \varphi^{-1} = \varphi.$$

By simple manipulations we can get φ.

$$\varphi + 1 = \varphi^2,$$
$$\varphi^2 - \varphi - 1 = 0,$$
$$\varphi = \frac{1 \pm \sqrt{5}}{2}.$$

We call the positive one the ” golden ratio ”. We immediately see that by theorem 3 we can state the following theorem.

Theorem 10. *The golden ratio* φ *is irrational.*

Moreover, we can say that

Theorem 11. *The golden ratio* φ *is algebraic.*

Theorem 11 is trivial by definition.

It should be said, that φ is tied closely to the Fibbonacci sequence (sequence defined by recursion $x_0 = 0, x_1 = 1, x_{n+1} = x_n + x_{n-1}$). In fact, the ratio of two consecutive terms of the Fibbonacci sequence tend to the golden ratio, as *n* goes to infinity.

5. The irrationality and value of *e*

The following chapters are the most important in this paper. In this chapter we will firstly find the value of *e* and then prove its irrationality. This is very important result in irrational numbers. We assume some prior knowledge of the function e^x.

Here we will calculate the value of *e* using 2 different methods. One will involve Taylor series. Firstly, we need to define what *e* is. We define *e* to be the value of the limit (we omit the proof that this limit converges):

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

or equivalently

$$e = \lim_{n\to 0}(1+n)^{\frac{1}{n}}.$$

We now consider the natural logarithm. We are going to calculate its derivative.

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0}\left(\frac{\ln(x+h)-\ln x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\ln\left(1+\frac{h}{x}\right)}{h}\right) = \lim_{h\to 0}\left(\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right).$$

Here we use substitution

$$u = \frac{h}{x},$$

such that

$$h = ux.$$

Now it follows that

$$\lim_{u\to 0}\left(\ln(1+u)^{\frac{1}{ux}}\right) = \lim_{u\to 0}\left(\frac{1}{x}\ln(1+u)^{\frac{1}{u}}\right).$$

And by definition of *e* and manipulating with the limit it follows that

$$\lim_{u\to 0}\left(\frac{1}{x}\ln(1+u)^{\frac{1}{u}}\right) = \frac{1}{x}\ln\left(\lim_{u\to 0}(1+u)^{\frac{1}{u}}\right) = \frac{1}{x}\ln e = \frac{1}{x}.$$

And so,

Theorem 12. *The derivative of* $\ln(x)$ *with respect to x is*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Now suppose the derivative of e^x exists. We know that $x = \ln e^x$, and so $\frac{d}{dx}(x) = \frac{d}{dx}(\ln e^x) = 1.$ Using the chain rule we see that $\frac{1}{e^x}\frac{d}{dx}(e^x) = 1$. Which means that also $\frac{d}{dx}(e^x) = e^x$. And as a matter of fact (which we will unfortunately not prove here), the derivative of e^x exists.

Theorem 13. *The number e can be represented as the following series*

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

This is the main reason I introduced Taylor series. Using it we can relatively easily find the series for *e*. Using the fact that $\frac{d}{dx}(e^x) = e^x$, we can easily calculate the Taylor series (in this case, MacLaurin series) of e^x and then set $x = 1$ to get the theorem 13. We will prove that the remainder goes to zero (hence, that the exponential function is analytic) later.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

This, in fact, converges for every *x*. Now, setting $x = 1$, we get

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

Now, for the remainder. Observe that

$$e^{|x|} > |e^{sx}|.$$

And hence, because the derivative of e^x is itself, we get

$$0 \leq \lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left|\frac{x^n}{n!} f^{(n)}(sx)\right| = \lim_{n\to\infty} \left|\frac{x^n}{n!} s^n e^{sx}\right| < \lim_{n\to\infty} \left|\frac{x^n}{n!} s^n e^x\right|$$
$$= e^x \lim_{n\to\infty} \left|\frac{(sx)^n}{n!}\right|.$$

Now notice that for sufficiently large k,

$$\frac{(|x|+k)!}{|x|^k} = \frac{(k+|x|)(k+|x|-1)\ldots k!}{|x|^k} \geq \frac{k^{|x|}k!}{|x|^k} = \frac{k^{|x|}k(k-1)\ldots|x|!}{|x|^k}$$
$$\geq \frac{k^{|x|}|x|^{k-|x|}|x|!}{|x|^k} = \frac{k^{|x|}x!}{|x|^{|x|}} > |x|^{|x|}.$$

Hence

$$\frac{(|x|+k)!}{|x|^k} > |x|^{|x|}.$$

So that

$$(|x|+k)! > |x|^{|x|+k}.$$

And so for sufficiently large n holds the inequality

$$n! > |x|^n.$$

It follows that there exists $N > |x|$ such that

$$0 \leq \lim_{n\to\infty} \frac{|x|^n}{n!} \leq \lim_{n\to\infty} \frac{|x|^n}{N^n} = \lim_{n\to\infty} \left(\frac{|x|}{N}\right)^n = 0.$$

And therefore by the squeeze theorem

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0,$$

which means that also

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

From that it is obvious that $\lim_{n\to\infty} R_n(x) = 0$, thus we have proved theorem 13.

Moving on to the second method where we do not use Taylor series, we are going to start by definition we used for *e*. I want to note that this method is incorrect because it can lead to a paradox (the problem step is in calculating the limit of a sum, where with the changing *n*, the number of summands also changes), but it is still an interesting one since it leads to a correct result. However with some additional help from theorems such as The Monotone Convergence Theorem we can make this method work. Now we will have to introduce another theorem called The Binomial Theorem.

Theorem 14 (The Binomial Theorem).

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

The *n* in The Binomial Theorem is, of course, integral. We are not going to present a proof of this theorem in this paper.

We will calculate the series for *e* by combining theorem 14 and the definition of *e*. In order to do that, notice, that the definition of *e* is a limit of a sequence, so the *n* in the limit is integral. Having that in mind

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k.$$

We have turned the limit into one involving a series. Now consider the limit of the general term of this series (assuming that the limit and the sum can be interchanged).

$$\lim_{n\to\infty}\binom{n}{k}\left(\frac{1}{n}\right)^k = \lim_{n\to\infty}\left(\frac{1}{n}\right)^k \frac{n!}{k!\,(n-k)!}.$$

Now it is only about calculating this limit.

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^k \frac{n!}{k!\,(n-k)!} = \frac{1}{k!}\lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k} = \frac{1}{k!}\lim_{n\to\infty}\frac{\prod_{j=0}^{k-1}(n-j)}{n^k}.$$

But why do we rewrite it like this? We can see that the series in the numerator has *k* factors. But also the denominator of the limit has *k* factors (each of which is *n*). So we can turn the last expression into

$$\frac{1}{k!}\lim_{n\to\infty}\frac{\prod_{j=0}^{k-1}(n-j)}{n^k} = \frac{1}{k!}\lim_{n\to\infty}\prod_{j=0}^{k-1}\left(\frac{n-j}{n}\right) = \frac{1}{k!}\lim_{n\to\infty}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right).$$

The last expression converges to $\frac{1}{k!}$.

$$\frac{1}{k!}\lim_{n\to\infty}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)=\frac{1}{k!}.$$

So we have successfully calculated the general term of the series for *e*. And thus

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}.$$

It is now very easy to prove the irrationality of *e*. Firstly suppose, that *e* is rational, such that $e=\frac{m}{n}$. Now from theorem 13 we know that

$$e=1+1+\frac{1}{2!}+\cdots$$

Then if we multiply *e* by *n!* the result should be an integer.

$$n!\,e=n!+n!+\frac{n!}{2}+\cdots+1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots$$

We know that every term up to $\frac{1}{n+1}$ is integral. We now observe one fact.

$$0<\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots<\frac{1}{n+1}+\frac{1}{(n+1)^2}+\cdots$$

And the sum of the right hand side of the inequality is equal by theorem 8 to

$$\sum_{k=1}^{\infty}\left(\frac{1}{n+1}\right)^k=\frac{1}{1-\frac{1}{n+1}}-1=\frac{n+1}{n}-1=\frac{1}{n}.$$

So by this calculation the sum should not be an integer, which is a contradiction. We have thus proved

Theorem 15. *The number e is irrational.*

That was the main goal of this chapter.

6. The irrationality of π

This chapter is about proving the irrationality of π. We assume the reader is familiar with the number π and its value. There are many methods to calculate π, but we shall not include here any of them. Our main goal in this chapter thus will be proving the irrationality of π.

The proof starts by proving that π^2 is irrational and then by applying the properties of irrational numbers we will conclude that π is irrational.

Suppose that $\pi^2 = \frac{a}{b}$, i.e. that π is rational. Observe that when n gets bigger, then

$$\frac{\pi a^n}{n!} < 1.$$

Theorem 16. *The number* π^2 *is irrational.*

Theorem 17. *The number* π *is irrational.*

We begin by defining a function.

$$f(x) = \frac{x^n(1-x)^n}{n!} = \frac{1}{n!}\sum_{m=n}^{2n} c_m x^m.$$

Now for $0 < x < 1$,

$$0 < f(x) < \frac{1}{n!}.$$

Observe that when plugging 1 or 0 into any derivative of $f(x)$, we get integer values. Now we define a new function.

$$F(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2(n-k)} f^{(2k)}(x).$$

Here, $f^{(2k)}(x)$ means the *(2k)-th* derivative of $f(x)$. Since any derivative of $f(x)$ evaluated at 1 or 0 is an integer, we can see that *F(0)* and *F(1)* are both integers. This will become important in a moment.

$$\frac{d}{dx}\{F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x)\} = \{F''(x) + \pi^2 F(x)\}\sin(\pi x).$$

$$\{F''(x) + \pi^2 F(x)\}\sin(\pi x) = b^n \pi^{2n+2} f(x)\sin(\pi x) = \pi^2 a^n f(x)\sin(\pi x).$$

And so

$$\pi \int_0^1 a^n f(x) \sin(\pi x)\, dx = \left[\frac{F'(x)\sin(\pi x)}{\pi} - F(x)\cos(\pi x)\right]_0^1 = F(0) + F(1),$$

an integer. But

$$0 < f(x) < \frac{1}{n!}.$$

So

$$0 < \pi \int_0^1 a^n f(x) \sin(\pi x)\, dx < \frac{\pi a^n}{n!} < 1.$$

This is a contradiction. So π^2 and π are irrational. Many similar versions of this proof can be found.

7. Some other (ir)rational numbers

There are also many other interesting irrational numbers which I have not yet introduced and also many which we do not know whether they are rational or not. In this quick and last chapter I want to show some of those.

One such number is the *Euler-Mascheroni constant* γ. It is defined as

$$\gamma = \lim_{n\to\infty} (H_n - \ln(n)).$$

Where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. The logarithm in this case is with base *e* (the natural logarithm). The value of γ is approximately 0.577. It is not known whether or not is γ irrational. But there are also numbers like $e + \pi, e - \pi$ or πe, which we also do not know if they are irrational. It has been shown that the sum

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3},$$

Converges to an irrational number.

There is also a beautiful method to show that at least one of the numbers $\pi + e, \pi e$ is irrational. Unfortunately this method uses the fact that both π and *e* are transcendental, which we will not prove here. Suppose the polynomial

$$(x + \pi)(x + e) = x^2 + (\pi + e)x + \pi e.$$

The roots of this polynomial are $-\pi$ and $-e$. Both of which are transcendental. That means that at least one of the coefficients of the polynomial is irrational. Since the coefficient of x^2 is rational, that means that at least one of the numbers $\pi + e, \pi e$ is irrational.

8. Epilogue

Although this paper was short I hope it contained some nice ideas and basics of irrational numbers and that the explanation of the topic was clear and understandable. And I hope the paper was helpful for the reader. The goal of this paper was to present basic ideas of the irrational numbers. In some cases I went into the more advanced maths of this topic, but I hope it was readable.

References

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