Tema 6
- CN -

6). $\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1 x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases}$ (1)

a). x^* soluție pt (1) $\Leftrightarrow$ x^* punct fix pt $G(x_1, x_2) =$

$= \left(\frac{x_1^2 + x_2^2 + 8}{10}, \frac{x_1 x_2^2 + x_1 + 8}{10} \right)$

"$\Rightarrow$" ipoteza x^* soluție pt (1) $\Rightarrow$

$\begin{cases} {x_1^*}^2 - 10x_1^* + {x_2^*}^2 + 8 = 0 \\ x_1^* {x_2^*}^2 + x_1^* - 10x_2^* + 8 = 0 \end{cases}$ unde $x^* = (x_1^*, x_2^*)$

$\Leftrightarrow \begin{cases} {x_1^*}^2 + {x_2^*}^2 + 8 = 10x_1^* \\ x_1^* {x_2^*}^2 + x_1^* + 8 = 10x_2^* \end{cases} \Leftrightarrow \begin{cases} \frac{{x_1^*}^2 + {x_2^*}^2 + 8}{10} = x_1^* \\ \frac{x_1^* {x_2^*}^2 + x_1^* + 8}{10} = x_2^* \end{cases}$

$\Rightarrow G(x_1^*, x_2^*) = (x_1^*, x_2^*) \Leftrightarrow G(x^*) = x^* \Rightarrow x^*$ punct fix pt G.

$\Leftarrow$ "Ipoteza x^* punct fix pt G. $\Rightarrow$

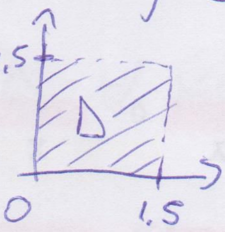
$$\Rightarrow G(x^*) = x^* \Leftrightarrow G(x_1^*, x_2^*) = \left(\frac{x_1^{*2} + x_2^{*2} + 8}{10}, \frac{x_1^* x_2^{*2} + x_1^* + 8}{10}\right)$$

$$= (x_1^*, x_2^*) \Rightarrow \begin{cases} \dfrac{x_1^{*2} + x_2^{*2} + 8}{10} = x_1^* \\[2ex] \dfrac{x_1^* x_2^{*2} + x_1^* + 8}{10} = x_2^* \end{cases} \Rightarrow$$

$$\begin{cases} x_1^{*2} - 10\cancel{x_1}x_1^* + x_2^{*2} + 8 = 0 \\ x_1^* x_2^{*2} + x_1^* - 10x_2^* + 8 = 0 \end{cases} \Rightarrow (x_1^*, x_2^*) = x^* \text{ soluție pt (1).}$$

5) Dem cf. T III.2. G admite un unic pct fix pe $D = \{(x_1, x_2) \mid 0 \le x_1, x_2 \le 1.5\} \subset \mathbb{R}^2$

Evident D este convexă



convexă (1).

$$G(x_1, x_2) = \left(\frac{x_1^2 + x_2^2 + 8}{10}, \frac{x_1 x_2^2 + x_1 + 8}{10}\right) = (y_1, y_2)$$

Fie $f_1 : D \to \mathbb{R}$, $f_1(x_1, x_2) = \dfrac{x_1^2 + x_2^2 + 8}{10}$

$f_2 : D \to \mathbb{R}$, $f_2(x_1, x_2) = \dfrac{x_1 x_2^2 + x_1 + 8}{10}$

$$G(x) = G(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (f_1(x), f_2(x)) = (y_1, y_2) \quad (*)$$

Vrem. $(\forall)\, x \in D$, $f_i(x) \in [0, 1.5]$ $(\forall)$ $i = \overline{1,2}$.

Fie $x \in D$ unde $x = (x_1, x_2)$

Cum $x \in D \Rightarrow 0 \le x_1, x_2 \le 1{,}5 \Rightarrow$

$$0 \le x_1 \le 1.5 \quad |^2 \qquad\qquad 0 \le x_2 \le 1.5 \quad |^2$$

$$0 \le x_1^2 \le 2.25. \qquad\qquad 0 \le x_2^2 \le 2.25 \quad \Big| \Rightarrow$$

$$0 \le x_1^2 + x_2^2 \le 4.5. \quad |+8$$

$$8 \le x_1^2 + x_2^2 + 8 \le 12.5. \quad |/10$$

$$0,8 \le f_1(x) \le 1,25. \Rightarrow f_1(x) \in [0,8,\ 1.25] \ \forall x \in D$$

$$\Leftrightarrow f_1(x) \in [0, 1.5] \ \forall x \in D. \ (2)$$

$$\left.\begin{array}{l} 0 \le x_1 \le 1.5 \\ 0 \le x_2^2 \le 2.25 \end{array}\right| \Rightarrow \left.\begin{array}{l} 0 \le x_1 x_2^2 \le 3,375. \\ 0 \le x_1 \le 1,5 \end{array}\right| +$$

$$0 \le x_1 x_2^2 + x_1 \le 4,875 \ + 8$$

$$8 \le 10 f_2(x) \le 12,875 \ /10$$

$$0,8 \le f_2(x) \le 1.2875.$$

$$\Leftrightarrow f_2(x) \in [0, 1.5] \ \forall x \in D \ (3)$$

$$\text{Din}^{(*)} (2) \text{ și } (3) \Rightarrow G(x) \in D \ \forall x \in D \ (4)$$

Vom considera $\|\cdot\|_\infty$

Calculăm derivatele parțiale $\frac{\partial f_i}{\partial x_j}(x)$ $i,j = \overline{1,2}$

$$\frac{\partial f_1}{\partial x_1}(x) = 2x_1/10 \qquad\qquad \frac{\partial f_2}{\partial x_1}(x) = \frac{x_2^2+1}{10}$$

$$\frac{\partial f_1}{\partial x_2}(x) = 2x_2/10 \qquad\qquad \frac{\partial f_2}{\partial x_2}(x) = 2x_1 x_2/10$$

$$G'(x) = J_{(x_1,x_2)} = \begin{pmatrix} \frac{2x_1}{10} & \frac{2x_2}{10} \\ \frac{x_2^2+1}{10} & \frac{2x_1 x_2}{10} \end{pmatrix}$$

$$\| G'(x) \|_\infty = \max_{i=1}^{2} \sum_{j=1}^{2} \left| \frac{\partial f_i}{\partial x_j}(x) \right|$$

$$\left| \frac{\partial f_1}{\partial x_1}(x) \right| + \left| \frac{\partial f_1}{\partial x_2}(x) \right| = \left| \frac{2x_1}{10} \right| + \left| \frac{2x_2}{10} \right| \xlongequal{x_1, x_2 \in [0, 1.5]}$$

$$\frac{2x_1}{10} + \frac{2x_2}{10} = \frac{1}{5}(x_1 + x_2) \leq \frac{1}{5} \cdot (1.5 + 1.5) = \frac{3}{5} = 0.6 < 1$$

$$\left| \frac{\partial f_2}{\partial x_1}(x) \right| + \left| \frac{\partial f_2}{\partial x_2}(x) \right| = \left| \frac{x_2^2 + 1}{10} \right| + \left| \frac{2x_1 x_2}{10} \right| \xlongequal{\quad " \quad}$$

$$\frac{x_2^2 + 2x_1 x_2 + 1}{10} \leq \frac{1}{10}\left(1.5^2 + 2 \cdot 1.5^2 + 1\right) = 0,755 < 1$$

$$\| G'(x) \|_\infty = \max\{0,6, 0,755\} = 0,755 < 1 \Rightarrow$$

$\Rightarrow$ deci $\rho = 0,755 < 1$ (5)

Din (1), (4) și (5) $\xrightarrow{T.\ III.2}$ $\forall\, x \in \mathring{D} = \{(x_1, x_2) \mid 0 < x_1, x_2 < 1.5\}$

atunci G admite un unic punct fix pe D (x^*)

iar șirul definit prin

$$x^{(k)} = G(x^{(k-1)}), \; k \geq 1, \; x^{(0)} \in D \text{ arbitrar}$$

este convergent la x^*