Q3 Derivation of $xRy, R, y:A \Rightarrow \Delta', z:\square B$
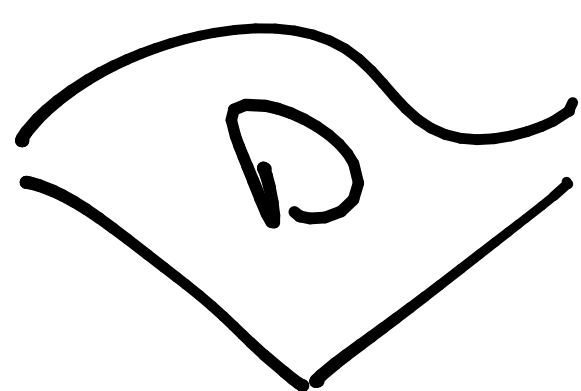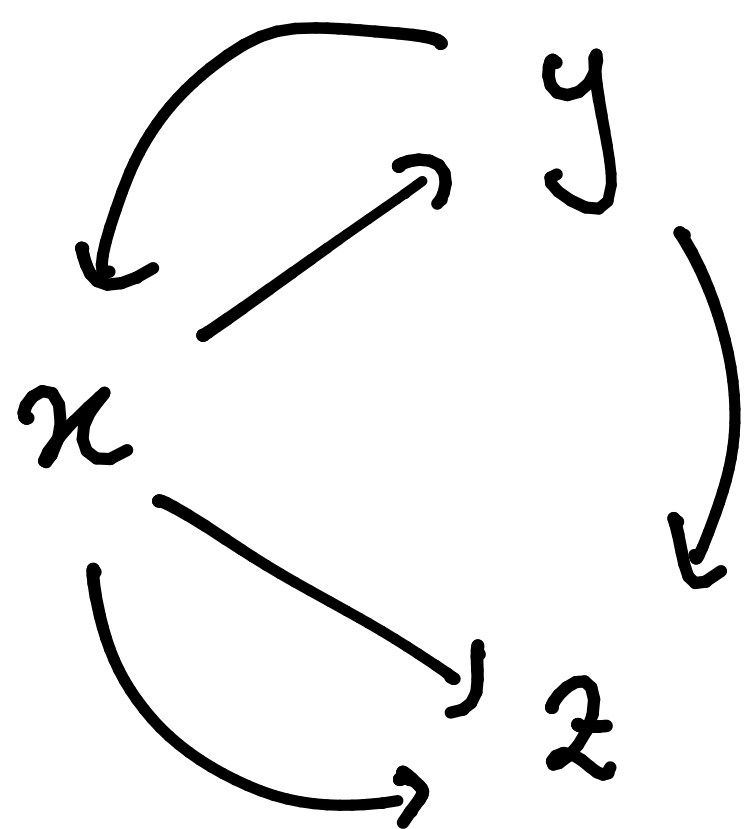
$$\overbrace{}^{D}$$

$$\frac{zRy, R, x:\lozenge A, \Gamma \Rightarrow \Delta', y:B}{zRk, R, x:\lozenge A, \Gamma \Rightarrow \Delta', k:B} \text{ subst } [y/k]$$

$$\frac{zRk, R, x:\lozenge A, \Gamma \Rightarrow \Delta', k:B}{xRy, zRk, R, y:A, \Gamma \Rightarrow \Delta', k:B} \text{ IH}$$

$$\frac{xRy, zRk, R, y:A, \Gamma \Rightarrow \Delta', k:B}{xRy, R, y:A, \Gamma \Rightarrow \Delta', z:\square B} \square_R$$

In the substitution, we need to choose a variable k which does not occur in Γ or Δ', and which is different from y and x (otherwise we cannot apply the IH, nor we can apply rule $\square_R$.

Q4

$$\frac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{yRz, xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta} \text{ wk}$$

$$\frac{yRz, xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta}{xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta} \text{ tr}$$

$$\frac{xRz, yRx, xRy, xRz, \Gamma \Rightarrow \Delta}{yRx, xRy, xRz, \Gamma \Rightarrow \Delta} \text{ sym}$$

$$\frac{yRx, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta} \text{ sym}$$

Derivation of $\Box A \to \Box\Box A$

(To be precise, the topmost sequent is not an instance of init but it is derivable! I should have written the axiom as $\Box p \to \Box\Box p$)

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\;}{xRz, xRz, zRy, yRy, yRx, xRx, xRy, yRz, x:\Box A, z:A \Rightarrow z:A}\;\text{init}
}{xRz, zRy, yRy, yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\;\Box_L
}{zRy, yRy, yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\;\text{euc}
}{yRy, yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\;\text{euc}
}{yRx, xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\;\text{ref}
}{xRx, xRy, yRz, x:\Box A \Rightarrow z:A}\;\text{euc}
}{xRy, yRz, x:\Box A \Rightarrow z:A}\;\text{ref}
}{xRy, x:\Box A \Rightarrow y:\Box A}\;\Box R
}{x:\Box A \Rightarrow x:\Box\Box A}\;\Box R
}{\Rightarrow x:\Box A \to x:\Box\Box A}\;\to R
$$

Solutions of Homework 3

Q1

$$\frac{yRk,\, zRk,\, xRy,\, xRz,\, R,\, \Gamma \Rightarrow \Delta}{xRy,\, xRz,\, R,\, \Gamma \Rightarrow \Delta}\ conf$$

where label k is fresh

$$\frac{}{yRk,\, zRk,\, xRy,\, xRz,\, z:\Box p,\, k:p \Rightarrow y:\Diamond p,\, k:p}\ init$$

$$\frac{}{yRk,\, zRk,\, xRy,\, xRz,\, z:\Box p,\, k:p \Rightarrow y:\Diamond p}\ \Diamond R$$

$$\frac{}{yRk,\, zRk,\, xRy,\, xRz,\, z:\Box p \Rightarrow y:\Diamond p}\ \Box_L$$

$$\frac{}{xRy,\, xRz,\, z:\Box p \Rightarrow y:\Diamond p}\ conf$$

$$\frac{}{xRy,\, x:\Diamond\Box p \Rightarrow y:\Diamond p}\ \Diamond_L$$

$$\frac{}{x:\Diamond\Box p \Rightarrow x:\Box\Diamond p}\ \Box R$$

$$\frac{}{\Rightarrow x:\Diamond\Box p \to \Box\Diamond p}\ \to R$$

Q2.

Rule tr is derivable:

$$\frac{xRz,\, xRy,\, yRz,\, \Gamma \Rightarrow \Delta}{xRz,\, zRy,\, yRy,\, yRx,\, xRx,\, xRy,\, yRz,\, \Gamma \Rightarrow \Delta}\ wk_L\ 4x$$

$$\frac{}{zRy\ \ yRy,\, yRx,\, xRx,\, xRy,\, yRz,\, \Gamma \Rightarrow \Delta}\ euc$$

$$\frac{}{yRy,\, yRx,\, xRx,\, xRy,\, yRz,\, \Gamma \Rightarrow \Delta}\ euc$$

$$\frac{}{yRx,\, xRx,\, xRy,\, yRz,\, \Gamma \Rightarrow \Delta}\ rf$$

$$\frac{}{xRx,\, xRy,\, yRz,\, \Gamma \Rightarrow \Delta}\ euc$$

$$\frac{}{xRy,\, yRz,\, \Gamma \Rightarrow \Delta}\ rf$$