Proof Theory of Modal Logic

Lecture 5
Beyond the S5-cube

Marianna Girlando

ILLC, Universtiy of Amsterdam

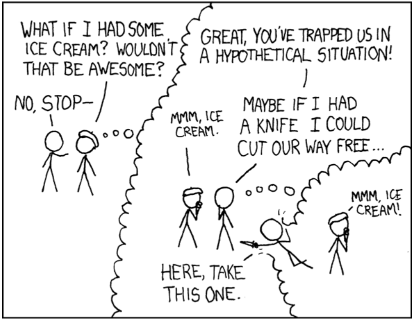
5th Tsinghua Logic Summer School
Beijing, 14 - 18 July 2025

Today's lecture: Beyond the S5 – *cube*

- Conditional logics
- Labelled proof systems for conditional logics
- Structured proof systems for conditional logics

Conditionals

If A *then* B



Counterfactuals

- If I had some ice cream, then life would be awesome.
- If kangaroos had no tails, then they would topple over.
- If Sam saw a lunar eclipse, then she would no longer believe that Earth is flat.

Paradoxes of material implication

Paradoxes of material implication

- If I had some ice cream, then life would be awesome.

Paradoxes of material implication

- If I had some ice cream, then life would be awesome.

A	B	$A \to B$
1	1	1
1	0	0
0	1	1
0	0	1

Paradoxes of material implication

- If I had some ice cream, then life would be awesome.

A	B	$A \to B$
1	1	1
1	0	0
0	1	1
0	0	1

Beyond material implication

Beyond material implication

- Co-tenability theories of conditionals [Goodman, 1955]

Beyond material implication

- Co-tenability theories of conditionals [Goodman, 1955]
- Probabilistic models [Adams, 1975]

Beyond material implication

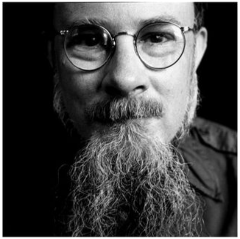
- Co-tenability theories of conditionals [Goodman, 1955]
- Probabilistic models [Adams, 1975]
- Belief revision [Gärdenfors, 1978]

Beyond material implication

- Co-tenability theories of conditionals [Goodman, 1955]
- Probabilistic models [Adams, 1975]
- Belief revision [Gärdenfors, 1978]
- Causal models [Galles and Pearl, 1998]

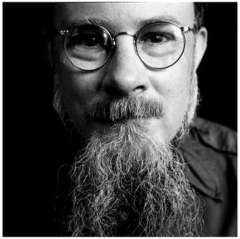
Beyond material implication

- Co-tenability theories of conditionals [Goodman, 1955]
- Probabilistic models [Adams, 1975]
- Belief revision [Gärdenfors, 1978]
- Causal models [Galles and Pearl, 1998]
- Possible world models [Stalnaker, 1968], [Lewis, 1973]



Beyond material implication

- Co-tenability theories of conditionals [Goodman, 1955]
- Probabilistic models [Adams, 1975]
- Belief revision [Gärdenfors, 1978]
- Causal models [Galles and Pearl, 1998]
- Possible world models [Stalnaker, 1968], [Lewis, 1973], [Nute, 1980], [Chellas, 1975], [Burgess, 1981], and many more!



Conditionals in a modal framework

Conditionals in a modal framework

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B$$

Conditionals in a modal framework

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B$$
$$\neg A := A \rightarrow \bot$$

Conditionals in a modal framework

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\neg A := A \rightarrow \bot$$

$$\Diamond A := \neg \Box \neg A$$

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\neg A := A \rightarrow \bot$$

$$\Diamond A := \neg \Box \neg A$$

Conditional logics

Conditionals in a modal framework

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot$$

$$\Diamond A := \neg \Box \neg A$$

Conditional logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B$$

$$\neg A := A \to \bot$$

Conditionals in a modal framework

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\neg A \quad := \quad A \rightarrow \bot$$

$$\Diamond A \quad := \quad \neg \Box \neg A$$

Conditional logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\neg A \quad := \quad A \rightarrow \bot$$

Conditionals in a modal framework

Modal logics

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\begin{aligned} \neg A &:= A \rightarrow \bot \\ \Diamond A &:= \neg \Box \neg A \end{aligned}$$

Conditional logics

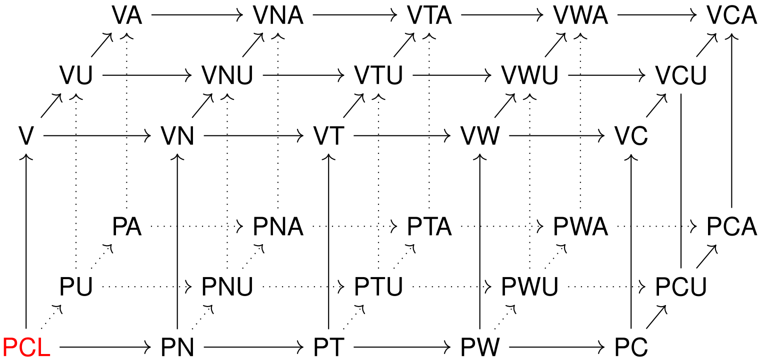
$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\begin{aligned} \neg A &:= A \rightarrow \bot \\ \Box A &:= \neg A > \bot \\ \Diamond A &:= \neg (A > \bot) \end{aligned}$$

Conditional logics

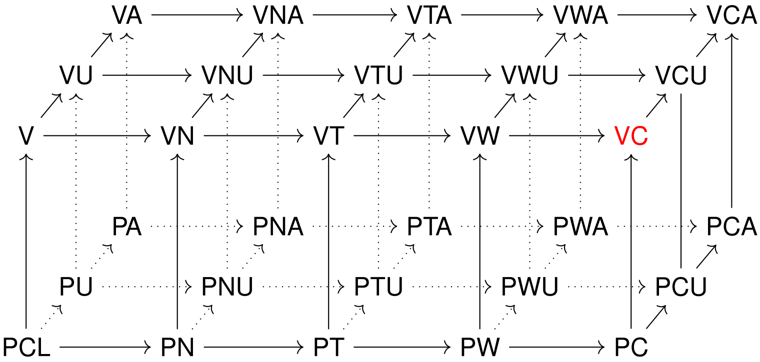
$$
\begin{tikzcd}[column sep=small]
& & \mathsf{VA} \arrow[rrr] & & & \mathsf{VNA} \arrow[rrr] & & & \mathsf{VTA} \arrow[rrr] & & & \mathsf{VWA} \arrow[rrr] & & & \mathsf{VCA} \\
& \mathsf{VU} \arrow[ur] \arrow[rrr] & & & \mathsf{VNU} \arrow[ur] \arrow[rrr] & & & \mathsf{VTU} \arrow[ur] \arrow[rrr] & & & \mathsf{VWU} \arrow[ur] \arrow[rrr] & & & \mathsf{VCU} \arrow[ur] & \\
\mathsf{V} \arrow[ur] \arrow[rrr] & & & \mathsf{VN} \arrow[ur] \arrow[rrr] & & & \mathsf{VT} \arrow[ur] \arrow[rrr] & & & \mathsf{VW} \arrow[ur] \arrow[rrr] & & & \mathsf{VC} \arrow[ur] & & \\
& & \mathsf{PA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PNA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PTA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PWA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PCA} \arrow[uuu] \\
& \mathsf{PU} \arrow[ur, dotted] \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PNU} \arrow[ur, dotted] \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PTU} \arrow[ur, dotted] \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PWU} \arrow[ur, dotted] \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathsf{PCU} \arrow[ur] \arrow[uuu] & \\
\mathsf{PCL} \arrow[ur, dotted] \arrow[uuu] \arrow[rrr] & & & \mathsf{PN} \arrow[ur, dotted] \arrow[uuu] \arrow[rrr] & & & \mathsf{PT} \arrow[ur, dotted] \arrow[uuu] \arrow[rrr] & & & \mathsf{PW} \arrow[ur, dotted] \arrow[uuu] \arrow[rrr] & & & \mathsf{PC} \arrow[ur] \arrow[uuu] & &
\end{tikzcd}
$$

Conditional logics



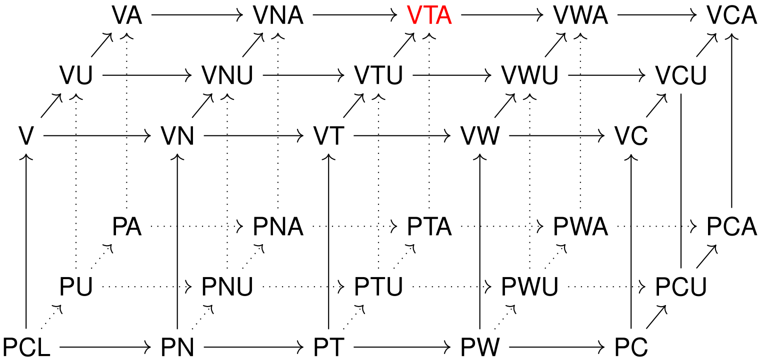
- Prototypical properties [KLM, 1990], [Burgess, 1981]

Conditional logics



- Prototypical properties [KLM, 1990], [Burgess, 1981]
- Counterfactuals [Lewis,1973]

Conditional logics



- Prototypical properties [KLM, 1990], [Burgess, 1981]
- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets, 2006, 2008]

Axioms

PCL: classical propositional logic plus

rcea $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$ rck $\dfrac{A \rightarrow B}{(C > A) \rightarrow (C > B)}$

id $A > A$

r.and $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

cm $(A > B) \wedge (A > C) \rightarrow ((A \wedge C) > B)$

rt $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

or $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

V: PCL plus

cv $(A > C) \wedge \neg(A > \neg B) \rightarrow ((A \wedge B) > C)$

Extensions of PCL and V

n $\neg(\top > \bot)$

t $A \rightarrow \neg(A > \bot)$

w $(A > B) \rightarrow (A \rightarrow B)$

c $(A \wedge B) \rightarrow (A > B)$

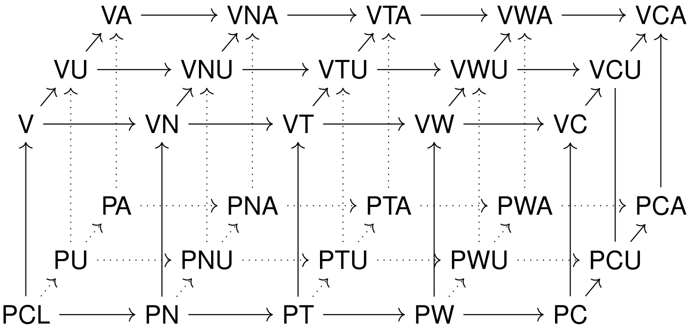
u_1 $(\neg A > \bot) \rightarrow (\neg(\neg A > \bot) > \bot)$

u_2 $\neg(A > \bot) \rightarrow ((A > \bot) > \bot)$

a_1 $(A > B) \rightarrow (C > (A > B))$

a_2 $\neg(A > B) \rightarrow (C > \neg(A > B))$

Possible-world semantics for conditionals

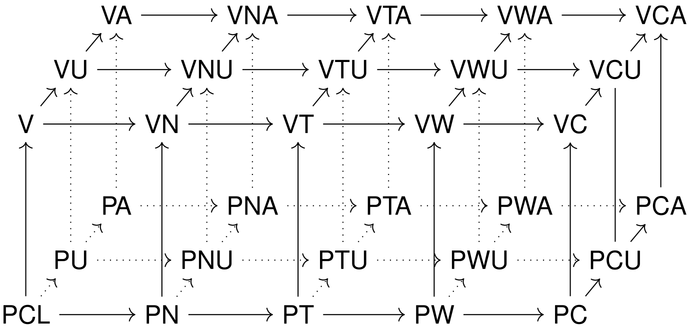


Possible-world semantics for conditionals

$$
\begin{tikzcd}[column sep=tiny]
 & & VA \arrow[rrr] & & & VNA \arrow[rrr] & & & VTA \arrow[rrr] & & & VWA \arrow[rrr] & & & VCA \\
 & VU \arrow[ur] \arrow[rrr] & & & VNU \arrow[ur] \arrow[rrr] & & & VTU \arrow[ur] \arrow[rrr] & & & VWU \arrow[ur] \arrow[rrr] & & & VCU \arrow[ur] & \\
V \arrow[ur] \arrow[rrr] & & & VN \arrow[ur] \arrow[rrr] & & & VT \arrow[ur] \arrow[rrr] & & & VW \arrow[ur] \arrow[rrr] & & & VC \arrow[ur] & & \\
 & & PA \arrow[uuu, dotted] \arrow[rrr, dotted] & & & PNA \arrow[uuu, dotted] \arrow[rrr, dotted] & & & PTA \arrow[uuu, dotted] \arrow[rrr, dotted] & & & PWA \arrow[uuu, dotted] \arrow[rrr, dotted] & & & PCA \arrow[uuu] \\
 & PU \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & PNU \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & PTU \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & PWU \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & PCU \arrow[uuu, no head] \arrow[ur] & \\
PCL \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & PN \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & PT \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & PW \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & PC \arrow[uuu] \arrow[ur] & &
\end{tikzcd}
$$

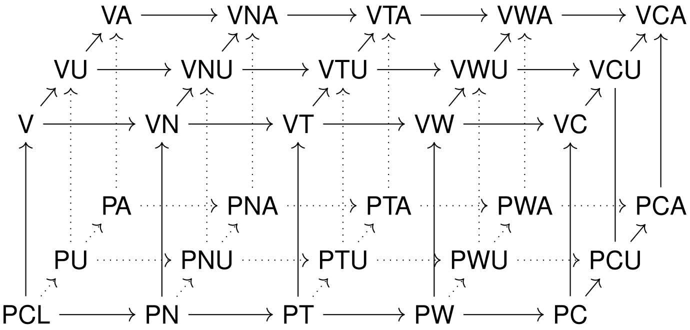
- Selection function semantics [Chellas, 1975]

Possible-world semantics for conditionals



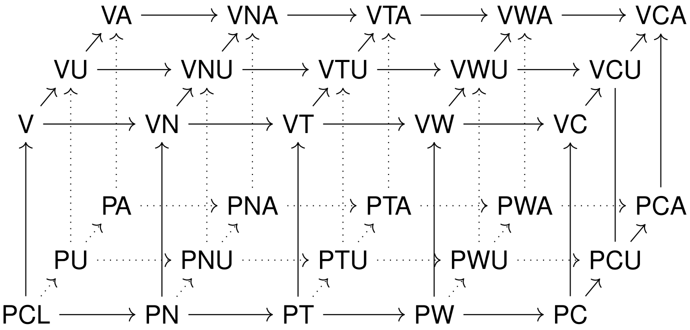
- Selection function semantics [Chellas, 1975]
- Sphere semantics [Lewis,1973]

Possible-world semantics for conditionals



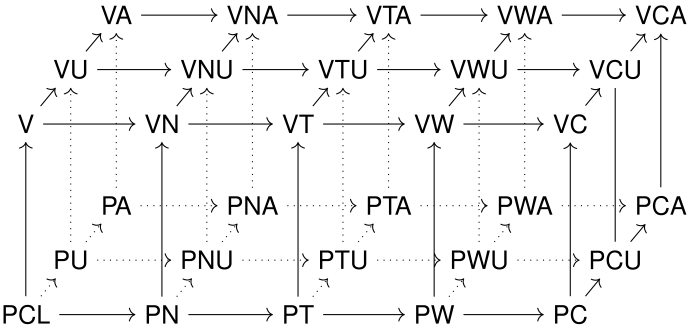
- Selection function semantics [Chellas, 1975]
- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]

Possible-world semantics for conditionals



- Selection function semantics [Chellas, 1975]
- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]
- Neighbourhood semantics [Scott, 1970], [Montague, 1970]

Possible-world semantics for conditionals



- Selection function semantics [Chellas, 1975]
- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]
- Neighbourhood semantics [Scott, 1970], [Montague, 1970]

  Direct proof of soundness and completeness w.r.t. the axiomatization of PCL and extensions [G, Negri, Olivetti, 2021]

Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$$

Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$$

y

x

z

k

Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \{\emptyset\} \notin N(x)$$

y

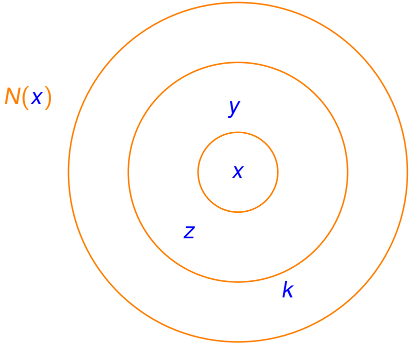
x

z

k

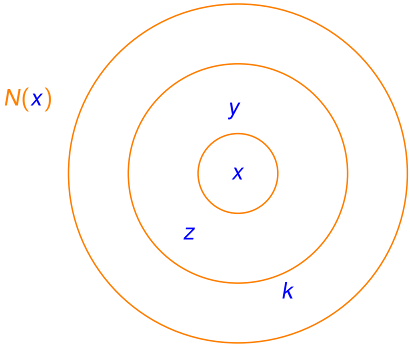
Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\{\emptyset\} \notin N(x)$



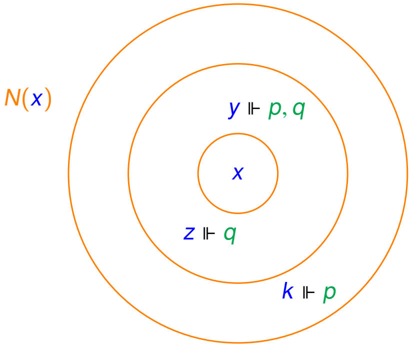
Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \{\emptyset\} \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$$



Neighbourhood models for VC

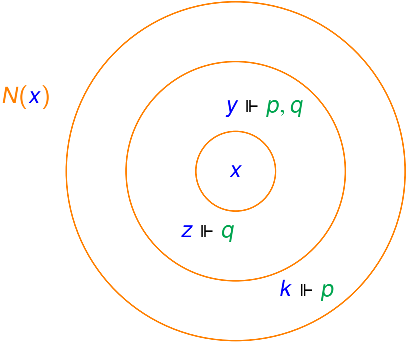
$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \{\emptyset\} \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$$



Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\{\emptyset\} \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x , for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

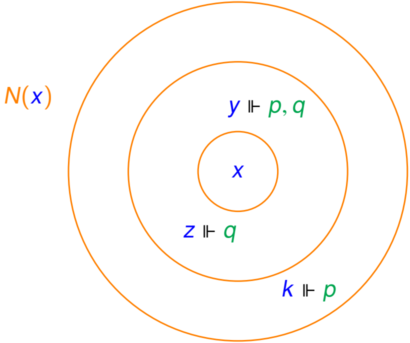


Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\{\emptyset\} \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x, for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

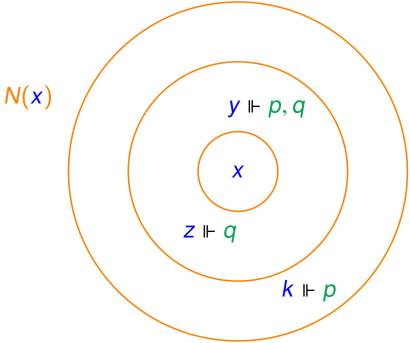
Centering for all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$ $\quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\{\emptyset\} \notin N(x)$ $\quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x, for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$
Centering for all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



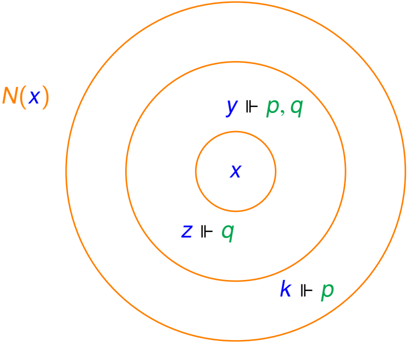
$x \Vdash p > q$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} p$ and $\beta \Vdash^{\forall} p \to q$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\{\emptyset\} \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x, for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering for all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



$x \Vdash p > q$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} p$ and $\beta \Vdash^{\forall} p \to q$

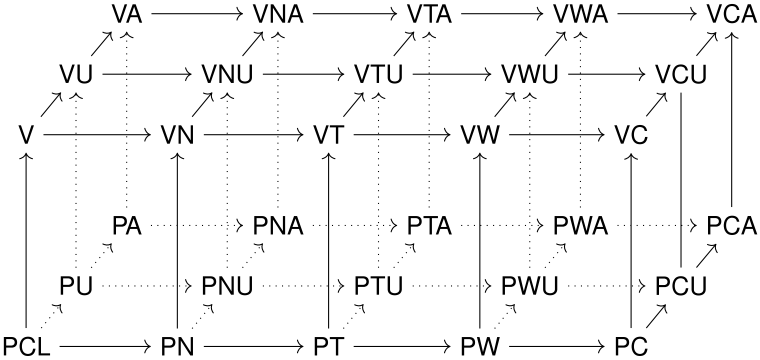
$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$

$\alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

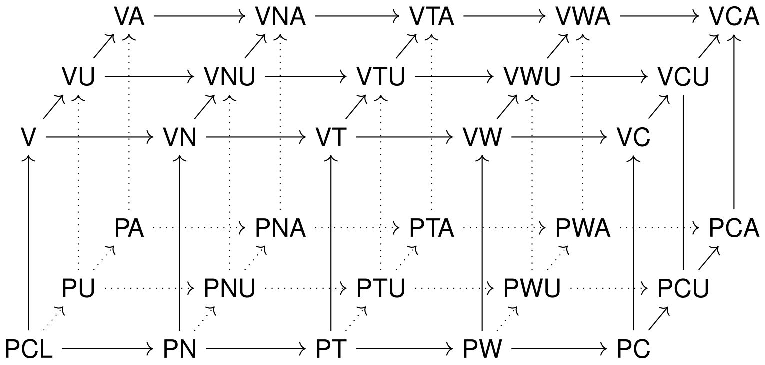
Conditional logics

$$
\begin{tikzcd}
 & & \mathsf{VA} \arrow[rrr] & & & \mathsf{VNA} \arrow[rrr] & & & \mathsf{VTA} \arrow[rrr] & & & \mathsf{VWA} \arrow[rrr] & & & \mathsf{VCA} \\
 & \mathsf{VU} \arrow[rrr] \arrow[ur] & & & \mathsf{VNU} \arrow[rrr] \arrow[ur] & & & \mathsf{VTU} \arrow[rrr] \arrow[ur] & & & \mathsf{VWU} \arrow[rrr] \arrow[ur] & & & \mathsf{VCU} \arrow[ur] & \\
\mathsf{V} \arrow[rrr] \arrow[ur] & & & \mathsf{VN} \arrow[rrr] \arrow[ur] & & & \mathsf{VT} \arrow[rrr] \arrow[ur] & & & \mathsf{VW} \arrow[rrr] \arrow[ur] & & & \mathsf{VC} \arrow[ur] & & \\
 & & \mathsf{PA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathsf{PNA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathsf{PTA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathsf{PWA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathsf{PCA} \arrow[uuu] \\
 & \mathsf{PU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathsf{PNU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathsf{PTU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathsf{PWU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathsf{PCU} \arrow[ur] \arrow[uuu, dash] & \\
\mathsf{PCL} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathsf{PN} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathsf{PT} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathsf{PW} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathsf{PC} \arrow[ur] \arrow[uuu] & &
\end{tikzcd}
$$

Conditional logics



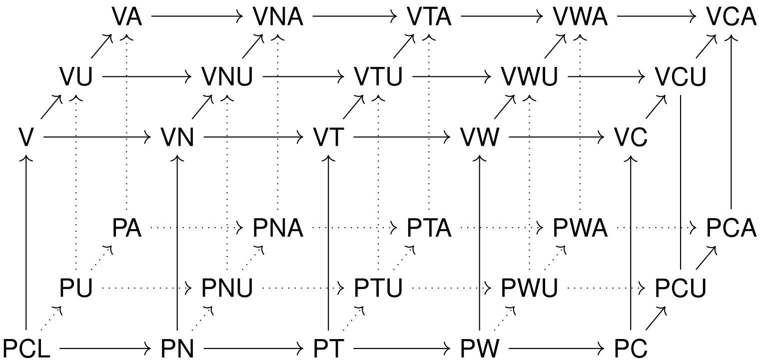
★ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V



★ *Centering* For all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$. C

★ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V

Conditional logics



- ⋆ *Normality* For all x, $N(x) \neq \emptyset$. N
- ⋆ *Total reflexivity* For all x, there is $\alpha \in N(x)$ such that $x \in \alpha$. T
- ⋆ *Weak centering* For all x, $N(x) \neq \emptyset$ and for all $\alpha \in N(x)$, $x \in \alpha$. W
- ⋆ *Centering* For all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$. C
- ⋆ *Uniformity* For all x, y, $\bigcup N(y) = \bigcup N(x)$. U
- ⋆ *Absoluteness* For all x, y, $N(x) = N(y)$. A
- ⋆ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V

Labelled calculi for conditional logics

$$
\begin{tikzcd}[column sep=small]
 & & \mathrm{VA} \arrow[rrr] & & & \mathrm{VNA} \arrow[rrr] & & & \mathrm{VTA} \arrow[rrr] & & & \mathrm{VWA} \arrow[rrr] & & & \mathrm{VCA} \\
 & \mathrm{VU} \arrow[rrr] \arrow[ur] & & & \mathrm{VNU} \arrow[rrr] \arrow[ur] & & & \mathrm{VTU} \arrow[rrr] \arrow[ur] & & & \mathrm{VWU} \arrow[rrr] \arrow[ur] & & & \mathrm{VCU} \arrow[ur] & \\
\mathrm{V} \arrow[rrr] \arrow[ur] & & & \mathrm{VN} \arrow[rrr] \arrow[ur] & & & \mathrm{VT} \arrow[rrr] \arrow[ur] & & & \mathrm{VW} \arrow[rrr] \arrow[ur] & & & \mathrm{VC} \arrow[ur] & & \\
 & & \mathrm{PA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PNA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PTA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PWA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PCA} \arrow[uuu] \\
 & \mathrm{PU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PNU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PTU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PWU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PCU} \arrow[ur] \arrow[uuu, no head] & \\
\mathrm{PCL} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PN} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PT} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PW} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PC} \arrow[ur] \arrow[uuu] & &
\end{tikzcd}
$$

Enriching the language: modal logics

Enriching the language: modal logics

☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \quad \rightsquigarrow \quad$ "x has access to y" (relational atoms)

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \;\rightsquigarrow\;$ "x has access to y" (relational atoms)
  - ▹ $x : A \;\rightsquigarrow\;$ "x satisfies A" (labelled formulas)

Enriching the language: modal logics

- Countably many variables for worlds: $x, y, z \ldots$ (labels)
- Labelled formulas
  - $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
  - $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)
- Labelled sequent: $\mathcal{R}, \Gamma \Rightarrow \Delta$

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
  - ▹ $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)
- ☞ Labelled sequent: $\mathcal{R}, \Gamma \Rightarrow \Delta$
- ☞ Rules for $\Box$

$$\Box_L \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} y! \qquad \Box_R \frac{xRy, \mathcal{R}, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}$$

$$x \Vdash \Box A \quad \textit{iff} \quad \textit{for all } y \textit{ s.t. } xRy, y \Vdash A$$

Enriching the language: modal logics

- Countably many variables for worlds: $x, y, z \ldots$ (labels)
- Labelled formulas
  - xRy $\rightsquigarrow$ "x has access to y" (relational atoms)
  - $x : A$ $\rightsquigarrow$ "x satisfies A" (labelled formulas)
- Labelled sequent: $\mathcal{R}, \Gamma \Rightarrow \Delta$
- Rules for $\Box$

$$\Box_L \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ y! \qquad \Box_R \frac{xRy, \mathcal{R}, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}$$

$$x \Vdash \Box A \quad \textit{iff} \quad \textit{for all } y \textit{ s.t. } xRy, y \Vdash A$$

- Rules for frame conditions, example: transitivity

$$\mathrm{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \dots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \dots$

- ☞ Relational atoms
  - ▹ $x \in a \rightsquigarrow$ "x is an element of a"

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a \quad \rightsquigarrow \quad$ "x is an element of a"
  - ▹ $a \in N(x) \quad \rightsquigarrow \quad$ "a is an element of $N(x)$"

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a$ $\rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x)$ $\rightsquigarrow$ "a is an element of $N(x)$"
  - ▹ $a \subseteq b$ $\rightsquigarrow$ "a is included in b"

Enriching the language: conditional logics

- Countably many variables for worlds $x, y, z, \ldots$
- Countably many variables for neighbourhoods $a, b, c, \ldots$

- Relational atoms
  - $x \in a$ $\rightsquigarrow$ "x is an element of a"
  - $a \in N(x)$ $\rightsquigarrow$ "a is an element of $N(x)$"
  - $a \subseteq b$ $\rightsquigarrow$ "a is included in b"

- Labelled formulas

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a \rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - ▹ $a \subseteq b \rightsquigarrow$ "a is included in b"

- ☞ Labelled formulas
  - ▹ $x : A \rightsquigarrow$ "x satisfies A"

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a \rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - ▹ $a \subseteq b \rightsquigarrow$ "a is included in b"

- ☞ Labelled formulas
  - ▹ $x : A \rightsquigarrow$ "x satisfies A"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \dots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \dots$

- ☞ Relational atoms
  - ▹ $x \in a \rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - ▹ $a \subseteq b \rightsquigarrow$ "a is included in b"

- ☞ Labelled formulas
  - ▹ $x : A \rightsquigarrow$ "x satisfies A"
  - ▹ $a \Vdash^{\exists} A \rightsquigarrow$ "A is satisfied at some world of a"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$

Enriching the language: conditional logics

- Countably many variables for worlds $x, y, z, \ldots$
- Countably many variables for neighbourhoods $a, b, c, \ldots$

- Relational atoms
  - $x \in a$ $\rightsquigarrow$ "x is an element of a"
  - $a \in N(x)$ $\rightsquigarrow$ "a is an element of $N(x)$"
  - $a \subseteq b$ $\rightsquigarrow$ "a is included in b"

- Labelled formulas
  - $x : A$ $\rightsquigarrow$ "x satisfies A"
  - $a \Vdash^{\exists} A$ $\rightsquigarrow$ "A is satisfied at some world of a"
  - $a \Vdash^{\forall} A$ $\rightsquigarrow$ "A is satisfied at all worlds of a"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$

Enriching the language: conditional logics

- Countably many variables for worlds $x, y, z, \ldots$
- Countably many variables for neighbourhoods $a, b, c, \ldots$

- Relational atoms
  - $x \in a$ $\rightsquigarrow$ "x is an element of a"
  - $a \in N(x)$ $\rightsquigarrow$ "a is an element of $N(x)$"
  - $a \subseteq b$ $\rightsquigarrow$ "a is included in b"

- Labelled formulas
  - $x : A$ $\rightsquigarrow$ "x satisfies A"
  - $a \Vdash^{\exists} A$ $\rightsquigarrow$ "A is satisfied at some world of a"
  - $a \Vdash^{\forall} A$ $\rightsquigarrow$ "A is satisfied at all worlds of a"
  - $x \Vdash_a A \mid B$ $\rightsquigarrow$ "there is a $b \in N(x)$ such that $b \subseteq a$, $b \Vdash^{\exists} A$ and $b \Vdash^{\forall} A \to B$"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$

Labelled rules

Labelled rules

- Rules for $>$

$$
>_R \ \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x A > B} \ (a!)
$$

$$
>_L \ \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \qquad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}
$$

$$
\Vdash_R \ \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \qquad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}
$$

$$
\Vdash_L \ \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta} \ (a!)
$$

Labelled rules

☞ Rules for $>$

$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x A > B} \text{ (a!)}$$

$$>_L \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_R \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

$$\Vdash_L \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta} \text{ (a!)}$$

☞ Rules for frame conditions: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

$$\text{C} \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled rules

☞ Rules for >

$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, xA > B} \text{ (a!)}$$

$$>_L \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_R \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \rightarrow B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

$$\Vdash_L \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \rightarrow B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta} \text{ (a!)}$$

☞ Rules for frame conditions: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

$$\text{C} \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{single} \frac{x \in \{x\}, \{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled rules

☞ Rules for $>$

$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, xA > B} \text{ (a!)}$$

$$>_L \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_R \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

$$\Vdash_L \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta} \text{ (a!)}$$

☞ Rules for frame conditions: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

$$\text{C} \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{single} \frac{x \in \{x\}, \{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{Repl}_1 \frac{y \in \{x\}, At(y), At(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Repl}_2 \frac{y \in \{x\}, At(x), At(y), \mathcal{R}, \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Example

Axiom c $(p \wedge q) \to (p > q)$

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\ldots x \in \{x\}, x : p \Rightarrow \{x\} \Vdash^{\exists} p, x : p$}
\LeftLabel{$\Vdash^{\exists}_{R}$}
\UnaryInfC{$\ldots x \in \{x\}, x : p \Rightarrow \{x\} \Vdash^{\exists} p$}
\LeftLabel{Single}
\UnaryInfC{$\ldots x : p \Rightarrow \{x\} \Vdash^{\exists} p$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$y \in \{x\}, \ldots, y : q, y : p \Rightarrow y : q$}
\LeftLabel{$\to_{R}$}
\UnaryInfC{$y \in \{x\}, \ldots, y : q \Rightarrow y : p \to q$}
\LeftLabel{Repl_1}
\UnaryInfC{$y \in \{x\}, \ldots, x : q \Rightarrow y : p \to q$}
\LeftLabel{$\Vdash^{\exists}_{R}$}
\UnaryInfC{$\ldots x : q \Rightarrow \{x\} \Vdash^{\forall} p \to q$}
\LeftLabel{$\Vdash_{L}$}
\BinaryInfC{$\{x\} \in a, \{x\} \subseteq a, a \in N(x), a \Vdash^{\exists} p, x : p, x : q \Rightarrow x \Vdash_{a} p \mid q$}
\LeftLabel{C}
\UnaryInfC{$a \in N(x), a \Vdash^{\exists} p, x : p, x : q \Rightarrow x \Vdash_{a} p \mid q$}
\LeftLabel{$>_{R}$}
\UnaryInfC{$x : p, x : q \Rightarrow x : p > q$}
\LeftLabel{$\wedge_{L}$}
\UnaryInfC{$x : p \wedge q \Rightarrow x : p > q$}
\LeftLabel{$\to_{R}$}
\UnaryInfC{$\Rightarrow x : (p \wedge q) \to (p > q)$}
\end{prooftree}
$$

Main results [G, Negri and Olivetti, 2021]

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Soundness). If the sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in the labelled calculus for L, then the sequent is valid in the logic L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Soundness). If the sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in the labelled calculus for L, then the sequent is valid in the logic L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Soundness). If the sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in the labelled calculus for L, then the sequent is valid in the logic L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice **without** absoluteness

Theorem (Completeness, II). If A is valid in the class of models for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Soundness). If the sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in the labelled calculus for L, then the sequent is valid in the logic L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice **without** absoluteness

Theorem (Completeness, II). If A is valid in the class of models for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. Show that if A is not provable, we can construct a finite countermodel for it (easy).

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Soundness). If the sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is provable in the labelled calculus for L, then the sequent is valid in the logic L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice **without** absoluteness

Theorem (Completeness, II). If A is valid in the class of models for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. Show that if A is not provable, we can construct a finite countermodel for it (easy). We need to show termination (difficult).

Structured proof systems for (some) Lewis' logics

$$
\begin{tikzcd}[column sep=tiny]
 & & \mathrm{VA} \arrow[rrr] & & & \mathrm{VNA} \arrow[rrr] & & & \mathrm{VTA} \arrow[rrr] & & & \mathrm{VWA} \arrow[rrr] & & & \mathrm{VCA} \\
 & \mathrm{VU} \arrow[rrr] \arrow[ur] & & & \mathrm{VNU} \arrow[rrr] \arrow[ur] & & & \mathrm{VTU} \arrow[rrr] \arrow[ur] & & & \mathrm{VWU} \arrow[rrr] \arrow[ur] & & & \mathrm{VCU} \arrow[ur] & \\
\mathrm{V} \arrow[rrr] \arrow[ur] & & & \mathrm{VN} \arrow[rrr] \arrow[ur] & & & \mathrm{VT} \arrow[rrr] \arrow[ur] & & & \mathrm{VW} \arrow[rrr] \arrow[ur] & & & \mathrm{VC} \arrow[ur] & & \\
 & & \mathrm{PA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PNA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PTA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PWA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PCA} \arrow[uuu] \\
 & \mathrm{PU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PNU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PTU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PWU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PCU} \arrow[ur] \arrow[uuu, dash] & \\
\mathrm{PCL} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PN} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PT} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PW} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PC} \arrow[ur] \arrow[uuu] & &
\end{tikzcd}
$$

Sequents with blocks

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preceq C) \vee (B \preceq C)$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preceq C) \vee (B \preceq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preceq C) \vee (B \preceq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \dots, [\Sigma_k \lhd C_k]$$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preceq C) \vee (B \preceq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_k \lhd C_k] \quad \rightsquigarrow$$

$$\bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Big(\bigvee_{B \in \Sigma_1} (B \preceq C_1) \Big) \vee \cdots \vee \Big(\bigvee_{B \in \Sigma_k} (B \preceq C_k) \Big)$$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preceq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preceq C) \vee (B \preceq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_k \lhd C_k] \quad \rightsquigarrow$$

$$\bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Big(\bigvee_{B \in \Sigma_1} (B \preceq C_1) \Big) \vee \cdots \vee \Big(\bigvee_{B \in \Sigma_k} (B \preceq C_k) \Big)$$

Example

$G_1, G_2 \Rightarrow D, [A, B \lhd C] \quad \rightsquigarrow \quad (G_1 \wedge G_2) \rightarrow \big(D \vee ((A \preceq C) \vee (B \preceq C))\big)$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_{\mathsf{L}}\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_{\mathsf{L}}\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_{\mathsf{L}}\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_{\mathsf{L}}\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B}$$

$$\preceq_{\mathsf{L}}\ \frac{\Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B}$$

$$\preceq_L\ \frac{\Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preceq_L\ \frac{\Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preceq_L\ \frac{\Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

$$>_L\ \frac{\bot \preceq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad >_R\ \frac{(A \wedge \neg B) \preceq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

$$A > B \;:=\; (\bot \preceq A) \vee \neg((A \wedge \neg B) \preceq (A \wedge B))$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preceq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preceq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preceq_L\ \frac{\Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

$$>_L\ \frac{\bot \preceq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \land \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad >_R\ \frac{(A \land \neg B) \preceq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

$$A > B \ := \ (\bot \preceq A) \lor \neg((A \land \neg B) \preceq (A \land B))$$

☞ Rules for extensions: centering

$$\text{C}\ \frac{A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B}{A \preceq B, \Gamma \Rightarrow \Delta}$$

Examples

Axiom $(A \preceq B) \vee (B \preceq A)$

$$
\text{com}\,\dfrac{
\text{jump}\,\dfrac{\text{init}\,\dfrac{}{b \Rightarrow a, b}}{\Rightarrow a \preceq b, b \preceq a, [a, b \lhd b], [b \lhd a]}
\qquad
\text{jump}\,\dfrac{\text{init}\,\dfrac{}{a \Rightarrow a, b}}{\Rightarrow a \preceq b, b \preceq a, [a \lhd b], [a, b \lhd a]}
}{
\preceq_R\,\dfrac{\Rightarrow a \preceq b, b \preceq a, [a \lhd b], [b \lhd a]}{
\preceq_R\,\dfrac{\Rightarrow a \preceq b, b \preceq a, [a \lhd b]}{
\vee_R\,\dfrac{\Rightarrow a \preceq b, b \preceq a}{\Rightarrow (a \preceq b) \vee (b \preceq a)}
}}}
$$

Axiom c $(p \wedge q) \rightarrow (p > q)$

$$
\text{C}\,\dfrac{
\wedge_L\,\dfrac{\neg_L\,\dfrac{\text{init}\,\dfrac{}{p, p, q \Rightarrow [\bot \lhd p], q}}{p, \neg q, p, q \Rightarrow [\bot \lhd p]}}{p \wedge \neg q, p, q \Rightarrow [\bot \lhd p]}
\qquad
\text{init}\,\dfrac{}{p, q \Rightarrow [\bot \lhd p], p}
}{
>_R\,\dfrac{(p \wedge \neg q) \preceq p, p, q \Rightarrow [\bot \lhd p]}{
\wedge_L\,\dfrac{p, q \Rightarrow p > q}{
\rightarrow_R\,\dfrac{p \wedge q \Rightarrow p > q}{\Rightarrow (p \wedge q) \rightarrow (p > q)}
}}}
$$

Main results [G Lellmann, Olivetti, Pozzato, 2016]

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For $L \in \{\mathrm{V}, \mathrm{VN}, \mathrm{VT}, \mathrm{VW}, \mathrm{VC}, \mathrm{VA}, \mathrm{VNA}\}$

Theorem (Soundness). If the sequent $\Gamma \Rightarrow \Delta$ is provable in the sequent calculus w. blocks for L, then the formula interpretation of the sequent is derivable from the axioms for L.

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For $L \in \{V, VN, VT, VW, VC, VA, VNA\}$

Theorem (Soundness). If the sequent $\Gamma \Rightarrow \Delta$ is provable in the sequent calculus w. blocks for L, then the formula interpretation of the sequent is derivable from the axioms for L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For $L \in \{\mathrm{V}, \mathrm{VN}, \mathrm{VT}, \mathrm{VW}, \mathrm{VC}, \mathrm{VA}, \mathrm{VNA}\}$

Theorem (Soundness). If the sequent $\Gamma \Rightarrow \Delta$ is provable in the sequent calculus w. blocks for L, then the formula interpretation of the sequent is derivable from the axioms for L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

Theorem (Completeness, II). If A is valid in the class of models for L, then A is provable in the labelled calculus for L.

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For $L \in \{V, VN, VT, VW, VC, VA, VNA\}$

Theorem (Soundness). If the sequent $\Gamma \Rightarrow \Delta$ is provable in the sequent calculus w. blocks for L, then the formula interpretation of the sequent is derivable from the axioms for L.

Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

Theorem (Completeness, II). If A is valid in the class of models for L, then A is provable in the labelled calculus for L.

Proof. Show that if A is not provable, we can construct a finite countermodel for it (difficult). We need to show termination (easy).

Summing up

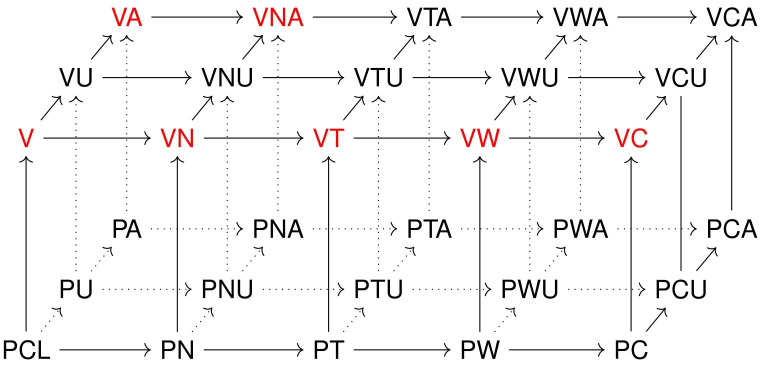
$$
\begin{tikzcd}
 & & \mathrm{VA} \arrow[rrr] & & & \mathrm{VNA} \arrow[rrr] & & & \mathrm{VTA} \arrow[rrr] & & & \mathrm{VWA} \arrow[rrr] & & & \mathrm{VCA} \\
 & \mathrm{VU} \arrow[ur] \arrow[rrr] & & & \mathrm{VNU} \arrow[ur] \arrow[rrr] & & & \mathrm{VTU} \arrow[ur] \arrow[rrr] & & & \mathrm{VWU} \arrow[ur] \arrow[rrr] & & & \mathrm{VCU} \arrow[ur] & \\
\mathrm{V} \arrow[ur] \arrow[rrr] & & & \mathrm{VN} \arrow[ur] \arrow[rrr] & & & \mathrm{VT} \arrow[ur] \arrow[rrr] & & & \mathrm{VW} \arrow[ur] \arrow[rrr] & & & \mathrm{VC} \arrow[ur] & & \\
 & & \mathrm{PA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathrm{PNA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathrm{PTA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathrm{PWA} \arrow[uuu, dotted] \arrow[rrr, dotted] & & & \mathrm{PCA} \arrow[uuu] \\
 & \mathrm{PU} \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & \mathrm{PNU} \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & \mathrm{PTU} \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & \mathrm{PWU} \arrow[uuu, dotted] \arrow[ur, dotted] \arrow[rrr, dotted] & & & \mathrm{PCU} \arrow[uuu] \arrow[ur] & \\
\mathrm{PCL} \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & \mathrm{PN} \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & \mathrm{PT} \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & \mathrm{PW} \arrow[uuu] \arrow[ur, dotted] \arrow[rrr] & & & \mathrm{PC} \arrow[uuu] \arrow[ur] & &
\end{tikzcd}
$$

Summing up

$$
\begin{tikzcd}
 & & \mathrm{VA} \arrow[rrr] & & & \mathrm{VNA} \arrow[rrr] & & & \mathrm{VTA} \arrow[rrr] & & & \mathrm{VWA} \arrow[rrr] & & & \mathrm{VCA} \\
 & \mathrm{VU} \arrow[rrr] \arrow[ur] & & & \mathrm{VNU} \arrow[rrr] \arrow[ur] & & & \mathrm{VTU} \arrow[rrr] \arrow[ur] & & & \mathrm{VWU} \arrow[rrr] \arrow[ur] & & & \mathrm{VCU} \arrow[ur] & \\
\mathrm{V} \arrow[rrr] \arrow[ur] & & & \mathrm{VN} \arrow[rrr] \arrow[ur] & & & \mathrm{VT} \arrow[rrr] \arrow[ur] & & & \mathrm{VW} \arrow[rrr] \arrow[ur] & & & \mathrm{VC} \arrow[ur] & & \\
 & & \mathrm{PA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PNA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PTA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PWA} \arrow[rrr, dotted] \arrow[uuu, dotted] & & & \mathrm{PCA} \arrow[uuu] \\
 & \mathrm{PU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PNU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PTU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PWU} \arrow[rrr, dotted] \arrow[ur, dotted] \arrow[uuu, dotted] & & & \mathrm{PCU} \arrow[ur] \arrow[uuu] & \\
\mathrm{PCL} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PN} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PT} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PW} \arrow[rrr] \arrow[ur, dotted] \arrow[uuu] & & & \mathrm{PC} \arrow[ur] \arrow[uuu] & &
\end{tikzcd}
$$

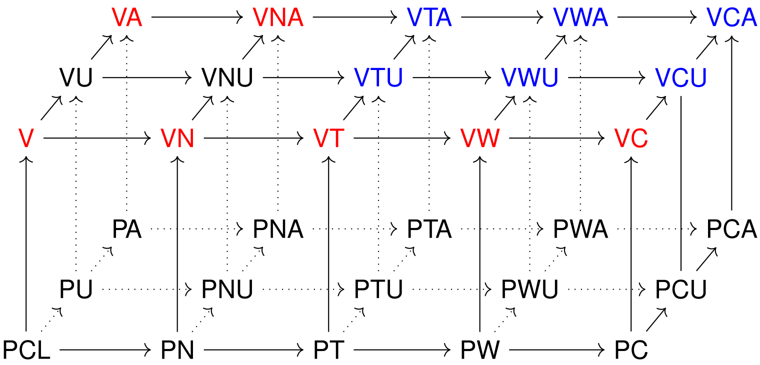
- Labelled calculi for all the logics [G, Negri, Olivetti, 2021]

Summing up



- Labelled calculi for all the logics [G, Negri, Olivetti, 2021]
- Sequent calculus with blocks for V, VN, VT, VW, VC, VA, VNA [G, Lellmann, Olivetti, Pozzato, 2016]

Summing up



- Labelled calculi for all the logics [G, Negri, Olivetti, 2021]
- Sequent calculus with blocks for V, VN, VT, VW, VC, VA, VNA [G, Lellmann, Olivetti, Pozzato, 2016]
- Hypersequent calculus with blocks for logics VTU, VWU, VCU, VTA, VWA, VCA [G, Lellmann, Olivetti, Pozzato, 2017]

Summing up

	fml. interpr.	invertible rules	analyticity	termination proof search	counterm. constr.	modularity
G3cp	yes	yes	yes	yes, easy!	yes, easy!	n/a
G3K	yes	no	yes	yes, easy!	yes, not easy	no
$\mathrm{NK} \cup \mathrm{X}^{\diamond}$	yes	yes	yes	yes	yes, easy!	45-clause
labK ∪ X	no	yes	yes	yes, for most	yes, easy!	yes
lab, cond	no	yes	easy	difficult	easy	yes
str, cond	yes	no	difficult	easy	difficult	no