# Homework 3

Proof Theory of Modal Logic
Tsinghua Logic Summer School, July 2025

Exercises marked with ($\star$) are not mandatory, but allow you to gain extra points.

**Question 1 (3 points).** Write down the labelled rule $\mathsf{conf}$ corresponding to the frame condition of *confluence*, that is:

$$\forall x \forall y \big( (xRy \land xRz) \to \exists k (yRk \land zRk) \big)$$

Then, derive the formula $\Diamond\Box p \to \Box\Diamond p$ in $\mathsf{labK} \cup \{\mathsf{conf}\}$.

**Question 2 (3.5 points).** Derive axiom 4, that is, $\Box A \to \Box\Box A$, in $\mathsf{labK} \cup \{\mathsf{t}, 5\}$. Then, show that rule $\mathsf{tr}$ is derivable in $\mathsf{labK} \cup \{\mathsf{t}, 5\} \cup \{\mathsf{wk_L}, \mathsf{wk_R}\}$.

Recall that $\mathsf{labK} \cup \{\mathsf{t}, 5\}$ consists of the rules of $\mathsf{labK}$, plus the following rules ($\mathsf{euc}'$ is the contracted instance of $\mathsf{euc}$):

$$\mathsf{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{euc}'\ \frac{yRy, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

The rule $\mathsf{tr}$ is the following rule:

$$\mathsf{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

**Question 3 (3.5 points).** We want to show that the rule $\Diamond_{\mathsf{L}}$ is *invertible*, that is, we want to prove the following statement:

> *If* $\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta$ *is derivable in* $\mathsf{labK}$*, then for every label* $y \neq x$ *which does not occur in* $\mathcal{R} \cup \Gamma \cup \Delta$*, it holds that* $xRy, \mathcal{R}, x{:}A, \Gamma \Rightarrow \Delta$ *is derivable in* $\mathsf{labK}$.

The proof proceeds by induction on the height of the derivation of $\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta$, and by distinguishing cases according to the last rule applied in the derivation. Prove the statement for the case in which the last rule applied is $\Box_{\mathsf{R}}$, that is, the derivation has the following form:

$$\Box_{\mathsf{R}}\ \frac{\overset{\mathcal{D}}{zRy, \mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta', y{:}B}}{\mathcal{R}, x{:}\Diamond A, \Gamma \Rightarrow \Delta', z{:}\Box B}$$

Show how to construct a derivation of sequent $xRy, \mathcal{R}, y{:}A, \Gamma \Rightarrow \Delta', z{:}\Box B$. You can use (without proof) admissibility of substitution and weakening, seen in the lecture.

**Question 4 ($\star$) (2 points).** Show that rule $\mathsf{euc}$ is derivable in $\mathsf{labK} \cup \{\mathsf{b}, 4\} \cup \{\mathsf{wk_L}, \mathsf{wk_R}\}$.

Recall that $\mathsf{labK} \cup \{\mathsf{b}, 4\}$ consists of the rules of $\mathsf{labK}$, plus the following rules:

$$\mathsf{sym}\,\frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{tr}\,\frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

The rule $\mathsf{euc}$ is the following rule:

$$\mathsf{euc}\,\frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Is rule $\mathsf{euc}$ derivable in $\mathsf{labK} \cup \{\mathsf{b}, 4\}$? Is it admissible in $\mathsf{labK} \cup \{\mathsf{b}, 4\}$? Why?