

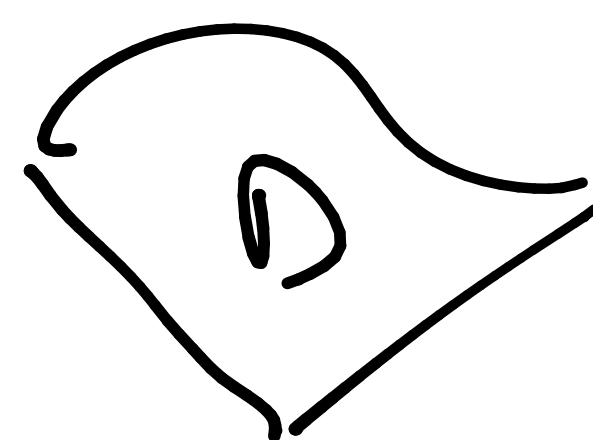
as follows (changing the context):

$\Gamma'\{B, \bar{B}\}$, where $\Gamma'\{\} = \Gamma\{\}[\{\}], \Diamond \bar{B}\}$.

Since the statement says "for any context", we can still apply the inductive hypothesis, and conclude that $\Gamma'\{B, \bar{B}\}$ is derivable.

Q4.

$$\frac{\Gamma\{\Diamond A, [\Delta], [\Diamond A, \Sigma]\}}{\Gamma\{\Diamond A, [\Delta], [\Sigma]\}} \quad \text{4}^\Diamond$$



By IH applied to $\Gamma\{\Diamond A, [\Delta], [\Diamond A, \Sigma]\}$, we have that $\Gamma\{\Diamond A, [\Delta, [\Diamond A, \Sigma]]\}$ is derivable.

We construct the following derivation:

$$\frac{\frac{\frac{\Gamma\{\Diamond A, [\Delta, [\Diamond A, \Sigma]]\}}{\Gamma\{\Diamond A, [\Delta, \Diamond A, [\Diamond A, \Sigma]]\}} \quad \text{wk}}{\Gamma\{\Diamond A, [\Delta, \Diamond A, [\Sigma]]\}} \quad \text{4}^\Diamond}{\Gamma\{\Diamond A, [\Delta, [\Sigma]]\}} \quad \text{4}^\Diamond$$

all the worlds $w \in W$ s.t. $f(\varepsilon) R w$. By transitivity of R , we have that $f(\delta) R w$, and so $w \models A$. Since this holds for any world accessible from $f(\varepsilon)$, conclude that $\mathcal{M}, f(\varepsilon) \models \Diamond A$.

We thus conclude that \mathcal{M} and f refute $\Gamma \{ \Diamond A, [\Diamond A, \Delta] \}$

Q3. we distinguish cases according to the form of A .

▷ $A := p$. Then $\Gamma \{ p, \bar{p} \}$ is an instance of init, and we are done.

▷ $A := B \wedge C$. We construct the following derivation:

$$\frac{\frac{\Gamma \{ B, \bar{B} \} \quad \Gamma \{ C, \bar{C} \}}{\Gamma \{ B \wedge C, \bar{B}, \bar{C} \}} \wedge}{\Gamma \{ B \wedge C, \bar{B} \vee \bar{C} \}} \vee$$

Since $ch(B) < ch(B \wedge C)$ and $ch(C) < ch(B \wedge C)$, both $\Gamma \{ B, \bar{B} \}$ and $\Gamma \{ C, \bar{C} \}$ are derivable by inductive hypothesis.

▷ $A := B \vee C$. Similar to the previous case.

▷ $A := \Box B$. We construct the derivation:

$$\frac{\frac{\Gamma \{ [B, \bar{B}], \Diamond \bar{B} \}}{\Gamma \{ [B], \Diamond \bar{B} \}} \Diamond}{\Gamma \{ \Box B, \Diamond \bar{B} \}} \Box$$

The sequent $\Gamma \{ [B, \bar{B}], \Diamond \bar{B} \}$ can be written

Solutions of Homework 2

Q1. We need to show that, if there is a derivation of $\Gamma \{ \Diamond A, [\Delta] \}$ in NK having height n , then there is a derivation of $\Gamma \{ \Diamond A, [A, \Delta] \}$ whose height is at most n .

We can use height-preserving admissibility of weakening in NK to construct our desired derivation:

$$\frac{\begin{array}{c} \vdots \\ \Diamond \\ \hline \Gamma \{ \Diamond A, [\Delta] \} \end{array}}{\Gamma \{ \Diamond A, [A, \Delta] \}} \text{wk}$$

The preservation of the height is guaranteed by the fact that weakening is hp-admissible.

Q2. We prove the contrapositive statement: if $\Gamma \{ \Diamond A, [\Delta] \}$ is not valid in transitive models, then $\Gamma \{ \Diamond A, [\Diamond A, \Delta] \}$ is not valid in transitive models.

Assume $\Gamma \{ \Diamond A, [\Delta] \}$ is not valid. Then there is a transitive model \mathcal{M} and a \mathcal{M} -map f s.t.:

$$\mathcal{M}, f(\delta) \not\models B, \text{ for all } \delta \in \text{tr}(\Gamma \{ \Diamond A, [\Delta] \}),$$
$$\text{for all } B \in \delta.$$

Let γ be s.t. $\Diamond A \in \gamma$ and $\varepsilon = \Delta$. Both γ and ε are nodes of $\text{tr}(\Gamma \{ \Diamond A, [\Delta] \})$. We have that $\mathcal{M}, f(\delta) \not\models \Diamond A$, and $\mathcal{M}, f(\varepsilon) \not\models D$, for all $D \in \Delta$.

Moreover, since \mathcal{M} is transitive, it holds that $\mathcal{M}, f(\varepsilon) \not\models \Diamond A$. To see this, consider