Homework 2

Proof Theory of Modal Logic
Tsinghua Logic Summer School, July 2025

Exercises marked with $(\star)$ are not mandatory, but allow you to gain extra points.

Question 1 (3 points). Show that the rule $\Diamond$ is height-preserving invertible in NK. You can use (without proof) admissibility of the structural rules for NK as stated in the lecture.

$$\Diamond\,\frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Question 2 (3 points). Prove soundness of the rule $\mathsf{4}^{\Diamond}$ reasoning semantically, that is: show that if the premiss of the rule is valid in all transitive models, then the conclusion of the rule is valid in all transitive models.

$$\mathsf{4}^{\Diamond}\,\frac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Question 3 (4 points). Consider the one-sided language used to define nested sequents, that is: $A, B ::= p \mid \overline{p} \mid A \wedge B \mid A \vee B$, for $p \in Atm$. We define the complexity of a formula A, denoted by $cp(A)$, as follows:

$$\begin{array}{rcl} cp(p) = cp(\bar{p}) & := & 0 \\ cp(A \wedge B) = cp(A \vee B) & := & cp(A) + cp(B) + 1 \\ cp(\Box A) = cp(\Diamond A) & := & cp(A) + 1 \end{array}$$

It is easy to verify that $cp(A) = cp(\bar{A})$, for any formula A of the language.

Prove that, for any context $\Gamma\{\ \}$ and any formula A, the nested sequent $\Gamma\{A, \bar{A}\}$ is derivable in NK. Prove the statement reasoning by induction on the complexity of A.

Question 4 $(\star)$ (2 points). Consider the rules $\mathsf{4}^{\Diamond}$ and $\mathsf{4}^{[\,]}$:

$$\mathsf{4}^{\Diamond}\,\frac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \mathsf{4}^{[\,]}\,\frac{\Gamma\{[\Sigma], [\Delta]\}}{\Gamma\{[\Sigma, [\Delta]]\}}$$

We want to show that $\mathsf{4}^{[\,]}$ is admissible in $\mathsf{NK} \cup \{\mathsf{4}^{\Diamond}\}$. The proof proceeds by induction on the height of the derivation $\mathcal{D}$ of the premiss of $\mathsf{S4}$, and by distinguishing cases according to the last rule applied in $\mathcal{D}$.

Prove the statement, for the following subcase: The last rule applied in the derivation of the premiss of $\mathsf{4}^{[\,]}$ is an application of $\mathsf{4}^{\Diamond}$ of the following form:

$$\mathsf{4}^{\Diamond}\ \frac{\overset{\mathcal{D}}{\Gamma\{\Diamond A, [\Delta], [\Diamond A, \Sigma]\}}}{\Gamma\{\Diamond A, [\Delta], [\Sigma]\}}$$

Show how to construct a derivation of the conclusion of $\mathsf{4}^{[\,]}$, that is, sequent $\Gamma\{\Diamond A, [\Delta, [\Sigma]]\}$. You can use (without proof) admissibility of the structural rules for $\mathsf{NK} \cup \{\mathsf{4}^{\Diamond}\}$ as stated in the lecture.