Exercises for Lecture 1

Proof theory for modal and non-classical logics

June 2023

Rules of **labK**

$$\mathsf{init}\ \frac{}{\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \bot\ \frac{}{\mathcal{R}, x : \bot, \Gamma \Rightarrow \Delta}$$

$$\to_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \to B, \Gamma \Rightarrow \Delta} \qquad \to_{\mathsf{R}}\ \frac{x : A, \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \to B}$$

$$\Box_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, y : A, x : \Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ (\mathrm{y!})$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : \Diamond A, \Gamma \Rightarrow \Delta}\ (\mathrm{y!}) \qquad \Diamond_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A}$$

Structural rules

$$\mathsf{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ (\mathrm{y!}) \qquad \mathsf{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\mathsf{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \mathsf{Euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Where $(y!)$ means that y does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$.

Calculi **labKX**, for $X \subseteq \{D, T, B, 4, 5\}$, are defined by adding to **labK** the structural rules corresponding to the semantic conditions in X. Thus, for instance, **labDB** is **labK** + ser + sym and **labT5** is **labK** + ref + Euc.

We denote by $\vdash_{\mathbf{labKX}} \mathcal{R}, \Gamma \Rightarrow \Delta$ derivability of sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ in the labelled sequent calculus **labK**.

For the definition of validity of a sequent in a class of frames refer, e.g., to S. Negri, *Kripke completeness revisited*, in G. Primiero and S. Rahman (eds.), ”Acts of Knowledge - History, Philosophy and Logic”, College Publications, 2009. Pdf: https://drive.google.com/file/d/1FQQZmz4yNw33eXfqlyIqtgOn4FdHFBRa/view

Exercise 1. Prove the following:

a) $\vdash_{\textbf{labKT}} \Rightarrow x : \Box p \to p$

b) $\vdash_{\textbf{labKB}} \Rightarrow x : p \to \Box\Diamond p$

c) $\vdash_{\textbf{labK4}} \Rightarrow x : \Box p \to \Box\Box p$

d) $\vdash_{\textbf{labK5}} \Rightarrow x : \Diamond p \to \Box\Diamond p$

e) $\vdash_{\textbf{labKT5}} \Rightarrow x : \Box p \to \Box\Box p$

f) $\vdash_{\textbf{labKB4}} \Rightarrow x : \Diamond p \to \Box\Diamond p$

g) $\vdash_{\textbf{labK}} \Rightarrow x : \Box(p \to q) \to (\Box p \to \Box q)$

h) $\vdash_{\textbf{labK}} \Rightarrow x : \Box(p \to q) \to (\Diamond p \to \Diamond q)$

Exercise 2.

a) In the proof of soundness for **labK**, prove the case of the $\Box_{\mathsf{L}}$ rule.

b) In the proof of soundness for **labKT**, prove the case of the ref rule.