Proof theory for modal and non-classical logics (a biased introduction)

A quest for termination

Proof systems for intuitionistic logic

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MoL project course
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Outline

Intuitionistic logic

Sequent-style calculi

Beyond sequent-style calculi

Intuitionistic propositional logic and its models

$$A, B ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \qquad\qquad \neg A := A \to \bot$$

Intuitionistic propositional logic and its models

$A, B ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B$ $\qquad \neg A := A \rightarrow \bot$

$\mathcal{M} = \langle W, \leq, v \rangle$ where

- $W \neq \emptyset$
- $\leq$ is reflexive and transitive
- $v : Atm \rightarrow \mathcal{P}(W)$ s.t. if $x \leq y$ and $x \in v(p)$, then $y \in v(p)$

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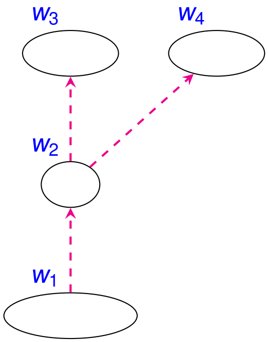

Intuitionistic propositional logic and its models

$A, B ::= p \mid \bot \mid A \land B \mid A \lor B \mid A \to B$ $\qquad \neg A := A \to \bot$

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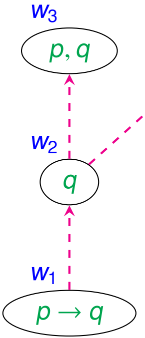

Intuitionistic propositional logic and its models

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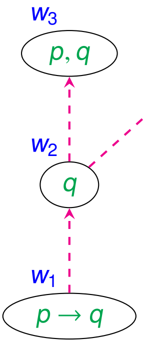

$x \models A \rightarrow B$ *iff* for all y such that $x \leq y$, if $y \models A$ then $y \models B$

Intuitionistic propositional logic and its models

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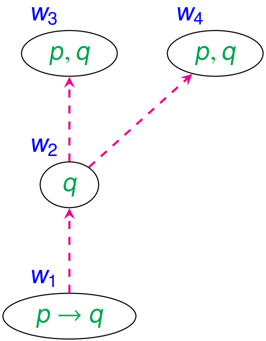

$x \models A \rightarrow B$ *iff* for all y such that $x \leq y$, if $y \models A$ then $y \models B$

Monotonicity: if $x \leq y$ and $x \models A$, then $y \models A$

Outline

Intuitionistic logic

Sequent-style calculi

Beyond sequent-style calculi

Single-conclusion calculi

[Gentzen, 1935], [Troelstra & Schwichtenberg, 2000]

$$\Gamma \Rightarrow C$$

$$fm(\Gamma \Rightarrow C) = \bigwedge \Gamma \to C$$

The rules of **G3i**

$$\text{init}\ \dfrac{}{p, \Gamma \Rightarrow p} \qquad \bot\ \dfrac{}{\bot, \Gamma \Rightarrow C}$$

$$\wedge_{\mathrm{R}}\ \dfrac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \wedge_{\mathrm{L}}\ \dfrac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_{\mathrm{L}}\ \dfrac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad \vee_{\mathrm{R}}\ \dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ i \in \{0, 1\}$$

$$\to_{\mathrm{L}}\ \dfrac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad \to_{\mathrm{R}}\ \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

Structural properties, **G3i**

- The weakening rule is admissible

$$wk\,\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}$$

- All the rules are invertible, **except** for the following cases:
  - ★ If $\vdash \Gamma \Rightarrow A_0 \vee A_1$, then $\vdash \Gamma \Rightarrow A_i$, for $i \in \{0, 1\}$
  - ★ If $\vdash A \rightarrow B, \Gamma \Rightarrow C$, then $\vdash A \rightarrow B, \Gamma \Rightarrow A$

- The contraction rule is admissible

$$ctr\,\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}$$

- The cut rule is admissible

$$cut\,\frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$$

Proof search in **G3i**

Proof search in **G3i**

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\LeftLabel{$\to_{\mathrm{L}}^{m}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow a \to (b \to c)$}
\LeftLabel{$\to_{\mathrm{L}}^{m}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow c$}
\LeftLabel{$\to_{\mathrm{R}}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot, a, b, a \Rightarrow b \to c$}
\LeftLabel{$\to_{\mathrm{R}}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot, a, b \Rightarrow a \to (b \to c)$}
\LeftLabel{$\to_{\mathrm{L}}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot, a, b \Rightarrow c$}
\LeftLabel{$\to_{\mathrm{R}}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot, a \Rightarrow b \to c$}
\LeftLabel{$\to_{\mathrm{R}}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot \Rightarrow a \to (b \to c)$}
\LeftLabel{$\to_{\mathrm{L}}$}
\UnaryInfC{$(a \to (b \to c)) \to \bot \Rightarrow \bot$}
\LeftLabel{$\to_{\mathrm{R}}$}
\UnaryInfC{$\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot$}
\end{prooftree}
$$

Proof search in **G3i**

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow a \to (b \to c)}\;{\to}^m_L}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow c}\;{\to}^m_L}{(a \to (b \to c)) \to \bot, a, b, a \Rightarrow b \to c}\;{\to}_R}{(a \to (b \to c)) \to \bot, a, b \Rightarrow a \to (b \to c)}\;{\to}_R}{(a \to (b \to c)) \to \bot, a, b \Rightarrow c}\;{\to}_L}{(a \to (b \to c)) \to \bot, a \Rightarrow b \to c}\;{\to}_R}{(a \to (b \to c)) \to \bot \Rightarrow a \to (b \to c)}\;{\to}_R}{(a \to (b \to c)) \to \bot \Rightarrow \bot}\;{\to}_L
$$

$$
\frac{(a \to (b \to c)) \to \bot \Rightarrow \bot}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}\;{\to}_R
$$

Proof search in **G3i**

$$
\to_{\mathrm{L}}^{m}\dfrac{\to_{\mathrm{L}}^{m}\dfrac{\vdots}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow a \to (b \to c)}}{\to_{\mathrm{R}}\dfrac{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow c}{\to_{\mathrm{R}}\dfrac{(a \to (b \to c)) \to \bot, a, b, a \Rightarrow b \to c}{\to_{\mathrm{L}}\dfrac{(a \to (b \to c)) \to \bot, a, b \Rightarrow a \to (b \to c)}{\to_{\mathrm{R}}\dfrac{(a \to (b \to c)) \to \bot, a, b \Rightarrow c}{\to_{\mathrm{R}}\dfrac{(a \to (b \to c)) \to \bot, a \Rightarrow b \to c}{\to_{\mathrm{L}}\dfrac{(a \to (b \to c)) \to \bot \Rightarrow a \to (b \to c)}{\to_{\mathrm{R}}\dfrac{(a \to (b \to c)) \to \bot \Rightarrow \bot}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}}}}}}}}
$$

Proof search in **G3i**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow a \to (b \to c)}\,\textit{fail}}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow c}\,{\to}_{\mathrm{L}}^{m}}{(a \to (b \to c)) \to \bot, a, b, a \Rightarrow b \to c}\,{\to}_{\mathrm{R}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow a \to (b \to c)}\,{\to}_{\mathrm{R}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow c}\,{\to}_{\mathrm{L}}}{(a \to (b \to c)) \to \bot, a \Rightarrow b \to c}\,{\to}_{\mathrm{R}}}{(a \to (b \to c)) \to \bot \Rightarrow a \to (b \to c)}\,{\to}_{\mathrm{R}}}{(a \to (b \to c)) \to \bot \Rightarrow \bot}\,{\to}_{\mathrm{L}}}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}\,{\to}_{\mathrm{R}}
$$

Multi-conclusion calculi

[Maehara, 1954], [Dragalin, 1988], [Troelstra & Schwichtenberg, 2000]

$$\Gamma \Rightarrow \Delta$$

$$fm(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$$

The rules of **m-G3i**

$$\text{init}^m \; \frac{}{p, \Gamma \Rightarrow \Delta p} \qquad \bot^m \; \frac{}{\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge^m_{\mathrm{R}} \; \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge^m_{\mathrm{L}} \; \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee^m_{\mathrm{L}} \; \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee^m_{\mathrm{R}} \; \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\to^m_{\mathrm{L}} \; \frac{A \to B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to^m_{\mathrm{R}} \; \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}$$

Structural properties, **m-G3i**

- The weakening rules are admissible

$$wk_{\mathsf{L}}\,\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad wk_{\mathsf{R}}\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

- All the rules are invertible, **except** for the following case:
  - ⋆ If $\vdash \Gamma \Rightarrow \Delta, A \to B$, then $\vdash A, \Gamma \Rightarrow \Delta$

- The contraction rules are admissible

$$ctr_{\mathsf{L}}\,\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad ctr_{\mathsf{R}}\,\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

- The cut rule is admissible

$$cut\,\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof search in **m-G3i**

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$$
{\scriptstyle \rightarrow^{m}_{\mathrm{R}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{L}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{R}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{R}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{L}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{R}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{R}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{L}}}\dfrac{{\scriptstyle \rightarrow^{m}_{\mathrm{L}}}\dfrac{\vdots}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot, a, b, a, b \Rightarrow \bot, c, c, a \rightarrow (b \rightarrow c)}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot, a, b, a, b \Rightarrow \bot, c, c}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot, a, b, a \Rightarrow \bot, c, b \rightarrow c}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot, a, b \Rightarrow \bot, c, a \rightarrow (b \rightarrow c)}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot, a, b \Rightarrow \bot, c}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot, a \Rightarrow \bot, b \rightarrow c}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot \Rightarrow \bot, a \rightarrow (b \rightarrow c)}}{(a \rightarrow (b \rightarrow c)) \rightarrow \bot \Rightarrow \bot}}{\Rightarrow ((a \rightarrow (b \rightarrow c)) \rightarrow \bot) \rightarrow \bot}
$$

Proof search in **m-G3i**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow \bot, c, c, a \to (b \to c)}\,{\to}^m_{\mathrm{L}}}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow \bot, c, c}\,{\to}^m_{\mathrm{L}}}{(a \to (b \to c)) \to \bot, a, b, a \Rightarrow \bot, c, b \to c}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow \bot, c, a \to (b \to c)}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow \bot, c}\,{\to}^m_{\mathrm{L}}}{(a \to (b \to c)) \to \bot, a \Rightarrow \bot, b \to c}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot \Rightarrow \bot, a \to (b \to c)}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot \Rightarrow \bot}\,{\to}^m_{\mathrm{L}}}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}\,{\to}^m_{\mathrm{R}}
$$

Proof search in **m-G3i**

$$
\to^m_R\,\dfrac{\to^m_L\,\dfrac{\to^m_R\,\dfrac{\to^m_R\,\dfrac{\to^m_L\,\dfrac{\to^m_R\,\dfrac{\to^m_R\,\dfrac{\to^m_L\,\dfrac{\to^m_L\,\dfrac{\vdots}{\color{purple}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow \bot, c, c, a \to (b \to c)}}}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow \bot, c, c}}{(a \to (b \to c)) \to \bot, a, b, a \Rightarrow \bot, c, b \to c}}{\color{purple}{(a \to (b \to c)) \to \bot, a, b \Rightarrow \bot, c, a \to (b \to c)}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow \bot, c}}{(a \to (b \to c)) \to \bot, a \Rightarrow \bot, b \to c}}{(a \to (b \to c)) \to \bot \Rightarrow \bot, \color{purple}{a \to (b \to c)}}}{(a \to (b \to c)) \to \bot \Rightarrow \bot}}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}
$$

Proof search in **m-G3i**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow \bot, c, c, a \to (b \to c)}\,\mathit{fail}}{(a \to (b \to c)) \to \bot, a, b, a, b \Rightarrow \bot, c, c}\,{\to}^m_{\mathrm{L}}}{(a \to (b \to c)) \to \bot, a, b, a \Rightarrow \bot, c, b \to c}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow \bot, c, a \to (b \to c)}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot, a, b \Rightarrow \bot, c}\,{\to}^m_{\mathrm{L}}}{(a \to (b \to c)) \to \bot, a \Rightarrow \bot, b \to c}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot \Rightarrow \bot, a \to (b \to c)}\,{\to}^m_{\mathrm{R}}}{(a \to (b \to c)) \to \bot \Rightarrow \bot}\,{\to}^m_{\mathrm{L}}}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}\,{\to}^m_{\mathrm{R}}
$$

A terminating sequent calculus

[Vorob'ev, 1952], [Hudelmaier, 1988] [Dyckhoff, 1992], [Dyckhoff and Negri, 2000]

$$\Gamma \Rightarrow C$$

$$fm(\Gamma \Rightarrow C) = \bigwedge \Gamma \rightarrow C$$

The rules of **G4ip**

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow p} \qquad \bot\ \frac{}{\bot, \Gamma \Rightarrow C}$$

$$\wedge_{\mathrm{R}}\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \wedge_{\mathrm{L}}\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_{\mathrm{L}}\ \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad \vee_{\mathrm{R}}\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ i \in \{0, 1\}$$

$$\rightarrow_{\mathrm{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$0{\rightarrow}_{\mathrm{L}}\ \frac{B, p, \Gamma \Rightarrow C}{p \rightarrow B, p, \Gamma \Rightarrow C} \qquad \wedge{\rightarrow}_{\mathrm{L}}\ \frac{E \rightarrow (F \rightarrow B), \Gamma \Rightarrow C}{(E \wedge F) \rightarrow B, \Gamma \Rightarrow C}$$

$$\vee{\rightarrow}_{\mathrm{L}}\ \frac{E \rightarrow B, F \rightarrow B, \Gamma \Rightarrow C}{(E \vee F) \rightarrow B, \Gamma \Rightarrow C} \qquad {\rightarrow}{\rightarrow}_{\mathrm{L}}\ \frac{E, F \rightarrow B, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{(E \rightarrow F) \rightarrow B, \Gamma \Rightarrow C}$$

Structural properties, **G4ip**

- [Dyckhoff, 1992]: cut-free completeness obtained by simulating proofs in **G3i**.
- [Dyckhoff and Negri, 2000]: direct proof of completeness, by showing admissibility of contraction and cut.

$$\mathit{ctr}\,\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C} \qquad \mathit{cut}\,\frac{\Gamma \Rightarrow A \quad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma', \Rightarrow C}$$

Proof search in **G4ip**

$$\dfrac{B, p, \Gamma \Rightarrow C}{p \to B, p, \Gamma \Rightarrow C}\; 0{\to_L} \qquad \dfrac{E, F \to B, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{(E \to F) \to B, \Gamma \Rightarrow C}\; {\to}{\to_L}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{c \to \bot, b, a, b \Rightarrow c}\,\mathit{fail} \quad \dfrac{}{\bot, a, b, \Rightarrow c}\,\mathrm{init}}{(b \to c) \to \bot, a, b \Rightarrow c}\,{\to}{\to_L}}{(b \to c) \to \bot, a \Rightarrow b \to c}\,{\to_R} \quad \dfrac{}{\bot \Rightarrow \bot}\,\mathrm{init}}{\dfrac{(a \to (b \to c)) \to \bot \Rightarrow \bot}{\Rightarrow ((a \to (b \to c)) \to \bot) \to \bot}\,{\to_R}}\,{\to}{\to_L}$$

Outline

Nested sequents for IPL [Fitting, 2014]

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Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

Nested sequents for IPL [Fitting, 2014]

Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

Nested sequent, for Γ, $\Delta_1, \ldots, \Delta_n$ multisets of polarised formulas

$$\Gamma, \langle \Delta_1 \rangle, \ldots, \langle \Delta_n \rangle$$

Nested sequents for IPL [Fitting, 2014]

Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

Nested sequent, for Γ, $\Delta_1, \ldots, \Delta_n$ multisets of polarised formulas

$$\Gamma, \langle \Delta_1 \rangle, \ldots, \langle \Delta_n \rangle$$

The rules of **Nipl**

$$\text{init}\ \frac{}{\Gamma\{p^\bullet, p^\circ\}} \qquad \bot\ \frac{}{\Gamma\{\bot^\bullet\}} \qquad \wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ\ \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \to^\bullet\ \frac{\Gamma\{A \to B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to B^\bullet\}} \qquad \to^\circ\ \frac{\Gamma\{\langle A^\bullet, B^\circ \rangle\}}{\Gamma\{A \to B^\circ\}}$$

$$\text{lift}\ \frac{\Gamma\{A^\bullet, \langle A^\bullet, \Delta \rangle\}}{\Gamma\{A^\bullet, \langle \Delta \rangle\}}$$

Nested sequents for IPL [Fitting, 2014]

Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

Nested sequent, for Γ, $\Delta_1, \ldots, \Delta_n$ multisets of polarised formulas

$$\Gamma, \langle \Delta_1 \rangle, \ldots, \langle \Delta_n \rangle$$

The rules of **Nipl**

$$\text{init}\ \frac{}{\Gamma\{p^\bullet, p^\circ\}} \qquad \bot\ \frac{}{\Gamma\{\bot^\bullet\}} \qquad \wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ\ \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \to^\bullet\ \frac{\Gamma\{A \to B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to B^\bullet\}} \qquad \to^\circ\ \frac{\Gamma\{\langle A^\bullet, B^\circ \rangle\}}{\Gamma\{A \to B^\circ\}}$$

$$\text{lift}\ \frac{\Gamma\{A^\bullet, \langle A^\bullet, \Delta \rangle\}}{\Gamma\{A^\bullet, \langle \Delta \rangle\}}$$

- Soundness If $\vdash_{\mathbf{Nipl}} A$, then $\vdash_{\mathrm{IPL}} A$
- Completeness Cut-free completeness obtained by simulating proofs in a tableaux calculus for IPL.

Labelled calculus for intuitionistic logic

[Dyckhoff and Negri, 2011]

☞ Relational atoms and labelled formulas

- $x \leq y \rightsquigarrow$ "y is accessible from x in the preorder"
- $x : A \rightsquigarrow$ "x satisfies A"

Labelled calculus for intuitionistic logic

[Dyckhoff and Negri, 2011]

- Relational atoms and labelled formulas
  - $x \leq y \rightsquigarrow$ "y is accessible from x in the preorder"
  - $x : A \rightsquigarrow$ "x satisfies A"

- Some labelled rules

$$\text{Ref}\,\frac{x \leq x, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Tr}\,\frac{x \leq z, x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled calculus for intuitionistic logic

[Dyckhoff and Negri, 2011]

- Relational atoms and labelled formulas
  - $x \leq y \rightsquigarrow$ "y is accessible from x in the preorder"
  - $x : A \rightsquigarrow$ "x satisfies A"

- Some labelled rules

$$\text{init}\ \frac{}{x \leq y, \mathcal{R}, x : p, \Gamma \Rightarrow \Delta, y : p} \qquad \supset_R \frac{x \leq y, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta, y : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \supset B}\ (y!)$$

$$\supset_L \frac{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta, y : A \quad x \leq y, \mathcal{R}, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta}$$

$$\text{Ref}\ \frac{x \leq x, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Tr}\ \frac{x \leq z, x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled calculus for intuitionistic logic

[Dyckhoff and Negri, 2011]

- Relational atoms and labelled formulas
  - $x \leq y$ $\rightsquigarrow$ "y is accessible from x in the preorder"
  - $x : A$ $\rightsquigarrow$ "x satisfies A"

- Some labelled rules

$$\text{init}\ \frac{}{x \leq y, \mathcal{R}, x : p, \Gamma \Rightarrow \Delta, y : p} \qquad \supset_R \frac{x \leq y, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta, y : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \supset B}\ (y!)$$

$$\supset_L \frac{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta, y : A \qquad x \leq y, \mathcal{R}, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leq y, \mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta}$$

$$\text{Ref}\ \frac{x \leq x, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Tr}\ \frac{x \leq z, x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

- Termination [Negri, 2014]