Proof theory for modal and non-classical logics

Labelled calculi for modal logics

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MoL project course
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(a biased introduction)

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Plan of the course

Week 1 (5-9 June)

- Lecture 1: Labelled calculi for modal logics (today)
- Lecture 2: Nested calculi for modal logics (8th June, 13-15, F1.43)

Week 2 (12-16 June)

- Lecture 3: Proof systems for intuitionistic logic (12th June, 11-13, F1.15)
- Lecture 4: Proof systems for intuitionistic modal logic (14th June, 11-13, F1.15)

Week 3 (19-23 June): individual study / oral presentations

Week 4 (26 - 30 June) and Week 4+1 (3 - 7 July): individual study / oral presentations

Sequent calculus

Gentzen-style calculus for classical logic

$$A, B \quad ::= \quad p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \qquad\qquad \neg A := A \supset \bot$$

The rules of **G3c** [Troelstra & Schwichtenberg, 2000]

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \qquad\qquad \bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta}$$

$$\wedge_{\mathrm{R}}\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad\qquad \wedge_{\mathrm{L}}\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\vee_{\mathrm{L}}\ \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad \vee_{\mathrm{R}}\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\supset_{\mathrm{L}}\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \qquad\qquad \supset_{\mathrm{R}}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}$$

$\vdash_{\mathbf{G3c}} \Gamma \Rightarrow \Delta \quad \rightsquigarrow \quad$ there is a derivation of $\Gamma \Rightarrow \Delta$ in **G3c**

Structural properties

- The rules of weakening are admissible in **G3c**.

$$wk_{\mathrm{L}}\,\frac{A,\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad wk_{\mathrm{R}}\,\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}$$

- All the rules of **G3c** are invertible.
- The rules of contraction are admissible in **G3c**.

$$ctr_{\mathrm{L}}\,\frac{A, A,\Gamma \Rightarrow \Delta}{A,\Gamma \Rightarrow \Delta} \qquad ctr_{\mathrm{R}}\,\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

- The rule of cut is admissible in **G3c**.

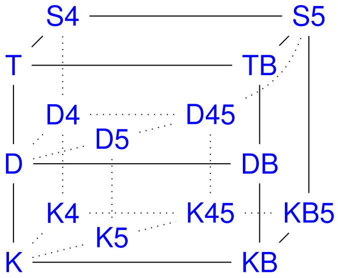
$$cut\,\frac{\Gamma \Rightarrow \Delta, A \quad A,\Gamma' \Rightarrow \Delta'}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta'}$$

Modal logics

The S5 cube of modal logics

$$A, B ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A$$

$$\neg A := A \supset \bot$$
$$\Diamond A \leftrightarrow \neg\Box\neg A$$
$$\Box A \leftrightarrow \neg\Diamond\neg A$$



Axioms / rules			Frame conditions	
k	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	K	$\emptyset$	
nec	if A is provable, $\Box A$ is provable		$\emptyset$	
d	$\Box A \supset \Diamond A$	D	Ser.	$\forall x \exists y (xRy)$
t	$\Box A \supset A$	T	Refl.	$\forall x (xRx)$
b	$A \supset \Box\Diamond A$	B	Sym.	$\forall x \forall y (xRy \rightarrow yRx)$
ax4	$\Box A \supset \Box\Box A$	4	Tran.	$\forall x \forall y \forall z (xRy \,\&\, yRz \rightarrow xRz)$
ax5	$\Diamond A \supset \Box\Diamond A$	5	Eucl.	$\forall x \forall y \forall z (xRy \,\&\, xRz \rightarrow yRz)$

Basic terminology

Frame: $\mathcal{F} = \langle W, R \rangle$

Model based on a frame: $\mathcal{M} = \langle \mathcal{F}, V \rangle$

A formula B is satisfiable if there are a world and a model s.t. $\mathcal{M}, x \models B$.

A formula B is satisfiable at a model, $\mathcal{M} \models B$, iff for all worlds x of a model $\mathcal{M}$, it holds that $\mathcal{M}, x \models B$.

A formula B is valid at a frame, $\mathcal{F} \models B$, iff for all models $\mathcal{M}$ based on $\mathcal{F}$, it holds that $\mathcal{M} \models B$.

For a subset $X \subseteq \{D, T, B, 4, 5\}$, an X-frame is a frame that satisfies all the conditions determined by the names in X.

A formula B is valid at the class of X-frames (or X-valid), notation $\models_X B$, iff for all X-frames $\mathcal{F}$, it holds that $\mathcal{F} \models B$.

Sequent calculus + Modal logics

???

Gentzen-style calculi for modal logics

In the literature: [Fitting, 1983], [Takano, 1992], ...

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$$\text{k}\ \frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta}$$

- Sequent calculus for K: **G3c** $+$ k

Gentzen-style calculi for modal logics

In the literature: [Fitting, 1983], [Takano, 1992], ...

$$\mathrm{k}\ \frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \qquad \mathrm{t}\ \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

- Sequent calculus for K: **G3c** $+$ k
- Sequent calculus for T: **G3c** $+$ k $+$ t

Gentzen-style calculi for modal logics

In the literature: [Fitting, 1983], [Takano, 1992], ...

$$\text{k}\ \frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \qquad \text{t}\ \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

$$4\ \frac{\Box B_1, \ldots, \Box B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta}$$

- Sequent calculus for K: **G3c** $+$ k
- Sequent calculus for T: **G3c** $+$ k $+$ t
- Sequent calculus for S4: **G3c** $+ 4 +$ t

Gentzen-style calculi for modal logics

In the literature: [Fitting, 1983], [Takano, 1992], ...

$$\text{k}\ \frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \qquad \text{t}\ \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

$$4\ \frac{\Box B_1, \ldots, \Box B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \qquad 45\ \frac{\Box B_1, \ldots, \Box B_n \Rightarrow A, \Box C_1, \ldots, \Box C_m}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Box C_1, \ldots, \Box C_m, \Delta}$$

- Sequent calculus for K: **G3c** $+$ k
- Sequent calculus for T: **G3c** $+$ k $+$ t
- Sequent calculus for S4: **G3c** $+$ 4 $+$ t
- Sequent calculus for S5: **G3c** $+$ 45 $+$ t

Gentzen-style calculi for modal logics

In the literature: [Fitting, 1983], [Takano, 1992], ...

$$\text{k}\,\frac{B_1, \ldots, B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \qquad \text{t}\,\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

$$4\,\frac{\Box B_1, \ldots, \Box B_n \Rightarrow A}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Delta} \quad 45\,\frac{\Box B_1, \ldots, \Box B_n \Rightarrow A, \Box C_1, \ldots, \Box C_m}{\Gamma, \Box B_1, \ldots, \Box B_n \Rightarrow \Box A, \Box C_1, \ldots, \Box C_m, \Delta}$$

- Sequent calculus for K: **G3c** $+$ k
- Sequent calculus for T: **G3c** $+$ k $+$ t
- Sequent calculus for S4: **G3c** $+$ 4 $+$ t
- Sequent calculus for S5: **G3c** $+$ 45 $+$ t

But.. the sequent calculus for S5 is **not** cut-free complete

Gentzen-style calculi for modal logics

In the literature: [Fitting, 1983], [Takano, 1992], ...

$$\text{k}\ \frac{B_1, \dots, B_n \Rightarrow A}{\Gamma, \Box B_1, \dots, \Box B_n \Rightarrow \Box A, \Delta} \qquad \text{t}\ \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}$$

$$4\ \frac{\Box B_1, \dots, \Box B_n \Rightarrow A}{\Gamma, \Box B_1, \dots, \Box B_n \Rightarrow \Box A, \Delta} \qquad 45\ \frac{\Box B_1, \dots, \Box B_n \Rightarrow A, \Box C_1, \dots, \Box C_m}{\Gamma, \Box B_1, \dots, \Box B_n \Rightarrow \Box A, \Box C_1, \dots, \Box C_m, \Delta}$$

- Sequent calculus for K: **G3c** + k
- Sequent calculus for T: **G3c** + k + t
- Sequent calculus for S4: **G3c** + 4 + t
- Sequent calculus for S5: **G3c** + 45 + t

But.. the sequent calculus for S5 is **not** cut-free complete

☞ "*A cut-free sequent calculus for* S5 *will require additional machinery in the rule format or a very different, possibly semantic, proof of cut admissibility.*" [Lellmann & Pattinson, 2013]

Labelled calculi for modal logics

In the literature

- [Kanger, 1957] Spotted formulas for S5
- [Fitting, 1983], [Goré 1998] Tableaux + labels
- [Simpson, 1994], [Viganò, 1998] Natural deduction + labels
- [Mints, 1997], [Viganò, 2000] Sequent calculus + labels
- ...

- [Negri, 2005], [Negri, 2003]

Labelled calculi for the S5 cube

$$\text{init}\ \frac{}{\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p} \qquad \bot\ \frac{}{\mathcal{R}, x : \bot, \Gamma \Rightarrow \Delta}$$

$$\supset_L \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta} \qquad \supset_R \frac{x : A, \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \supset B}$$

$$\Box_L \frac{xRy, \mathcal{R}, y : A, x : \Box A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ (y!)$$

$$\Diamond_L \frac{xRy, \mathcal{R}, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : \Diamond A, \Gamma \Rightarrow \Delta}\ (y!) \qquad \Diamond_R \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, x : \Diamond A}$$

$$\text{ser}\ \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\ (y!) \qquad \text{ref}\ \frac{xRx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{tr}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, xRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$y! \rightsquigarrow y$ does not occur in $\mathcal{R} \cup \Gamma \cup \Delta$

Beyond the S5 cube

From (geometric) axioms to rules

Geometric axiom [Simpson, 1994], [Negri, 2003]

$$\forall \vec{x}\Big((P_1 \wedge \ldots \wedge P_n) \rightarrow \bigvee_{i=1}^{m} \exists \vec{y}_i \big(Q_{i1} \wedge \ldots \wedge Q_{ik_i}\big)\Big)$$

- $\vec{x}$, $\vec{y}_i$ are (possibly empty) vectors of variables
- $n, m \geq 0, \quad k_1, \ldots, k_m \geq 1$
- $P_1, \ldots, P_n$, $Q_{i1}, \ldots, Q_{ik_i}$ atomic formulas
- $\vec{y}_1, \ldots, \vec{y}_m$ do not occur in any of $P_1, \ldots, P_n$

Labelled rule

$$\mathsf{r}\,\frac{\Xi_1[\vec{z_1}/\vec{y_1}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta \quad \cdots \quad \Xi_m[\vec{z_m}/\vec{y_m}], \Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}{\Pi, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

- $\Pi = \{P_1, \ldots, P_n\}$ and $\Xi_i = \{Q_{i1}, \ldots, Q_{ik_i}\}$ are multisets
- $\Xi[\vec{z}/\vec{y}]$: multiset obtained by substituting the free variables $\vec{y}$ with variables $\vec{z}$ in every formula of Ξ
- $\vec{z_1}, \ldots, \vec{z_m}$ do not occur in $\mathcal{R}, \Gamma \cup \Delta$

Appendix

Derivation example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}
\LeftLabel{init}
\UnaryInfC{$xRy, y : p \Rightarrow y : q, x : \Diamond p, y : p$}
\LeftLabel{$\Diamond_R$}
\UnaryInfC{$xRy, y : A \Rightarrow y : q, x : \Diamond p$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$xRy, x : \Box q, y : q, y : p \Rightarrow y : q$}
\LeftLabel{$\Box_L$}
\UnaryInfC{$xRy, x : \Box q, y : p \Rightarrow y : q$}
\LeftLabel{$\supset_L$}
\BinaryInfC{$xRy, x : \Diamond p \supset \Box q, y : p \Rightarrow y : q$}
\LeftLabel{$\supset_R$}
\UnaryInfC{$xRy, x : \Diamond p \supset \Box q \Rightarrow y : p \supset q$}
\LeftLabel{$\Box_R$}
\UnaryInfC{$x : \Diamond p \supset \Box q \Rightarrow x : \Box(p \supset q)$}
\LeftLabel{$\supset_R$}
\UnaryInfC{$\Rightarrow x : (\Diamond p \supset \Box q) \supset \Box(p \supset q)$}
\end{prooftree}
$$

Derivation example

$$
\supset_R\dfrac{\Box_R\dfrac{\supset_R\dfrac{\supset_L\dfrac{\Diamond_R\dfrac{\text{init}\dfrac{}{xRy, y : p \Rightarrow y : q, x : \Diamond p, y : p}}{xRy, y : A \Rightarrow y : q, x : \Diamond p} \qquad \Box_L\dfrac{\text{init}\dfrac{}{xRy, x : \Box q, y : q, y : p \Rightarrow y : q}}{xRy, x : \Box q, y : p \Rightarrow y : q}}{xRy, x : \Diamond p \supset \Box q, y : p \Rightarrow y : q}}{xRy, x : \Diamond p \supset \Box q \Rightarrow y : p \supset q}}{x : \Diamond p \supset \Box q \Rightarrow x : \Box(p \supset q)}}{\Rightarrow x : (\Diamond p \supset \Box q) \supset \Box(p \supset q)}
$$

Exercise. Construct a derivation of the following:

$$\Rightarrow x : \Box(p \supset q) \supset (\Box p \supset \Box q)$$
$$\Rightarrow x : \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$$
$$\Rightarrow x : \Diamond(p \vee q) \supset (\Diamond p \vee \Diamond q)$$

Summing up

	G3c	G3c+ modal r.	Labelled	Nested
☞ Formula interpretation	yes	yes	no	yes
☞ Analyticity	yes	no	subterm	yes
☞ Termination	yes	yes	yes	yes
☞ Invertibility	yes	no	yes	yes
☞ Modularity	n.a.	no	yes*	yes

* Even beyond the S5 cube!

Main references

- Blackburn, de Rijke and Venema, Modal Logic, Cambridge University Press, 2001.
- Gentzen, Untersuchungen über das logische schliessen, Matematische Zeitschrift, 39, 1934-35.
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- Negri, Proof analysis in modal logic, Journal of Philosophical Logic 34.5, 2005.
- Takano, Subformula property as a substitute for cut-elimination in modal propositional logics, Mathematica japonica, 37, 1992.
- Troelstra and Schwichtenberg, Basic proof theory, vol. 43, Cambridge University Press, 2000.