Exercises for Lecture 2

Proof theory for modal and non-classical logics

June 2023

Rules of **NK**

$$\mathsf{init}\ \frac{}{\Gamma\{p, \bar{p}\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Rules of **NKX**, for X 45-closed and $X \subseteq \{D, T, B, 4, 5\}$

$$\mathsf{d}\ \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}\ \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

$$4\ \frac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5\ \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ depth(\Gamma\{\ \}\{\emptyset\}) > 0$$

Calculi **NKX**, for X 45-closed and $X \subseteq \{D, T, B, 4, 5\}$, are defined by adding to **NK** the rules corresponding to the letters in X. For instance, **NKDB** is $\mathbf{NK} + \mathsf{d} + \mathsf{b}$. We denote by $\vdash_{\mathbf{NKX}}$ derivability of sequent Δ in the nested calculus **NKX**.

Exercise 1. Prove the following:

a) $\vdash_{\mathbf{NK}} \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$

b) $\vdash_{\mathbf{NK}} \Diamond(A \supset B) \supset \Box(A \supset B)$

c) $\vdash_{\mathbf{NKT}} \Box p \supset p$

d) $\vdash_{\mathbf{NKB}} p \supset \Box\Diamond p$

e) $\vdash_{\mathbf{NK4}} \Box p \supset \Box\Box p$

$f)$ $\vdash_{\mathbf{NK5}} \Diamond p \supset \Box\Diamond p$

Exercise 2. (optional)

Have a look at the countermodel construction from a failed proof search detailed in the following paper (subsection "Completeness", Theorem 3):

Brnnler, Deep sequent systems for modal logic. Arch. Math. Logic 48, 551577 (2009). https://link.springer.com/article/10.1007/s00153-009-0137-3

Construct a Kripke model $\mathcal{M}$ containing a world x such that $\mathcal{M}, x \not\models p \supset \Box\Diamond p$ from a failed branch of a proof search tree for the formula in **NK**, applying the strategy from the paper. Hint: the strategy can be simplified, as proof search in **NK** always terminates.