

Exercises for Lecture 2

Proof theory for modal and non-classical logics

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Rules of **NK**

$$\begin{array}{c}
 \text{init} \frac{}{\Gamma\{p, \bar{p}\}} \quad \vee_R \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \wedge_R \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \\
 \\
 \Box_R \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond_R \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}
 \end{array}$$

Rules of **NKX**, for X 45-closed and $X \subseteq \{D, T, B, 4, 5\}$

$$\begin{array}{c}
 d \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} \quad t \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \quad b \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \\
 \\
 4 \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad 5 \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}} \text{ } depth(\Gamma\{\}\{\emptyset\}) > 0
 \end{array}$$

Calculi **NKX**, for X 45-closed and $X \subseteq \{D, T, B, 4, 5\}$, are defined by adding to **NK** the rules corresponding to the letters in X . For instance, **NKDB** is **NK** + d + b. We denote by $\vdash_{\mathbf{NKX}}$ derivability of sequent Δ in the nested calculus **NKX**.

Exercise 1. Prove the following:

- a) $\vdash_{\mathbf{NK}} \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
- b) $\vdash_{\mathbf{NK}} \Diamond(A \supset B) \supset \Box(A \supset B)$
- c) $\vdash_{\mathbf{NKT}} \Box p \supset p$
- d) $\vdash_{\mathbf{NKB}} p \supset \Box \Diamond p$
- e) $\vdash_{\mathbf{NK4}} \Box p \supset \Box \Box p$

f) $\vdash_{\mathbf{NK5}} \Diamond p \supset \Box \Diamond p$

Exercise 2. (optional)

Have a look at the countermodel construction from a failed proof search detailed in the following paper (subsection "Completeness", Theorem 3):

Brünnler, Deep sequent systems for modal logic. Arch. Math. Logic 48, 551–577 (2009). <https://link.springer.com/article/10.1007/s00153-009-0137-3>

Construct a Kripke model \mathcal{M} containing a world x such that $\mathcal{M}, x \not\models p \supset \Box \Diamond p$ from a failed branch of a proof search tree for the formula in \mathbf{NK} , applying the strategy from the paper. Hint: the strategy can be simplified, as proof search in \mathbf{NK} always terminates.