Proof theory for modal and non-classical logics (a biased introduction)

Proof theory for intuitionistic modal logics

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MoL project course
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Outline

Intuitionistic modal logics

Nested-style sequents

Labelled sequents

Constructive modal logics

$$\mathcal{L} ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A \qquad \neg A \equiv A \supset \bot$$

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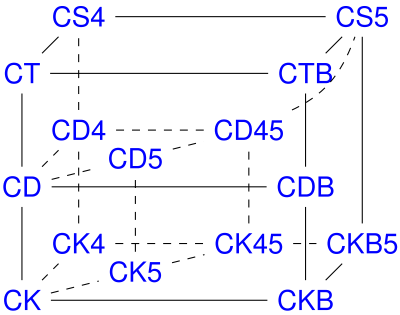

Constructive modal logics in the literature

- Type systems for constructive modal logics [Bierman, de Paiva, 2000]

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  Rules of **G3i**, plus

$$k_{\Box}\ \frac{C_1, \ldots, C_n \Rightarrow A}{\Box C_1, \ldots, \Box C_n, \Gamma \Rightarrow \Box A} \qquad k_{\Diamond}\ \frac{C_1, \ldots, C_n, A \Rightarrow B}{\Box C_1, \ldots, \Box C_n, \Diamond A, \Gamma \Rightarrow \Diamond B}$$

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- Cut-free nested sequents for CK (and for some extensions) [Arisaka, Das, Straßburger, 2015]

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IK

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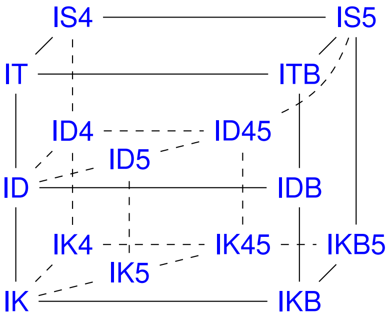

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Requirements for an intuitionistic modal logic (IML) [Sec. 3.2]:

1. IML is conservative over IPL.
2. IML contains all substitution instances of theorems of IPL and is closed under modus ponens.
3. The addition of the schema $A \vee \neg A$ to IML yields a standard classical modal logic.
4. If $A \vee B$ is a theorem of IML then either A is a theorem of IML or B is.
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$$\mathsf{IML} \vdash A \quad \Longleftrightarrow \quad \mathsf{IFOL} \vdash \mathsf{ST}(A)(x)$$

Some observations

- The $\Diamond$-free fragment of CK and IK are different
  - ★ $\neg\neg\Box\neg p \to \Box\neg p$ (inspired from the double negation translation!)
  - ★ $(\neg\Box\bot \to \Box\bot) \to \Box\bot$ (attributed to Carsten Grefe)

☞ https://prooftheory.blog/2022/08/19/brouwer-meets-kripke-constructivising-modal-logic/#paperkey_43

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- There are no known Gentzen-style calculi for IK

Bi-relational models for IK (and extensions)

- [Fisher Servi, 1984], soundness and completeness proof
- [Simpson, 1994]

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$x \Vdash \Box A$	iff	for all y, z s.t. $x \leq y$ and yRz, $z \Vdash A$
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☞ *Monotonicity* if $x \leq y$ and $x \Vdash A$, then $y \Vdash A$

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Single-conclusion nested sequents [Straßburger, 2013]

Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

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$$\Gamma ::= \Lambda, \Pi \qquad \Lambda ::= A_1^{\bullet}, \ldots, A_n^{\bullet}, [\Lambda_1], \ldots, [\Lambda_k] \qquad \Pi ::= A^{\circ} \mid [\Gamma]$$

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fm(\Lambda, \Pi) & = & fm(\Lambda) \supset fm(\Pi) \\
fm(\emptyset) & = & \bot \\
fm(A_1^\bullet, \ldots, A_n^\bullet, [\Lambda_1], \ldots, [\Lambda_k]) & = & A_1 \wedge \cdots \wedge A_n \wedge \Diamond[\Lambda_1] \wedge \cdots \wedge \Diamond[\Lambda_k] \\
fm(A^\circ) & = & A \\
fm([\Gamma]) & = & \Box fm(\Gamma)
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$\Gamma\{\,\}$ context; $\Gamma^*\{\,\}$ context with the unique output formula removed

Single-conclusion nested rules $\mathsf{NIK_s}$ (and extensions)

$$\bot^\bullet\,\frac{}{\Gamma\{\bot^\bullet\}}\quad \mathsf{id}\,\frac{}{\Gamma\{a^\bullet, a^\circ\}}\quad \wedge^\bullet\,\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A\wedge B^\bullet\}}\quad \wedge^\circ\,\frac{\Gamma\{A^\circ\}\qquad \Gamma\{B^\circ\}}{\Gamma\{A\wedge B^\circ\}}\quad \vee^\bullet\,\frac{\Gamma\{A^\bullet\}\qquad \Gamma\{B^\bullet\}}{\Gamma\{A\vee B^\bullet\}}$$

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$$\Box^\bullet\,\frac{\Gamma\{\Box A^\bullet, [A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}}\quad \Box^\circ_s\,\frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}\quad \Diamond^\bullet\,\frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}}\quad \Diamond^\circ\,\frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

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$$\mathsf{5}^\circ_s \frac{\Gamma\{\Diamond A^\circ\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0 \qquad \mathsf{5}^\bullet \frac{\Gamma\{\emptyset\}\{\Box A^\bullet\}}{\Gamma\{\Box A^\bullet\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0$$

Main results

☞ [Straßburger, 2013]

$\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, for $\mathsf{X} \subseteq \{D, T, B, 4, 5\}$

Soundness

Whenever a sequent Γ is provable in $\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, then $fm(\Gamma)$ is X-valid.

Cut-admissibility for $\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$

$$\mathsf{w}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \qquad \mathsf{c}\,\frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \qquad \mathsf{cut}\,\frac{\Gamma^*\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}$$

Completeness

Let $\mathsf{X} \subseteq \{D, T, B, 4, 5\}$ be 45-closed. Then every theorem of the logic $\mathsf{IK} + \mathsf{X}$ is provable in $\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$.

Main results

☞ [Straßburger, 2013]

$\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, for $\mathsf{X} \subseteq \{D, T, B, 4, 5\}$

Soundness
Whenever a sequent Γ is provable in $\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, then $fm(\Gamma)$ is X-valid.

Cut-admissibility for $\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$

$$\mathsf{w}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \qquad \mathsf{c}\,\frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \qquad \mathsf{cut}\,\frac{\Gamma^*\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}$$

Completeness
Let $\mathsf{X} \subseteq \{D, T, B, 4, 5\}$ be 45-closed. Then every theorem of the logic $\mathsf{IK} + \mathsf{X}$ is provable in $\mathsf{NIK_s} + \mathsf{X}^\bullet + \mathsf{X}^\circ$.

☞ [Marin, Straßburger, 2014]: full modularity by adding rules

Multi-conclusion nested sequents [Kuznets, Straßburger, 2019]

Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

Multi-conclusion nested sequents [Kuznets, Straßburger, 2019]

Polarised formulas: $A^\bullet$ and A°

$$A, B \Rightarrow C, D \quad \rightsquigarrow \quad A^\bullet, B^\bullet, C^\circ, D^\circ$$

Multi-conclusion nested sequents

$$B_1^\bullet, \ldots, B_h^\bullet, C_1^\circ, \ldots, C_k^\circ, [\Gamma_1], \ldots, [\Gamma_m]$$

where $B_1, \ldots, B_h$, $C_1, \ldots, C_k$ are formulas and $\Gamma_1, \ldots, \Gamma_m$ are multi-conclusion nested sequents.

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$\Gamma\{\}$ context; $\Gamma^*\{\}$ context with **all** output formula removed

Multi-conclusion rules $\mathsf{NIK_m}$ (and extensions)

$$\perp^{\bullet}\,\frac{}{\Gamma\{\perp^{\bullet}\}} \qquad \mathsf{id}\,\frac{}{\Gamma\{a^{\bullet}, a^{\circ}\}} \qquad \wedge^{\bullet}\,\frac{\Gamma\{A^{\bullet}, B^{\bullet}\}}{\Gamma\{A \wedge B^{\bullet}\}} \qquad \wedge^{\circ}\,\frac{\Gamma\{A^{\circ}\} \qquad \Gamma\{B^{\circ}\}}{\Gamma\{A \wedge B^{\circ}\}} \qquad \vee^{\bullet}\,\frac{\Gamma\{A^{\bullet}\} \qquad \Gamma\{B^{\bullet}\}}{\Gamma\{A \vee B^{\bullet}\}}$$

$$\vee^{\circ}_{m}\,\frac{\Gamma\{A^{\circ}, B^{\circ}\}}{\Gamma\{A \vee B^{\circ}\}} \qquad \supset^{\bullet}_{m}\,\frac{\Gamma\{A \supset B^{\bullet}, A^{\circ}\} \quad \Gamma\{B^{\bullet}\}}{\Gamma\{A \supset B^{\bullet}\}} \qquad \supset^{\circ}_{m}\,\frac{\Gamma^{*}\{A^{\bullet}, B^{\circ}\}}{\Gamma\{A \supset B^{\circ}\}}$$

$$\Box^{\bullet}\,\frac{\Gamma\{\Box A^{\bullet}, [A^{\bullet}, \Delta]\}}{\Gamma\{\Box A^{\bullet}, [\Delta]\}} \qquad \Box^{\circ}_{m}\,\frac{\Gamma^{*}\{[A^{\circ}]\}}{\Gamma\{\Box A^{\circ}\}} \qquad \Diamond^{\bullet}\,\frac{\Gamma\{[A^{\bullet}]\}}{\Gamma\{\Diamond A^{\bullet}\}} \qquad \Diamond^{\circ}\,\frac{\Gamma\{[A^{\circ}, \Delta]\}}{\Gamma\{\Diamond A^{\circ}, [\Delta]\}}$$

Multi-conclusion rules $\mathsf{NIK_m}$ (and extensions)

$$
\bot^\bullet\ \frac{}{\Gamma\{\bot^\bullet\}} \quad
\mathsf{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}} \quad
\wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \quad
\wedge^\circ\ \frac{\Gamma\{A^\circ\} \qquad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \quad
\vee^\bullet\ \frac{\Gamma\{A^\bullet\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}}
$$

$$
\vee^\circ_m\ \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad
\supset^\bullet_m\ \frac{\Gamma\{A \supset B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \quad
\supset^\circ_m\ \frac{\Gamma^*\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}
$$

$$
\Box^\bullet\ \frac{\Gamma\{\Box A^\bullet, [A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \quad
\Box^\circ_m\ \frac{\Gamma^*\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \quad
\Diamond^\bullet\ \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \quad
\Diamond^\circ\ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}
$$

$$
\mathsf{d}^\circ_m\ \frac{\Gamma\{\Diamond A^\circ, [A^\circ]\}}{\Gamma\{\Diamond A^\circ\}} \quad
\mathsf{t}^\circ_m\ \frac{\Gamma\{\Diamond A^\circ, A^\circ\}}{\Gamma\{\Diamond A^\circ\}} \quad
\mathsf{b}^\circ_m\ \frac{\Gamma\{[\Delta, \Diamond A^\circ], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}} \quad
\mathsf{4}^\circ_m\ \frac{\Gamma\{\Diamond A^\circ, [\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}
$$

$$
\mathsf{d}^\bullet\ \frac{\Gamma\{\Box A^\bullet, [A^\bullet]\}}{\Gamma\{\Box A^\bullet\}} \quad
\mathsf{t}^\bullet\ \frac{\Gamma\{\Box A^\bullet, A^\bullet\}}{\Gamma\{\Box A^\bullet\}} \quad
\mathsf{b}^\bullet\ \frac{\Gamma\{[\Delta, \Box A^\bullet], A^\bullet\}}{\Gamma\{[\Delta, \Box A^\bullet]\}} \quad
\mathsf{4}^\bullet\ \frac{\Gamma\{\Box A^\bullet, [\Box A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}}
$$

$$
\mathsf{5}^\circ_m\ \frac{\Gamma\{\Diamond A^\circ\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0 \qquad
\mathsf{5}^\bullet\ \frac{\Gamma\{\Box A^\bullet\}\{\Box A^\bullet\}}{\Gamma\{\Box A^\bullet\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) > 0
$$

Main results: Soundness

☞ For a sequent Γ, let $tr(\Gamma)$ denote its sequent tree.

☞ For a sequent Γ and a birelational model $\mathcal{M} = \langle W, \leq, R, V\rangle$, define a $\mathcal{M}$-map $f : tr(\Gamma) \to W$ such that, whenever δ is a children of γ in $tr(\Gamma)$, then $f(\gamma) R f(\delta)$.

☞ A sequent Γ is satisfied by a $\mathcal{M}$-map f iff

$$\mathcal{M}, f(\gamma) \models A \text{ for all } A^{\bullet} \in \gamma \in tr(\Gamma) \implies$$
$$\implies \mathcal{M}, f(\delta) \models B \text{ for some } B^{\circ} \in \delta \in tr(\Gamma)$$

☞ For X $\subseteq \{D, T, B, 4, 5\}$, a sequent is X-valid iff it is satisfiable by all $\mathcal{M}$-maps for all X-models $\mathcal{M}$.

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$\mathsf{NIK_m} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, for $\mathsf{X} \subseteq \{D, T, B, 4, 5\}$

Soundness

Whenever a sequent Γ is provable in $\mathsf{NIK_m} + \mathsf{X}^\bullet + \mathsf{X}^\circ$, then Γ is X-valid.

Main results: Completeness

Completeness

Let $\mathsf{X} \subseteq \{D, T, B, 4, 5\}$ be 45-closed. Then if Γ is X-valid, then Γ is provable in $\mathsf{NIK_m} + \mathsf{X}^\bullet + \mathsf{X}^\circ$.

Main results: Completeness

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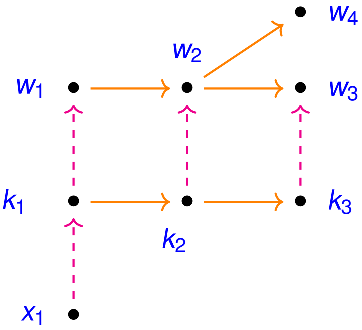
Let X $\subseteq \{D, T, B, 4, 5\}$ be 45-closed. Then if Γ is X-valid, then Γ is provable in $\mathrm{NIK_m} + \mathrm{X}^\bullet + \mathrm{X}^\circ$.

$$\supset_m^\circ \frac{\Gamma^*\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}} \qquad \Box_m^\circ \frac{\Gamma^*\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \qquad \Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}}$$

$$\to_m^\circ \dfrac{\wedge^\bullet \dfrac{\Diamond^\bullet \dfrac{\Diamond^\bullet \dfrac{\Box_m^\bullet \dfrac{\supset_m^\bullet \dfrac{\Box_m^\circ \dfrac{[\Box s \to q^\bullet, [\, s^\circ\,], [\, r^\bullet\,]\,], \Box(\Box s \to q)^\bullet}{[\Box s \to q^\bullet, \Box s^\circ, [\, r^\bullet\,]\,], \Box(\Box s \to q)^\bullet, c^\circ}}{[\Box s \to q^\bullet, [\, r^\bullet\,]\,], \Box(\Box s \to q)^\bullet, c^\circ}}{[\,[\, r^\bullet\,]\,], \Box(\Box s \to q)^\bullet, c^\circ}}{[\,\Diamond r^\bullet\,], \Box(\Box s \to q)^\bullet, c^\circ}}{\Diamond\Diamond r^\bullet, \Box(\Box s \to q)^\bullet, c^\circ}}{(\Diamond\Diamond r \wedge \Box(\Box s \to q))^\bullet, c^\circ}}{((\Diamond\Diamond r \wedge \Box(\Box s \to q)) \to c)^\circ}$$



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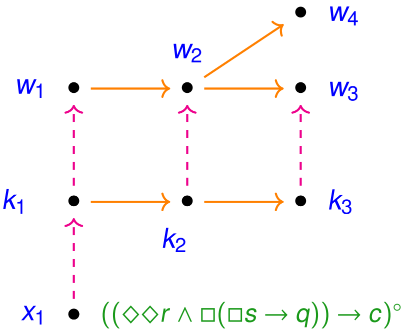
$$\rightarrow^\circ_m \dfrac{\wedge^\bullet \dfrac{\Diamond^\bullet \dfrac{\Diamond^\bullet \dfrac{\Box^\bullet_m \dfrac{\supset^\bullet_m \dfrac{\Box^\circ_m \dfrac{[\Box s \rightarrow q^\bullet, [\, s^\circ \,], [\, r^\bullet \,]\,], \Box(\Box s \rightarrow q)^\bullet}{[\Box s \rightarrow q^\bullet, \Box s^\circ, [\, r^\bullet \,]\,], \Box(\Box s \rightarrow q)^\bullet, c^\circ}}{[\Box s \rightarrow q^\bullet, [\, r^\bullet \,]\,], \Box(\Box s \rightarrow q)^\bullet, c^\circ}}{[\,[\, r^\bullet \,]\,], \Box(\Box s \rightarrow q)^\bullet, c^\circ}}{[\, \Diamond r^\bullet \,], \Box(\Box s \rightarrow q)^\bullet, c^\circ}}{\Diamond\Diamond r^\bullet, \Box(\Box s \rightarrow q)^\bullet, c^\circ}}{(\Diamond\Diamond r \wedge \Box(\Box s \rightarrow q))^\bullet, c^\circ}}{((\Diamond\Diamond r \wedge \Box(\Box s \rightarrow q)) \rightarrow c)^\circ}$$



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$$\begin{array}{rc}
 & [\Box s \to q^\bullet, [\,s^\circ\,], [\,r^\bullet\,]\,], \Box(\Box s \to q)^\bullet \\ \hline
\Box^\circ_m & [\Box s \to q^\bullet, \Box s^\circ, [\,r^\bullet\,]\,], \Box(\Box s \to q)^\bullet, c^\circ \\ \hline
\supset^\bullet_m & [\Box s \to q^\bullet, [\,r^\bullet\,]\,], \Box(\Box s \to q)^\bullet, c^\circ \\ \hline
\Box^\bullet_m & [\,[\,r^\bullet\,]\,], \Box(\Box s \to q)^\bullet, c^\circ \\ \hline
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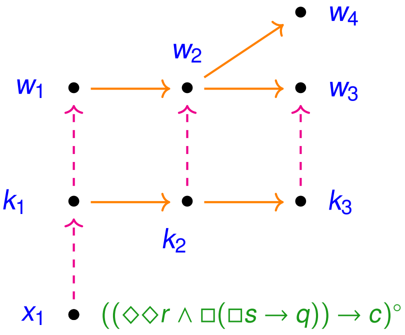
\to^\circ_m & ((\Diamond\Diamond r \wedge \Box(\Box s \to q)) \to c)^\circ
\end{array}$$



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$$
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\qquad
\Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}}
$$

$$
\begin{array}{ll}
 & [\Box s \to q^\bullet, [\, s^\circ \,], [\, r^\bullet \,]], \Box(\Box s \to q)^\bullet \\
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\Box^\bullet_m & \overline{[\,[\, r^\bullet \,]\,], \Box(\Box s \to q)^\bullet, c^\circ} \\
\Diamond^\bullet & \overline{[\, \Diamond r^\bullet \,], \Box(\Box s \to q)^\bullet, c^\circ} \\
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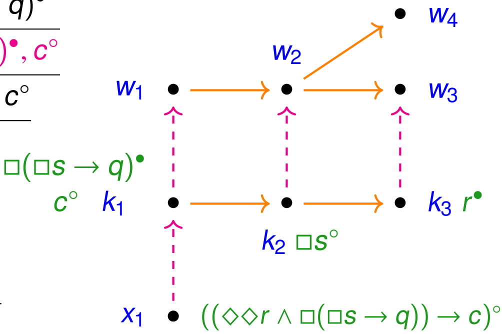
\end{array}
$$



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$$\Diamond^\bullet \frac{[\,[\, r^\bullet \,]\,], \Box(\Box s \to q)^\bullet, c^\circ}{[\, \Diamond r^\bullet \,], \Box(\Box s \to q)^\bullet, c^\circ}$$

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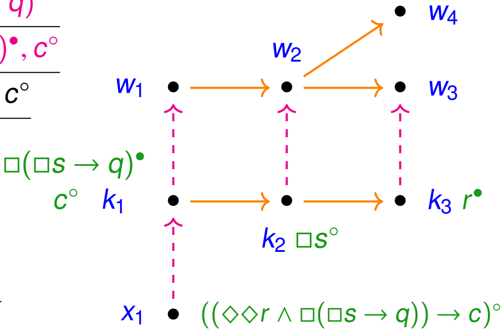

Main results: Completeness

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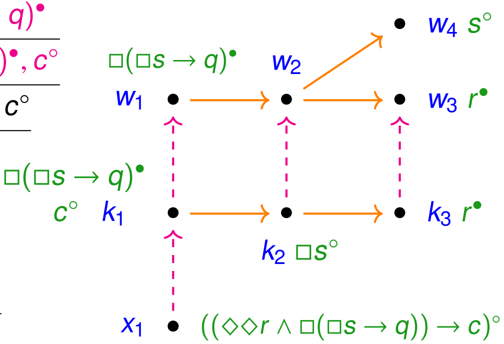

Outline

Labelled calculi for IK (and extensions)

[Marin, Morales and Straßburger, 2021]

☞ Relational atoms and labelled formulas

- $x \leq y \;\; \rightsquigarrow$ "y is accessible from x in the preorder"
- $xRy \;\; \rightsquigarrow$ "y is accessible from x"
- $x : A \;\; \rightsquigarrow$ "x satisfies A"

☞ Some labelled rules

Labelled calculi for IK (and extensions)

[Marin, Morales and Straßburger, 2021]

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  - $x \leq y \quad \rightsquigarrow$ "y is accessible from x in the preorder"
  - $xRy \quad \rightsquigarrow$ "y is accessible from x"
  - $x : A \quad \rightsquigarrow$ "x satisfies A"

- Some labelled rules

$$\text{init}\ \frac{}{x \leq y, \mathcal{R}, x : p, \Gamma \Rightarrow \Delta, y : p}$$

$$\Box_L\ \frac{x \leq y, xRy, \mathcal{R}, x : \Box A, z : A, \Gamma \Rightarrow \Delta}{x \leq y, xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta} \qquad \Box_R\ \frac{x \leq y, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta, z : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ \text{(y,z!)}$$

$$\text{F1}\ \frac{x'Ru, y \leq u, x \leq x', xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}{x \leq x', xRy, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \text{(u!)}$$

$$\text{F2}\ \frac{x \leq u, uRy', xRy, y \leq y', \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, y \leq y', \mathcal{R}, \Gamma \Rightarrow \Delta}\ \text{(u!)}$$