Course on Proof Theory - Lecture 1

Introduction to Propositional and First-order Logic

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From logic to proof theory

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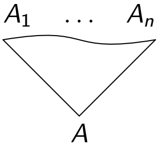

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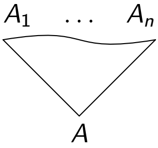

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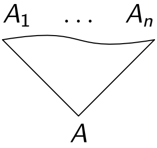

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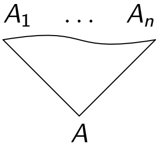

- ▷ Gödel's Incompleteness Theorem, 1931

  We cannot prove every true sentence in certain proof systems[1].

- ▷ Gentzen's *Hauptsatz* (Cut-elimination Theorem), 1934

  Every logical theorem can be proved analytically, that is, has a proof which uses only "elements" occurring in its statement.

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5. Beyond classical logic

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Language

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Formulas (Form_p) $A, B, C, \dots$ are inductively generated as follows:

- ▷ Propositional variables and constants are formulas
- ▷ If A, B are formulas then $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ are formulas.

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A	B	$A \wedge B$
1	1	1
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A	B	$A \vee B$
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A	B	$A \to B$
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Equivalently: Define $\alpha \models A$ "α satisfies A"

$$
\begin{array}{lll}
\alpha \not\models \bot & & \\
\alpha \models \top & & \\
\alpha \models p & \textit{iff} & \alpha(p) = 1 \\
\alpha \models \neg A & \textit{iff} & \alpha \not\models A \\
\alpha \models A \wedge B & \textit{iff} & \alpha \models A \text{ and } \alpha \models B \\
\alpha \models A \vee B & \textit{iff} & \alpha \models A \text{ or } \alpha \models B \\
\alpha \models A \to B & \textit{iff} & \alpha \not\models A \text{ or } \alpha \models B
\end{array}
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Satisfiability, semantic entailment and tautologies

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- A is a propositional tautology if $\varnothing \vDash A$. We write $\vDash A$.

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- ▷ Quantifiers: $\exists$ (*existential*) and $\forall$ (*universal*)

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Informally, terms denote individual entities.

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- If A is a formula then $\exists x A$ and $\forall x A$ are formulas.

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Formally, we define $\mathsf{FV}(A)$ by induction over the structure of a formula A:

$$\begin{aligned}
\mathsf{FV}(x) &:= \{x\} \\
\mathsf{FV}(f(t_1, \ldots, t_k)) = \mathsf{FV}(P(t_1, \ldots, t_k)) &:= \bigcup_{i=1}^{k} \mathsf{FV}(t_i) \\
\mathsf{FV}(s = t) &:= \mathsf{FV}(s) \cup \mathsf{FV}(t) \\
\mathsf{FV}(\neg A) &:= \mathsf{FV}(A) \\
\mathsf{FV}(A \star B) &:= \mathsf{FV}(A) \cup \mathsf{FV}(B) \qquad \text{for } \star \in \{\vee, \wedge, \rightarrow\} \\
\mathsf{FV}(\forall x.A) = \mathsf{FV}(\exists x.A) &:= \mathsf{FV}(A) \backslash \{x\}
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If $x \in \mathsf{FV}(A)$ then x is free in A. Otherwise it is bound in A.

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We write $A[t/x]$ for the operation of substituting all free occurrences of x with term t.

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A variable assignment $\sigma : \mathrm{Var} \to D$.

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Satisfiability relation for formulas $\mathcal{M}, \sigma \vDash A$ "$\mathcal{M}, \sigma$ satisfies A"

Valuation of terms and formulas

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$$\mathcal{M}, \sigma \vDash P(t_1, \ldots, t_k) \quad \textit{iff} \quad \langle [t_1]^{\sigma}_{\mathcal{M}}, \ldots, [t_k]^{\sigma}_{\mathcal{M}} \rangle \in P^{\mathcal{M}}$$

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Satisfiability relation for formulas $\mathcal{M}, \sigma \models A$ "$\mathcal{M}, \sigma$ satisfies A"

$$
\begin{array}{rcl}
\mathcal{M}, \sigma \models P(t_1, \ldots, t_k) & \textit{iff} & \langle [t_1]^{\sigma}_{\mathcal{M}}, \ldots, [t_k]^{\sigma}_{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\
\mathcal{M}, \sigma \models \neg A & \textit{iff} & \mathcal{M}, \sigma \not\models A \\
\mathcal{M}, \sigma \models A \wedge B & \textit{iff} & \mathcal{M}, \sigma \models A \text{ and } \mathcal{M}, \sigma \models B \\
\mathcal{M}, \sigma \models A \vee B & \textit{iff} & \mathcal{M}, \sigma \models A \text{ or } \mathcal{M}, \sigma \models B \\
\mathcal{M}, \sigma \models A \to B & \textit{iff} & \mathcal{M}, \sigma \not\models A \text{ or } \mathcal{M}, \sigma \models B
\end{array}
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\mathcal{M}, \sigma \vDash \exists x.A & \textit{iff} & \mathcal{M}, \sigma[x \mapsto d] \vDash A, \ \textbf{for some } d \in D \\
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[c]^\sigma_\mathcal{M} &:= c^\mathcal{M}
\end{aligned}
$$

Satisfiability relation for formulas $\mathcal{M}, \sigma \vDash A$ "$\mathcal{M}, \sigma$ satisfies A"

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\begin{aligned}
\mathcal{M}, \sigma \vDash P(t_1, \ldots, t_k) \quad & \textit{iff} \quad \langle [t_1]^\sigma_\mathcal{M}, \ldots, [t_k]^\sigma_\mathcal{M} \rangle \in P^\mathcal{M} \\
\mathcal{M}, \sigma \vDash \neg A \quad & \textit{iff} \quad \mathcal{M}, \sigma \nvDash A \\
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\end{aligned}
$$

$\sigma[x \mapsto d]$ denotes the assignment which agrees with σ for all variables which are different from x, and maps x to d.

Satisfiability, semantic entailment, validity

$\mathcal{M}, \sigma \vDash A$ "$\mathcal{M}, \sigma$ satisfies A"

Satisfiability, semantic entailment, validity

$$\mathcal{M}, \sigma \vDash A \qquad \text{“}\mathcal{M}, \sigma \text{ satisfies } A\text{”}$$

- A structure $\mathcal{M}$ models a formula A if $\mathcal{M}, \sigma \vDash A$ for **every** variable assignment σ. We write $\mathcal{M} \vDash A$.

Satisfiability, semantic entailment, validity

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- ▷ For $\Gamma = \{A_1, \ldots, A_n\}$ is a set of formulas, Γ semantically entails A if, for any structure $\mathcal{M}$ such that $\mathcal{M} \vDash A_i$ for all $A_i \in \Gamma$, then $\mathcal{M} \vDash A$. We write $\Gamma \vDash A$.

Satisfiability, semantic entailment, validity

$$\mathcal{M}, \sigma \vDash A \qquad \text{“}\mathcal{M}, \sigma \text{ satisfies } A\text{”}$$

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- For $\Gamma = \{A_1, \ldots, A_n\}$ is a set of formulas, Γ semantically entails A if, for any structure $\mathcal{M}$ such that $\mathcal{M} \vDash A_i$ for all $A_i \in \Gamma$, then $\mathcal{M} \vDash A$. We write $\Gamma \vDash A$.
- A sentence A is valid if $\varnothing \vDash A$. We write $\vDash A$.

What are proof systems?

- Deduction (or derivation) of a statement A from a (possibly empty) set of hypothesis (or assumptions) $\Gamma = \{A_1, \ldots, A_n\}$ can be seen as an argumentation construed by applying a series of "elementary" inference steps.
- Deduction gives us evidence that A follows "logically" from Γ.

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- Deduction gives us evidence that A follows "logically" from Γ.
- Proof systems: formal (syntactic) notion of deduction $\Gamma \vdash A$ to model logical (and mathematical) reasoning.
- The formal deducibility relation $\vdash$ should satisfy two desiderata:
  - Soundness: $\Gamma \vdash A \implies \Gamma \vDash A$
  - Completeness: $\Gamma \vDash A \implies \Gamma \vdash A$
- Soundness prevents fallacious reasoning, completeness guarantees that $\vdash$ is sufficiently powerful to model any valid reasoning.

Two approaches to proof systems

Hilbert-Frege proof systems:

- *Axiomatic approach*: logical laws are expressed as axioms, a few inference rules
- Makes easier to establish meta-properties (easy to prove things about the system) but ad hoc and artificial notion of deduction (hard to prove things within the system).

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Gentzen proof systems:

- *Inferential approach*: only inference rules (e.g. natural deduction, sequent calculus)
- Models logical and mathematical reasoning in a natural way (easy to prove things within the system) but less suitable for establishing meta-properties (hard to prove things about the system)

A simpler language

Semantic equivalences:

- $\mathcal{M}, \sigma \models \neg A \iff \mathcal{M}, \sigma \models A \to \bot$
- $\mathcal{M}, \sigma \models A \lor B \iff \mathcal{M}, \sigma \models \neg A \to B$
- $\mathcal{M}, \sigma \models A \land B \iff \mathcal{M}, \sigma \models \neg(\neg A \lor \neg B)$
- $\mathcal{M}, \sigma \models \top \iff \mathcal{M}, \sigma \models \neg\bot$
- $\mathcal{M}, \sigma \models \exists x.A \iff \mathcal{M}, \sigma \models \neg\forall x \neg A$ (Exercise)

Example:

$$
\begin{aligned}
\mathcal{M}, \sigma \models \neg(\neg A \lor \neg B) &\iff \mathcal{M}, \sigma \not\models \neg A \lor \neg B \\
&\iff \mathcal{M}, \sigma \not\models \neg A \quad \textbf{and} \quad \mathcal{M}, \sigma \not\models \neg B \\
&\iff \mathcal{M}, \sigma \models A \quad \textbf{and} \quad \mathcal{M}, \sigma \models B
\end{aligned}
$$

Language of predicate logic over the logical basis $\{\bot, \to, \forall\}$, using the abbreviation $\neg A := A \to \bot$

Hilbert-Frege proof system for predicate logic

Axioms of HF:

- HF1. $A \to (B \to A)$
- HF2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- HF3. $\neg\neg A \to A$
- HF4. $\forall x.A \to A[t/x]$
- HF5. $(\forall x.(A \to B)) \to (A \to \forall x.B)$ where $x \notin \mathrm{FV}(A)$
- HF6. $\forall x.(x = x)$
- HF7. $\forall x.\forall y.(x = y \to (A[x/z] \to A[y/z]))$

Inference rules of HF:

$$\text{mp}\ \frac{A \quad A \to B}{B} \qquad\qquad \text{gen}\ \frac{A}{\forall x.A}$$

Condition on gen: x **not free** in the assumptions A depends on.

Proofs and derivations

A derivation of A from Γ is a list of formulae $(A_1, \ldots, A_n)$ with $A_n = A$ (called the conclusion) such that each member of the sequence is either:

- an axiom;
- a hypothesis i.e. an element of Γ;
- obtained from previous formulae by an inference rule.

We write $\Gamma \vdash A$ if there exists a derivation of A from Γ.

A proof is a derivation from an empty set of premises. We write $\vdash A$ if there is a proof of A ("A is a theorem").

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Deduction theorem (Herbrand): $\vdash$ and $\rightarrow$ are "equivalent"

$$\Gamma \cup \{A\} \vdash B \iff \Gamma \vdash A \rightarrow B$$

NB: It allows us to move from derivations to proofs and vice versa

Propositional examples

Example: A proof of $\vdash A \to A$

1.	$A \to (B \to A) \to A$	HF1
2.	$(A \to (B \to A) \to A) \to (A \to (B \to A)) \to A \to A$	HF2
3.	$(A \to B \to A) \to A \to A$	mp(1, 2)
4.	$A \to B \to A$	HF1
5.	$A \to A$	mp(4, 3)

Propositional examples

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4.	$A \to B \to A$	HF1
5.	$A \to A$	mp$(4, 3)$

Example: A derivation of $\{A \to B, B \to C\} \vdash A \to C$

1.	$B \to C$	hyp
2.	$(B \to C) \to A \to (B \to C)$	HF1
3.	$A \to (B \to C)$	mp$(1, 2)$
4.	$(A \to (B \to C)) \to (A \to B) \to (A \to C)$	HF2
5.	$(A \to B) \to (A \to C)$	mp(3,4)
6.	$A \to B$	hyp
7.	$A \to C$	mp$(6, 5)$

An example with quantifiers

Example: A derivation of $\{\forall x.(A \rightarrow B), \forall x.A\} \vdash \forall x.B$

1.	$\forall x.(A \rightarrow B)$	hyp
2.	$\forall x.(A \rightarrow B) \rightarrow A \rightarrow B$	HF4
3.	$A \rightarrow B$	$\text{mp}(1,2)$
4.	$\forall x.A$	hyp
5.	$\forall x.A \rightarrow A$	HF4
6.	A	$\text{mp}(4,5)$
7.	B	$\text{mp}(3,6)$
8.	$\forall x.B$	$\text{gen}(7)$

An example with quantifiers

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1.	$\forall x.(A \to B)$	hyp
2.	$\forall x.(A \to B) \to A \to B$	HF4
3.	$A \to B$	mp(1, 2)
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5.	$\forall x.A \to A$	HF4
6.	A	mp(4, 5)
7.	B	mp(3, 6)
8.	$\forall x.B$	gen(7)

NB: $\{\forall x.(A \to B), A\} \vdash \forall x.B$ does **not** hold

1.	$\forall x.(A \to B)$	hyp
2.	$\forall x.(A \to B) \to A \to B$	HF4
3.	$A \to B$	mp(1, 2)
4.	A	hyp
5.	B	mp(3, 6)
6.	$\forall x.B$	???

Theories and models

A (first-order) theory T consists of:

- A language of predicate logic $\mathcal{L}_\mathsf{T}$
- A set of formulas Γ_T in the language $\mathcal{L}_\mathsf{T}$ called (non-logical) axioms.

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A model of a theory T is a structure (for $\mathcal{L}_\mathsf{T}$) $\mathcal{M}$ such that $\mathcal{M} \vDash A$ for all $A \in \Gamma_\mathsf{T}$. We write $\mathcal{M} \vDash \Gamma_\mathsf{T}$ in this case.

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Example The theory G

- $\mathcal{L}_\mathsf{G}$ = constant e, unary function symbol i, binary function symbol $\circ$.
- The axioms are the following:

$$\forall x.(x \circ e = x)$$

$$\forall x.(x \circ i(x) = e)$$

$$\forall x.\forall y.\forall z.(x \circ (y \circ z) = (x \circ y) \circ z)$$

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NB: $\mathcal{M}$ is a model of G precisely when $\mathcal{M}$ is a group!

The theory of Peano arithmetic

Peano Arithmetic (PA) = first-order theory of natural numbers and arithmetical operations.

The theory PA. $\mathcal{L}_{\mathrm{PA}}$ = constant 0 (zero), unary function symbol s (successor), binary function symbols $+$ (addition) and $\cdot$ (multiplication).

The theory of Peano arithmetic

Peano Arithmetic (PA) = first-order theory of natural numbers and arithmetical operations.

The theory PA. $\mathcal{L}_{\mathrm{PA}}$ = constant 0 (zero), unary function symbol s (successor), binary function symbols $+$ (addition) and $\cdot$ (multiplication).

- PA1. $\forall x.\neg(0 = s(x))$
- PA2. $\forall x.\forall y.(s(x) = s(y) \to x = y)$
- PA3. $\forall x.(x + 0 = x)$
- PA4. $\forall x.\forall y.(x + s(y) = s(x + y))$
- PA5. $\forall x.(x \cdot 0 = 0)$
- PA6. $\forall x.\forall y.(x \cdot s(y) = (x \cdot y) + x)$
- Induction scheme: $A[0/x] \to \forall x.(A \to A[s(x)/x]) \to \forall x.A$

NB: Induction scheme = infinite axioms, one for each $A \in \mathcal{L}_{\mathrm{PA}}$.

Proving $\forall x.(0 + x = x)$

Induction scheme with $A :\equiv 0 + x = x$:

$$\underbrace{0 + 0 = 0}_{A[0/x]} \rightarrow \forall x(\underbrace{0 + x = x}_{A} \rightarrow \underbrace{0 + s(x) = s(x)}_{A[s(x)/x]}) \rightarrow \forall x(\underbrace{0 + x = x}_{A})$$

Goal: show $\vdash A[0/x]$ and $\vdash \forall x.(A \rightarrow A[s(x)/x])$.

1.	$\forall x.(x + 0 = x)$	PA3
2.	$\forall x.(x + 0 = x) \rightarrow 0 + 0 = 0$	HF4
3.	$\underbrace{0 + 0 = 0}_{A[0/x]}$	mp$(1, 2)$

Proving $\forall x.(0 + x = x)$ continued

1.	$\underbrace{B[x/z]}_{s(x)=s(x)} \to (0 + x = x) \to B[0 + x/z]$	HF7, $B := s(z) = s(x)$
2.	$s(x) = s(x)$	HF4, HF7
3.	$(0 + x = x) \to \underbrace{B[0 + x/z]}_{s(0+x)=s(x)}$	mp(1, 2)
4.	$\underbrace{C[0 + x/z]}_{0+s(x)=s(0+x)} \to (s(0 + x) = s(x)) \to C[x/z]$	HF6, $C := 0 + s(x) = s(z)$
5.	$0 + s(x) = s(0 + x)$	PA4
6.	$s(0 + x) = s(x) \to \underbrace{C[x/z]}_{0+s(x)=s(x)}$	mp(4, 5)
7.	$\underbrace{(0 + x = x)}_{A} \to \underbrace{0 + s(x) = s(x)}_{A[s(x)/x]}$	transitivity of $\to$ (3,4)
8.	$\forall x.(A \to A[s(x)/x])$	gen(7)

Models of PA

Standard model of PA. $\mathcal{N}$ defined by:

- $D := \mathbb{N}$
- $0^{\mathcal{N}} := 0 \in \mathbb{N}$
- $s^{\mathcal{N}} := n \mapsto n + 1$
- $+^{\mathcal{N}} := (n, m) \mapsto n + m$
- $\cdot^{\mathcal{N}} := (n, m) \mapsto n \cdot m$

Theorem $\mathcal{N} \vDash \mathrm{PA}$

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Theorem $\mathcal{N} \models \mathrm{PA}$

Question: Is any model of PA **isomorphic**? Does PA "capture" structure of natural numbers?

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Theorem $\mathcal{N} \models \mathrm{PA}$

Question: Is any model of PA **isomorphic**? Does PA "capture" structure of natural numbers?

Answer: No, unfortunately! We can find non-standard models of PA:

- As application of the compactness theorem for first-order logic
- As application of Gödel's incompleteness theorem for PA

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Course on Proof Theory - Lecture 1

Exercises - Introduction to Propositional and First-order Logic

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Nottingham, 10-14 April 2022

Part 1: Propositional logic

1. Show $\vdash \bot \to A$ (*ex falso quodlibet*)

2. Show $\vdash (A \to (B \to C)) \to B \to A \to C$ (*exchange*).

3. We can extend HF to include conjunction $\wedge$ by adding the following axioms:

   $$\begin{array}{rl} \text{HF8.} & A \to B \to (A \wedge B) \\ \text{HF9.} & (A \wedge B) \to A \\ \text{HF10.} & (A \wedge B) \to B \end{array}$$

   Show $\vdash (A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$, where

   $$A \leftrightarrow B := (A \to B) \wedge (B \to A)$$

4. We haven't yet used the axiom (*neg*)! Show that HF proves:

   $$\begin{array}{ll} (a) & \neg A \to (A \to B) \\ (b) & (\neg B \to \neg A) \to (A \to B) \\ (c) & ((A \to B) \to A) \to A \end{array}$$

Hint for part 1: Use the deduction theorem to reduce any proof of $A \to B$ to a proof of B with hypothesis A.

Part 2: Predicate logic

1. We can extend HF to include existential quantifier $\exists$ by adding the following axioms:

$$
\begin{array}{lll}
\text{HF11.} & A[t/x] \to \exists x.A & \\
\text{HF12.} & \forall x.(A \to B) \to (\exists x.A \to B) & x \notin FV(B)
\end{array}
$$

   Show the following equivalences:

$$
\begin{array}{lll}
(a) & \mathcal{M}, \sigma \vDash \exists x.A \iff \mathcal{M}, \sigma \vDash \neg\forall x.\neg A & \\
(b) & \mathcal{M}, \sigma \vDash (\exists x.A \to B) \iff \mathcal{M}, \sigma \vDash \forall x.(A \to B) & x \notin FV(B) \\
(c) & \vdash (\exists x.A \to B) \leftrightarrow \forall x.(A \to B) & x \notin FV(B)
\end{array}
$$

2. Outline a first-order theory whose models are the partial orders. Adapt this theory to characterise
   - total orders e.g. $(\mathbb{Z}, \leq)$
   - total orders with a minimum element e.g. $(\mathbb{N}, \leq, 0)$
   - dense total orders e.g. $(\mathbb{Q}, \leq)$
   - Is it possible to characterise well-founded partial orders?