Course on Proof Theory - Lecture 2

Exercises - Soundness, completeness and other metalogical results

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Exercises for Lecture 2

1. Show that axiom HF5. is valid ($x \notin \mathrm{FV}(A)$):

$$(\forall x(A \to B)) \to (A \to \forall x B)$$

2. Show that the inference rule gen preserves validity:

$$\text{gen}\ \frac{A}{\forall x.A}$$

Exercises for Lecture 2

3. For a set of formulas Γ, if $\Gamma \not\models \bot$ then Γ is satisfiable. Why?

4. For a set of formulas Γ, show that the following are equivalent:
   - If $A \notin \Gamma$, then $\Gamma \cup A$ is not consistent.
   - For all formulas A, either $A \in \Gamma$ or $A \notin \Gamma$

5. Show that the following forms of consistency are equivalent:
   - $\Gamma \nvdash \bot$
   - There is no formula A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$
   - There is a formula A such that $\Gamma \nvdash A$

6. In Lindenbaum Lemma, prove that every Γ_k in the construction is consistent.

Exercises for Lecture 2

7. **(Difficult)** A graph is a structure $\mathfrak{G} = \langle V, E \rangle$ such that $E \subseteq V \times V$. For $k \in \mathbb{N}$ we say that $\mathfrak{G}$ is k-colourable if there is a function $c : V \to \{1, \ldots, k\}$ such that, whenever $(u, v) \in E$ then $c(u) \neq c(v)$.

   Using the compactness theorem, show that a (possibly infinite) graph is k-colourable iff every finite subgraph of it is k-colourable.