Course on Proof Theory - Lecture 4

A proof of the cut-elimination theorem

Gianluca Curzi, Marianna Girlando

University of Birmingham

Midlands Graduate School
Nottingham, 10-14 April 2022

The rules of LK

$$\text{init}\ \frac{}{p, \Gamma \vdash \Delta, p} \qquad \text{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

$$\neg_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \neg_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad \wedge_{\mathbf{L}}\ \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{\mathbf{L}}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \to_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta} \qquad \to_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}$$

$$\forall_{\mathbf{L}}\ \frac{A[t/x], \forall x.A, \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta} \qquad \forall_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}\ * \qquad \exists_{\mathbf{L}}\ \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}\ * \qquad \exists_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A[t/x], \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

$*$ y does not occur free in Γ, Δ, A

Derivation example

Is $(a \to b) \to ((\overset{\neg a}{a \to \bot}) \vee b)$ a theorem of LK?

$$\text{init}\ \frac{}{p, \Gamma \vdash \Delta, p} \qquad \left\{ \text{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \right\}$$

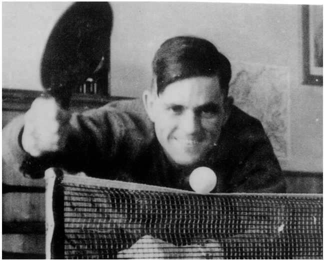
$$\vee_{\mathbf{L}}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \quad \vee_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \quad \to_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta} \quad \to_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}$$

$c = (a \wedge (b \to \bot)) \to \bot$

$$\frac{\text{init}\ \dfrac{}{a \vdash \bot, b, a} \qquad \text{init}\ \dfrac{}{a, b \vdash b}}{\dfrac{a \to b, a \vdash \bot, b}{\dfrac{a \to b \vdash a \to \bot, b}{\dfrac{a \to b \vdash (a \to \bot) \vee b}{\vdash (a \to b) \to ((a \to \bot) \vee b)}\ \to R}\ \vee R}\ \to R}$$

$$\frac{\dfrac{a \to b \vdash c \qquad c \vdash (a \to \bot) \vee b}{a \to b \vdash (a \to \bot) \vee b}\ \text{cut}}{\vdash (a \to b) \to ((a \to \bot) \vee b)}$$

Today goal



Theorem (*Hauptsatz*, Gentzen 1934)

Every theorem of LK *has a proof that does not use the cut rule.*

Corollary (Analyticity)

Every theorem of LK *has a proof that contains only subformulas of it (up to substitution of free variables).*

Informal example 1

$$\dfrac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\text{wk}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash \Delta, A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash \Delta, B\end{array}}{\Gamma \vdash \Delta, {\color{red}A \wedge B}}\,\wedge_R \qquad \dfrac{\begin{array}{c}\mathcal{D}_3\\ A, B, \Gamma \vdash \Delta\end{array}}{{\color{red}A \wedge B}, \Gamma \vdash \Delta}\,\wedge_L}{\Gamma \vdash \Delta}\,\text{cut}\ {\color{red}*}$$

Let's eliminate the occurrence of cut marked by $*$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash \Delta, \underline{A}\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash \Delta, B\end{array}}{A, \Gamma \vdash \Delta, \underline{B}}\,\text{wk} \qquad \begin{array}{c}\mathcal{D}_3\\ A, \underline{B}, \Gamma \vdash \Delta\end{array}}{\underline{A}, \Gamma \vdash \Delta}\,\text{cut}}{\Gamma \vdash \Delta}\,\text{cut}$$

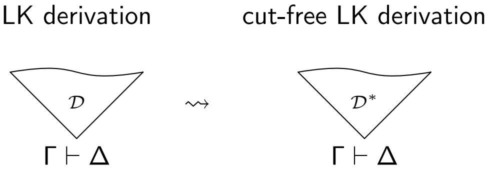
Informal example 2

$$
\dfrac{
  {\scriptstyle\forall_R}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash \Delta, B[x/y]}}{\Gamma \vdash \Delta, \forall x B}
  \qquad
  {\scriptstyle\forall_L}\,\dfrac{\dfrac{\mathcal{D}_2}{B[x/t], \forall x.B, \Gamma \vdash \Delta}}{\forall x.B, \Gamma \vdash \Delta}
}{\Gamma \vdash \Delta}\,{\scriptstyle\text{cut}}\ *
$$

Let's eliminate the occurrence of cut marked by $*$

$$
\dfrac{
  {\scriptstyle\text{subst}}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash \Delta, B[x/y]}}{\Gamma \vdash \Delta, B[x/t]}
  \qquad
  {\scriptstyle\text{cut}}\,\dfrac{
    {\scriptstyle\text{wk}}\,\dfrac{{\scriptstyle\forall_R}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash \Delta, B[x/y]}}{\Gamma \vdash \Delta, \forall x B}}{B[x/t], \Gamma \vdash \Delta, \forall x.B}
    \qquad
    \dfrac{\mathcal{D}_2}{B[x/t], \forall x.B, \Gamma \vdash \Delta}
  }{B[x/t], \Gamma \vdash \Delta}
}{\Gamma \vdash \Delta}\,{\scriptstyle\text{cut}}
$$

General strategy of the proof



- Apply the cut on smaller formulas, until they disappear
- Push the cuts upwards in the proof, and deal with them using IH
- We need a "measure" on formulas and on derivations, to ensure that the cut-elimination procedure terminates.

... The cut-elimination proof is quite complex.

We are going to sketch the proof for propositional LK (no quantifiers rules).

References

Several proofs of cut-elimination exist in the literature, using slightly different procedures and for slightly different systems:

- [Buss, 1998]. *Handbook of Proof Theory*.
- [Negri and von Plato, 2001]. *Structural Proof Theory*.
- [Sørensen, Urzyczyn]. *Lectures on the Curry-Howard Isomorphism*.
- [Troelstra and Schwichtenberg, 1996]. *Basic Proof Theory*.
- ...

A measure of formulas

The degree of a formula A, $\mathrm{deg}(A)$, is the number of logical connectives occurring in it.

Inductive definition on the structure of the formula:

$$
\begin{aligned}
\mathrm{deg}(p) &:= 0 \\
\mathrm{deg}(\neg A) &:= \mathrm{deg}(A) + 1 \\
\mathrm{deg}(A \star B) &:= \mathrm{deg}(A) + \mathrm{deg}(B) + 1 \qquad \text{for } \star \in \{\wedge, \vee, \rightarrow\}
\end{aligned}
$$

A measure of derivations

The height of $\mathcal{D}$, $\mathrm{ht}(\mathcal{D})$, is the length of its longest branch, minus one.

The rank of $\mathcal{D}$, $\mathrm{rk}(\mathcal{D})$, is the maximal degree of the cut formulas occurring in $\mathcal{D}$, plus 1.

$$\dfrac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathit{cut}$$

We write

$$\boxed{\Gamma \vdash^{m}_{p} \Delta}$$

meaning

> *There is a derivation of $\Gamma \vdash \Delta$ of height **at most** m and rank **at most** p.*

A measure of derivations (more formally)

Height and rank can be inductively defined on the structure of $\mathcal{D}$:

$$\mathcal{D} = \text{init}\,\dfrac{}{\Gamma \vdash \Delta} \qquad \mathsf{ht}(\mathcal{D}) = \mathsf{rk}(\mathcal{D}) = 0$$

$$\mathcal{D} = \text{R}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \vdash \Delta_1\end{array}}{\Gamma \vdash \Delta} \qquad \mathsf{ht}(\mathcal{D}) = \mathsf{ht}(\mathcal{D}_1) + 1 \quad \mathsf{rk}(\mathcal{D}) = \mathsf{rk}(\mathcal{D}_1)$$

$$\mathcal{D} = \text{R}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \vdash \Delta_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2 \vdash \Delta_2\end{array}}{\Gamma \vdash \Delta} \qquad \begin{array}{l}\mathsf{ht}(\mathcal{D}) = \max(\mathsf{ht}(\mathcal{D}_1), \mathsf{ht}(\mathcal{D}_2)) + 1\\ \mathsf{rk}(\mathcal{D}) = \max(\mathsf{rk}(\mathcal{D}_1), \mathsf{rk}(\mathcal{D}_2))\end{array}$$

$$\mathcal{D} = \text{cut}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash \Delta, A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ A, \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \Delta} \qquad \begin{array}{l}\mathsf{ht}(\mathcal{D}) = \max(\mathsf{ht}(\mathcal{D}_1), \mathsf{ht}(\mathcal{D}_2)) + 1\\ \mathsf{rk}(\mathcal{D}) = \max(\mathsf{rk}(\mathcal{D}_1), \mathsf{rk}(\mathcal{D}_2), \underline{\mathsf{deg}(A) + 1})\end{array}$$

Some preliminary lemmas

1. Lemma: Closure under weakening

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

 - ▷ If $\Gamma \vdash_p^m \Delta$, then $\Gamma', \Gamma \vdash_p^m \Delta, \Delta'$, for any Γ', Δ'.

Proof. Easy induction on the height m of the derivation.

2. Lemma: Invertibility All the rules are invertible:
 - $(\wedge_L)$ If $A \wedge B, \Gamma \vdash_p^m \Delta$, then $A, B, \Gamma \vdash_p^m \Delta$.
 - $(\wedge_R)$ If $\Gamma \vdash_p^m \Delta, A \wedge B$, then $\Gamma \vdash_p^m \Delta, A$ and $\Gamma \vdash_p^m \Delta, A$.
 - (…and so on for all the rules)

$$\frac{A, B, \Gamma \vdash \Delta}{\Rightarrow A \wedge B, \Gamma \vdash \Delta} \quad \text{inv.} \quad A, B, \Gamma \vdash \Delta$$

Proof. Induction on m, using closure under weakening.

3. Lemma: Closure under contraction

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

 - ▷ If $A, A, \Gamma \vdash_p^m \Delta$, then $A, \Gamma \vdash_p^m \Delta, \cancel{A}$.
 - ▷ If $\Gamma \vdash_p^m \Delta, A, A$, then $\Gamma \vdash_p^m \Delta, A$.

Proof. Induction on m, using invertibility.

NB: all the above preserve height and rank of the derivation.

The plan

- Principal Lemma (most of the work)

  If $\Gamma \vdash_p^m \Delta, A$ and $A, \Gamma \vdash_p^n \Delta$ for $p = \mathrm{deg}(A)$, then $\Gamma \vdash_p^{m+n} \Delta$.

- Reduction Lemma (uses PrL)

  If $\Gamma \vdash_{p+1}^m \Delta$, then $\Gamma \vdash_p^{2^m} \Delta$.

- Cut-elimination Theorem (uses RedL)

  If $\Gamma \vdash_p^m \Delta$ then $\Gamma \vdash_0^{2_p(m)} \Delta$.

Principal Lemma

Lemma Let $\Gamma \vdash_{p}^{m} \Delta, A$ and $A, \Gamma \vdash_{p}^{n} \Delta$ with $p = \deg(A)$:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash_{p}^{m} \Delta, A} \qquad \dfrac{\mathcal{D}_2}{A, \Gamma \vdash_{p}^{n} \Delta}}{\Gamma \vdash_{p+1}^{\max(m,n)+1} \Delta}$$

Then, we can construct a derivation $\Gamma \vdash_{p}^{m+n} \Delta$:

$$\dfrac{\mathcal{D}}{\Gamma \vdash_{p}^{m+n} \Delta}$$

Proof of the Principal Lemma

$$\mathcal{D}_1 \left\{ \begin{array}{c} \mathcal{D}_1 \\ \mathrm{R}_1 \dfrac{\Gamma' \vdash_p^{m1} \Delta'}{\Gamma \vdash_p^{m} \Delta, A} \end{array} \right. \qquad \left. \begin{array}{c} \mathcal{D}_2 \\ \mathrm{R}_2 \dfrac{\Gamma'' \vdash_p^{n1} \Delta''}{A, \Gamma \vdash_p^{n} \Delta} \end{array} \right\} \mathcal{D}_2$$

Induction on $m + n$. We distinguish cases:

1. R_1 is init (R$_2$ is init, symmetric)
2. A is principal in both R_1 and R_2
3. A is not principal in R_1 (A is not principal in R_2, symmetric)

R_1 is init

$$\mathcal{D}_1 = \dfrac{}{\underbrace{A, \Gamma'}_{\Gamma} \vdash_p^m \Delta, A}\ \text{init} \qquad\qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma'' \vdash_p^{n-1} \Delta''}}{A, \Gamma \vdash_p^n \Delta}\ \mathsf{R}_2$$

with $\Gamma = A, \Gamma'$ We construct the following derivation $\mathcal{D}$ of $\Gamma \vdash_p^{m+n} \Delta$:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{\Gamma'' \vdash_p^{n-1} \Delta''}}{A, A, \Gamma' \vdash_p^n \Delta}\ R_2}{\underbrace{A, \Gamma'}_{\Gamma} \vdash_p^n \Delta}\ c_1$$

$$\Gamma \vdash_p^n \Delta \qquad\qquad n \leq m + n$$

A is principal in both R_1 and R_2

R_1 is $\to_\mathsf{R}$ and R_2 is $\to_\mathsf{L}$

$$\dfrac{\overset{\mathcal{D}_1}{B, \Gamma \vdash_p^{m-1} \Delta, C}}{\Gamma \vdash_p^{m} \Delta, B \to C}\ \to_\mathsf{R} \qquad\qquad \dfrac{\overset{\mathcal{D}_2'}{\Gamma \vdash_p^{n1} \Delta, B} \qquad \overset{\mathcal{D}_2''}{C, \Gamma \vdash_p^{n2} \Delta}}{B \to C, \Gamma \vdash_p^{n} \Delta}\ \to_\mathsf{L}$$

with $n1, n2 < n$ and $n, m \geq 1$.

We construct the following derivation $\mathcal{D}$ of $\Gamma \vdash_p^{m+n} \Delta$:

$$k = \max(n1, m-1) + 1 \leq \max(m, n)$$

$$k' = \max(k, n2) + 1 \leq \max(m, n) + 1 \leq m + n$$

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_2'}{\Gamma \vdash_p^{n1} \Delta, B}}{\Gamma \vdash_p^{n1} \Delta, B, C}\ \mathrm{wk} \qquad \overset{\mathcal{D}_1}{B, \Gamma \vdash_p^{m-1} \Delta, C}}{\Gamma \vdash_p^{k} \Delta, C}\ \mathrm{cut} \qquad \overset{\mathcal{D}_2''}{C, \Gamma \vdash_p^{n2} \Delta}}{\Gamma \vdash_p^{k'} \Delta}\ \mathrm{cut}$$

A is not principal in R_1

R_1 is a one-premiss rule

$$
\begin{array}{c}\mathcal{D}_1\\ \vdots\end{array}
\qquad\qquad\qquad
\begin{array}{c}\mathcal{D}_2\\ \vdots\end{array}
$$

$$
R_1\ \dfrac{\Gamma' \vdash_p^{m-1} \Delta', A}{\Gamma \vdash_p^{m} \Delta, A}
\qquad\qquad
R_2\ \dfrac{\Gamma'' \vdash_p^{n-1} \Delta''}{A, \Gamma \vdash_p^{n} \Delta}
$$

We construct the following derivation $\mathcal{D}$ of $\Gamma \vdash_p^{m+n} \Delta$:

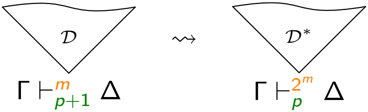
$$
\text{ctr}\ \dfrac{R_1\ \dfrac{\text{IH}\ \dfrac{\text{wk}\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma' \vdash_p^{m-1} \Delta', A\end{array}}{\Gamma', \Gamma \vdash_p^{m-1} \Delta, \Delta', A} \qquad \text{wk}\ \dfrac{R_2\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma'' \vdash_p^{n-1} \Delta''\end{array}}{A, \Gamma \vdash_p^{n} \Delta}}{A, \Gamma', \Gamma \vdash_p^{n} \Delta, \Delta'}}{\Gamma', \Gamma \vdash_p^{(m-1)+n} \Delta, \Delta'}}{\Gamma, \Gamma \vdash_p^{m+n} \Delta, \Delta}}{\Gamma \vdash_p^{m+n} \Delta}
$$

R_1 is a two-premisses rule ...

End of the proof of PrL □

Reduction Lemma

Reduction Lemma If $\Gamma \vdash^{m}_{p+1} \Delta$, we can construct $\Gamma \vdash^{2^m}_{p} \Delta$.



Proof. Induction on m

Base case Just set $\mathcal{D} = \mathcal{D}^*$

Induction step Case distinction according to the last rule R applied in $\mathcal{D}$.

We show just one case: R is cut.

Proof of the Reduction Lemma: key case

The last rule R applied in $\mathcal{D}$ is cut, with $\deg(A) = p$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash^{m1}_{p+1} \Delta, A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ A, \Gamma \vdash^{m2}_{p+1} \Delta\end{array}}{\Gamma \vdash^{m}_{p+1} \Delta}\ \text{cut} \qquad \rightsquigarrow \qquad \begin{array}{c}\mathcal{D}^*\\ \Gamma \vdash^{2^m}_{p} \Delta\end{array}$$

We construct $\mathcal{D}^*$ as follows:

Since $m1, m2 < m$, by IH, we have:

$$\dfrac{\begin{array}{c}\mathcal{D}_1^*\\ \Gamma \vdash^{2^{m1}}_{p} \Delta, A\end{array} \qquad \begin{array}{c}\mathcal{D}_2^*\\ A, \Gamma \vdash^{2^{m2}}_{p} \Delta\end{array}}{\Gamma \vdash^{2^{m1}+2^{m2}}_{p} \Delta}\ \text{PrL}$$

$$2^{m1} + 2^{m2} \leq 2^{m-1} + 2^{m-1} = 2\left(2^{m-1}\right) = 2^m$$

We have constructed $\mathcal{D}^*$ of $\Gamma \vdash^{2^m}_{p} \Delta$. End of the proof of RedL $\square$

Cut-elimination Theorem

Cut-elimination Theorem If $\Gamma \vdash^m_p \Delta$, we can construct $\Gamma \vdash^{2_p(m)}_{\underline{0}} \Delta$, that is, a derivation **where** cut **does not occur.**

$$\mathcal{D} \;\; \Gamma \vdash^m_p \Delta \quad \rightsquigarrow \quad \mathcal{D}^* \;\; \Gamma \vdash^{2_p(m)}_0 \Delta$$

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^m}}}}}}_{p \text{ times}}$$

Proof. Induction on p

Base case $p = 0$ Just set $\mathcal{D} = \mathcal{D}^*$

Induction step $p = \underline{k+1}$

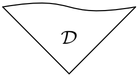
$$2_k(2^m) = \underbrace{2^{2^{\cdot^{\cdot^{2^{2^m}}}}}}_{k} = 2_{k+1}(m)$$

$$\mathcal{D} \;\; \Gamma \vdash^m_{k+1} \Delta \quad \overset{Red.Lemma}{\rightsquigarrow} \quad \mathcal{D}^* \;\; \Gamma \vdash^{2^m}_{\underline{k}} \Delta \quad \overset{I.H.}{\rightsquigarrow} \quad \mathcal{D}^* \;\; \Gamma \vdash^{2_k(2^m)}_0 \Delta$$

□

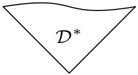
Putting it all together

LK derivation — cut-free LK derivation



$\Gamma \vdash_{p}^{m} \Delta$

$\rightsquigarrow$



$\Gamma \vdash_{0}^{2_p(m)} \Delta$

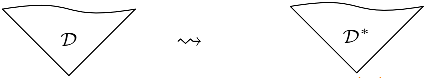
Proof sketch of the Cut-elimination Theorem

By induction on the rank od a proof:

- Identify the cuts of highest rank, say $p + 1$, in the proof, and apply the Reduction Lemma to them.
- The Principal Lemma ensures that we might only introduce cuts of rank at most p in the process.
- Thus, the rank of the derivation decreases to p.
- We may conclude by the inductive hypothesis.

The cost of cut-elimination

LK derivation | cut-free LK derivation



$$\Gamma \vdash^{m}_{p} \Delta \quad \rightsquigarrow \quad \Gamma \vdash^{2_p(m)}_{0} \Delta$$

Eliminating cuts from a propositional LK derivation leads to a hyperexponential blow-up of the size of the proof.

Can we do better?

- For propositional LK: yes, we can get an exponential bound in proof size.
- For full LK (with quantifiers rules): no.

  Theorem (Statman '79, Orevkov '82). Cut-elimination for predicate logic necessarily has a non-elementary cost in proof size.

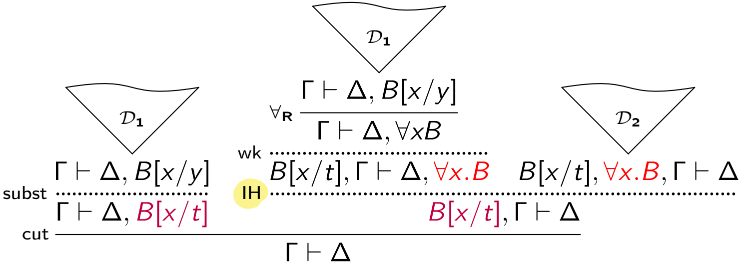
Cut-elimination for predicate logic

The method presented before can be extended to full LK, modulo assuming a renaming of variables and the following:

Substitution Lemma If $\Gamma \vdash_p^m \Delta$, then for each x variable and t term, $\Gamma[x/t] \vdash_p^m \Delta[x/t]$.

Proof. Easy induction on p.

The case of Principal Lemma in which the cut formula is $\forall x B$ and is principal in both subderivations is the following:



Summing up: is it worth to eliminate cuts?

Drawbacks:

- Exponential or hyper-exponential blow-up of proof size w.r.t. input size
- Headache proof

Benefits:

- Analyticity: automated proof search
- Consistency
- Interpolation
- Herbrand's theorem
- ...

References

- [Buss, 1998]. *Handbook of Proof Theory*.
- [Negri and von Plato, 2001]. *Structural Proof Theory*.
- [Sørensen, Urzyczyn]. *Lectures on the Curry-Howard Isomorphism*.
- [Troelstra and Schwichtenberg, 1996]. *Basic Proof Theory*.

Exercises for Lecture 4

1. Prove that the following are derivable:
   - ▷ $(a \to b) \vdash ((a \wedge (b \to \bot)) \to \bot)$
   - ▷ $((a \wedge (b \to \bot)) \to \bot) \vdash ((a \to \bot) \vee b)$

2. Prove the following cases of the Principal Lemma:
   - ▷ The cut formula is $B \vee C$ and it is principal in both derivations:

$$\vee_{\mathbf{R}}\ \dfrac{\begin{array}{c}\mathcal{D}_{\mathbf{1}}\\ \Gamma \vdash_p^{m-1} \Delta, B, C\end{array}}{\Gamma \vdash_p^{m} \Delta, B \vee C} \qquad\qquad \vee_{\mathbf{L}}\ \dfrac{\begin{array}{c}\mathcal{D}_{\mathbf{2}}'\\ B, \Gamma \vdash_p^{n1} \Delta\end{array} \quad \begin{array}{c}\mathcal{D}_{\mathbf{2}}''\\ C, \Gamma \vdash_p^{n2} \Delta\end{array}}{B \vee C, \Gamma \vdash_p^{n} \Delta}$$

Exercises for Lecture 4

- The cut formula is A and it is principal in the leftmost derivation, with $\deg(C) < p$:

$$\text{cut}\ \dfrac{\begin{array}{c}\mathcal{D}'_1\\ \Gamma \vdash^{m1}_{p} \Delta, A, C\end{array} \qquad \begin{array}{c}\mathcal{D}''_1\\ C, \Gamma \vdash^{m2}_{p} \Delta, A\end{array}}{\Gamma \vdash^{m}_{p} \Delta, A} \qquad\qquad \text{R}_1\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ C, \Gamma' \vdash^{n-1}_{p} \Delta'\end{array}}{A, \Gamma \vdash^{n}_{p} \Delta}$$

- The cut formula is $\forall x B$ and it is principal in both R_1 and R_2

$$\forall_{\mathbf{R}}\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash \Delta, B[x/y]\end{array}}{\Gamma \vdash \Delta, \forall x B} \qquad\qquad \forall_{\mathbf{L}}\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ B[x/t], \forall x.B, \Gamma \vdash \Delta\end{array}}{\forall x.B, \Gamma \vdash \Delta}$$