Course on Proof Theory - Lecture 3

Gentzen's sequent calculus

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How mathematicians prove theorems

Proof of a theorem T by introducing lemmas $L_1, \dots, L_n$:

$$\frac{\overset{?}{\vdash L_1} \quad \dots \quad \overset{?}{\vdash L_n} \qquad L_1, \dots, L_n \vdash T}{T}$$

- ▷ Two approaches in proof search:

	pros	cons
mathematician	short & clever	"guess" right lemmas
computer scientist	algorithmic	long & tedious

The problem with proof search in HF

- Hilbert-Frege systems are closer to mathematicians' approach to proof search:

$$\frac{\vdash L \quad L \vdash T}{\vdash T} \quad \underset{\text{deduction thm}}{\Longleftrightarrow} \quad \text{mp}\,\frac{L \quad L \to T}{T}$$

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- **Key concept** in proof theory is analyticity (Bolzano, 1781-1848):

  "A proof is analytic if it does not use concepts beyond its subject matter"

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- **Problem**: Find alternative proof systems that trade complexity for a more algorithmic treatment of proof search.

- **Key concept** in proof theory is analyticity (Bolzano, 1781-1848):

  "A proof is analytic if it does not use concepts beyond its subject matter"

- In proof theory analyticity is implemented by proof systems that satisfy the subformula property:

  Proofs of T can be construed using only subformulas of T

Gentzen's sequent calculus

Three fundamental steps to sequent calculus:

- ▷ From axioms to rules:

$$(A \wedge B) \rightarrow A \qquad \rightsquigarrow \qquad \frac{A \wedge B}{A}$$

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Three fundamental steps to sequent calculus:

- From axioms to rules:

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- From formulas to sequents: manipulate directly the meta-level deducibility relation $\Gamma \vdash A$

- From one to multiple conclusions: full symmetry between hypothesis and conclusions $\Gamma \vdash \Delta$ for expressing classical logic dualities

$$\Gamma, A \vdash \Delta \qquad \Longleftrightarrow \qquad \Gamma \vdash \neg A, \Delta$$

NB: Crucial use of the deduction theorem!

Sequents

- A sequent is an expression $\Gamma \vdash \Delta$, where Γ, Δ are multisets of formulae.
- A sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ can be interpreted as:

$$\bigwedge_{i=1}^{n} A_i \rightarrow \bigvee_{j=1}^{m} B_j$$

where $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

- Intuitively:

$$\begin{array}{rcccl} \Gamma \vdash & \iff & \Gamma \vdash \bot & \iff & \Gamma \text{ inconsistent} \\ \vdash \Delta & \iff & \top \vdash \Delta & \iff & \text{some formulas in } \Delta \text{ are true} \end{array}$$

The rules of LK

$$\text{init}\ \frac{}{A, \Gamma \vdash \Delta, A}$$

$$\neg_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \neg_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\wedge_{\mathbf{L}}\ \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{\mathbf{L}}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\to_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta} \qquad \to_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}$$

$$\text{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

The rules identity and cut

$$\text{init}\,\frac{}{A, \Gamma \vdash \Delta, A} \qquad\qquad \text{cut}\,\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

- The initial sequent mixes two principles:
  - A can be derived assuming A, i.e. $A \to A$
  - Weakening the reasoning:

$$\Gamma \vdash \Delta \qquad \Longleftrightarrow \qquad \Gamma, A \vdash \Delta \qquad \text{and} \qquad \Gamma \vdash A, \Delta$$

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- The cut rule corresponds to *modus ponens* via deduction theorem:

$$\text{cut}\ \frac{\vdash L \quad L \vdash T}{\vdash T} \qquad \underset{\text{deduction thm}}{\Longleftrightarrow} \qquad \text{mp}\ \frac{L \quad L \to T}{T}$$

Propositional rules

$$\neg_L \; \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \neg_R \; \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\wedge_L \; \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_R \; \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_L \; \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_R \; \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\rightarrow_L \; \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta} \qquad \rightarrow_R \; \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

- Symmetries between left introduction and right introduction rules
- Hypothesis are shared by the premises of rules
- $\rightarrow_R \quad = \quad$ deduction theorem
- $\rightarrow_R$ and $\rightarrow_L$ derivable ($A \rightarrow B := \neg A \vee B$).

Quantifier rules

$$\forall_{\mathbf{L}}\ \frac{A[t/x], \forall x.A, \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta} \qquad \forall_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_{\mathbf{L}}\ \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta} \qquad \exists_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A[t/x], \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

- **Condition:** y does not occur free in Γ, Δ, A
- Symmetry between $\forall$ and $\exists$
- $\forall_L$ and $\exists_R$ extra copy of the quantified formula.

Propositional example: Pierce's law

$$
\dfrac{\dfrac{\dfrac{}{A \vdash A, B}\,\text{init}}{\vdash A, A \to B}\,{\to_{\mathbf{R}}} \qquad \dfrac{}{A \vdash A}\,\text{init}}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\,{\to_{\mathbf{R}}}}\,{\to_{\mathbf{L}}}
$$

NB : $\vdash A, A \to B$ has 2 conclusions. Classical principles rely on the symmetries of $\vdash$.

First-order example: drinker's paradox

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D(u), D(v) \vdash D(v), \forall y. D(y)}\ \text{init}}{D(u) \vdash D(v), D(v) \to \forall y. D(y)}\ {\to_{\mathbf{R}}}}{D(u) \vdash D(v), \exists x. (D(x) \to \forall y. D(y))}\ \exists_{\mathbf{R}}}{D(u) \vdash \forall x. D(x), \exists x. (D(x) \to \forall y. D(y))}\ \forall_{\mathbf{R}}}{\vdash D(u) \to \forall y. D(y), \exists x. (D(x) \to \forall y. D(y))}\ {\to_{\mathbf{R}}}}{\vdash \exists x. (D(x) \to \forall y. D(y))}\ \exists_{\mathbf{R}}$$

- "There is someone in the pub such that, if he is drinking, then everyone in the pub is drinking." (Smullyan, 1978)

Soundness and completeness of LK

Theorem. A formula A is derivable from hypotheses Γ in HF iff there is an LK proof of $\Gamma \vdash B$.

Proof idea. By structural induction (exercise). Fundamental role of deduction theorem. An example for the direction HF $\mapsto$ LK:

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma \vdash A \to B \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathsf{init} \qquad \dfrac{}{B \vdash B}\,\mathsf{init}}{A, A \to B \vdash B}\,{\to_{\mathsf{L}}}}{\Gamma, A \vdash B}\,\mathsf{cut}}{\Gamma \vdash B}\,\mathsf{cut}$$

Soundness and completeness. $\Gamma \vdash \Delta$ provable in LK if and only if $\Gamma \models \Delta$.

Structural rules

- Structural rules. "hidden" in our presentation, do not manipulate logical connectives, but the structure of sequents:

$$\mathit{c}_{\mathbf{L}}\,\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \mathit{c}_{\mathbf{R}}\,\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \quad \mathit{w}_{\mathbf{L}}\,\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \mathit{w}_{\mathbf{R}}\,\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

- Weakening. Formulas can be **discarded**. It corresponds to the implications:

$$A \to \top \qquad\qquad \bot \to A$$

- Contraction. Formula can be **reused** at will. It corresponds to the implications:

$$A \to (A \wedge A) \qquad\qquad (A \vee A) \to A$$

Two equivalent presentations of connectives

	multiplicative	additive
right	$\wedge_R \dfrac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B}$	$\wedge_R \dfrac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$
left	$\wedge_L \dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$	$\wedge_L \dfrac{\Gamma, A_i \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta}$

In presence of the structural rules the multiplicative and the additive formulation of $\wedge/\vee$ are equivalent:

$$\begin{tikzcd} \text{multiplicative} \arrow[r, Rightarrow, bend left, "\textit{contraction}"] & \text{additive} \arrow[l, Rightarrow, bend left, "\textit{weakening}"] \end{tikzcd}$$

Exercise: show the equivalence!

Gentzen's Hauptsatz

All rules of LK but cut satisfy the subformula property: only subformulae of the conclusion appear in premisses (up to substitution):

$$\wedge_{\mathbf{R}}\,\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad \forall_{\mathbf{L}}\,\frac{A[t/x], \forall x.A, \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta} \qquad \mathbf{cut}\,\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

The cut-formula A has to be guessed in proof search.

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The cut-formula A has to be guessed in proof search.

Theorem (*Hauptsatz*, Gentzen '34). Every sequent provable in LK has a proof that does not use the cut rule.

- Gentzen proved the above theorem by showing an effective procedure for eliminating the cut rule
- The procedure turns a clever and short proof into an algorithmic but long proof (non-elementary time).

Subformula property and consistency of LK

Cut-free proofs allow for automatic proof search:

Corollary (Subformula property) Every sequent $\Gamma \vdash \Delta$ provable in LK has a proof that contains only subformulae of Γ, Δ, up to substitution of variables for terms.

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A straightforward corollary of cut-elimination is consistency of LK:

Corollary (Consistency). For some formula A, $\vdash A$ is not provable in LK.

E.g.

$$\wedge_R \frac{\dfrac{???}{\vdash P(x)} \quad \dfrac{???}{\vdash \neg P(x)}}{\vdash P(x) \wedge \neg P(x)}$$

Perspectives

Applications of Gentzen's Hauptsatz:

- ▷ Automated reasoning: proof search & theorem proving
- ▷ Structural proof theory: metalogical results
- ▷ Mathematical logic: consistency of theories
- ▷ λ-calculus & type theory: cut-elimination as a computational process (Curry-Howard correspondence).

A stronger Hauptsatz

Theorem (Prenex normal form). Any formula A can be turned into a semantically equivalent formula

$$\underbrace{Q_1 x_1 . \ldots . Q_n x_n .}_{Q_i \in \{\forall, \exists\}} \underbrace{B(x_1, \ldots, x_n)}_{\bot, \neg, \wedge, \vee, \to}$$

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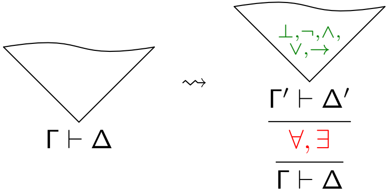
$$\underbrace{Q_1 x_1. \dots Q_n x_n.}_{Q_i \in \{\forall, \exists\}} \underbrace{B(x_1, \dots, x_n)}_{\perp, \neg, \wedge, \vee, \rightarrow}$$

Theorem (Strong *Hauptsatz*). If Γ, Δ are multisets of formulas in prenex normal form:



Idea: Apply Hauptsatz + convert a cut-free proof into a bipartite proof.

Herbrand's theorem

Herbrand's theorem permits to reduce the question of the deducibility (validity) of a formula F of first-order classical logic to the question of the deducibility (validity) of a quantifier-free ("propositional") formula:

Theorem (Herbrand).

$$\vdash \exists x.\ \underbrace{A(x)}_{\substack{\bot,\neg,\wedge,\\ \vee,\rightarrow}} \quad \Longrightarrow \quad \exists n \geq 0, \quad \exists t_1, \ldots, t_n, \quad \underbrace{\vdash A(t_1/x), \ldots, A(t_n/x)}_{\bigvee_{i=1}^{n} A(t_i/x_i)}$$

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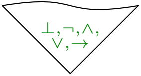
Proof idea. By strong Hauptsatz:



$$\dfrac{\dfrac{\vdash A(t_1/x), \dots, A(t_n/x)}{\exists\text{-steps}}}{\vdash \exists x. A(x)}$$

Exercise: Is there a recursive procedure to compute $n \geq 0$?

Craig's interpolation

Theorem (Craig's interpolation, 1957). If $\vdash A \to B$ then there is a C in the common language of A and B such that $\vdash A \to C$ and $\vdash C \to B$.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & ??? \arrow[ur] & \end{tikzcd}$$

Analyticity: When deducing B from A I use only the notions that are relevant for both A and B.

Applications of interpolation to complexity

- A disjoint NP-pair (A, B) is a pair of nonempty sets A and B such that $A, B \in \text{NP}$ and $A \cap B = \varnothing$. A separator for (A, B) is a set S such that $A \subseteq S$ and $B \subset \bar{S}$. E.g. $(\text{CLIQUE}_{n,k}, \text{COLOUR}_{n,k})$.
- Propositional formulas $\text{clique}_{n,k}(\vec{e}, \vec{p})$ and $\text{colour}_{n,k}(\vec{e}, \vec{q})$ with variables $\vec{e}$, $\vec{p}$, $\vec{q}$ such that, intuitively:
  - $e_{u,v}$ iff there is an edge between u and v
  - $p_{u,i}$ iff u is the i-th node in the clique $(1 \leq i \leq k)$
  - $q_{u,i}$ iff u is a node with colour $1 \leq i \leq k$.
- $\vdash \text{clique}_{n,k+1}(\vec{e}, \vec{p}) \to \neg\text{colour}_{n,k}(\vec{e}, \vec{q})$.
- By Craig's interpolation theorem there exists $I(\vec{e})$ such that:

  $\vdash \text{clique}_{n,k+1}(\vec{e}, \vec{p}) \to I(\vec{e})$ $\qquad$ $\vdash I(\vec{e}) \to \neg\text{colour}_{n,k}(\vec{e}, \vec{q})$
- Formula $I(\vec{e})$ represents a separator S for $(\text{CLIQUE}_{n,k}, \text{COLOUR}_{n,k})$:
  - If G has a $k+1$-clique then $G \in S$
  - If G is k-colourable then $G \in \bar{S}$.

Incompleteness theorems and consistency of PA

A theory T of predicate logic is consistent if there is no formula A (in the language of T) such that both A and $\neg A$ are provable. It is complete if for any formula A (in the language of T) either A or $\neg A$ is provable in T.

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Godel's first incompleteness theorems (1931). If PA is consistent there are formulas of PA such that neither A nor $\neg A$ are provable in the theory.

Gödel was a forerunner of proof theory: proofs are encoded into natural numbers so that meta theoretical properties such as provability and consistency can be manipulated within the theory arithmetic.

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Gentzen's approach 1938:

- Consistency of PA using transfinite induction up to the ordinal ϵ_0
- He initiated proof theoretic ordinal analysis.

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Course on Proof Theory - Lecture 3

Exercises - Gentzen's sequent calculus

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Exercises for Lecture 3

1. We saw that modus ponens could be simulated in LK. Give cut-free LK proofs for each of the three propositional axioms of HF:

$$\begin{array}{rl} (wk) & A \to (B \to A) \\ (dist) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ (neg) & ((A \to \bot) \to \bot) \to A \end{array}$$

2. What about the quantifier axioms and rule:

$$\text{gen}\,\frac{A}{\forall x A} \qquad \begin{array}{l} \forall x.A \to A[t/x] \\ \forall x.(A \to B) \to (A \to \forall x.B) \quad \text{as long as } x \notin \mathrm{FV}(A) \end{array}$$

(**NB:** this concludes the proof of completeness of LK).

3. What possible rules might we need in LK if we had $\bot$ in our language?

4. Show that the multiplicative formulation of the rules for $\wedge$ can be derived from the additive formulation of the rules for $\wedge$ using the structural rules, and vice versa (see slide 13).