Course on Proof Theory - Lecture 2

Soundness, completeness and other metalogical results

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Recap: syntax and semantics

Soundness (for predicate logic)

Completeness (for propositional logic)

Compactness (for propositional logic)

References

Exercises

Recap: semantics for first-order logic

Language   Logical basis: $\{\bot, \to, \forall\}$ with $\neg A := A \to \bot$

Satisfiability relation for formulas

$$\mathcal{M}, \sigma \vDash A$$

- ▷ Structure $\mathcal{M}$: a non-empty set D (the *domain*) and, for each function symbol f, constant c and predicate P, their interpretations $f^{\mathcal{M}}$, $c^{\mathcal{M}}$ and $P^{\mathcal{M}}$.
- ▷ Variable assignment $\sigma : \mathrm{Var} \to D$.

$$\begin{aligned}
\mathcal{M}, \sigma \vDash P(t_1, \dots, t_k) \quad & \textit{iff} \quad \langle [t_1]^{\sigma}_{\mathcal{M}}, \dots, [t_k]^{\sigma}_{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\
\mathcal{M}, \sigma \not\vDash \bot \quad & \\
\mathcal{M}, \sigma \vDash A \to B \quad & \textit{iff} \quad \mathcal{M}, \sigma \not\vDash A \text{ or } \mathcal{M}, \sigma \vDash B \\
\mathcal{M}, \sigma \vDash \forall x A \quad & \textit{iff} \quad \mathcal{M}, \sigma[x \mapsto d] \vDash A, \ \textbf{for all } d \in D
\end{aligned}$$

$\sigma[x \mapsto d]$ denotes the assignment which agrees with σ for all variables which are different from x, and maps x to d.

Recap: Hilbert-Frege system for first-order logic

$$\Gamma \vdash A$$

Axioms of HF

- ▷ HF1. $A \rightarrow (B \rightarrow A)$
- ▷ HF2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
- ▷ HF3. $\neg\neg A \rightarrow A$
- ▷ HF4. $\forall x.A \rightarrow A[t/x]$
- ▷ HF5. $\forall x.(A \rightarrow B) \rightarrow A \rightarrow \forall x.B$ where $x \notin \mathrm{FV}(A)$
- ▷ HF6. $\forall x.(x = x)$
- ▷ HF7. $\forall x.\forall y.(x = y \rightarrow A[x/z] \rightarrow A[y/z])$

Inference rules of HF

$$\text{mp}\ \frac{A \quad A \rightarrow B}{B} \qquad\qquad \text{gen}\ \frac{A}{\forall x A}$$

Relationship between syntax and semantics

$$\Gamma = \{A_1, \ldots, A_n\} \qquad A$$

- Γ semantically entails A if, for any structure $\mathcal{M}$ such that $\mathcal{M} \models A_i$ for all $A_i \in \Gamma$, then $\mathcal{M} \models A$. We write $\Gamma \models A$.
- Γ is derivable from A if there exists a *derivation* of A from Γ. We write $\Gamma \vdash A$.

$$\Gamma \vdash A \quad \Longleftrightarrow \quad \Gamma \models A$$

- Soundness: If $\Gamma \vdash A$ then $\Gamma \models A$
  "the proof system only proves true things"
- Completeness: If $\Gamma \models A$ then $\Gamma \vdash A$
  "the proof system is *sufficient*: every true thing has a proof"

Recap: syntax and semantics

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Soundness

$$\text{If } \Gamma \vdash A, \text{ then } \Gamma \vDash A$$

Proof.
Suppose that there is a derivation $\mathcal{D}$ of A from assumptions Γ.
The proof proceeds by structural induction on the derivation:

- ▷ Base cases If A is an axiom, A is valid.
- ▷ Inductive step If $\mathcal{M}$ models the premiss(es) of an inference rule, then $\mathcal{M}$ models the conclusion.

Base cases HF1. $A \to (B \to A)$ is valid

A	B	$B \to A$	$A \to (B \to A)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Soundness

Proof (continued).
HF4. $\forall x.A \rightarrow A[t/x]$ is valid

We show that for any $\mathcal{M}, \sigma$, it holds that $\mathcal{M}, \sigma \vDash \forall x.A \rightarrow A[t/x]$.

By definition, either $\mathcal{M}, \sigma \nvDash \forall x.A$ or $\mathcal{M}, \sigma \vDash A[t/x]$.

If $\mathcal{M}, \sigma \nvDash \forall x.A$ we're done.

If $\mathcal{M}, \sigma \vDash \forall x.A$, then by definition **for all** $d \in D$ it holds that $\mathcal{M}, \sigma[x \mapsto d] \vDash A$.

$[t]^{\sigma}_{\mathcal{M}} = c$, **for some** $c \in D$

Thus $\mathcal{M}, \sigma[x \mapsto c] \vDash A$.

And we can conclude that $\mathcal{M}, \sigma \vDash A[t/x]$.

Exercise: prove validity of the other axioms.

Soundness

Proof (continued).

modus ponens If $\mathcal{M}$ models the premiss(es) of an inference rule, then $\mathcal{M}$ models the conclusion.

$$\text{mp}\,\frac{A \to B \quad A}{B}$$

Assume that for an arbitrary $\mathcal{M}$ it holds that:

1. $\mathcal{M} \models A \to B$; and
2. $\mathcal{M} \models A$.

We need to show that $\mathcal{M} \models B$.

From 1, by definition it holds that either $\mathcal{M} \not\models A$ or $\mathcal{M} \models B$.

Since $\mathcal{M} \models A$ (by 2), we conclude (by 1) that $\mathcal{M} \not\models B$.

Exercise: show that gen preserves validity.

This conclues the soundness proof. $\square$

Completeness

$$\text{If } \Gamma \vDash A, \text{ then } \Gamma \vdash A$$

Significantly **more difficult** than soundness!

- Gödel proved completeness for predicate logic in 1929.
- Henkin showed in his Ph.D. thesis that the completeness proof for predicate logic could be simplified, reformulating it in terms of model existence. The result was published in 1949.

We'll show completeness for propositional logic.

Interlude: Deduction Theorem

Theorem (Deduction theorem)

Let A be a formula and Γ a set of formulas. $\Gamma \vdash A \to B$ iff $\Gamma \cup \{A\} \vdash B$.

$$
\begin{array}{c} \Gamma \\ \vdots \\ A \to B \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \Gamma \cup \{A\} \\ \vdots \\ B \end{array}
$$

Proof.

If $\Gamma \vdash A \to B$ then $\Gamma \cup \{A\} \vdash B$. (easy)

$$
\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ A \to B \end{array} \qquad A}{B}\ \mathrm{mp}
$$

If $\Gamma \cup \{A\} \vdash B$ then $\Gamma \vdash A \to B$. (difficult)

Statement of the theorem

$$(\star) \quad \text{If } \Gamma \vDash A, \text{ then } \Gamma \vdash A$$

$$(\circ) \quad \text{If } \Gamma \text{ is consistent, then } \Gamma \text{ is satisfiable}$$

$$\text{If } \Gamma \nvdash \bot, \text{ then } \alpha \vDash A, \text{ for all } A \in \Gamma$$

$(\star) \implies (\circ)$. Assume $(\star)$. By contraposition,

$$(\star') \quad \text{If } \Gamma \nvdash A, \text{ then } \Gamma \nvDash A$$

Assume $\Gamma \nvdash \bot$. Then by $(\star')$, $\Gamma \nvDash \bot$. Then by definition Γ is satisfiable.

$(\circ) \implies (\star)$. Assume $(\circ)$. By contraposition,

$$(\circ') \quad \text{If } \Gamma \text{ is not satisfiable, then } \Gamma \text{ is not consistent}$$

Assume $\Gamma \vDash A$. Then, $\Gamma, \neg A$ is not satisfiable (Q: Why?). Then by $(\circ')$, $\Gamma \cup \{\neg A\}$ is not consistent, i.e., $\Gamma, \neg A \vdash \bot$. By the deduction theorem, and since $\neg\neg A \to A$, we conclude that $\Gamma \vdash A$.

Completeness theorem

$(\circ)$ If Γ is consistent, then Γ is satisfiable.

If $\Gamma \nvdash \bot$, then $\alpha \vDash A$, for all $A \in \Gamma$

The proof is modular and composed of the following ingredients:

1. Definition of maximally consistent sets of formulas (maxcons)
2. Lemma: Every maxcons is satisfiable.
3. Lemma (Lindembaum): For every consistent set of formulas Γ, there is a maxcons $\Gamma^* \supseteq \Gamma$.

$$\Gamma \text{ consistent} \overset{3}{\Longrightarrow} \Gamma^* \text{ maxcons} \overset{2}{\Longrightarrow} \Gamma^* \text{ satisfiable} \Longrightarrow \Gamma \text{ satisfiable}$$

Maximally consistent sets (1/3)

Let Γ be a set of propositional formulas.

A set of formulas Γ is maximally consistent if:

- It is consistent $\Gamma \nvdash \bot$
- If $A \notin \Gamma$, then $\Gamma \cup \{A\}$ is not consistent, i.e., $\Gamma \cup \{A\} \vdash \bot$.

  Or, equivalently

  For all formulas A, either $A \in \Gamma$ or $\neg A \in \Gamma$.

Satisfiability of maxcons (2/3)

$$\Gamma \text{ consistent} \Longrightarrow \Gamma^* \text{ maxcons} \Longrightarrow \Gamma^* \text{ satisfiable} \Longrightarrow \Gamma \text{ satisfiable}$$

Lemma (Satisfiability of maxcons). Every maxcons is satisfiable.

Proof.
Let Γ^* be a maxcons.
Then, for each formula p, either $p \in \Gamma^*$ or $\neg p \in \Gamma^*$.

We construct a propositional assignment α^Γ as follows:

$$\alpha^\Gamma(p) := \begin{cases} 1 & \text{if } p \in \Gamma^* \\ 0 & \text{if } p \notin \Gamma^* \end{cases}$$

For all $A \in \Gamma^*$, it holds that $\alpha^\Gamma \models A$. Thus, Γ^* is satisfiable. □

NB: Not all consistent sets are are maxcons!
Example: $\varnothing$ is consistent but it is not maxcons (Q: Why?)

Lindembaum Lemma (3/3)

$$\Gamma \text{ consistent} \stackrel{3}{\Longrightarrow} \Gamma^* \text{ maxcons} \stackrel{2}{\Longrightarrow} \Gamma^* \text{ satisfiable} \Longrightarrow \Gamma \text{ satisfiable}$$

Lemma (Lindembaum). For Γ consistent set of formulas, there is a maxcons Γ^* such that $\Gamma \subseteq \Gamma^*$.

Proof.
Fix an enumeration of all the formulas: $A_0, A_1, A_2, \ldots$

Define $\Gamma_0 := \Gamma$ and inductively define Γ_{k+1} as follows:

$$\Gamma_{k+1} := \begin{cases} \Gamma_k \cup \{A_k\} & \text{if } \Gamma \vdash A_k; \\ \Gamma_k \cup \{\neg A\} & \text{otherwise.} \end{cases}$$

Each Γ_k is consistent. Exercise: prove this.

Lindenbaum Lemma (3/3)

Proof (continued).

$$\Gamma^* := \bigcup_{n \geq 0} \Gamma_n$$

- Γ^* is consistent $\Gamma^* \nvdash \bot$. Suppose $\Gamma^* \vdash \bot$. Then, since derivations are finite objects, there exists $B_1, \ldots, B_k$ s.t.

$$B_1, \ldots, B_k \vdash \bot$$

Then, there exists Γ_n s.t. $\{B_1, \ldots, B_k\} \subseteq \Gamma_n$. Then $\Gamma_n \vdash \bot$, against consistency of each $\Gamma_n \subseteq \Gamma^*$.

Lindenbaum Lemma

$$\Gamma^* := \bigcup_{n \geq 0} \Gamma_n$$

- ▷ Γ^* is maximally consistent   If $A \notin \Gamma^*$, then $\Gamma^* \cup \{A\} \vdash \bot$.

  Suppose $A \notin \Gamma^*$ but $\Gamma^* \cup \{A\} \nvdash \bot$. Then, for some $k \in \mathbb{N}$, we have that $A = A_k$ and:

  $$\Gamma_{k+1} := \begin{cases} \Gamma_k \cup \{A_k\} & \text{if } \Gamma_k \vdash A_k; \\ \Gamma_k \cup \{\neg A\} & \text{otherwise.} \end{cases}$$

  If $\Gamma_k \vdash A$ then $A \in \Gamma_{k+1} \subseteq \Gamma^*$ contradiction

  If $\Gamma_k \vdash A$ then $\neg A \in \Gamma_{k+1} \subseteq \Gamma^*$. Then $\Gamma^* \vdash \neg A$, and so $\Gamma^* \cup \{A\} \vdash \bot$ contradiction

Thus, Γ^* is maxcons.

This concludes the proof of Lindenbaum Lemma. □

Some remarks

$$\Gamma \text{ consistent} \implies \Gamma^* \text{ maxcons} \implies \Gamma^* \text{ satisfiable} \implies \Gamma \text{ satisfiable}$$

This method is an overkill for propositional logic. However, it is a general method, that can be adapted to

- Predicate logic with respect to Henkin's models
- Modal logics with respect to frame properties
- Intuitionistic logic with respect to frame properties

What happens with predicate logic?

Γ consistent $\Longrightarrow$ Γ^* maxcons $\Longrightarrow$ Γ^* satisfiable $\Longrightarrow$ Γ satisfiable

1. Definition of maximally consistent sets of formulas (maxcons)
2. Definition of Henkin structures
3. Lemma: Every maxcons is satisfiable in the associated Henkin structure
4. Lemma (Lindenbaum-Henkin): For every consistent set of formulas Γ, there is a maxcons $\Gamma^* \supseteq \Gamma$.

Henkin structures

Intuitively, a Henkin structure is built from the syntactic elements of a set of Γ.

For predicate languages without equality

The canonical structure $\mathcal{M}_\Gamma$ induced by Γ consists of:

- $D_\Gamma = \text{Ter}$ (the domain is composed of all the terms of $\mathcal{L}$)
- For each k-ary function symbol f of $\mathcal{L}$, a function $f^{\mathcal{M}_\Gamma} : D_\Gamma^k \to D$ s.t.
$$f^{\mathcal{M}_\Gamma}(t_1, \dots, t_k) = f(t_1, \dots, t_k)$$
- For each k-ary predicate symbol P of $\mathcal{L}$, a relation $P^{\mathcal{M}_\Gamma} \subseteq D_\Gamma^k$ s.t.
$$P^{\mathcal{M}_\Gamma} := \{\langle t_1, \dots, t_k \rangle \mid \Gamma \vdash P(t_1, \dots, t_k)\}$$

For predicate languages with equality

- $D_\Gamma = \text{Ter}/\approx$ where $s \approx t$ iff $\Gamma \vdash s = t$ (the domain is composed of equivalence classes of terms)

Lemma For all $A \in \Gamma$, it holds that $\mathcal{M}_\Gamma, \sigma_\Gamma \vDash A$.

A corollary of completeness

We say that a set Γ is finitely satisfiable if every finite subset $\Gamma' \subseteq \Gamma$ is satisfiable.

Theorem (Compactness). Every finitely satisfiable set Γ is satisfiable.

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We say that a set Γ is finitely satisfiable if every finite subset $\Gamma' \subseteq \Gamma$ is satisfiable.

Theorem (Compactness). Every finitely satisfiable set Γ is satisfiable.

Proof idea.

- We show the contrapositive: we assume that Γ is not satisfiable and we want to prove that it is not finitely satisfiable
- By definition, we want to prove that there exists Γ_0 finite subset Γ such that Γ_0 is not satisfiable
- $\Gamma \vDash \bot$ iff Γ is not satisfiable.
- By soundness, completeness and finiteness of derivations:

$$\Gamma \vDash \bot \underset{\textit{completeness}}{\Longrightarrow} \Gamma \vdash \bot \underset{\textit{finiteness}}{\Longrightarrow} \Gamma_0 \vdash \bot \quad (\Gamma_0 \subseteq_{fin} \Gamma) \underset{\textit{soundness}}{\Longrightarrow} \Gamma_0 \vDash \bot$$

Recap: Peano arithmetic

The language of arithmetic is $\mathcal{L}_A := \{0, s, +, \cdot\}$.

The standard structure is $\mathcal{N} = (\mathbb{N}, 0^{\mathcal{N}} := 0 \in \mathbb{N}, s^{\mathcal{N}} : n \mapsto n+1, +^{\mathcal{N}} : (m, n) \mapsto m + n, \cdot^{\mathcal{N}} : (m, n) \mapsto m \times n)$.

The theory PA.

- PA1. $\forall x. \neg(0 = s(x))$
- PA2. $\forall x. \forall y. (s(x) = s(y) \to x = y)$
- PA3. $\forall x. (x + 0 = x)$
- PA4. $\forall x. \forall y. (x + s(y) = s(x + y))$
- PA5. $\forall x. (x \cdot 0 = 0)$
- PA6. $\forall x. \forall y. (x \cdot s(y) = x \cdot y + x)$
- Induction scheme: $A[0/x] \to \forall x. (A \to A[s(x)/x]) \to \forall x. A$

Nonstandard models of PA

- Theory PA^*:
  - Extend language of PA with a new constant symbol ω
  - Extend axioms of PA with all axioms of the form:

$$
\begin{array}{rc}
PA^*0. & \neg(\omega = 0) \\
PA^*1. & \neg(\omega = s(0)) \\
PA^*2. & \neg(\omega = s(s(0))) \\
 & \vdots
\end{array}
$$

Nonstandard models of PA

- Theory PA^*:
  - Extend language of PA with a new constant symbol ω
  - Extend axioms of PA with all axioms of the form:

$$
\begin{aligned}
&PA^*0. && \neg(\omega = 0)\\
&PA^*1. && \neg(\omega = s(0))\\
&PA^*2. && \neg(\omega = s(s(0)))\\
& && \vdots
\end{aligned}
$$

- Let Γ_0 be a finite set of axioms of PA^*. Γ_0 is satisfied by $\mathcal{N}$ by interpreting ω with some natural number sufficiently large
- By the compactness theorem, we must have $\mathcal{N}^* \models \mathrm{PA}^*$ for some structure $\mathcal{N}^*$, and hence also $\mathcal{N}^* \models \mathrm{PA}$.

NB: in $\mathcal{N}^*$, the interpretation of ω must be distinct from any number!

Second order and beyond

In the first-order setting, we have to accept the existence of nonstandard models $\mathcal{N}^*$ (see *Lowenheim-Skolem theorem*).

This problem can be avoided by formulating of Peano's postulates using second-order logic, which in addition to talking about things $x, y, z, \ldots$ allows us reason about sets of things $X, Y, Z, \ldots$.

The price to pay: second-order logic has no complete axiomatisation.

References

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Course on Proof Theory - Lecture 2

Exercises - Soundness, completeness and other metalogical results

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Exercises for Lecture 2

1. Show that axiom HF5. is valid ($x \notin \mathrm{FV}(A)$):

$$(\forall x(A \to B)) \to (A \to \forall x B)$$

2. Show that the inference rule gen preserves validity:

$$\text{gen}\ \frac{A}{\forall x.A}$$

Exercises for Lecture 2

3. For a set of formulas Γ, if $\Gamma \not\models \bot$ then Γ is satisfiable. Why?
4. For a set of formulas Γ, show that the following are equivalent:
   - If $A \notin \Gamma$, then $\Gamma \cup A$ is not consistent.
   - For all formulas A, either $A \in \Gamma$ or $A \notin \Gamma$
5. Show that the following forms of consistency are equivalent:
   - $\Gamma \nvdash \bot$
   - There is no formula A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$
   - There is a formula A such that $\Gamma \nvdash A$
6. In Lindenbaum Lemma, prove that every Γ_k in the construction is consistent.

Exercises for Lecture 2

7. **(Difficult)** A graph is a structure $\mathfrak{G} = \langle V, E \rangle$ such that $E \subseteq V \times V$. For $k \in \mathbb{N}$ we say that $\mathfrak{G}$ is k-colourable if there is a function $c : V \to \{1, \dots, k\}$ such that, whenever $(u, v) \in E$ then $c(u) \neq c(v)$.

   Using the compactness theorem, show that a (possibly infinite) graph is k-colourable iff every finite subgraph of it is k-colourable.