Course on Proof Theory - Lecture 1

Exercises - Introduction to Propositional and First-order Logic

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Part 1: Propositional logic

1. Show $\vdash \bot \to A$ (*ex falso quodlibet*)
2. Show $\vdash (A \to (B \to C)) \to B \to A \to C$ (*exchange*).
3. We can extend HF to include conjunction $\wedge$ by adding the following axioms:

$$\begin{array}{rl} \text{HF8.} & A \to B \to (A \wedge B) \\ \text{HF9.} & (A \wedge B) \to A \\ \text{HF10.} & (A \wedge B) \to B \end{array}$$

   Show $\vdash (A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$, where

$$A \leftrightarrow B := (A \to B) \wedge (B \to A)$$

4. We haven't yet used the axiom (*neg*)! Show that HF proves:

$$\begin{array}{rl} (a) & \neg A \to (A \to B) \\ (b) & (\neg B \to \neg A) \to (A \to B) \\ (c) & ((A \to B) \to A) \to A \end{array}$$

Hint for part 1: Use the deduction theorem to reduce any proof of $A \to B$ to a proof of B with hypothesis A.

Part 2: Predicate logic

1. We can extend HF to include existential quantifier $\exists$ by adding the following axioms:

$$\begin{array}{lll} \text{HF11.} & A[t/x] \to \exists x.A & \\ \text{HF12.} & \forall x.(A \to B) \to (\exists x.A \to B) & \quad x \notin FV(B) \end{array}$$

   Show the following equivalences:

$$\begin{array}{lll} (a) & \mathcal{M}, \sigma \vDash \exists x.A \iff \mathcal{M}, \sigma \vDash \neg\forall x.\neg A & \\ (b) & \mathcal{M}, \sigma \vDash (\exists x.A \to B) \iff \mathcal{M}, \sigma \vDash \forall x.(A \to B) & x \notin FV(B) \\ (c) & \vdash (\exists x.A \to B) \leftrightarrow \forall x.(A \to B) & x \notin FV(B) \end{array}$$

2. Outline a first-order theory whose models are the partial orders. Adapt this theory to characterise
   - total orders e.g. $(\mathbb{Z}, \leq)$
   - total orders with a minimum element e.g. $(\mathbb{N}, \leq, 0)$
   - dense total orders e.g. $(\mathbb{Q}, \leq)$
   - Is it possible to characterise well-founded partial orders?