Course on Proof Theory - Lecture 5

Beyond classical logic

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On classical reasoning

- Classical logic based on the notion of truth, where not false = true:

$$A \vee \neg A \quad (\textit{excluded middle}) \qquad \neg\neg A \to A \quad (\textit{involutivity})$$

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- Truth-value of a formula is "absolute" (either true or false), whether or not we know it, prove it, or verify it in any possible way. E.g.

There are seven 7's in a row somewhere in the decimal representation of the real number R

Intuitionistic logic

Figure: From left, L.E.J. Brower (1881-1966) and Arend Heyting (1898-1980).

- Intuitionistic logic: there is no absolute truth, there is only the knowledge and intuitive construction of the idealised mathematician. A logical judgement is only considered "true" if we can verify its correctness.
- Intuitionistic logic rejects the excluded middle and involutivity.

An example

Theorem. There exist irrational numbers x and y, such that x^y is rational.

First proof. If $\sqrt{2}^{\sqrt{2}}$ is a rational number then we take $x = y = \sqrt{2}$; otherwise, we take $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$.

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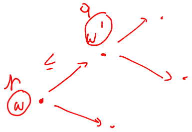
Drawback: which of the two possibilities actually holds?

Second proof. Take $x = \sqrt{2}$ and $y = 2\log_2 3$, so that $x^y = 3$.

Benefit: Proof is constructive, it exhibits a "witness".

Kripke semantics

(Intuitionistic) Kripke model. $(W, \leq, \Vdash)$ where:

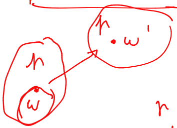


- W nonempty set of states (or worlds)
- $\leq$ is partial order on W

$w \Vdash p$

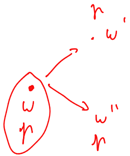
- $\Vdash$ called forcing relation is binary relation between states and propositional variables such that:

$$w \leq w' \quad \text{and} \quad w \Vdash p \quad \Longrightarrow \quad w' \Vdash p \qquad \boxed{\text{(monotonicity)}}$$



Idea:

- $w \in W$ represents a state of knowledge
- $\leq$ represents gaining of knowledge
- $\Vdash$ determines which facts are true at a given state.



Kripke semantics, continued

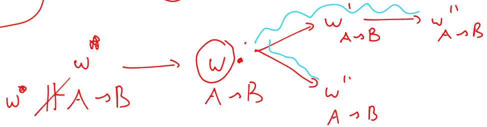
$w \Vdash p$

$w \Vdash A$

Given $(W, \leq, \Vdash)$, we extend $\Vdash$ to any formula:

$$
\begin{array}{lll}
w \Vdash A \wedge B & \text{iff} & w \Vdash A \textbf{ and } w \Vdash B \\
w \Vdash A \vee B & \text{iff} & w \Vdash A \textbf{ or } w \Vdash B \\
w \Vdash A \to B & \text{iff} & \textbf{for all } w' \geq w,\ w' \Vdash A \textbf{ implies } w' \Vdash B \\
w \nVdash \bot & &
\end{array}
$$

$\Gamma \Vdash A$ iff for every $(W, \leq, \Vdash)$ and every $w \in W$, if $w \Vdash \Gamma$ then $w \Vdash A$.



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- Monotonicity "lifts" to formulas: $w \leq w'$ and $w \Vdash A \implies w' \Vdash A$

$$\begin{array}{ccc} w & \longrightarrow & w' \\ A & & A \end{array}$$

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- Monotonicity "lifts" to formulas: $w \leq w'$ and $w \Vdash A \implies w' \Vdash A$
- Intuitionistic logic breaks dualites of classical logic:
  - Negation is not involutive:

$$w \Vdash \neg A := A \to \bot \quad \text{iff} \quad w' \nVdash A \quad \text{for all } w' \geq w$$

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  - The equivalence $w \Vdash A \to B \iff w \Vdash \neg A \vee B$ does not hold.

An example

Example. Let $(W, \leq, \Vdash)$ be such that:

- $W = \{w, w', w''\}$
- $w \leq w'$, $w \leq w''$ with w', w'' incomparable.
- $w' \Vdash p$, $\quad w'' \Vdash q$, $\boxed{w \nVdash p,}$ $\boxed{w \nVdash q}$

$$
\begin{tikzcd}
 & \overset{p}{w'} \\
w \arrow[ur] \arrow[dr] & \\
 & \overset{q}{w''}
\end{tikzcd}
$$

We have $w \Vdash \neg\neg(p \vee q)$ and $w \Vdash (p \to q) \to q$. Notice that $w \nVdash p \vee \neg p$.

From classical logic to intuitionistic logic

Semantics of intuitionistic logic

Sequent calculus LJ

LK

Meta-properties of LJ

Automated proof search

References

Sequent calculus for intuitionistic logic, naive attempt

LJ = restriction of LK where all sequents have exactly one formula on the right-hand side of the sequent.

$$\frac{\Gamma \vdash \Delta}{\boxed{\Gamma \vdash A}}$$

$$\text{init}\ \frac{}{A, \Gamma \vdash A} \qquad \bot_{\mathbf{L}}\ \frac{}{\bot, \Gamma \vdash A}$$

$$\wedge_{\mathbf{L}}\ \frac{A_1, A_2, \Gamma \vdash C}{A_1 \wedge A_2, \Gamma \vdash C} \qquad \wedge_{\mathbf{R}}\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{\mathbf{L}}\ \frac{A_1, \Gamma \vdash C \quad A_2, \Gamma \vdash C}{A_1 \vee A_2, \Gamma \vdash C} \qquad \vee_{\mathbf{R}}\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}$$

$$\frac{\Gamma \vdash A_1, A_2}{\Gamma \vdash A_1 \wedge A_2}$$

$$\to_{\mathbf{L}}\ \frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \to B, \Gamma \vdash C} \qquad \to_{\mathbf{R}}\ \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}$$

$$\text{cut}\ \frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C}$$

Examples

Double negation law is not intuitionistically (cut-free) provable (Exercise).

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Peirce's law is not intuitionistically (cut-free) provable:

$$
\rightarrow_{\mathbf{R}}\dfrac{\rightarrow_{\mathbf{L}}\dfrac{\rightarrow_{\mathbf{R}}\dfrac{\text{init}\,\dfrac{}{A \vdash \cancel{A}, B}}{\vdash \cancel{A}, A \rightarrow B} \qquad \text{init}\,\dfrac{}{A \vdash A}}{(A \rightarrow B) \rightarrow A \vdash A}}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}
$$

A counterexample to completeness

$$\to_{\mathbf{R}}\ \frac{\to_{\mathbf{L}}\ \dfrac{\dfrac{???}{\vdash A \vee (A \to \bot)} \qquad \text{init}\ \dfrac{}{\bot \vdash \bot}}{(A \vee (A \to \bot)) \to \bot \vdash \bot}}{\vdash \underbrace{((A \vee (A \to \bot)) \to \bot) \to \bot}_{\neg\neg(A \vee \neg A)}}$$

A counterexample to completeness

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}\LeftLabel{init}
\UnaryInfC{$((A \vee (A \to \bot)) \to \bot, A \vdash A$}
\LeftLabel{$\vee_{\mathbf{R}}$}
\UnaryInfC{$((A \vee (A \to \bot)) \to \bot, A \vdash A \vee (A \to \bot)$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\bot \vdash \bot$}
\LeftLabel{$\to_{\mathbf{L}}$}
\BinaryInfC{$((A \vee (A \to \bot)) \to \bot, A \vdash \bot$}
\LeftLabel{$\to_{\mathbf{R}}$}
\UnaryInfC{$((A \vee (A \to \bot)) \to \bot \vdash A \to \bot$}
\LeftLabel{$\vee_{\mathbf{R}}$}
\UnaryInfC{$((A \vee (A \to \bot)) \to \bot \vdash A \vee (A \to \bot)$}
\AxiomC{}
\LeftLabel{init}
\UnaryInfC{$\bot \vdash \bot$}
\LeftLabel{$\to_{\mathbf{L}}$}
\BinaryInfC{$\Rightarrow \quad (A \vee (A \to \bot)) \cancel{\to} \bot \vdash \bot$}
\LeftLabel{$\to_{\mathbf{R}}$}
\UnaryInfC{$\vdash \underbrace{((A \vee (A \to \bot)) \to \bot) \to \bot}_{\neg\neg(A \vee \neg A)}$}
\end{prooftree}
$$

Idea: To prove intuitionistically valid formula $\neg\neg(A \vee \neg A)$ we need to apply twice $\to_{\mathbf{L}}$ to the formula $((A \vee (A \to \bot)) \to \bot$, so we need to "save" a copy of it in our set of hypothesis.

Sequent calculus for intuitionistic logic

$$\text{init}\ \frac{}{A, \Gamma \vdash A} \qquad \bot_{\mathbf{L}}\ \frac{}{\bot, \Gamma \vdash A}$$

$$\wedge_{\mathbf{L}}\ \frac{A_1, A_2, \Gamma \vdash C}{A_1 \wedge A_2, \Gamma \vdash C} \qquad \wedge_{\mathbf{R}}\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

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$$\text{cut}\ \frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C}$$

Sequent calculus for intuitionistic logic

$$\frac{A[{}^{t}/_{x}], \forall x.A \Rightarrow}{\forall x.A \Rightarrow}$$

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$\forall$

$$\text{cut}\ \frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C}$$

N.B. Recall that $A \to B$ is a primitive connective ($A \to B \neq \neg A \vee B$).

Soundness and completeness for LJ

Theorem. LJ is sound and complete for Kripke models.

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Example.

$$\vdash \exists x.(\underline{3} + x = \underline{5}) \quad \rightsquigarrow \quad \exists_{\mathsf{R}}\,\frac{\vdash \underline{3} + \underline{2} = \underline{5}}{\vdash \exists x.(\underline{3} + x = \underline{5})}$$

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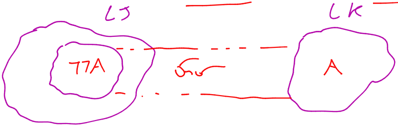
Glivenko's theorem (1929). $\vdash_{\mathsf{LK}} A \iff \vdash_{\mathsf{LJ}} \neg\neg A$.



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- Gödel and Gentzen's glasses, or double negation translation:

$$
\begin{aligned}
p^G &:= \neg\neg p \\
(\neg A)^G &:= \neg A^G \\
(A \wedge B)^G &:= A^G \wedge B^G \\
(A \vee B)^G &:= \neg(\neg A^G \wedge \neg B^G) \\
(A \to B)^G &:= A^G \to B^G
\end{aligned}
$$

$(\_)^G$

Gödel & Gentzen's theorem (1933). $\vdash_{\mathsf{LK}} A \iff \vdash_{\mathsf{LJ}} A^G$.

Intermediate logics

- Are there logics in-between LJ and LK?

$$\mathrm{LJ} \overset{???}{\dashrightarrow} \mathrm{LK}$$

Intermediate logics

$\underbrace{LJ + \{\ldots\ldots\}}$

- Are there logics in-between LJ and LK?

$$\mathrm{LJ} \xrightarrow{???} \mathrm{LK}$$

$B \vee \neg B$

$B := \neg A$

- Yes ... and they are called intermediate logics!

$$\begin{aligned} \text{Weak excluded middle logic} &:= \mathrm{LJ} + \{\neg A \vee \neg\neg A\} \\ \text{Gödel-Dummett logic} &:= \mathrm{LJ} + \{\underbrace{(A \to B) \vee (B \to A)}_{(A \to B) \vee (B \to A)}\} \\ &\vdots \end{aligned}$$

Intermediate logics

- ▷ Are there logics in-between LJ and LK?

$$\mathsf{LJ} \overset{???}{\dashrightarrow} \mathsf{LK}$$

- ▷ Yes . . . and they are called intermediate logics!

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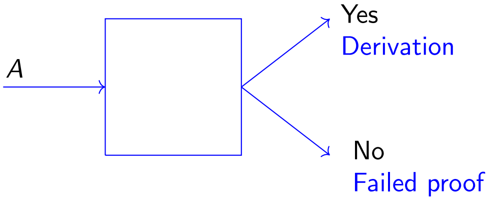
- ▷ Question for the audience: How many intermetiate logics are there?

Automated proof search

How to check if A is a theorem of classical or intuitionistic logic?



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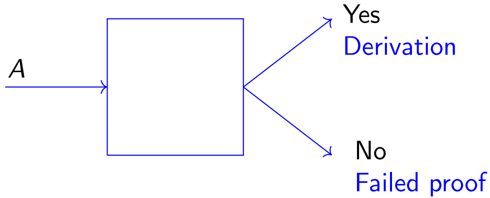
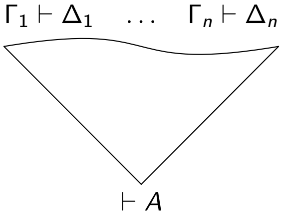


Sequent calculus can be used to implement a decision procedure, that is, a terminating and effective procedure to check if A is a theorem in LK or LJ.



Desirable features for root-first proof search, I

Wish 1: Don't have to make a "guess" when going from the conclusion to the premiss of a rule.

$$\frac{\Gamma \Rightarrow \Delta, \bar{A} \qquad \bar{A}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Cut-free sequent calculus

$$\begin{array}{ccc} \Gamma \vdash_{S} \Delta & \overset{\text{soundness}}{\underset{\text{completeness}}{\Longleftrightarrow}} & \Gamma \models \Delta \\ \Downarrow & & \\ \Gamma \vdash_{S^-} \Delta & & \end{array}$$

for $S = \{\mathsf{LK}, \mathsf{LJ}\}$ and S^- denoting $\mathsf{S} \backslash \{\mathsf{cut}\}$

Soundness $\quad \Gamma \vdash_{S} \Delta \Longrightarrow \Gamma \models \Delta$

Completeness $\quad \Gamma \models \Delta \Longrightarrow \Gamma \vdash_{S} \Delta \Longrightarrow \Gamma \vdash_{S^-} \Delta$

Desirable features for root-first proof search, II

Wish 2: don't have to make a "choice" when going from the conclusion to the premiss of a rule

$\rightsquigarrow$ choices introduce backtrack points in proof search

Example

$$\vee_{\mathbf{R}}1\,\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{\mathbf{R}}2\,\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{\mathbf{R}}\,\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\vee_{\mathbf{L}}2\,\frac{a \vdash b}{a \vdash a \vee b}$$

Desirable features for root-first proof search, II

Wish 2: don't have to make a "choice" when going from the conclusion to the premiss of a rule

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	multiplicative	additive
right	$\wedge_R \dfrac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B}$	$\wedge_R \dfrac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$
left	$\wedge_L \dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$	$\wedge_L \dfrac{\Gamma, A_i \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta}$

When possible:

- Choose the multiplicative version of one premisses-rules
- Choose the additive version of the two-premisses rules

Classical first-order logic

LK^- meet our two desiderata. Is this enough to ensure termination of root-first proof search? no

$$\forall_L \frac{A[t/x], \forall x.A, \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta} \qquad \forall_R \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} * \qquad \exists_L \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta} * \qquad \exists_R \frac{\Gamma \vdash \Delta, A[t/x], \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

$*$ y does not occur free in Γ, Δ, A

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\vdash \exists x \forall y (P(x,y)), P(z,k), P(x,z)}}{\vdash \exists x \forall y (P(x,y)), \forall y (P(z,y)), P(x,z)}\,\forall_R}{\vdash \exists x \forall y (P(x,y)), P(x,z)}\,\exists_R}{\cfrac{\vdash \exists x \forall y (P(x,y)), \forall y (P(x,y))}{\vdash \exists x \forall y (P(x,y))}\,\exists_R}\,\forall_R$$

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$$\exists_R \cfrac{\forall_R \cfrac{\exists_R \cfrac{\forall_R \cfrac{\vdots}{\vdash \exists x \forall y (P(x,y)), P(z,k), P(x,z)}}{\vdash \exists x \forall y (P(x,y)), \forall y (P(z,y)), P(x,z)}}{\vdash \exists x \forall y (P(x,y)), P(x,z)}}{\vdash \exists x \forall y (P(x,y)), \forall y (P(x,y))}}{\vdash \exists x \forall y (P(x,y))}$$

First-order logic is semi-decidable: for every formula A,

- ▷ If A is a theorem, the algorithm produces a proof;
- ▷ Otherwise, the algorithm either produces a failed proof or does not terminate.

Classical propositional logic

Propositional LK$^-$ meets our two desiderata. Is this enough to ensure termination of root-first proof search? yes

$$\text{init}\ \frac{}{p, \Gamma \vdash \Delta, p} \qquad \neg_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \neg_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\wedge_{\mathbf{L}}\ \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \qquad \vee_{\mathbf{L}}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_{\mathbf{R}}\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\rightarrow_{\mathbf{L}}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta} \qquad \rightarrow_{\mathbf{R}}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$$

Proof search strategy

- Apply the rules of LK$^-$ in whatever order
- The calculus has the subformula property: all formulas get decomposed into smaller ones
- Proof search comes to an end in a finite number of steps.

Classical propositional logic is decidable.

Intuitionistic propositional logic

Propositional LJ$^-$ is cut-free, but we have to make choices on some rules.

Is this enough to ensure termination of root-first proof search? no

$$\text{init}\ \frac{}{p, \Gamma \vdash p} \qquad \bot_{\mathbf{L}}\ \frac{}{\bot, \Gamma \vdash A}$$

$$\wedge_{\mathbf{L}}\ \frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \qquad \wedge_{\mathbf{R}}\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \vee_{\mathbf{L}}\ \frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C} \qquad \vee_{\mathbf{R}} i\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ i \in \{1, 2\}$$

$$\to_{\mathbf{L}}\ \frac{A \to B, \Gamma \vdash A \quad B, \Gamma \vdash C}{A \to B, \Gamma \vdash C} \qquad \to_{\mathbf{R}}\ \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}$$

Intuitionistic propositional logic

Propositional LJ^- is cut-free, but we have to make choices on some rules.

Is this enough to ensure termination of root-first proof search? no

$$\text{init}\;\frac{}{p, \Gamma \vdash p} \qquad \bot_{\mathbf{L}}\;\frac{}{\bot, \Gamma \vdash A}$$

$$\wedge_{\mathbf{L}}\;\frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C} \quad \wedge_{\mathbf{R}}\;\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \vee_{\mathbf{L}}\;\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C} \quad \vee_{\mathbf{R}}i\;\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\; i \in \{1, 2\}$$

$$\to_{\mathbf{L}}\;\frac{A \to B, \Gamma \vdash A \quad B, \Gamma \vdash C}{A \to B, \Gamma \vdash C} \qquad \to_{\mathbf{R}}\;\frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}$$

$$\to_{\mathbf{R}}\;\cfrac{\to_{\mathbf{L}}\;\cfrac{\to_{\mathbf{L}}\;\cfrac{\to_{\mathbf{L}}\;\cfrac{\vdots}{a \to \bot \vdash a} \quad \bot \vdash a}{a \to \bot \vdash a} \quad \bot \vdash a}{a \to \bot \vdash a} \quad \bot \vdash \bot}{\cfrac{a \to \bot \vdash \bot}{\vdash (a \to \bot) \to \bot}}$$

Solution

Proof search strategy [Troelstra, Schwichtenberg, Basic proof theory]

- ▷ Apply rule $\rightarrow_\mathsf{L}$ after all the other rules.
- ▷ Before applying rule $\rightarrow_\mathsf{L}$ to a sequent $\Gamma \vdash \Delta$, check if there is a sequent in the branch containing the same formulas as $\Gamma \vdash \Delta$.
- ▷ The calculus has the subformula property: all formulas get decomposed into smaller ones.
- ▷ Proof search comes to an end in a finite number of steps.
- ▷ If a proof has not been found, backtrack on all choice points.

Intuitionistic propositional logic is decidable.

Many other solutions exist: [Dyckhoff, Intuitionistic decision procedures since Gentzen, 2015]

References

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