El Filtro CIC

May 22, 2024

1 Building blocks

1.1 The decimator

Fig. 1 shows the operation of a decimator block: given the decimation rate R, only those samples of the input $x[n]$ that occur at time equal to multiples of R are presented on the output. Notice that R must be a positive integer.

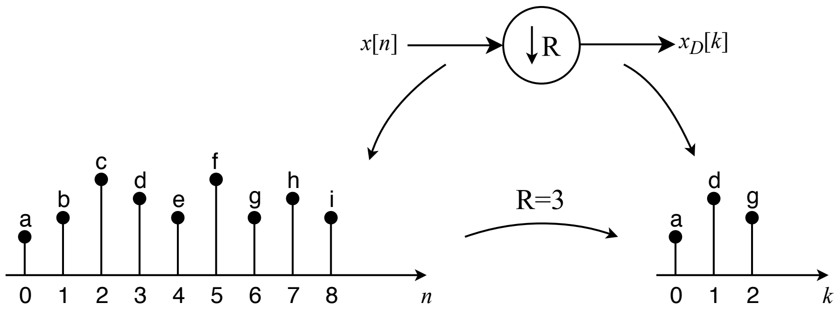

Figure 1: Decimator block with decimation rate $R = 3$.

When the input samples $x[n]$ are taken by sampling a continuous time signal $x(t)$ with a sampling period T_S or equivalentely with a sampling frequency $f_S = 1/T_S$, the decimator changes the sampling frequency from f_S to $\tilde{f}_S = f_S/R$.

Mathematically, the operation of the decimator can be written as:

$$x_D[k] = x[Rk]; \quad n = Rk; \quad k = n/R \tag{1}$$

Hence, the Z-transform of $x_D[k]$ follows:

$$X_D(z) = \sum_{k=-\infty}^{\infty} x_D[k]z^{-k} = \sum_{k=-\infty}^{\infty} x[Rk]z^{-k} = \sum_{n=-\infty}^{\infty} \tilde{x}[n]z^{-n/R} \tag{2}$$

where

$$\tilde{x}[n] = \begin{cases} x[n] & \text{if } n = kR \text{ (i.e. n is integer multiple of R)} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Thus,

$$X_D(z) = \tilde{X}(z^{1/R}) \tag{4}$$

Let it be

$$c_R[n] = \begin{cases} 1 & \text{if } n = kR \text{ (i.e. n is integer multiple of R)} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

then

$$c_R[n] = \frac{1}{R}\sum_{k=0}^{R-1} W_R^{-kn} \tag{6}$$

where

$$W_R = e^{-j\frac{2\pi}{R}} \tag{7}$$

Notice that $\tilde{x}[n] = c_R[n]x[n]$ then

$$\begin{aligned} \tilde{X}(z) &= \textstyle\sum_{n=-\infty}^{\infty} \left(\frac{1}{R}\sum_{k=0}^{R-1} W_R^{-kn}\right) x[n]z^{-n} \\ &= \textstyle\frac{1}{R}\sum_{k=0}^{R-1} \left(\sum_{n=-\infty}^{\infty} x[n]W_R^{-kn}\right) z^{-n} \\ &= \textstyle\frac{1}{R}\sum_{k=0}^{R-1} \sum_{n=-\infty}^{\infty} x[n] \left(zW_R^k\right)^{-n} \\ &= \textstyle\frac{1}{R}\sum_{k=0}^{R-1} X\left(zW_R^k\right) \end{aligned} \tag{8}$$

By replacing eq. (8) in eq. (4), following is obtained:

$$X_D(z) = \tilde{X}\left(z^{1/R}\right) = \frac{1}{R}\sum_{k=0}^{R-1} X\left(z^{1/R}W_R^k\right) \tag{9}$$

The frequency spectrum of X_D is given by evaluating $z = e^{j\tilde{\Omega}}$:

$$X_D(e^{j\tilde{\Omega}}) = \frac{1}{R}\sum_{k=0}^{R-1} X\left(e^{\frac{j\tilde{\Omega}-2\pi k}{R}}\right) \tag{10}$$

Assume that $x[n]$ such that $X(\Omega) = 0\ \forall|\Omega| \geq \Omega_M$ and $\Omega_M R < \pi$. Thus, if $\tilde{\Omega} = \Omega R$ then

$$X_D(e^{j\tilde{\Omega}R}) = \frac{1}{R}X\left(e^{j\Omega}\right) \tag{11}$$

Under the same assumptions, suppose that $X_M = \max|X(\Omega)|$, then

$$X_{D_M} = \max|X_D(\tilde{\Omega})| = \frac{1}{R}X_M \tag{12}$$

Fig. 2 shows the effects of the decimator when $R = 3$ under the above assumptions. Observe that under these conditions, $X_{D_m} = X_M/R$ and $\tilde{\Omega}_M = R\Omega_M$.

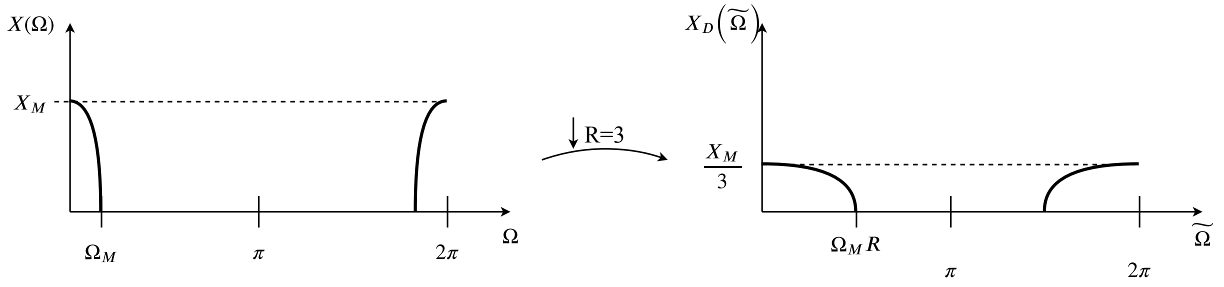

Figure 2: Decimator block effect when $R = 3$.

1.2 The expander

Fig. 3 shows the operation of an expander block: given the expansion rate R, R zeros are added between input samples Notice that R must be a positive integer.

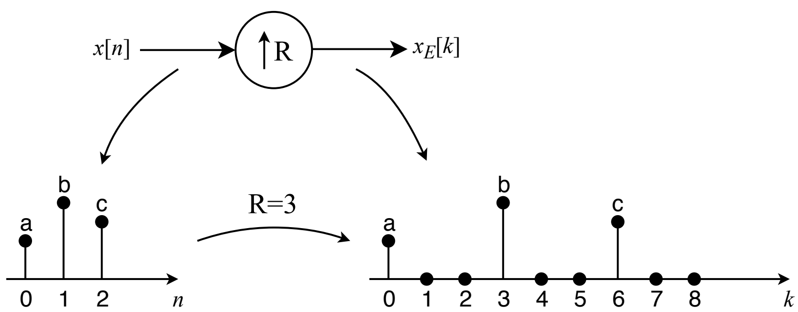

Figure 3: Expander block with expansion rate $R = 3$.

Mathematically, the operaton of the expander can be written as:

$$x_E[k] = \begin{cases} x[k/R] & \text{if } k \text{ is integer multiple of } R\text{, i.e. } k = nR \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Hence, the Z-transform of $x_E[k]$ follows:

$$X_E(z) = \sum_{k=-\infty}^{\infty} x_E[k]z^{-k} = \sum_{n=-\infty}^{\infty} x[n]z^{-nR} = X(z^R) \tag{14}$$

Thus,

$$X_E(e^{j\tilde{\Omega}}) = X(e^{j\tilde{\Omega}R}) \tag{15}$$

Let it be called $\max|X(\Omega)| = X_M$. If $X(\Omega) = 0$ for $|\Omega| \geq \Omega_M$, $\tilde{\Omega} = \Omega/R$ then $X_E(e^{j\Omega/R}) = X(e^{j\Omega})$ as seen in Fig. 4.

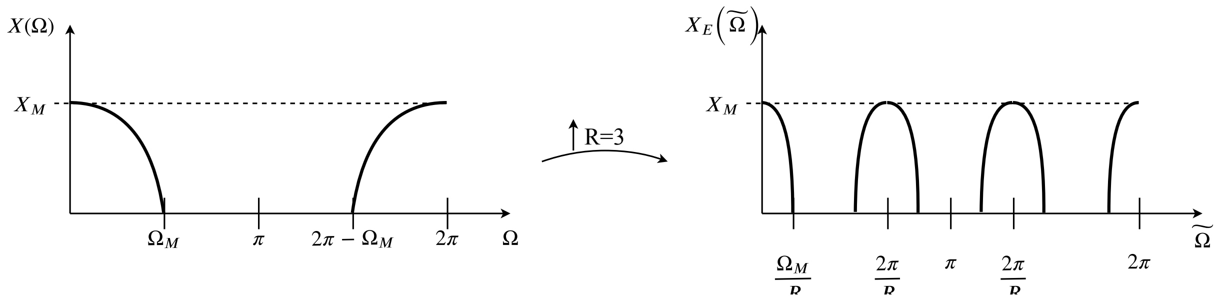

Figure 4: Expander block effect when $R = 3$.

1.3 Noble identities

Figure 5 illustrates the so-called noble identities for commuting expanders and decimators. Note that expanders and decimators are linear, time-varying operators. Therefore, operation order is important.

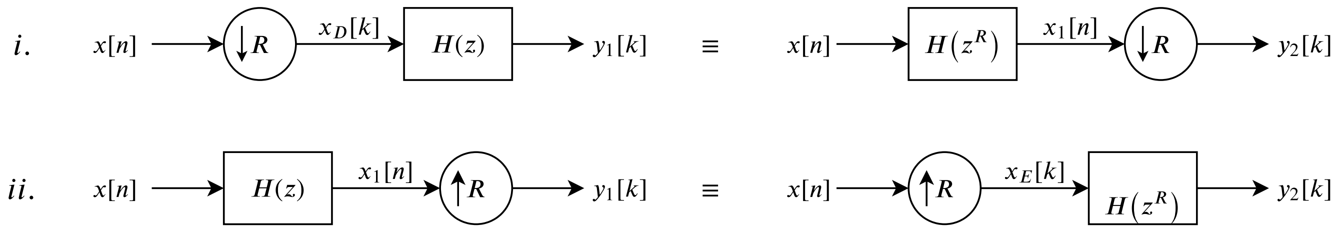

Figure 5: Noble identities.

Proof:

1. From eq. (9),

$$Y_2(z) = \frac{1}{R}\sum_{k=0}^{R-1} X_1\left(z^{1/R}W_R^k\right) \tag{16}$$

while

$$X_1(z) = X(z)H(z^R) \tag{17}$$

Thus,

$$X_1(z^{1/R}W_R^k) = X(z^{1/R}W_R^k)H(zW_R^{kR}) \tag{18}$$

Notice that $W_R^{kR} = (e^{-j\frac{2\pi}{R}})^{kR} = e^{-j2\pi k} = 1$ then

$$X_1(z^{1/R}W_R^k) = X(z^{1/R}W_R^k)H(z) \tag{19}$$

Eq. (16) can be written as

$$Y_2(z) = \frac{1}{R}\sum_{k=0}^{R-1} X\left(z^{1/R}W_R^k\right)H(z) \tag{20}$$

On the other hand,

$$Y_1(z) = X_D(z)H(z) \tag{21}$$

From eq. (9),

$$Y_1(z) = \frac{1}{R}\sum_{k=0}^{R-1} X(z^{1/R}W_R^k)H(z) \tag{22}$$

From eq. (20) and eq. (22), it is concluded that $Y_1(z) = Y_2(z)$ and then the identity holds.

2.

$$Y_2(z) = H(z^R)X_E(z) = H(z^R)X(z^R) \tag{23}$$

On the other hand,

$$X_1(z) = H(z)X(z) \tag{24}$$

From eq. (14), $Y_1(z)$ can be written as

$$Y_1(z) = X_1(z^R) = H(z^R)X(z^R) \tag{25}$$

From eq. (23) and eq. (25), it is concluded that $Y_1(z) = Y_2(z)$ and then the identity holds.

Noble identities can be re written as shown in Figure 6.

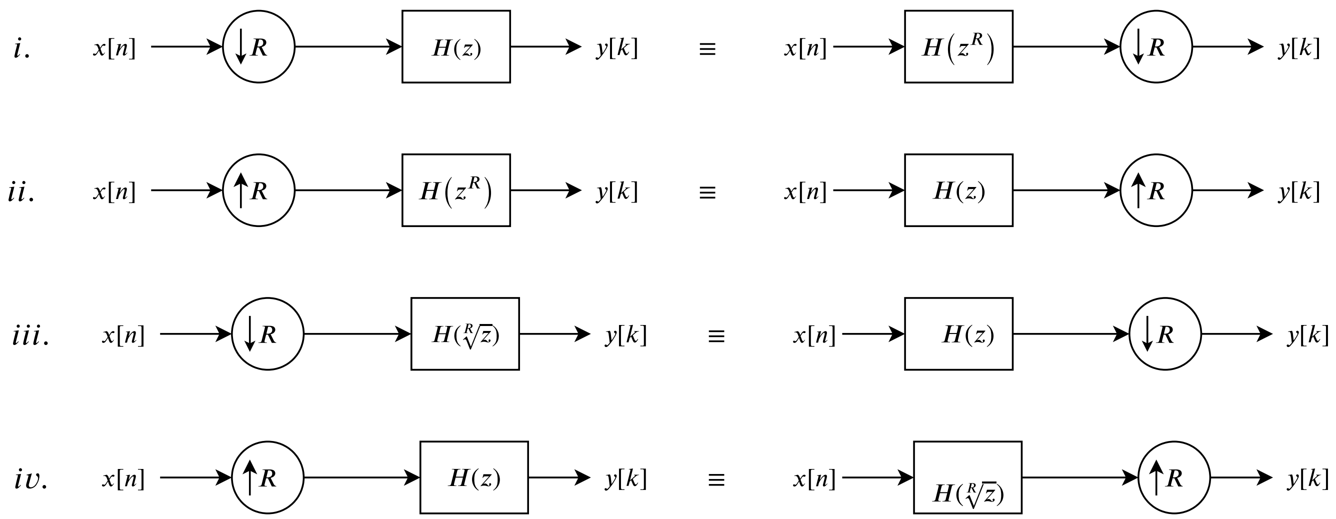

Figure 6: Noble identities.

1.4 The integrator

Figure 7 shows the block diagram of an integrator.

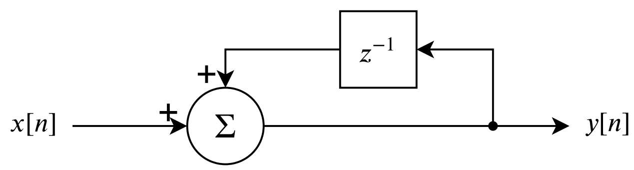

Figure 7: Integrator block diagram.

The integrator output $y[n]$ is given by

$$y[n] = x[n] + y[n-1] \tag{26}$$

Taking the Z-transform of both sides,

$$Y(z) = X(z) + z^{-1}Y(z) \tag{27}$$

Then, the integrator transfer function is

$$H_I(z) = \frac{1}{1 - z^{-1}} \tag{28}$$

Notice that the integrator is not stable since it has a pole on the unit circle. Figure 8 and Figure 9 show the impulse response and the pole-zero diagram of the integrator.

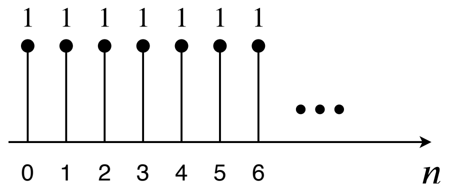

Figure 8: Integrator impulse response.

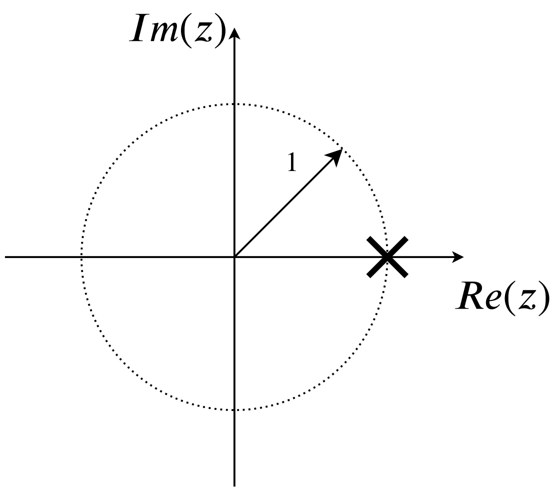

Figure 9: Integrator zero-pole diagram.

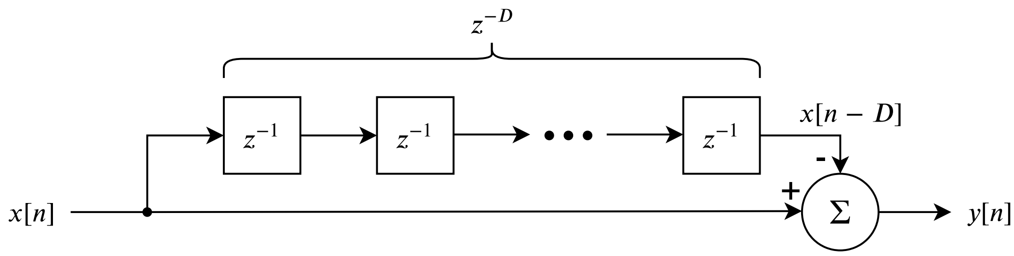

Figure 10: Comb filter block diagram.

1.5 The comb filter

Figure 10 shows the block diagram of a comb filter. The parameter D is known as the differential delay of the comb filter.

The comb filter output $y[n]$ is given by

$$y[n] = x[n] - x[n-D] \tag{29}$$

Taking the Z-transform of both sides,

$$Y(z) = X(z) - z^{-D}X(z) \tag{30}$$

Then, the comb filter transfer function is

$$H_c(z) = 1 - z^{-D} \tag{31}$$

Notice that the comb is a stable system with no poles and zeros distributed symmetrically around the the unit circle. Figure 11 shows the impulse response of a comb filter.

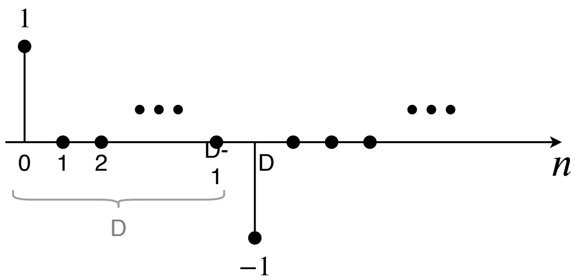

Figure 11: Comb filter impulse response.

Figure 12 shows the frequency response and 13 shows the pole-zero diagram of the comb filter.

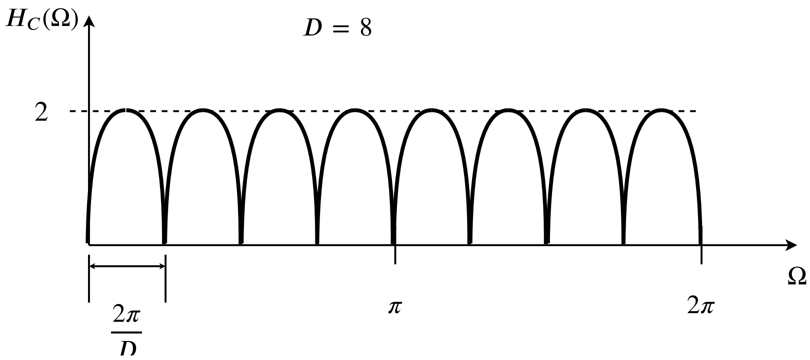

Figure 12: Comb frequency response diagram.

To see zeros are placed symmetrically around the unit circle, proceed as follows:

$$H_c(e^{j\Omega}) = 1 - e^{-j\Omega D} = 1 - \cos(\Omega D) + j\sin(\Omega D) \tag{32}$$

Then,

$$\begin{aligned} |H_c(e^{j\Omega})| &= \sqrt{[1-\cos(\Omega D)]^2 + \sin^2(\Omega D)} \\ &= \sqrt{1 - 2\cos(\Omega D) + \cos^2(\Omega D) + \sin^2(\Omega D)} \\ &= \sqrt{2 - 2\cos(\Omega D)} \end{aligned} \tag{33}$$

Hence, $|H_c(e^{j\Omega})| = 0$ if and only if $\cos(\Omega D) = 1$ and this happens only if $\Omega = 2\pi k/D$ with $k = 0, 1, 2, \ldots, D-1$.

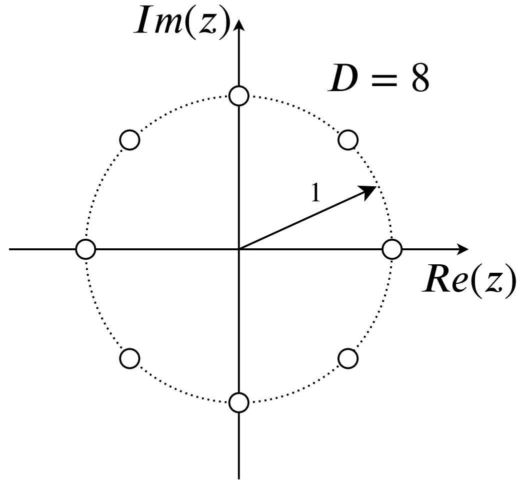

Figure 13: Comb zero-pole diagram.

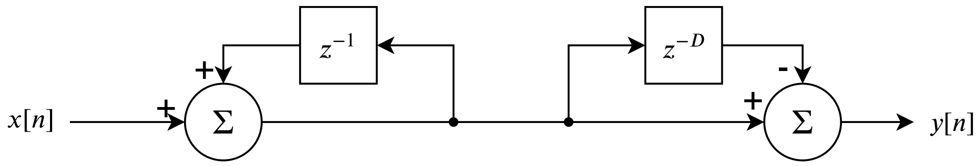

Figure 14: An integrator and a comb filter cascaded.

2 The CIC filter

Figure 14 shows an integrator and a comb filter cascaded.

From eqs. (28) and (31), the transfer function is given by

$$H(z) = H_I(z)H_c(z) = \frac{1 - z^{-D}}{1 - z^{-1}} \tag{34}$$

Notice that the pole at $\Omega = 0$ (i.e. $z = 1$) is perfectly cancelled out with one of the comb zeros and the system is then stable. Figure 15 shows the zero-pole diagram of the system. Also notice that

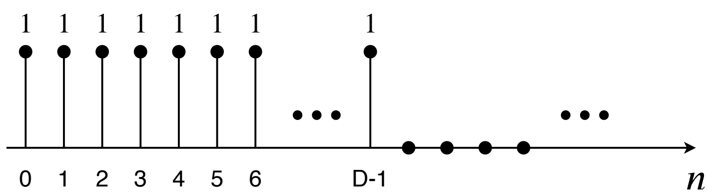

Figure 15: Zero-pole diagram of an integrator and a comb filter cascaded.

$$\begin{aligned}
1 - z^{-D} &= 1 + [(z^{-1} - z^{-1}) + (z^{-2} - z^{-2}) + \ldots + (z^{-D+1} - z^{-D+1})] + z^{-D} \\
&= (1 + z^{-1} + z^{-2} + \ldots + z^{-D+1}) - (z^{-1} + z^{-2} + \ldots + z^{-D+1} + z^{-D}) \\
&= (1 + z^{-1} + z^{-2} + \ldots + z^{-D+1}) - z^{-1}(1 + z^{-1} + \ldots + z^{-D+1}) \\
&= (1 - z^{-1})(1 + z^{-1} + z^{-2} + \ldots + z^{-D+1}) \\
&= (1 - z^{-1}) \textstyle\sum_{k=0}^{D-1} z^{-k}
\end{aligned} \tag{35}$$

Then, eq. (34) becomes

$$H(z) = \frac{1 - z^{-D}}{1 - z^{-1}} = \sum_{k=0}^{D-1} z^{-k} \tag{36}$$

Eq. (36) shows that the impulse response of the system corresponds to a moving average (MA) of length D as shown in Figure 16.

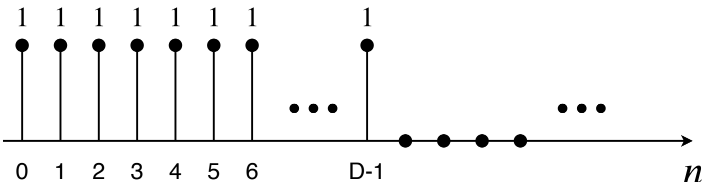

Figure 16: Impulse response of an integrator and a comb filter cascaded.

It is easy to see from the Figure 16 the impulse response is equivalent to a MA filter. Being a time finite impulse response, it is then a stable FIR system. Actually, it is the only FIR filter known which comprises a feedback loop. Notice also that since the impulse response is symmetric in time then the filter has linear phase.

To find the frequency response, let us start from eq. (34) and replace z by $e^{j\omega}$:

$$\begin{aligned} H(e^{j\Omega}) &= \frac{1-e^{-j\Omega D}}{1-e^{-j\Omega}} \\ &= \frac{e^{-j\Omega D/2}e^{j\Omega D/2}-e^{-j\Omega D}e^{-j\Omega D/2}e^{j\Omega D/2}}{e^{-j\Omega/2}e^{j\Omega/2}-e^{-j\Omega}e^{-j\Omega/2}e^{j\Omega/2}} \\ &= \frac{e^{-j\Omega D/2}\left(e^{j\Omega D/2}-e^{-j\Omega D/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega D/2}-e^{-j\Omega/2}\right)} \\ &= \frac{2je^{-j\Omega D/2}\sin(\Omega D/2)}{2je^{-j\Omega/2}\sin(\Omega/2)} \\ &= \frac{\sin(\Omega D/2)}{\sin(\Omega/2)}e^{-j\Omega(D-1)/2} \end{aligned} \tag{37}$$

From the above term $e^{-j\Omega(D-1)/2}$ it can be said that the system has linear phase as it was expected. The maximum gain is at DC ($\Omega = 0$):

$$\lim_{\Omega\to 0}|H(e^{j\Omega})| = \left|\frac{\sin(\Omega D/2)}{\sin(\Omega/2)}\right| = \lim_{\Omega\to 0}\left|\frac{\Omega D/2}{\Omega/2}\right| = D \tag{38}$$

To find the zeros in the transfer function recall that $sin(\theta) = 0$ if and only if $\theta = k\pi$ for k an integer. Then, $|H(\omega)| = 0$ iff $\omega D/2 = k\pi$ iff $\omega = 2\pi k/D$ for $k = 0, 1, 2, \ldots, D-1$.

Figure 17 shows the case when $D = 8$.

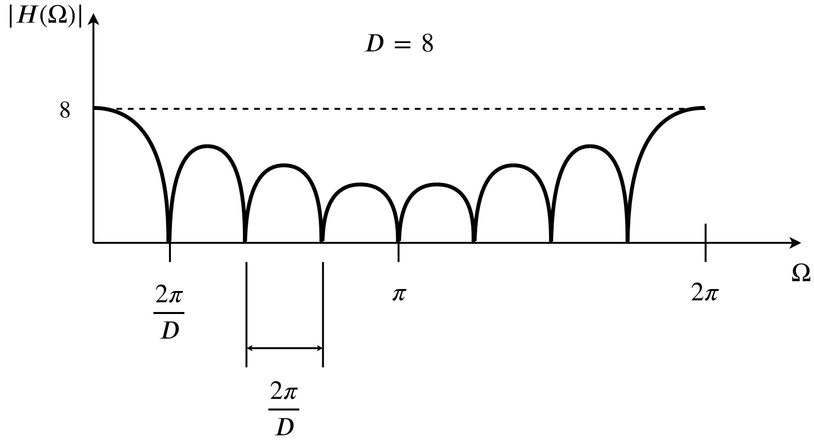

Figure 17: Impulse response of an integrator and a comb filter cascaded.

3 The CIC downsampler filter

Now, take the system of Figure 14 and add a decimator block at the output as seen in Figure 18. Then,

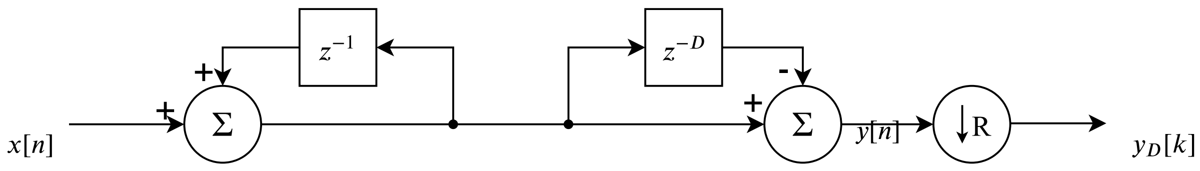

Figure 18: System of Figure 14 with a decimator at the output.

$$H_D(z) = \frac{1}{R}\sum_{k=0}^{R-1} H(z^{1/R}W_R^k) \tag{39}$$

where $H(z) = \sum_{k=0}^{D-1} z^{-k}$.

Applying noble identities, it yields the system in Figure 19 which is known as CIC downsampler with order 1. Notice that to be the system implementable, D must be an integer multiple of R, i.i. $D = MR$. Commonly, it is

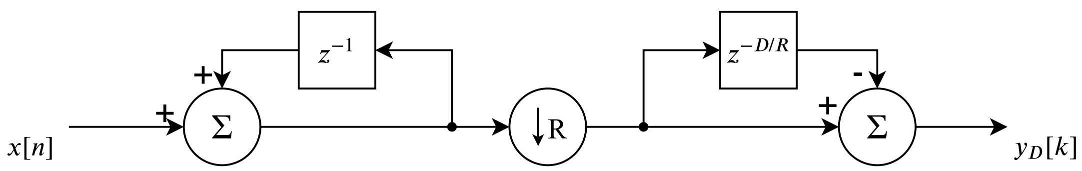

Figure 19: First order CIC downsampler.

chosen $M = 1$ and then $D = R$ and then the number of registers in the comb implementation is minimized getting the system in Figure 20. Actually from now on, it will consider $D = R$ unless otherwise stated. Another common choose is to take $R = 2^K$, being K a positive integer.

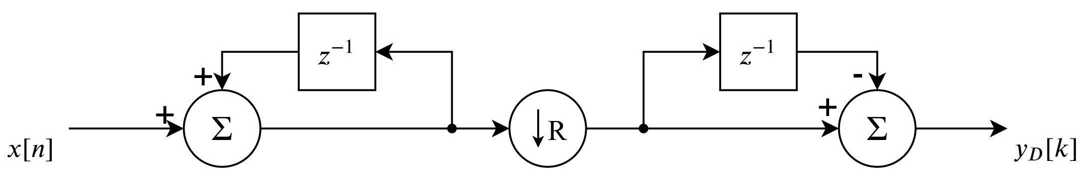

Figure 20: First order CIC downsampler with $D = R$.

An N-order CIC downsampler is shown in Fig 21, where I and C are integrators and conv filters respectively.

The structure of Figure 21 is equivalent to that one in Figure 22.

By applying noble identity, Figure 23 is obtained, where

$$H_{CIC}(z) = H_I^N(z)H_C^N(z^R) = \left(\frac{1-z^{-R}}{1-z^{-1}}\right)^N = \left(\sum_{k=0}^{R-1} z^{-k}\right)^N \tag{40}$$

is the N-order CIC downsampler transfer function at the input sampling rate.

From eq. (37) we can say that the zeros are placed in the same position of the first order CIC, i.e. at integer multiples of $2\pi/R$ (remember that it is being assumed that $D = R$). The DC gain is $H_{CIC}(e^{j0}) = R^N$ and the phase is linear givem by $e^{-j\Omega N(R-1)/2}$.

3.0.1 Aliasing effect

If the input signal $x[n]$ has a bandwidth wider than the first notch of H_{CIC}, then aliasing corrupts the output signal $y_D[k]$. See Figure 24. After decimation, eq. (10) shows that each side lobe of H_{CIC} is folded into the $[0, 2\pi]$ range.

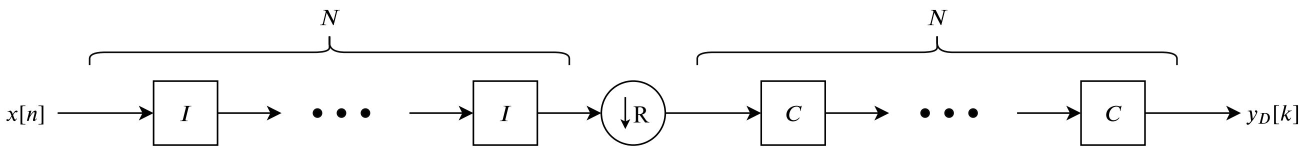

Figure 21: N-order CIC downsampler.

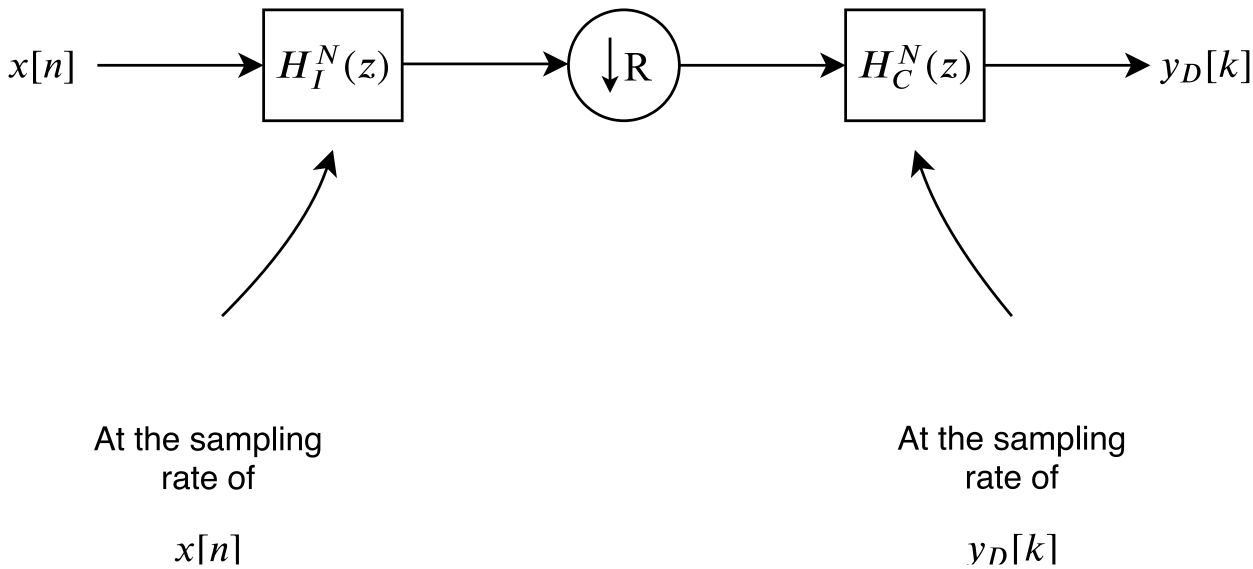

Figure 22: Equivalent system to Figure 21.

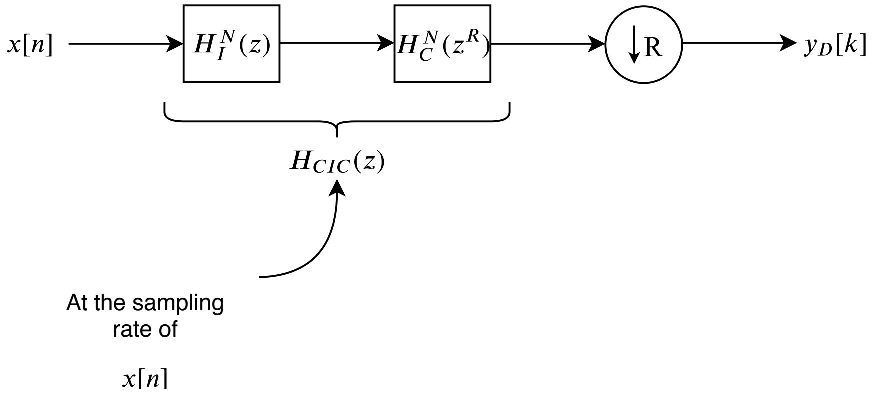

Figure 23: Equivalent system to Figure 23.

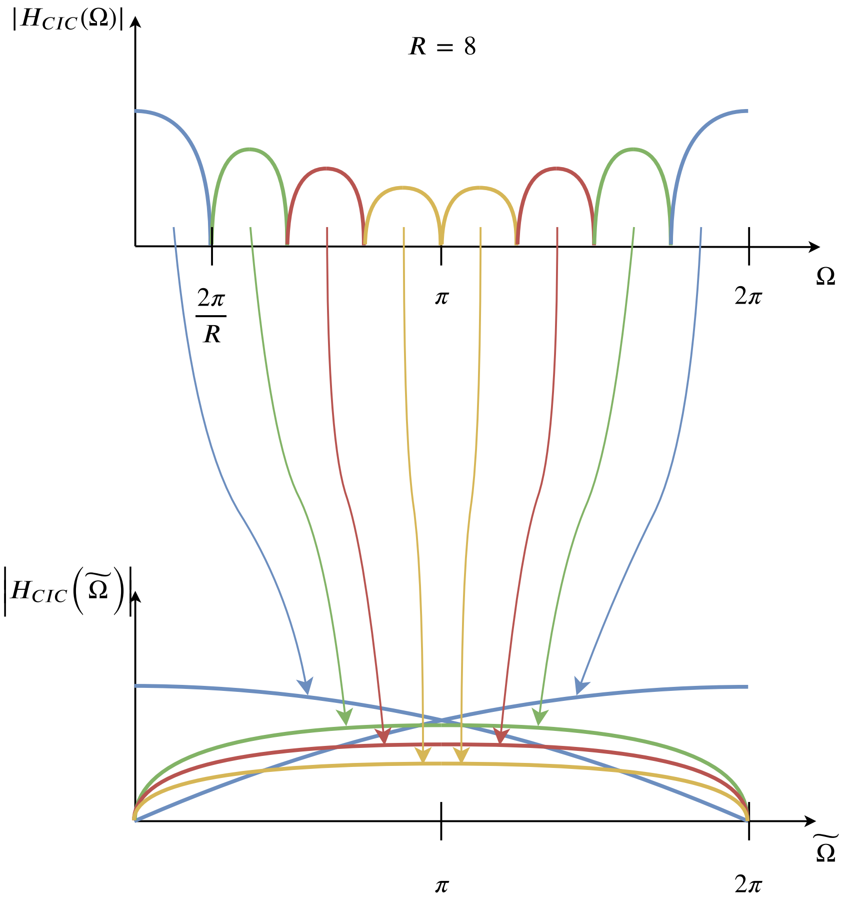

Figure 24: CIC frequency response side lobes are folded into the range $[0, 2\pi]$.

3.1 Step response

It is known that $H_{CIC}(z)$ is equivalent than N MA filters cascaded, i.e. $H_{CIC}(z) = \left(\sum_{k=0}^{R-1} z^{-k}\right)^N$. Each of these moving average filters has a inpulse response of length R:

$$h_{MA}[n] = \begin{cases} 1 & \text{if } n = 0, 1, \ldots, R-1 \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

Then, the step response of a MA filter is given by

$$g[n] = u[n] * h_{MA}[n] \tag{42}$$

Figure 25 shows the case for $R = 8$. At the sample $n = R - 1$ the output is settle (reaches its final value). In general, any FIR filter reaches the final value of the step response at the sample $n = L - 1$ then the length of the impulse response is L. For a N-order CIC downsampler, the impulse response of H_{CIC} has length $L = (R-1)N + 1$

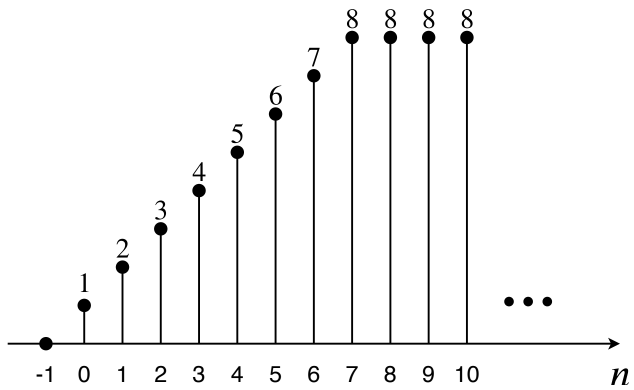

Figure 25: Step response of an MA filter of $L = 8$.

which means that for the sample $n_0 = (R-1)N$ the output reaches the final value to a input step. figure 26 shows an example of a 3rd order CIC downsampler. It can be observed that in the decimated domain, the final value is reached at $k_0 = \lceil n_0/R \rceil = \lceil (R-1)N/R \rceil$. Commonly $R \gg N$ and then $k_0 \approx N$.

3.2 Digital implementation

The first thing to consider in the CIC downsampler implementation is that a first order CIC can be replaced by an accumulate and dump (AaD) circuit. See Figure 27. At time $Rk - 1$, the value of $x_I[Rk-1]$ is given by:

$$x_I[Rk-1] = \sum_{i=R(k-1)}^{Rk-1} x[i] \tag{43}$$

When the R-module counter reaches $R - 1$, the value in $x_I[n]$ is dumped into the output register. Thus, the output register corresponds to the sequence $x_I[Rk-1]$ given by eq. (43).

Now, see Figure 28. The integrator output is given by:

$$x_I[n] = \sum_{i=0}^{n} x[i-1] \tag{44}$$

Then, after decimation, it becomes

$$x_{I_D}[k] = x_I[Rk] = \sum_{i=0}^{Rk} x[i-1]; \quad n = Rk; \quad k = n/R \tag{45}$$

Thus, by applying eq. (29), $y[k]$ is obtained:

$$\begin{aligned} y[k] &= x_{I_D}[k] - x_{I_D}[k-1] \\ &= \textstyle\sum_{i=0}^{Rk} x[i-1] - \sum_{i=0}^{R(k-1)} x[i-1] \\ &= \textstyle\sum_{i=-1}^{Rk-1} x[i] - \sum_{i=-1}^{R(k-1)-1} x[i] \\ &= \textstyle\sum_{i=R(k-1)}^{Rk-1} x[i] \end{aligned} \tag{46}$$

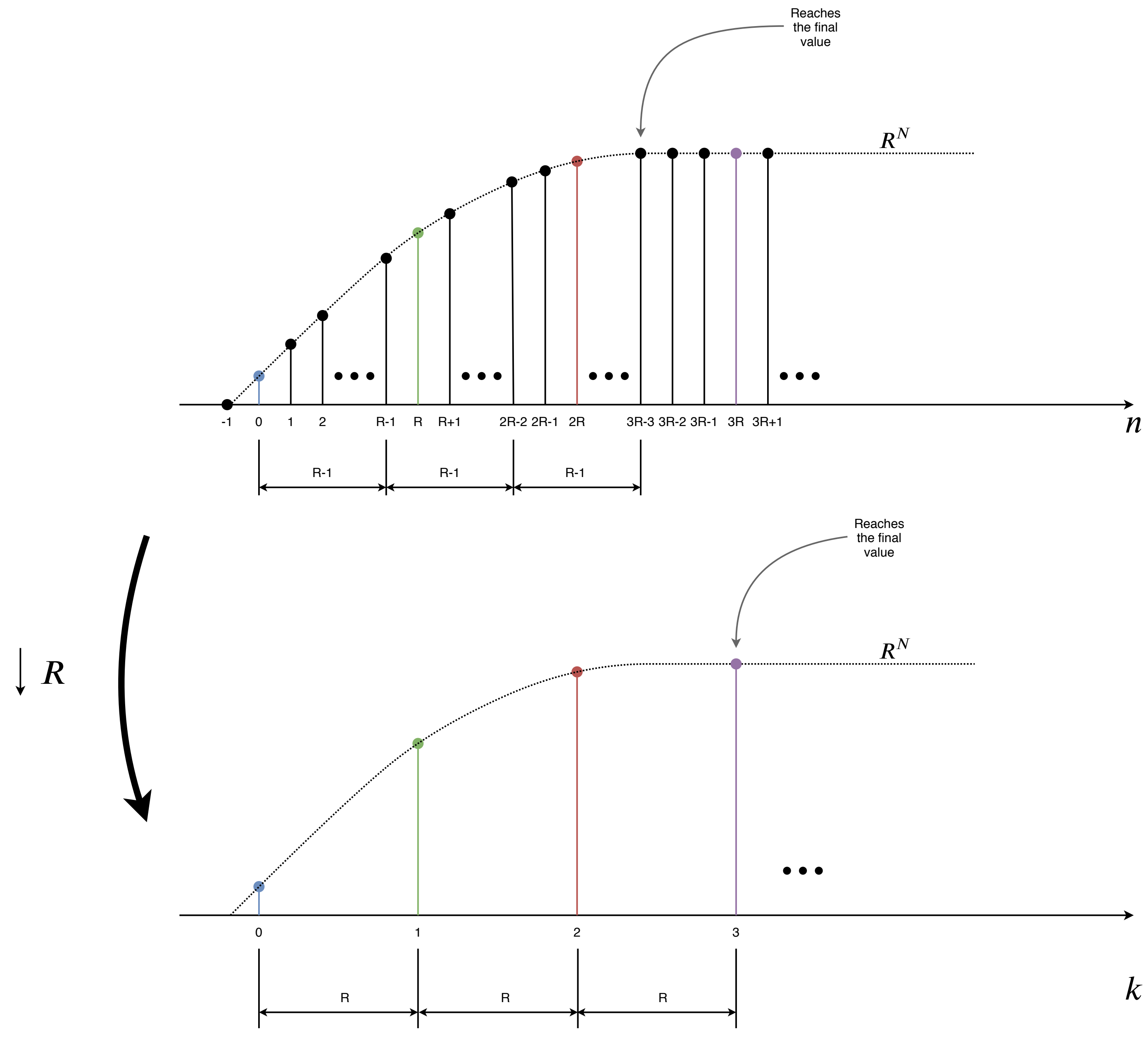

Figure 26: Step response of an 3rd order CIC downsampler.

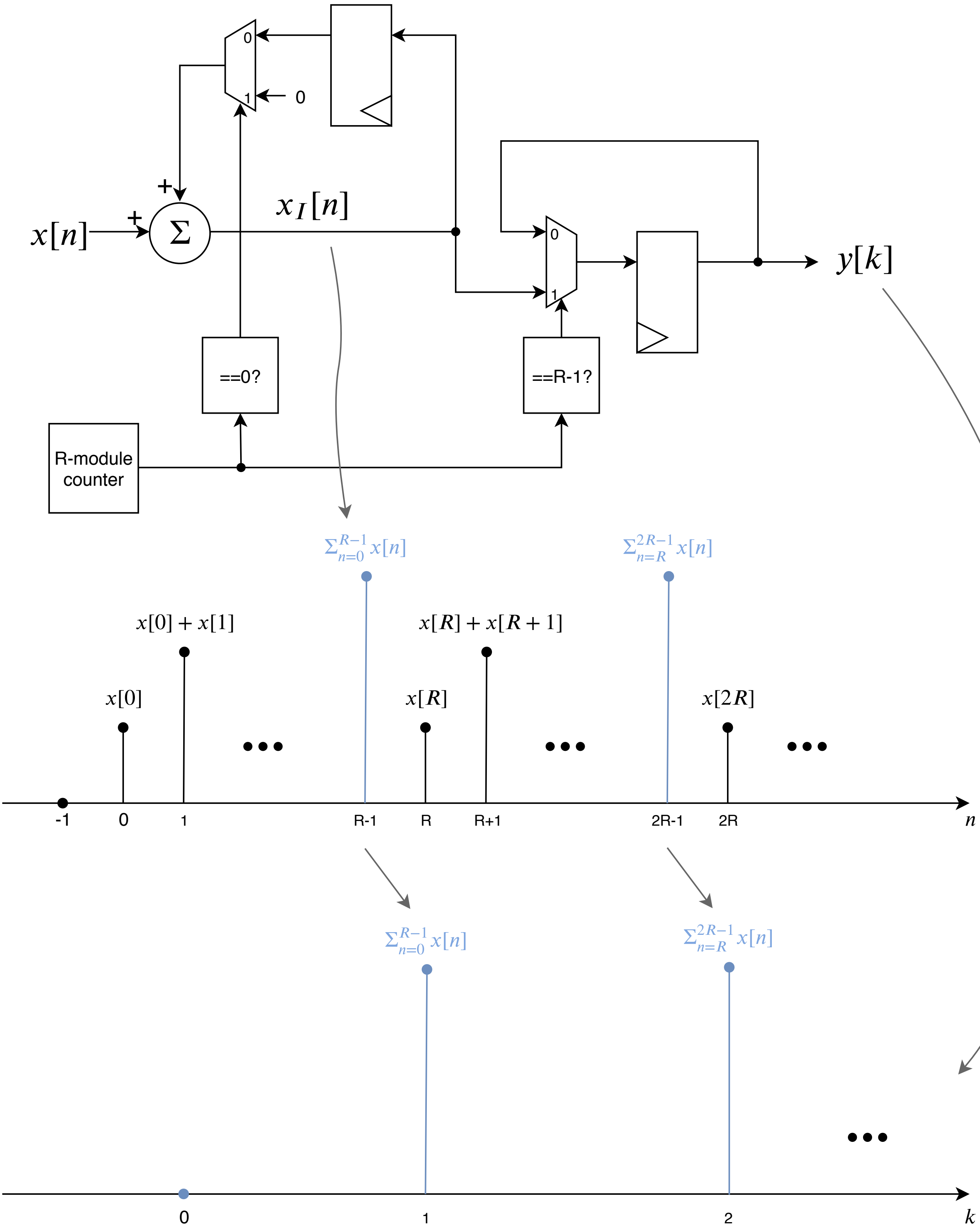

Figure 27: Accumulate and dump operation.

Notice that eqs. (43) and (46) are the same and then both systems are equivalent.

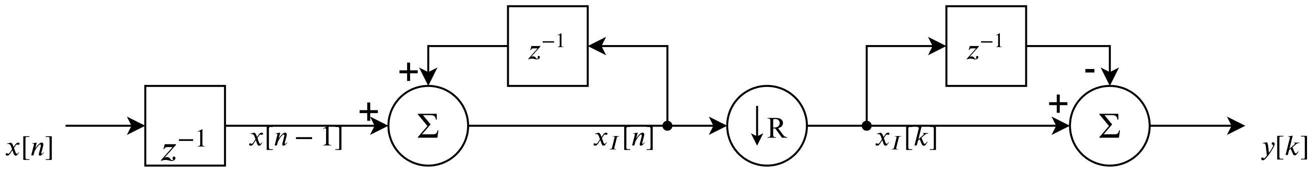

Figure 28: First order CIC downsampler with input delay.

In general, a N-order CIC downsampler is implemented in a real circuit as shown in Figure 29.

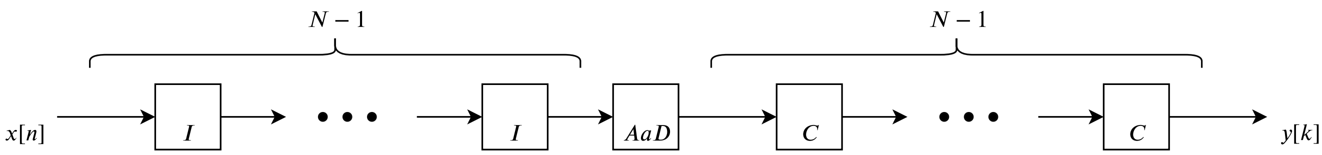

Figure 29: N-order CIC downsampler implemented with an accumulate and dump circuit.

3.3 Number of bits per stage

Suppose a set of integer numbers $\{x_0, x_1, \ldots, x_{N-1}\} \subset \mathbb{Z}$ and all of them are represented in two's complement. Since it is a finite set of numbers then exits a positive integer $B \in \mathbb{N}$ such that

$$-2^{B-1} \leq y = \sum_{i=0}^{N-1} x_i < 2^{B-1} - 1 \tag{47}$$

Let B_0 the smallest B that accomplishes eq. (47). Then it is easy to prove that it is sufficient to represent any partial term of the summation with B_0 bits to ensure correctness of the result value y.

Example 1:
$B_0 = 4$, then $-8 \leq y < 7$, $x_0 = 5$, $x1 = 4$, $x2 = -6$, then $y = x_0 + x_1 + x_2 = 3$
$x_0 + x_1 = 9 \equiv 1001_{2C} \equiv -7_{10}$
$(x_0 + x_1) + x_2 \equiv 1001_{2C} + 1010_{2C} = 0011_{2C} \equiv 3_{10}$

Example 2:
$B_0 = 4$, then $-8 \leq y < 7$, $x_7 = 5$, $x1 = -4$, $x2 = -5$, then $y = x_0 + x_1 + x_2 = -2$
$x_1 + x_2 = -9 \equiv 0111_{2C} \equiv +7_{10}$
$x_0 + (x_1 + x_2) \equiv 0111_{2C} + 0111_{2C} = 1100_{2C} \equiv -2_{10}$

Suppose now that the input $x_{[n]}$ to a N-order CIC downsampler has B_x bits in 2C representation. The minimum value that $x_{[n]}$ can be is $x_{min} = \min\{x[n]\} = -2^{B_x-1}$. The maximum filter gain is at $\Omega = 0$. Suppposse then that the input is constant over time having its minimum value, i.e. $x[n] = -2^{B_x-1}$ for all n. Then, the output value of the filter will be $y[n] = -2^{B_x-1}R^N$, where R is the decimation rate, N is the order of the filter and R^N is the maximum gain.

Commonly, R is choosen such that it is a power of 2, i.e. $R = 2^K$. Then the minimum value of the output $y[n]$ will be given by

$$y_{min} = \min\{y[n]\} = R^N x_{min} = -2^{KN}2^{B_x-1} \tag{48}$$

Thus, the number of bits to represent y_{min} is given by $\log_2 2^{KN}2^{B_x-1} = NK + B_x - 1$. Finally, an additional bit is needed to represent positive values of $y[n]$:

$$B_y = NK + B_x \tag{49}$$

As mentioned above, to ensure the integrity of the final result $y[n]$ with B_y it is enough to make sure that any internal adder,sustractor and register in the CIC downsampler is B_y bits wide.

To exemplify this, consider the CIC downsampler of Figure 20 and suppose the input $x[n] = x_{min} = -2^{B_x - 1}$ for $n \geq 0$. In such a case, the integrated value of $x[n]$ will be

$$x_I[n] = \sum_{i=0}^{n} x[i] = \sum_{i=0}^{n} x_{min} = n x_{min} \tag{50}$$

After decimation, it is obtained:

$$x_{I_D}[k] = x_I[Rk] = Rkx_{min} \tag{51}$$

Finally, the output is given by

$$\begin{aligned} y[k] &= x_{I_D}[k] - x_{I_D}[k-1] \\ &= x_I[Rk] - x_I[R(k-1)] \\ &= Rkx_{min} - R(k-1)x_{min} \\ &= Rx_{min} \end{aligned} \tag{52}$$

Being $R = 2^K$, then $y_{min} = 2^K x_{min}$. This result shows that to correctly represent the minimum value of $y[k]$, it is enough use $\log_2 2^K x_{min} = -\log_2 2^K 2^{B_x - 1} = K + B_x - 1$ and then $B_y = K + B_x$ to represent also the positive values of $y[k]$. Recall that, as seen before, it is sufficient to represent partial terms in the calculation of $y[k]$ with also B_y bits without arithmetic issues. This means that the intermediate result $x_{I_D}[k] = x_I[Rk] = Rkx_{min}$ must be represented with B_y bits and then the integrator too.

4 The CIC upsampler filter

Upsampling by a factor R is the process of inserting R-1 zerovalued samples between original samples in order to increase the sampling rate. The output sample rate is R times bigger than input rate. The circuit that performs this operation is known as *expander*.

The CIC upsampler is similar to the downsampler but integrators and comb filters are interchanged as shown in Fig 30. Basically, the operation principle consists of an expander and a low pass filter to remove undesired spectral images of the input signal $x[n]$ and taking advantage of the noble identities, the expander is moved between the comb stage and the interpolator stage saving registers in the implementation of the comb filters.

4.1 Anti-imaging effect

To illustrate the spectral effects of interpolation, Figure 31 shows an arbitrary baseband spectrum, with its spectral replications, of a signal applied to a interpolating CIC filter with R = 8. The interpolation filter's output spectrum shows howimperfect CIC filter lowpass filtering gives rise to the undesired spectral images.

After interpolation, the unwanted images of the B-width baseband spectrum reside at the null centers, located at integer multiples of f_out/R. If we follow the CIC filter with a traditional lowpass tapped-delay line FIR filter, whose stopband includes the first image band, fairly high image rejection can be achieved.

4.2 Digital implementation

When implementing a CIC interpolator, the first order CIC can be replaced by a holder circuit as shown in Figure 32. It means, the first input is sampled at rate f_{in} and the next R output values of the first stage will have this same value and will be outputed at $f_{out} = f_{in} \times R$.

Figure 33 shows the a sample and hold circuit for multirate operation.

5 Polyphase implementation

Recall the transfer function of an integrator block:

$$H_I(z) = \frac{1}{1 - z^{-1}} \tag{53}$$

Let M be a positive integer, from eq. (35) $H_I(z)$ can be decomposed as

$$H_I(z) = \frac{1}{1 - z^{-M}} + \frac{z^{-1}}{1 - z^{-M}} + \frac{z^{-2}}{1 - z^{-M}} + \ldots + \frac{z^{-M+1}}{1 - z^{-M}} \tag{54}$$

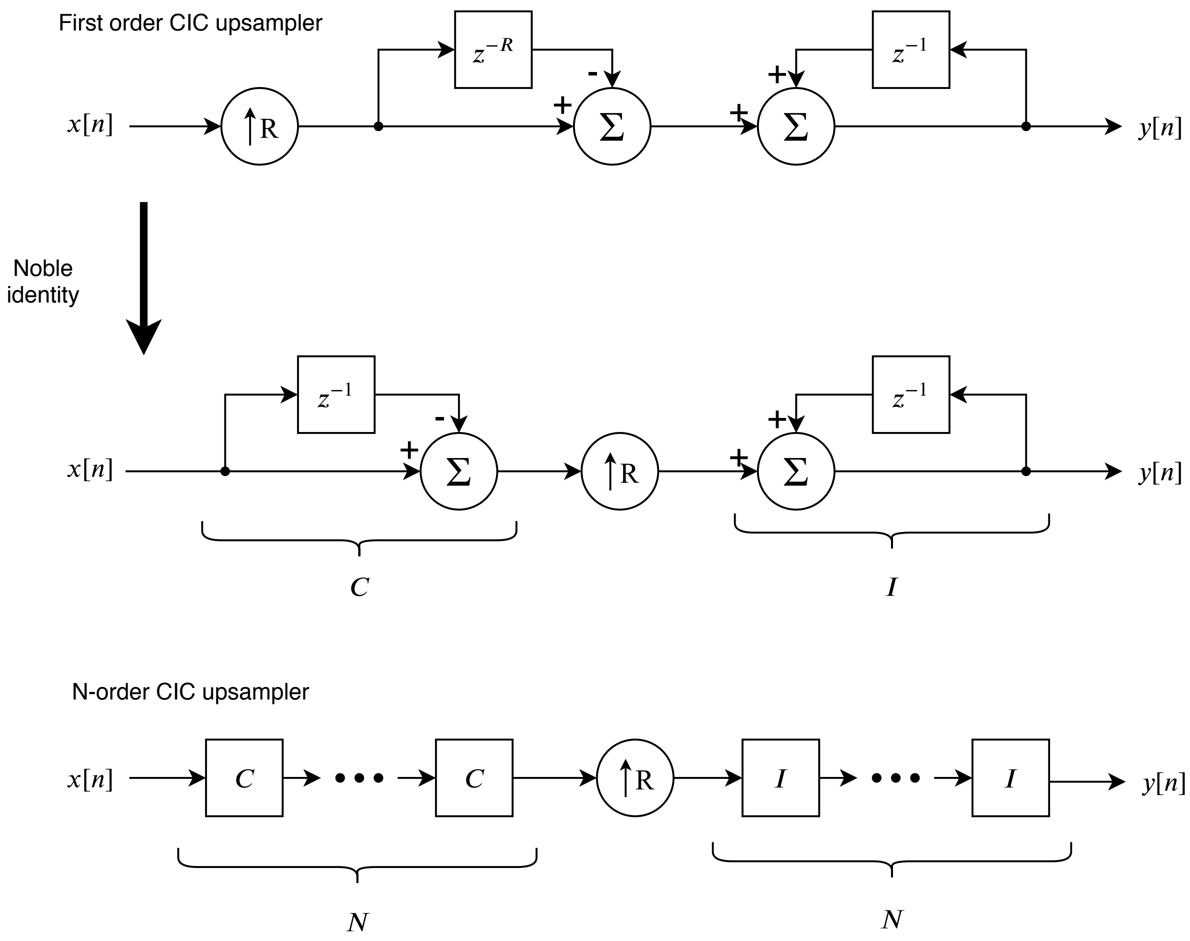

Figure 30: CIC upsampler block diagram.

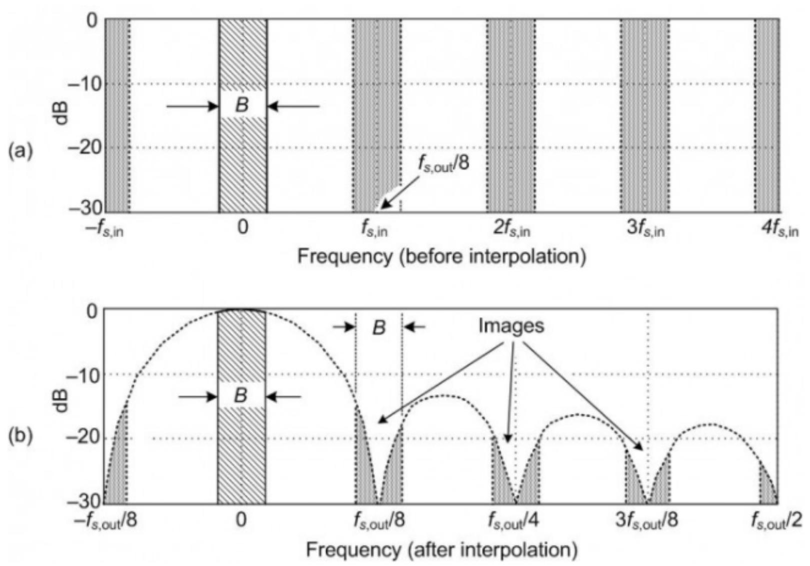

Figure 31: N-order CIC upsampler implemented with a sample and hold circuit.

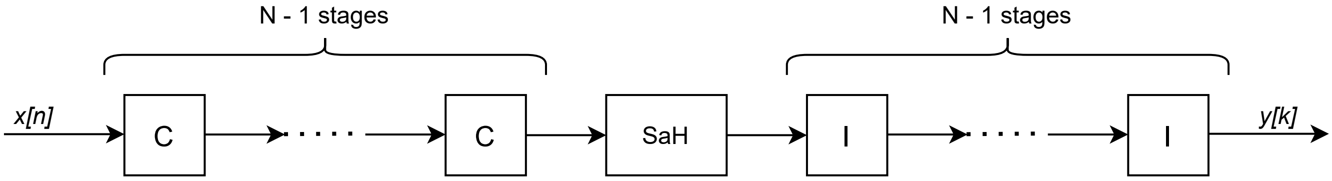

Figure 32: N-order CIC upsampler implemented with a sample and hold circuit.

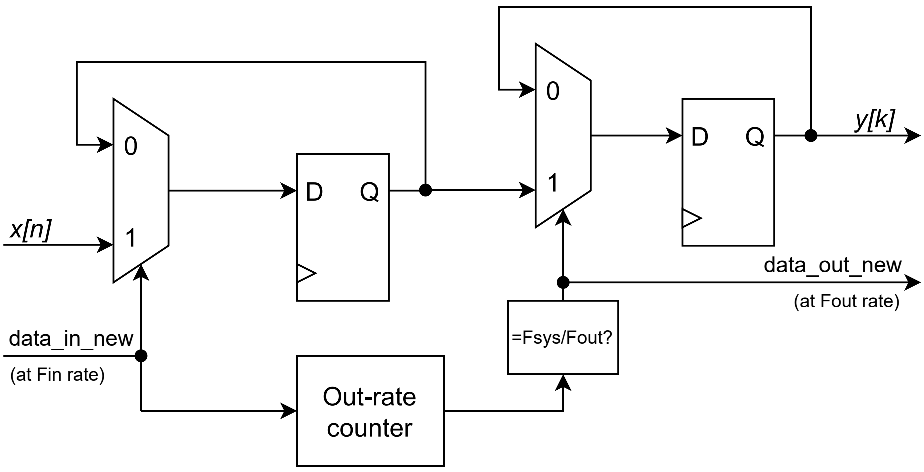

Figure 33: Sample and hold circuit.

This can be compactly written as

$$H_I(z) = \sum_{i=0}^{M-1} \frac{z^{-i}}{1 - z^{-M}} = \sum_{i=0}^{M-1} E_i(z^M) z^{-i} \tag{55}$$

where $E_i(z^M) = 1/(1 - z^{-M})$ for all $i = 1, 2, \dots, M$ is called the components of $H_I(z)$.

Figure 34 shows an efficient way to implement the polyphase decomposition for order $N = 1$. Notice that R and M must be integer multiples. Most commonly both are power of two.

5.1 Downsampler example for order $N = 2$ and number of phases $M = 2$

In this case, the system in Figure 21 when $N = 2$ is shown in Figure 35.

Equation (28) can be rewritten as follows:

$$H_I(z) = \frac{1}{1 - z^{-1}} = \frac{(1 + z^{-1})}{(1 - z^{-1})(1 + z^{-1})} = \frac{1}{(1 - z^{-2})}(1 + z^{-1}) = H_I(z^2)(1 + z^{-1}) \tag{56}$$

Thus,

$$H_I^2(z) = H_I^2(z^2)(1 + 2z^{-1} + z^{-2}) \tag{57}$$

Equation (57) turns out final implementation shown in Figure 36.

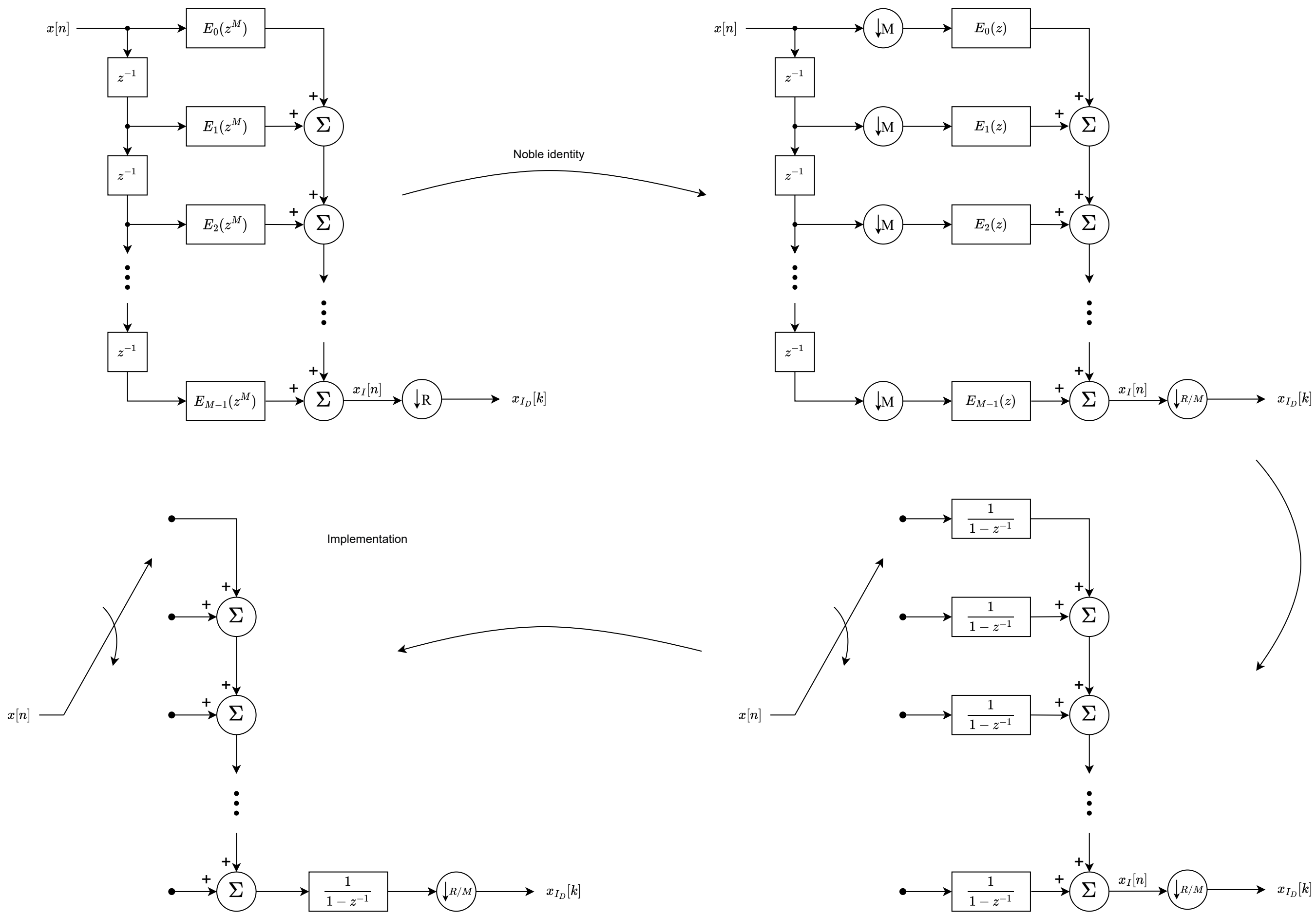

Figure 34: Polyphase implementation of a CIC downsampler, order $N = 1$.

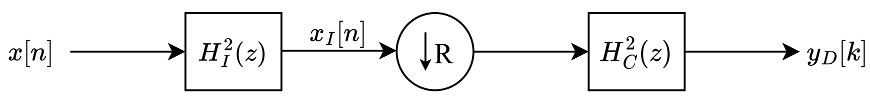

Figure 35: Second order downsampler, $N = 2$.

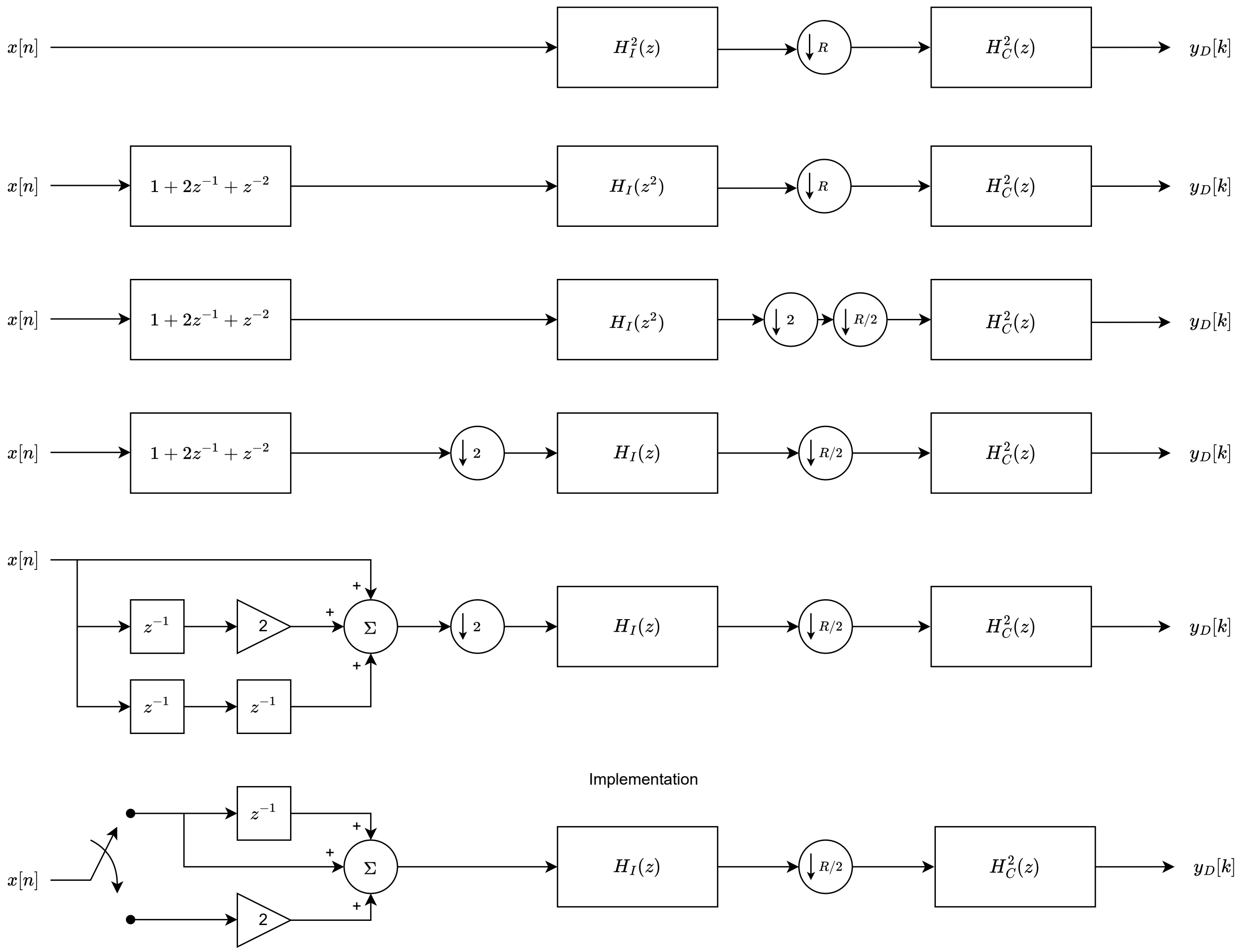

Figure 36: Two phases, second order downsampler, $M = 2$, $N = 2$.