

**Teaching goals:** After completing, the student

- understands the notion of substructure, generated substructure, can find them
- understands the notion of expansion and reduct of a structure, can define them formally and give examples
- understands the notions of [simple, conservative] extension, can formulate the definitions and the corresponding semantic criterion (for both expansions and reducts), and apply it to an example
- understands the notion of extension by definition, can define it formally and give examples
- can decide whether a given theory is a extension by definition, construct an extension by a given definition
- understands the notion of definability in a structure, can find definable subsets/relations

#### IN-CLASS PROBLEMS

**Problem 1.** Consider  $\underline{\mathbb{Z}}_4 = \langle \{0, 1, 2, 3\}; +, -, 0 \rangle$  where  $+$  is binary addition modulo 4 and  $-$  is the unary function returning the *inverse* for  $+$  with respect to the *neutral* element 0.

- Is  $\underline{\mathbb{Z}}_4$  a model of the theory of groups (i.e. is it a *group*)?
- Determine all substructures  $\underline{\mathbb{Z}}_4 \langle a \rangle$  generated by some  $a \in \mathbb{Z}_4$ .
- Does  $\underline{\mathbb{Z}}_4$  contain any other substructures?
- Is every substructure of  $\underline{\mathbb{Z}}_4$  a model of the theory of groups?
- Is every substructure of  $\underline{\mathbb{Z}}_4$  elementarily equivalent to  $\underline{\mathbb{Z}}_4$ ?

**Solution.** (a) Yes, one can check that  $\underline{\mathbb{Z}}_4$  satisfies all axioms of the theory of groups ( $+$  is associative, 0 is neutral for  $+$ ,  $-x$  is the inverse of  $x$  w.r.t.  $+$  and 0).

- $\underline{\mathbb{Z}}_4 \langle 0 \rangle = \underline{\mathbb{Z}}_4 \upharpoonright \{0\}$  (the trivial group),  $\underline{\mathbb{Z}}_4 \langle 1 \rangle = \underline{\mathbb{Z}}_4 \langle 3 \rangle = \underline{\mathbb{Z}}_4$ ,  $\underline{\mathbb{Z}}_4 \langle 2 \rangle = \underline{\mathbb{Z}}_4 \upharpoonright \{0, 2\}$  (a two-element group isomorphic to  $\underline{\mathbb{Z}}_2$ ).
- No, as soon as we have the element 1 or 3, the generated substructure is the whole  $\underline{\mathbb{Z}}_4$ .
- Yes, the theory of groups is universal (closed under substructures), hence substructures of models (groups) are also models (subgroups).
- No, the language of group theory is with equality, and any finite model size can be expressed by a sentence, so finite models of different sizes cannot be elementarily equivalent. However, we do not even need to express model size directly. It suffices to use “group properties” to distinguish them: e.g. the sentence  $(\forall x)x = 0$  distinguishes the trivial group  $\underline{\mathbb{Z}}_4 \upharpoonright \{0\}$  from the two-element group  $\underline{\mathbb{Z}}_4 \upharpoonright \{0, 2\}$  and from  $\underline{\mathbb{Z}}_4$ ; and e.g.  $(\forall x)x + x = 0$  is valid in  $\underline{\mathbb{Z}}_4 \upharpoonright \{0, 2\}$  but not in  $\underline{\mathbb{Z}}_4$ .

**Problem 2.** Let  $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, \cdot, 0, 1 \rangle$  be the field of rationals with the standard operations.

- Is there a reduct of  $\underline{\mathbb{Q}}$  that is a model of the theory of groups?
- Can the reduct  $\langle \mathbb{Q}, \cdot, 1 \rangle$  be extended to a model of the theory of groups?
- Does  $\underline{\mathbb{Q}}$  contain a substructure that is not elementarily equivalent to  $\underline{\mathbb{Q}}$ ?
- Let  $\text{Th}(\underline{\mathbb{Q}})$  denote the set of all sentences true in  $\underline{\mathbb{Q}}$ . Is  $\text{Th}(\underline{\mathbb{Q}})$  a complete theory?

**Solution.** (a) Yes,  $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, 0 \rangle$  (the additive group reduct).

- No, the element 1 (interpretation of the symbol 0 of the language of groups) is not a neutral element with respect to  $\cdot$  (the interpretation of  $+$ ), because  $1 \cdot 0 = 0 \neq 1$ .

- (c) Yes, for example  $\mathbb{Q} \upharpoonright \mathbb{Z} = \langle \mathbb{Z}; +, -, \cdot, 0, 1 \rangle$  (the ring of integers): in it not every nonzero element has a multiplicative inverse, so the sentence  $(\forall x)(\neg x = 0 \rightarrow (\exists y)x \cdot y = 1)$  fails (e.g. 2 has no inverse in  $\mathbb{Z}$ , while it does in  $\mathbb{Q}$ ). (From this it follows that the theory of fields is not openly axiomatizable, otherwise substructures of fields would be fields.)
- (d) Yes, the so-called theory of a structure is always complete: for every sentence  $\psi$  either  $\text{Th}(\mathbb{Q}) \models \psi$  or  $\text{Th}(\mathbb{Q}) \models \neg\psi$ , because  $\mathbb{Q} \models \psi$  or  $\mathbb{Q} \models \neg\psi$ .

**Problem 3.** Consider the theory  $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$  in the language  $L = \langle c_1, c_2, c_3 \rangle$  with equality.

- (a) Is  $T$  complete?
- (b) How many simple extensions of  $T$  are there, up to equivalence? How many are complete? Write down all complete ones and at least three incomplete ones.
- (c) Is the theory  $T' = T \cup \{x = c_1 \vee x = c_4\}$  in the language  $L' = \langle c_1, c_2, c_3, c_4 \rangle$  an extension of  $T$ ? Is  $T'$  a simple extension of  $T$ ? Is  $T'$  a conservative extension of  $T$ ?

**Solution.** The theory says that every element is one of the three constants. But these constants need not be distinct. First find all models up to isomorphism; there are five (draw them):

- $\mathcal{A}_1 = \langle \{0\}; 0, 0, 0 \rangle$  (one-element model,  $c_1^{\mathcal{A}_1} = c_2^{\mathcal{A}_1} = c_3^{\mathcal{A}_1} = 0$ )
- $\mathcal{A}_2 = \langle \{0, 1\}; 0, 0, 1 \rangle$  (two-element model,  $c_1^{\mathcal{A}_2} = c_2^{\mathcal{A}_2} \neq c_3^{\mathcal{A}_2}$ )
- $\mathcal{A}_3 = \langle \{0, 1\}; 0, 1, 1 \rangle$  (two-element model,  $c_1^{\mathcal{A}_3} \neq c_2^{\mathcal{A}_3} = c_3^{\mathcal{A}_3}$ )
- $\mathcal{A}_4 = \langle \{0, 1\}; 0, 1, 0 \rangle$  (two-element model,  $c_1^{\mathcal{A}_4} = c_3^{\mathcal{A}_4} \neq c_2^{\mathcal{A}_4}$ )
- $\mathcal{A}_5 = \langle \{0, 1, 2\}; 0, 1, 2 \rangle$  (three-element model, constants are distinct)

- (a) It is not complete; for example the sentence  $c_1 = c_2$  is independent in  $T$ : it is valid in  $\mathcal{A}_1$  but not in  $\mathcal{A}_3$ . (Equivalently, by the semantic criterion, models  $\mathcal{A}_1$  and  $\mathcal{A}_3$  are not elementarily equivalent.)
- (b) Simple extensions correspond to subsets of  $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$ , there are  $2^5 = 32$  of them; complete ones correspond to singletons (complete theories of individual models), so there are 5.

Simple extensions that are not complete:

- $T$  models  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$
  - $T \cup \{x = y \vee x = z\}$  models  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$
  - $T \cup \{(\exists x)(\exists y)\neg x = y\}$  models  $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$
- (Note:  $(\exists x)(\exists y)\neg x = y \sim \neg(\forall x)(\forall y)x = y \not\sim \neg x = y \sim (\forall x)(\forall y)\neg x = y$ .)

⋮

- $\{x = x \wedge \neg x = x\}$  the inconsistent theory, has no model

Complete simple extensions:

- $\text{Th}(\mathcal{A}_1) \sim \{x = y\}$
- $\text{Th}(\mathcal{A}_2) \sim \{(\exists x)(\exists y)\neg x = y, x = y \vee x = z, c_1 = c_2, \neg c_2 = c_3\}$
- $\text{Th}(\mathcal{A}_3) \sim \{(\exists x)(\exists y)\neg x = y, x = y \vee x = z, \neg c_1 = c_2, c_2 = c_3\}$
- $\text{Th}(\mathcal{A}_4) \sim \{(\exists x)(\exists y)\neg x = y, x = y \vee x = z, c_1 = c_3, \neg c_1 = c_2\}$
- $\text{Th}(\mathcal{A}_5) \sim \{x = c_1 \vee x = c_2 \vee x = c_3, \neg(c_1 = c_2 \vee c_1 = c_3 \vee c_2 = c_3)\}$

- (c) The theory additionally says that every element is either the interpretation of  $c_1$  or  $c_4$ . Thus models have at most two elements; up to isomorphism they are:

- $\mathcal{A}'_1 = \langle \{0\}; 0, 0, 0, 0 \rangle$
- $\mathcal{A}'_2 = \langle \{0, 1\}; 0, 0, 1, 1 \rangle$

- $\mathcal{A}'_3 = \langle \{0, 1\}; 0, 1, 1, 1 \rangle$
- $\mathcal{A}'_4 = \langle \{0, 1\}; 0, 1, 0, 1 \rangle$

The theory  $T'$  is an extension of  $T$ . All consequences of  $T$  are valid in  $T'$ ; semantically: the reducts of models of  $T'$  to the original language  $L$  are models of  $T$  (e.g. the reduct of  $\mathcal{A}'_1$  to  $L$  is  $\mathcal{A}_1$ ). It is not a simple extension, since we enlarged the language.

It is also not a conservative extension: for example the sentence  $(\forall x)(\forall y)(\forall z)(x = y \vee x = z)$  is a sentence in the original language  $L$ , it is valid in  $T'$  but it was not valid in  $T$ . Semantically: the three-element model  $\mathcal{A}_5$  of  $T$  cannot be expanded to an  $L'$ -structure that models  $T'$ , i.e. the reducts of models of  $T'$  to  $L$  do not yield all models of  $T$ .

**Problem 4.** Let  $T'$  be an extension of  $T = \{(\exists y)(x + y = 0), (x + y = 0) \wedge (x + z = 0) \rightarrow y = z\}$  in the language  $L = \langle +, 0, \leq \rangle$  with equality by definitions of  $<$  and unary  $-$  with axioms

$$\begin{aligned} -x = y &\leftrightarrow x + y = 0 \\ x < y &\leftrightarrow x \leq y \wedge \neg(x = y) \end{aligned}$$

Find formulas in the language  $L$  that are equivalent in  $T'$  to the following formulas.

- (a)  $(-x) + x = 0$       (b)  $x + (-y) < x$       (c)  $-(x + y) < -x$

**Solution.** Note that the axioms express existence and uniqueness for the definition of the function symbol  $-$ , so this is a proper extension by definition. We proceed according to the (proof of the) claim from the lecture:

- (a)  $(\exists z)(x + z = 0 \wedge z + x = 0)$  (The subformula  $x + z = 0$  says that ‘ $z$  is  $-x$ ’ and the second subformula says that ‘ $(-x) + x = 0$ ’.)  
 (b) First replace the term  $-y$  by its definition:

$$(\exists z)(y + z = 0 \wedge x + z < x)$$

Now replace the relation symbol  $<$ :

$$(\exists z)(y + z = 0 \wedge x + z \leq x \wedge \neg(x + z = x))$$

- (c)  $(\exists u)(\exists v)((x + y) + u = 0 \wedge x + v = 0 \wedge u \leq v \wedge \neg u = v)$  (Where ‘ $u$  is  $-(x + y)$ ’ and ‘ $v$  is  $-x$ ’.)

**Problem 5.** Consider the language  $L = \langle F \rangle$  with equality, where  $F$  is a binary function symbol. Find formulas defining the following sets (without parameters):

- (a) the interval  $(0, \infty)$  in  $\mathcal{A} = \langle \mathbb{R}, \cdot \rangle$  where  $\cdot$  is multiplication of real numbers  
 (b) the set  $\{(x, 1/x) \mid x \neq 0\}$  in the same structure  $\mathcal{A}$   
 (c) the set of all singleton subsets of  $\mathbb{N}$  in  $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup \rangle$   
 (d) the set of all prime numbers in  $\mathcal{C} = \langle \mathbb{N} \cup \{0\}, \cdot \rangle$

- Solution.** (a)  $(\exists y)F(y, y) = x \wedge \neg(\forall y)F(x, y) = x$  (The number  $x$  is a square, and it is not zero.)  
 (b)  $(\exists z)(F(x, y) = z \wedge (\forall u)F(z, u) = u)$  (The product equals one.)  
 (c)  $(\forall y)(\forall z)(F(y, z) = x \rightarrow y = x \vee z = x) \wedge \neg(\forall y)F(x, y) = y$  (Whenever the set is the union of two sets, it equals one of them. And it is not empty.)  
 (d) Same as (c),  $(\forall y)(\forall z)(F(y, z) = x \rightarrow y = x \vee z = x) \wedge \neg(\forall y)F(x, y) = x$  (Whenever the product of two numbers equals a prime, one of them equals the prime, and a prime is not zero.)

## EXTRA PRACTICE

**Problem 6.** Let  $T = \{\neg E(x, x), E(x, y) \rightarrow E(y, x), (\exists x)(\exists y)(\exists z)(E(x, y) \wedge E(y, z) \wedge E(x, z) \wedge \neg(x = y \vee y = z \vee x = z)), \varphi\}$  be a theory in the language  $L = \langle E \rangle$  with equality, where  $E$  is a binary relation symbol and  $\varphi$  expresses that “there are exactly four elements.”

- (a) Consider the expansion  $L' = \langle E, c \rangle$  of the language by a new constant symbol  $c$ . Determine the number (up to equivalence) of theories  $T'$  in  $L'$  that are extensions of  $T$ .
- (b) Does  $T$  have any *conservative* extension in the language  $L'$ ? Justify your answer.

**Problem 7.** Let  $T = \{x = f(f(x)), \varphi, \neg c_1 = c_2\}$  be a theory in the language  $L = \langle f, c_1, c_2 \rangle$  with equality, where  $f$  is a unary function symbol,  $c_1, c_2$  are constant symbols, and the axiom  $\varphi$  expresses that “there are exactly three elements.”

- (a) Determine how many pairwise nonequivalent complete simple extensions the theory  $T$  has. Write down two of them.
- (b) Let  $T' = \{x = f(f(x)), \varphi, \neg f(c_1) = f(c_2)\}$  be a theory in the same language, with  $\varphi$  same as above. Is  $T'$  an extension of  $T$ ? Is  $T$  an extension of  $T'$ ? If so, is it a conservative extension? Provide justification.

**Problem 8.** Consider  $L = \langle P, R, f, c, d \rangle$  with equality and the following two formulas:

$$\begin{aligned}\varphi : \quad & P(x, y) \leftrightarrow R(x, y) \wedge \neg x = y \\ \psi : \quad & P(x, y) \rightarrow P(x, f(x, y)) \wedge P(f(x, y), y)\end{aligned}$$

Consider the following  $L$ -theory:

$$\begin{aligned}T = \{ & \varphi, \psi, \neg c = d, \\ & R(x, x), \\ & R(x, y) \wedge R(y, x) \rightarrow x = y, \\ & R(x, y) \wedge R(y, z) \rightarrow R(x, z), \\ & R(x, y) \vee R(y, x)\}\end{aligned}$$

- (a) Find an expansion of the structure  $\langle \mathbb{Q}, \leq \rangle$  to the language  $L$  that is a model of  $T$ .
- (b) Is the sentence  $(\forall x)R(c, x)$  valid/contradictory/independent in  $T$ ? Justify all 3 answers.
- (c) Find two nonequivalent complete simple extensions of  $T$ , or justify why they do not exist.
- (d) Let  $T' = T \setminus \{\varphi, \psi\}$  be a theory in the language  $L' = \langle R, f, c, d \rangle$ . Is the theory  $T$  a conservative extension of the theory  $T'$ ? Provide justification.

## FOR FURTHER THOUGHT

**Problem 9.** Let  $T_n = \{\neg c_i = c_j \mid 1 \leq i < j \leq n\}$  be a theory in the language  $L_n = \langle c_1, \dots, c_n \rangle$  with equality, where  $c_1, \dots, c_n$  are constant symbols.

- (a) For a given finite  $k \geq 1$ , count  $k$ -element models of the theory  $T_n$  up to isomorphism.
- (b) Determine the number of countable models of the theory  $T_n$  up to isomorphism.
- (c) For which pairs of values  $n$  and  $m$  is  $T_n$  an extension of  $T_m$ ? For which pairs is it a conservative extension? Justify your answer.