

Teaching goals: After completing, the student

- understands the notion of substructure, generated substructure, can find them
- understands the notion of expansion and reduct of a structure, can define them formally and give examples
- understands the notions of [simple, conservative] extension, can formulate the definitions and the corresponding semantic criterion (for both expansions and reducts), and apply it to an example
- understands the notion of extension by definition, can define it formally and give examples
- can decide whether a given theory is an extension by definition, construct an extension by a given definition
- understands the notion of definability in a structure, can find definable subsets/relations

IN-CLASS PROBLEMS

Problem 1. Consider $\underline{\mathbb{Z}}_4 = \langle \{0, 1, 2, 3\}; +, -, 0 \rangle$ where $+$ is binary addition modulo 4 and $-$ is the unary function returning the *inverse* for $+$ with respect to the *neutral* element 0.

- (a) Is $\underline{\mathbb{Z}}_4$ a model of the theory of groups (i.e. is it a *group*)?
- (b) Determine all substructures $\underline{\mathbb{Z}}_4\langle a \rangle$ generated by some $a \in \underline{\mathbb{Z}}_4$.
- (c) Does $\underline{\mathbb{Z}}_4$ contain any other substructures?
- (d) Is every substructure of $\underline{\mathbb{Z}}_4$ a model of the theory of groups?
- (e) Is every substructure of $\underline{\mathbb{Z}}_4$ elementarily equivalent to $\underline{\mathbb{Z}}_4$?

Solution. (a) Yes, one can check that $\underline{\mathbb{Z}}_4$ satisfies all axioms of the theory of groups ($+$ is associative, 0 is neutral for $+$, $-x$ is the inverse of x w.r.t. $+$ and 0).

- (b) $\underline{\mathbb{Z}}_4\langle 0 \rangle = \underline{\mathbb{Z}}_4 \upharpoonright \{0\}$ (the trivial group), $\underline{\mathbb{Z}}_4\langle 1 \rangle = \underline{\mathbb{Z}}_4\langle 3 \rangle = \underline{\mathbb{Z}}_4$, $\underline{\mathbb{Z}}_4\langle 2 \rangle = \underline{\mathbb{Z}}_4 \upharpoonright \{0, 2\}$ (a two-element group isomorphic to \mathbb{Z}_2).
- (c) No, as soon as we have the element 1 or 3, the generated substructure is the whole $\underline{\mathbb{Z}}_4$.
- (d) Yes, the theory of groups is universal (closed under substructures), hence substructures of models (groups) are also models (subgroups).
- (e) No, the language of group theory is with equality, and any finite model size can be expressed by a sentence, so finite models of different sizes cannot be elementarily equivalent. However, we do not even need to express model size directly. It suffices to use “group properties” to distinguish them: e.g. the sentence $(\forall x)x = 0$ distinguishes the trivial group $\underline{\mathbb{Z}}_4 \upharpoonright \{0\}$ from the two-element group $\underline{\mathbb{Z}}_4 \upharpoonright \{0, 2\}$ and from $\underline{\mathbb{Z}}_4$; and e.g. $(\forall x)x + x = 0$ is valid in $\underline{\mathbb{Z}}_4 \upharpoonright \{0, 2\}$ but not in $\underline{\mathbb{Z}}_4$.

Problem 2. Let $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, \cdot, 0, 1 \rangle$ be the field of rationals with the standard operations.

- (a) Is there a reduct of $\underline{\mathbb{Q}}$ that is a model of the theory of groups?
- (b) Can the reduct $\langle \mathbb{Q}, \cdot, 1 \rangle$ be extended to a model of the theory of groups?
- (c) Does $\underline{\mathbb{Q}}$ contain a substructure that is not elementarily equivalent to $\underline{\mathbb{Q}}$?
- (d) Let $\text{Th}(\underline{\mathbb{Q}})$ denote the set of all sentences true in $\underline{\mathbb{Q}}$. Is $\text{Th}(\underline{\mathbb{Q}})$ a complete theory?

Solution. (a) Yes, $\underline{\mathbb{Q}} = \langle \mathbb{Q}; +, -, 0 \rangle$ (the additive group reduct).

- (b) No, the element 1 (interpretation of the symbol 0 of the language of groups) is not a neutral element with respect to \cdot (the interpretation of $+$), because $1 \cdot 0 = 0 \neq 1$.

- (c) Yes, for example $\underline{\mathbb{Q}} \upharpoonright \mathbb{Z} = \langle \mathbb{Z}; +, -, \cdot, 0, 1 \rangle$ (the ring of integers): in it not every nonzero element has a multiplicative inverse, so the sentence $(\forall x)(\neg x = 0 \rightarrow (\exists y)x \cdot y = 1)$ fails (e.g. 2 has no inverse in \mathbb{Z} , while it does in $\underline{\mathbb{Q}}$). (From this it follows that the theory of fields is not openly axiomatizable, otherwise substructures of fields would be fields.)
- (d) Yes, the so-called theory of a structure is always complete: for every sentence ψ either $\text{Th}(\underline{\mathbb{Q}}) \models \psi$ or $\text{Th}(\underline{\mathbb{Q}}) \models \neg\psi$, because $\underline{\mathbb{Q}} \models \psi$ or $\underline{\mathbb{Q}} \models \neg\psi$.

Problem 3. Consider the theory $T = \{x = c_1 \vee x = c_2 \vee x = c_3\}$ in the language $L = \langle c_1, c_2, c_3 \rangle$ with equality.

- (a) Is T complete?
- (b) How many simple extensions of T are there, up to equivalence? How many are complete? Write down all complete ones and at least three incomplete ones.
- (c) Is the theory $T' = T \cup \{x = c_1 \vee x = c_4\}$ in the language $L' = \langle c_1, c_2, c_3, c_4 \rangle$ an extension of T ? Is T' a simple extension of T ? Is T' a conservative extension of T ?

Solution. The theory says that every element is one of the three constants. But these constants need not be distinct. First find all models up to isomorphism; there are five (draw them):

- $\mathcal{A}_1 = \langle \{0\}; 0, 0, 0 \rangle$ (one-element model, $c_1^{\mathcal{A}_1} = c_2^{\mathcal{A}_1} = c_3^{\mathcal{A}_1} = 0$)
- $\mathcal{A}_2 = \langle \{0, 1\}; 0, 0, 1 \rangle$ (two-element model, $c_1^{\mathcal{A}_2} = c_2^{\mathcal{A}_2} \neq c_3^{\mathcal{A}_2}$)
- $\mathcal{A}_3 = \langle \{0, 1\}; 0, 1, 1 \rangle$ (two-element model, $c_1^{\mathcal{A}_3} \neq c_2^{\mathcal{A}_3} = c_3^{\mathcal{A}_3}$)
- $\mathcal{A}_4 = \langle \{0, 1\}; 0, 1, 0 \rangle$ (two-element model, $c_1^{\mathcal{A}_4} = c_3^{\mathcal{A}_4} \neq c_2^{\mathcal{A}_4}$)
- $\mathcal{A}_5 = \langle \{0, 1, 2\}; 0, 1, 2 \rangle$ (three-element model, constants are distinct)

- (a) It is not complete; for example the sentence $c_1 = c_2$ is independent in T : it is valid in \mathcal{A}_1 but not in \mathcal{A}_3 . (Equivalently, by the semantic criterion, models \mathcal{A}_1 and \mathcal{A}_3 are not elementarily equivalent.)
- (b) Simple extensions correspond to subsets of $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$, there are $2^5 = 32$ of them; complete ones correspond to singletons (complete theories of individual models), so there are 5.

Simple extensions that are not complete:

- T models $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$
 - $T \cup \{x = y \vee x = z\}$ models $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$
 - $T \cup \{(\exists x)(\exists y)\neg x = y\}$ models $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$
- (Note: $(\exists x)(\exists y)\neg x = y \sim \neg(\forall x)(\forall y)x = y \not\sim \neg x = y \sim (\forall x)(\forall y)\neg x = y$.)
- \vdots
- $\{x = x \wedge \neg x = x\}$ the inconsistent theory, has no model

Complete simple extensions:

- $\text{Th}(\mathcal{A}_1) \sim \{x = y\}$
- $\text{Th}(\mathcal{A}_2) \sim \{(\exists x)(\exists y)\neg x = y, x = y \vee x = z, c_1 = c_2, \neg c_2 = c_3\}$
- $\text{Th}(\mathcal{A}_3) \sim \{(\exists x)(\exists y)\neg x = y, x = y \vee x = z, \neg c_1 = c_2, c_2 = c_3\}$
- $\text{Th}(\mathcal{A}_4) \sim \{(\exists x)(\exists y)\neg x = y, x = y \vee x = z, c_1 = c_3, \neg c_1 = c_2\}$
- $\text{Th}(\mathcal{A}_5) \sim \{x = c_1 \vee x = c_2 \vee x = c_3, \neg(c_1 = c_2 \vee c_1 = c_3 \vee c_2 = c_3)\}$

- (c) The theory additionally says that every element is either the interpretation of c_1 or c_4 . Thus models have at most two elements; up to isomorphism they are:
- $\mathcal{A}'_1 = \langle \{0\}; 0, 0, 0, 0 \rangle$
 - $\mathcal{A}'_2 = \langle \{0, 1\}; 0, 0, 1, 1 \rangle$

- $\mathcal{A}'_3 = \langle \{0, 1\}; 0, 1, 1, 1 \rangle$
- $\mathcal{A}'_4 = \langle \{0, 1\}; 0, 1, 0, 1 \rangle$

The theory T' is an extension of T . All consequences of T are valid in T' ; semantically: the reducts of models of T' to the original language L are models of T (e.g. the reduct of \mathcal{A}'_1 to L is \mathcal{A}_1). It is not a simple extension, since we enlarged the language.

It is also not a conservative extension: for example the sentence $(\forall x)(\forall y)(\forall z)(x = y \vee x = z)$ is a sentence in the original language L , it is valid in T' but it was not valid in T . Semantically: the three-element model \mathcal{A}_5 of T cannot be expanded to an L' -structure that models T' , i.e. the reducts of models of T' to L do not yield all models of T .

Problem 4. Let T' be an extension of $T = \{(\exists y)(x+y=0), (x+y=0) \wedge (x+z=0) \rightarrow y=z\}$ in the language $L = \langle +, 0, \leq \rangle$ with equality by definitions of $<$ and unary $-$ with axioms

$$\begin{aligned} -x = y &\leftrightarrow x + y = 0 \\ x < y &\leftrightarrow x \leq y \wedge \neg(x = y) \end{aligned}$$

Find formulas in the language L that are equivalent in T' to the following formulas.

- (a) $(-x) + x = 0$ (b) $x + (-y) < x$ (c) $-(x+y) < -x$

Solution. Note that the axioms express existence and uniqueness for the definition of the function symbol $-$, so this is a proper extension by definition. We proceed according to the (proof of the) claim from the lecture:

- (a) $(\exists z)(x+z=0 \wedge z+x=0)$ (The subformula $x+z=0$ says that ‘ z is $-x$ ’ and the second subformula says that ‘ $(-x)+x=0$ ’.)
(b) First replace the term $-y$ by its definition:

$$(\exists z)(y+z=0 \wedge x+z < x)$$

Now replace the relation symbol $<$:

$$(\exists z)(y+z=0 \wedge x+z \leq x \wedge \neg(x+z=x))$$

- (c) $(\exists u)(\exists v)((x+y)+u=0 \wedge x+v=0 \wedge u \leq v \wedge \neg u=v)$ (Where ‘ u is $-(x+y)$ ’ and ‘ v is $-x$ ’.)

Problem 5. Consider the language $L = \langle F \rangle$ with equality, where F is a binary function symbol. Find formulas defining the following sets (without parameters):

- (a) the interval $(0, \infty)$ in $\mathcal{A} = \langle \mathbb{R}, \cdot \rangle$ where \cdot is multiplication of real numbers
(b) the set $\{(x, 1/x) \mid x \neq 0\}$ in the same structure \mathcal{A}
(c) the set of all singleton subsets of \mathbb{N} in $\mathcal{B} = \langle \mathcal{P}(\mathbb{N}), \cup \rangle$
(d) the set of all prime numbers in $\mathcal{C} = \langle \mathbb{N} \cup \{0\}, \cdot \rangle$

- Solution.** (a) $(\exists y)F(y, y) = x \wedge \neg(\forall y)F(x, y) = x$ (The number x is a square, and it is not zero.)
(b) $(\exists z)(F(x, y) = z \wedge (\forall u)F(z, u) = u)$ (The product equals one.)
(c) $(\forall y)(\forall z)(F(y, z) = x \rightarrow y = x \vee z = x) \wedge \neg(\forall y)F(x, y) = y$ (Whenever the set is the union of two sets, it equals one of them. And it is not empty.)
(d) Same as (c), $(\forall y)(\forall z)(F(y, z) = x \rightarrow y = x \vee z = x) \wedge \neg(\forall y)F(x, y) = x$ (Whenever the product of two numbers equals a prime, one of them equals the prime, and a prime is not zero.)

EXTRA PRACTICE

Problem 6. Let $T = \{\neg E(x, x), E(x, y) \rightarrow E(y, x), (\exists x)(\exists y)(\exists z)(E(x, y) \wedge E(y, z) \wedge E(x, z)) \wedge \neg(x = y \vee y = z \vee x = z)), \varphi\}$ be a theory in the language $L = \langle E \rangle$ with equality, where E is a binary relation symbol and φ expresses that “there are exactly four elements.”

- (a) Consider the expansion $L' = \langle E, c \rangle$ of the language by a new constant symbol c . Determine the number (up to equivalence) of theories T' in L' that are extensions of T .
- (b) Does T have any *conservative* extension in the language L' ? Justify your answer.

Problem 7. Let $T = \{x = f(f(x)), \varphi, \neg c_1 = c_2\}$ be a theory in the language $L = \langle f, c_1, c_2 \rangle$ with equality, where f is a unary function symbol, c_1, c_2 are constant symbols, and the axiom φ expresses that “there are exactly three elements.”

- (a) Determine how many pairwise nonequivalent complete simple extensions the theory T has. Write down two of them.
- (b) Let $T' = \{x = f(f(x)), \varphi, \neg f(c_1) = f(c_2)\}$ be a theory in the same language, with φ same as above. Is T' an extension of T ? Is T an extension of T' ? If so, is it a conservative extension? Provide justification.

Problem 8. Consider $L = \langle P, R, f, c, d \rangle$ with equality and the following two formulas:

$$\begin{aligned}\varphi : \quad &P(x, y) \leftrightarrow R(x, y) \wedge \neg x = y \\ \psi : \quad &P(x, y) \rightarrow P(x, f(x, y)) \wedge P(f(x, y), y)\end{aligned}$$

Consider the following L -theory:

$$\begin{aligned}T = \{&\varphi, \psi, \neg c = d, \\ &R(x, x), \\ &R(x, y) \wedge R(y, x) \rightarrow x = y, \\ &R(x, y) \wedge R(y, z) \rightarrow R(x, z), \\ &R(x, y) \vee R(y, x)\}\end{aligned}$$

- (a) Find an expansion of the structure $\langle \mathbb{Q}, \leq \rangle$ to the language L that is a model of T .
- (b) Is the sentence $(\forall x)R(c, x)$ valid/contradictory/independent in T ? Justify all 3 answers.
- (c) Find two nonequivalent complete simple extensions of T , or justify why they do not exist.
- (d) Let $T' = T \setminus \{\varphi, \psi\}$ be a theory in the language $L' = \langle R, f, c, d \rangle$. Is the theory T a conservative extension of the theory T' ? Provide justification.

FOR FURTHER THOUGHT

Problem 9. Let $T_n = \{\neg c_i = c_j \mid 1 \leq i < j \leq n\}$ be a theory in the language $L_n = \langle c_1, \dots, c_n \rangle$ with equality, where c_1, \dots, c_n are constant symbols.

- (a) For a given finite $k \geq 1$, count k -element models of the theory T_n up to isomorphism.
- (b) Determine the number of countable models of the theory T_n up to isomorphism.
- (c) For which pairs of values n and m is T_n an extension of T_m ? For which pairs is it a conservative extension? Justify your answer.