

# Nondisjoint CDSAT: arrays, maps, and vectors with abstract domain<sup>1</sup>

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# Arrays

- ▶ Data structure with **direct access** to values via indices
- ▶ Basic operations: **read/write** or **select/store**
- ▶ Theory of arrays:
  - ▶ Sorts: indices, values, arrays
  - ▶ **Select-over-store** axioms [McCarthy 1963]:
$$\forall a, v, i. \text{select}(\text{store}(a, i, v), i) \simeq v$$
$$\forall a, v, i, j. i \neq j \rightarrow \text{select}(\text{store}(a, i, v), j) \simeq \text{select}(a, j)$$
  - ▶ **Extensionality** axiom:
$$\forall a, b. (\forall i. \text{select}(a, i) \simeq \text{select}(b, i)) \rightarrow a \simeq b$$
- ▶ Not decidable, but the quantifier-free fragment is
- ▶ Considered useful to reason about computer memory

# Arrays: finite of infinite?

Arrays in programming languages:

- ▶ Integer-indexed
- ▶ **Finite**: indices in the interval  $[0, n - 1]$ , length  $n$   
Ada arrays: indices in the interval  $[n, m]$ , length  $m - n + 1$

Computer memory: finite

Arrays in the theory of arrays:

- ▶ Finite or infinite depending on the cardinality of the set used to interpret the sort of indices
- ▶ If integer-indexed: **infinite** arrays

# Array property fragment (APF) of the theory of arrays

- ▶ Limited usage of  $\forall$  over index variables
- ▶ Integer-indexed arrays are **infinite**, but it is possible to define:
  - ▶ **Bounded array equality**:  $beq(a, b, l, u)$  iff
$$\forall i. l \leq i \leq u \rightarrow \text{select}(a, i) \simeq \text{select}(b, i)$$
  - ▶ **Sortedness**:  $sorted(a, l, u)$  iff
$$\forall i, j. l \leq i \leq j \leq u \rightarrow \text{select}(a, i) \leq \text{select}(a, j)$$
assuming values are integers or rationals
- ▶ Decidable: finitely many instances of  $\forall$  + decision procedure for the disjoint union of UF, LIA, theory of values
- ▶ Efficient handling of  $\forall$  still a challenge in SMT

[Bradley, Manna, Sipma 2006] [Bradley, Manna 2007]

# How about adding a length function len?

- ▶ Maps every array to its length:  $\text{len}(a) \simeq n$
- ▶ Revised axiom of **extensionality** for integer-index arrays:  
$$\forall a, b. [ \text{len}(a) \simeq \text{len}(b) \wedge (\forall i. 0 \leq i < \text{len}(a) \rightarrow \text{select}(a, i) \simeq \text{select}(b, i)) ] \rightarrow a \simeq b$$
- ▶ Arrays and integers **no longer disjoint** theories:  
they share the symbol for the integer ordering
- ▶ Similar phenomenon for lists:
  - ▶  $\text{len}(\text{nil}) = 0$
  - ▶  $\text{len}(\text{cons}(x, y)) = 1 + \text{len}(y)$
  - ▶ 0, 1 and + become shared symbols

# Length is a bridging function

- ▶ **Bridging functions:** length of arrays, length of lists, size of trees, height of trees
- ▶ **Bridging axioms:**
  - ▶ RDS/AFDS (e.g., lists): define the bridging function over the **constructors**
  - ▶ Arrays: **extensionality** axiom
- ▶ Symbols other than equality become shared: **non-disjoint** theories
- ▶ Most methods for reasoning in theory unions require disjoint theories (equality is the only shared symbol)

[Ganzinger, Rueß, Shankar 2004] [Sofronie-Stokkermans 2009]

[Chocron, Fontaine, Ringeissen 2020]

# Other theories: strings and sequences

- ▶ Strings: sequences of elements from a finite alphabet (e.g., [Liang et al. 2014] [Berzish, Ganesh, Zheng 2017])
  - ▶ Sequences: generalization with generic and possibly infinite element sort
    - ▶ Empty sequence, binary associative concatenation: a **monoid**
    - ▶ Unary constructor wrapping single element into sequence
    - ▶ **Extract** function: returns the subsequence btw two positions
    - ▶ **Access** function: returns the element at a given position
    - ▶ **Length** function  $|x|$ : returns the number of elements in sequence  $x$
- (e.g., [Bjørner et al. 2012] [Jež et al. 2023])

# Theories of finite sequences to model finite arrays

- ▶ Theory Seq [Sheng et al. 2023] with integer indices  $[0, |x| - 1]$  and countably infinite element sort:
  - ▶ Add **update** function: **access/update** for **select/store**
  - ▶ **Extensionality** axiom as in arrays with length
  - ▶ **Nondisjointness**: conservative extension of the theory of integers into Seq
- ▶ Theory N-Seq [Ait-El-Hara, Bobot, Bury 2024] [Ait-El-Hara 2025]:
  - ▶ Integer indices  $[n, m]$  (Ada arrays)
  - ▶ Add functions: **first** and **last**, constant (sub)sequence, **relocate**, **subsequence update**
  - ▶ **Extensionality** axiom using first and last in place of length
- ▶ Decidability of quantifier-free fragment: unknown (soundness results)

# Summary of the issues and proposed solution

In order to model finite arrays:

deal with either  $\forall$  or non-disjointness or possibly undecidability

Solution: a new theory ArrAD of **arrays with abstract domain**:

- ▶ No need for quantifier reasoning
- ▶ Deal with the resulting **non-disjoint** theory unions by CDSAT:
  - ▶ Theory combination method that requires **neither stably infinite nor disjoint**
  - ▶ **Predicate-sharing theories**: either disjoint or sharing predicates other than equality
- ▶ The quantifier-free fragment of ArrAD is **decidable**: follows from fitting ArrAD in CDSAT + CDSAT completeness

# The theory of arrays with abstract domain: signature

- ▶ Sorts: indices  $I$ , values  $V$ , arrays  $A$ , lengths  $L$ , Booleans  $Prop$
- ▶  $\text{select} : A \times I \rightarrow V$      $\text{store} : A \times I \times V \rightarrow A$      $\text{len} : A \rightarrow L$
- ▶ Admissibility predicate:  $\text{Adm} : I \times L \rightarrow Prop$   
 $\text{Adm}(i, l)$ : index  $i$  is **admissible** wrt length  $l$
- ▶ **Abstract domain**: definition of  $\text{Adm}$
- ▶ **Concrete domain**: set of admissible indices given  $\text{Adm}$ 's definition and an interpretation of the sorts
- ▶  $\text{Adm}$  is **shared** by ArrAD, where it is free and another theory  $\mathcal{T}$  that provides its definition

# The theory of arrays with abstract domain: axioms

## ► Select-over-store axioms:

- $\forall a, v, i. \text{select}(\text{store}(a, i, v), i) \simeq v$  is replaced by  
 $\forall a, v, i. \text{Adm}(i, \text{len}(a)) \rightarrow \text{select}(\text{store}(a, i, v), i) \simeq v$   
a store at an inadmissible index has no effect
- $\forall a, v, i, j. i \neq j \rightarrow \text{select}(\text{store}(a, i, v), j) \simeq \text{select}(a, j)$

## ► Store does **not** change length:

$$\forall a, i, v. \text{len}(\text{store}(a, i, v)) \simeq \text{len}(a)$$

## ► Extensionality with length and admissibility:

$$\begin{aligned} &\forall a, b. [ \text{len}(a) \simeq \text{len}(b) \wedge \\ &(\forall i. \text{Adm}(i, \text{len}(a)) \rightarrow \text{select}(a, i) \simeq \text{select}(b, i)) ] \\ &\rightarrow a \simeq b \end{aligned}$$

## ► Congruence axioms for select, store, len, and Adm

# Example: the most common interpretation

- ▶ Let LIA be the theory defining **Adm**
- ▶ Interpreting indices and lengths as integers and defining admissibility by the axiom

$$\forall i, n. \text{Adm}(i, n) \leftrightarrow 0 \leq i < n$$

- ▶ The **set of admissible indices** is the interval  $[0, n)$
- ▶ Under this interpretation **extensionality** in ArrAD covers
  - ▶ Extensionality for arrays with length given above
  - ▶ Extensionality in the theory Seq of sequences

# Example: capturing bounded equality as in APF

- ▶ Let LIA be the theory defining **Adm**
- ▶ Interpreting indices as integers, lengths as pairs of integers, and defining admissibility by the axiom

$$\forall i, l, u. \text{Adm}(i, (l, u)) \leftrightarrow l \leq i \leq u$$

- ▶ The **set of admissible indices** is the interval  $[l, u]$
- ▶ Under this interpretation **extensionality** in ArrAD covers
  - ▶ Bounded equality in APF
  - ▶ Extensionality in the theory N-Seq of sequences

## Example: length with starting address

- ▶ The theory  $\mathcal{T}$  defining **Adm** interprets indices as integers, lengths as pairs  $(addr, n)$  where
  - ▶  $addr$  is a binary number – the starting address of the array in memory
  - ▶  $n$  is an integer – the number of admissible indicesand defines **Adm** by the axiom

$$\forall i, addr, n. \text{Adm}(i, (addr, n)) \leftrightarrow 0 \leq i < n$$

Starting address does not affect the admissibility of an index

- ▶ **Extensionality**: arrays  $a$  and  $b$  with same set of admissible indices, same values at all admissible indices, but different starting addresses are different (as it is in programming languages)

# Example: admissibility as membership

- ▶ The theory  $\mathcal{T}$  defining **Adm** interprets indices as elements of a set  $S$  and lengths as subsets of  $S$
- ▶  $\mathcal{T}$  defines admissibility by the axiom

$$\forall i, N. \text{Adm}(i, N) \leftrightarrow i \in N$$

- ▶ The **set of admissible indices** is the subset  $N \subseteq S$

The set  $S$  does not have to be a set of numbers, neither it is required to be (linearly) ordered

# Variant: a theory of maps with abstract domain

Same signature as **arrays with abstract domain**

- ▶ **Store at inadmissible index  $i$  makes only  $i$  admissible:**  
$$\forall a, j, i, v. \text{Adm}(j, \text{len}(\text{store}(a, i, v))) \leftrightarrow (\text{Adm}(j, \text{len}(a)) \vee j \simeq i)$$
- ▶ Store does **not** change length **if the index is admissible:**  
$$\forall a, i, v. \text{Adm}(i, \text{len}(a)) \rightarrow \text{len}(\text{store}(a, i, v)) \simeq \text{len}(a)$$
- ▶ **Select-over-store** axioms:
  - ▶ Restored:  $\forall a, v, i. \text{select}(\text{store}(a, i, v), i) \simeq v$
  - ▶  $\forall a, v, i, j. i \not\simeq j \rightarrow \text{select}(\text{store}(a, i, v), j) \simeq \text{select}(a, j)$
- ▶ **Extensionality** unchanged:  $\forall a, b. [\text{len}(a) \simeq \text{len}(b) \wedge (\forall i. \text{Adm}(i, \text{len}(a)) \rightarrow \text{select}(a, i) \simeq \text{select}(b, i))] \rightarrow a \simeq b$
- ▶ Congruence axioms for all symbols

# Another variant: a theory of vectors with abstract domain

Vectors are **dynamic** arrays: how to capture the change?

- ▶ Store at an inadmissible index makes that index and those in between (requires  $<$  on indices) admissible:  
$$\forall a, j, i, v. \text{Adm}(j, \text{len}(\text{store}(a, i, v))) \leftrightarrow (\text{Adm}(j, \text{len}(a)) \vee j \leq i)$$
- ▶ Everything else is as in the theory of **maps with abstract domain**, except that the signature for vectors adds an ordering  $<$  on indices (does not have to be linear)

Theories Seq and N-Seq do **not** capture the **dynamic** nature of vectors

Reasoning about **arrays, maps, and vectors with abstract domain?**

**CDSAT**

# What is CDSAT

- ▶ **CDSAT**: **C**onflict-**D**riven **SAT**isfiability in a union of theories
- ▶ Orchestrates **theory modules** in a **conflict-driven search**
- ▶ Generalizes **MCSAT** to **theory combination**:
  - ▶ Assignments of values to terms: both Boolean and **first-order**
  - ▶ Theory conflict explanation by theory inferences that can generate **new** terms
- ▶ Propositional logic is one of the theories: no hierarchy btw Boolean reasoning and theory reasoning
- ▶ Input first-order assignments:  
**Satisfiability Modulo Assignment**
- ▶ Sound, terminating, and complete for **predicate-sharing** theories **without** requiring **stable infiniteness**

# How to fit a component theory in CDSAT?

- ▶ A **theory module**  $\mathcal{I}_k$  for theory  $\mathcal{T}_k$ : an inference system (abstraction of a decision procedure)
- ▶ Requirements on a theory module:
  - ▶ **Soundness** (for the soundness of CDSAT)
  - ▶ **Finite local basis**:  $\text{basis}_k(X)$  – all the terms that  $\mathcal{I}_k$  can generate from set  $X$  of input terms  
Used to construct the **finite global basis** for the theory union (for the termination of CDSAT)
  - ▶ **Completeness** (for the completeness of CDSAT):
    - ▶ Leading theory  $\mathcal{T}_1$ : has all sorts and all shared predicates
    - ▶ Leading theory  $\mathcal{T}_1$ :  $\mathcal{I}_1$  is **complete**
    - ▶ All other theories  $\mathcal{T}_k$ :  $\mathcal{I}_k$  is **leading-theory complete**

# A theory module $\mathcal{I}_{\text{ArrAD}}$ for ArrAD

From **axioms** to **inference rules**, e.g.:

- ▶  $n \simeq m, i \simeq j, \text{Adm}(i, n), \neg \text{Adm}(j, m) \vdash \perp$
- ▶  $a \simeq b \vdash \text{len}(a) \simeq \text{len}(b)$
- ▶  $\text{len}(\text{store}(a, i, v)) \not\simeq \text{len}(a) \vdash \perp$
- ▶ Some rules generate  $\perp$  (**conflict detection**) others don't: balancing **finite basis design** and **completeness**
- ▶ From  $\forall a, v, i. \text{Adm}(i, \text{len}(a)) \rightarrow \text{select}(\text{store}(a, i, v), i) \simeq v$  to  $i \simeq j, \text{len}(a) \simeq n, \text{Adm}(i, n), b \simeq \text{store}(a, i, v), \text{select}(b, j) \not\simeq v \vdash \perp$
- ▶ It suffices to have  $b \simeq \text{store}(a, i, v)$  and  $\text{select}(b, j) \not\simeq v$  not necessarily  $\text{select}(\text{store}(a, i, v), j) \not\simeq v$

# How ArrAD fits in predicate-sharing completeness

The interpretation of arrays:

- ▶ Array sort  $A$ : **updatable function set**:  
a set of functions such that every function obtained by a finite number of updates to a member is a member

With abstract domain:

- ▶ **Partial** functions with domain of definition the set of admissible indices
- ▶ Array sort  $A$ : a **collection of updatable function sets**  $(X_n)_n$  for all values  $n$  in the interpretation of the sort  $L$  of lengths

# How ArrAD fits in predicate-sharing completeness

- ▶ **Theorem:** the module for ArrAD is **leading-theory-complete** for all **suitable** leading theories  $\mathcal{T}_1$
- ▶ A leading theory  $\mathcal{T}_1$  is **suitable** if:
  - ▶  $\mathcal{T}_1$  has **all the sorts** of ArrAD
  - ▶  $\mathcal{T}_1$  shares with ArrAD equality and **Adm**
  - ▶ For all  $\mathcal{T}_1$ -models  $\mathcal{M}_1$  there exists a collection of updatable function sets  $(X_n)_n$  such that
    - ▶  $n$  ranges over all possible values for lengths according to  $\mathcal{M}_1$
    - ▶  $f \in X_n$  is a function from admissible indices to values in the  $\mathcal{M}_1$ -interpretation of indices, admissibility, and values
    - ▶ the sum of the cardinalities of the  $X_n$  determines the cardinality of the sort  $A$  of arrays in  $\mathcal{M}_1$
- ▶ Suitability does not restrict combinability

- ▶ Proof objects in memory (checkable by proof checker)
  - ▶ The theory modules produce proofs
  - ▶ **Proof-carrying CDSAT** transition system
  - ▶ Proof reconstruction: from proof terms to proofs (e.g., resolution proofs)
- ▶ LCF style as in interactive theorem proving (correct by construction)
  - ▶ Trusted kernel of primitives

# Current and future work

## Current work:

- ▶ Theory modules for maps and vectors with abstract domain
- ▶ Leading theory completeness theorems for them

## Longer term:

- ▶ Arrays with abstract domain enriched with **concatenation** (may subsume sequences): QF decidability to be determined
- ▶ Sprout: a baby CDSAT-based verified solver written in Rust by Xavier Denis
- ▶ CDSAT and QSMA (for quantified satisfiability)

# References

- ▶ Conflict-driven satisfiability for theory combination: transition system and completeness.  
JAR 64(3):579–609, 2020 (Conference version at CADE 2017)
- ▶ Conflict-driven satisfiability for theory combination: modules, lemmas, and proofs.  
JAR 66(1):43–91, 2022 (Conference version at CPP 2018)
- ▶ The CDSAT method for satisfiability modulo theories and assignments: an exposition.  
Proc. CiE-21, LNAI 15764, 1–16, Springer, July 2025.
- ▶ CDSAT for predicate-sharing theories: arrays, maps, and vectors with abstract domain.  
In preparation (Short version at SMT 2022)

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