Operator Algebras and Algebraic Structures in Functional Analysis

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1 Introduction

Functional analysis is an area of mathematics which brings together many ideas from real and complex analysis with ideas from algebra and ring theory. Linear spaces and operator algebras in particular are algebraic structures which are fundamental to functional analysis and are used to extend ideas from real and complex analysis to more general algebraic structures.

An algebra is an algebraic structure which is both a module and a ring. As such, operator algebras inherit algebraic properties of rings which can be used in operator analysis. We will build up through definitions and examples of particular linear spaces and algebras, followed by a brief exploration of some of the intersections of algebra and analysis that arise in operator theory, and we will conclude with the functional calculus of an element of an algebra, depending upon maximal ideals and various algebra mappings.

We begin with some algebraic definitions which will be important for the rest of this paper:

An **additive abelian group** $(G, +)$ is a set of elements G together with binary operation of addition $(a, b) \mapsto a+b$ such that

1. Closure: $a + b \in G$ for all $a, b \in G$
2. Associativity: $(a + b) + c = a + (b + c)$ for all $a, b, c \in G$
3. Commutativity: $a + b = b + a$ for all $a, b \in G$
4. Identity: $a + 0 = 0 + a = a$ for all $a \in G$.
5. Inverses: for each $a \in G$, there is some element $d \in G$ such that $a + d = d + a = 0$ (such d is unique and denoted $-a$.)

Here 0 is the additive identity element of G. We denote the group $(G, +)$ by G unless there is ambiguity regarding the operation on the group G. We may refer to an additive abelian group G as an 'additive group' since addition is, by nature, a commutative operation.

A **ring** $\mathcal{R}$ is an additive group equipped with a second binary operation $(r, s) \mapsto rs$ satisfying

1. Associativity: $r(st) = (rs)t$ for $r, s, t \in \mathcal{R}$
2. Distributivity: $r(s+t) = rs + rt$ and $(r+s)t = rt + st$ for $r, s, t \in \mathcal{R}$.

A ring may or may not have a multiplicative identity element 1 satisfying $r1 = 1r = r$ for all $r \in \mathcal{R}$. Rings with a multiplicative identity element are called rings with unity, unital rings, or rings with identity.

2 Linear Operators and Spaces

A **linear space** is another name for a vector space. That is, a linear space X over a field $\mathbb{F}$ is an additive group of elements together with an operation of scalar multiplication with elements from the field $\mathbb{F}$ which satisfies

1. Scalar closure: $\alpha x \in X$ for all $x \in X, \alpha \in \mathbb{F}$
2. Scalar associativity: $\alpha(\beta x) = (\alpha\beta)x$ for $x \in X$, $\alpha, \beta \in \mathbb{F}$
3. Vector distributivity: $\alpha(x+y) = \alpha x + \alpha y$ for $\alpha \in \mathbb{F}, x, y \in X$
4. Scalar distributivity: $(\alpha+\beta)x = \alpha x + \beta x$ for $\alpha, \beta \in \mathbb{F}, x \in X$
5. Field identity: if 1 is the multiplicative identity of $\mathbb{F}$, then $1x = x$ for all $x \in X$.

If X, Y are linear spaces, then a mapping $T : X \to Y$ is a **linear operator** if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. That is, linear operators are just linear transformations.

A **norm** on a linear space X over a field $\mathbb{F}$ is a function $\nu : X \to [0, \infty)$ such that

1. Triangle inequality: $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in X$
2. Scalar multiplication: $\nu(\alpha x) = |\alpha|\nu(x)$ for $\alpha \in \mathbb{F}, x \in X$
3. Only the additive identity gets sent to zero: $\nu(x) = 0$ implies $x = 0 \in X$.

We will denote the norm ν by $\|\cdot\|$ (so $\nu(x) = \|x\|$.) A **normed space** is a linear space X together with a norm $\|\cdot\|$ on X.

Proposition 1. *A normed space is a special example of a metric space. That is, the norm on a linear space induces a metric on that space.*

Proof. A metric space is a set X together with a *metric* $d : X \times X \to [0, \infty)$ such that

1. Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$
2. Commutativity: $d(x, y) = d(y, x)$ for all $x, y \in X$
3. Zero: $d(x, y) = 0$ implies $x = y$.

The norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|} : X \times X \to [0, \infty)$ defined by

$$d_{\|\cdot\|}(x, y) = \|x - y\|$$

for $x, y \in X$. Then, we have the following:

1. Triangle inequality:

$$\begin{aligned} d_{\|\cdot\|}(x, y) &= \|x - y\| \\ &= \|x - z + z - y\| \\ &\leq \|x - z\| + \|z - y\| \\ &= d_{\|\cdot\|}(x, z) + d_{\|\cdot\|}(z, y) \end{aligned}$$

2. Commutativity:

$$\begin{aligned} d_{\|\cdot\|}(x, y) &= \|x - y\| \\ &= |-1|\|y - x\| \\ &= \|y - x\| \\ &= \ d_{\|\cdot\|}(y, x) \end{aligned}$$

3. Zero: $0 = d_{\|\cdot\|}(x, y) = \|x - y\|$ implies $x - y = 0 \in X$, which in turn implies $x = y$.

Thus $d_{\|\cdot\|}$ is indeed a metric on X, and a normed space is a metric space. $\square$

Let X, Y be normed spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. A linear operator $T : X \to Y$ is **bounded** if there exists a real number $M > 0$ such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$.

If $T : X \to Y$ is a bounded operator, then the **operator norm** of T is given by

$$\|T\| = \inf\{M : \|T(x)\|_Y \leq M\|x\|_X \text{ for all } x \in X\}.$$

We denote the set of all bounded operators $T : X \to Y$ by $\mathcal{B}(X, Y)$. We denote the set of all bounded operators from a normed space X to itself by $\mathcal{B}(X)$.

3 Banach Space

We know a sequence $\{x_n\}$ of real numbers is called **Cauchy** if for each $\epsilon > 0$ there is an index $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ whenever $m, n \geq N$. This definition extends to metric spaces, and thus to normed spaces.

Let X be a normed space. A sequence $\{x_n\}$ of elements of X is called a **Cauchy sequence** if for $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \epsilon \text{ whenever } m, n \geq N.$$

Every convergent sequence is a Cauchy sequence. In $\mathbb{R}$, the converse is also true: every Cauchy sequence is convergent. However, in a general metric space, this is not necessarily the case. Here we consider the idea of **completeness** to describe spaces in which this converse does hold. A metric space X is called **complete** if every Cauchy sequence $\{x_n\}$ in X converges to some point in X. A complete normed space is called a **Banach space**.

For normed spaces X and Y, we let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators $T : X \to Y$. We denote $\mathcal{B}(X, X)$ by $\mathcal{B}(X)$. The set $\mathcal{B}(X, \mathbb{F})$ is called the **dual space** of X. The dual space of any normed space is an example of a Banach space.

Proposition 2. *If X is a normed linear space, then the set $\mathcal{B}(X)$ of all bounded linear operators from X to itself is a normed linear space with norm given by the operator norm. If X is a Banach space, then $\mathcal{B}(X)$ is a Banach space as well.*

Proof. First, we state a lemma that will be useful for all proofs regarding operator norms.

Lemma 1. *Since $\|T\| = \inf\{M : \|T(x)\| \leq M\|x\| \text{ for } x \in X\}$ for $T \in \mathcal{B}(X)$, we have that $\|T(x)\| \leq \|T\|\|x\|$ for all $x \in X$.*

Proof. Note that the set $\{M : \|T(x)\| \leq M\|x\|\}$ is equivalent to the set $\{M \geq \frac{\|T(x)\|}{\|x\|} : x \in X\}$, so $\|T\| = \inf\{M : \|T(x)\| \leq M\|x\|\} = \inf\{\frac{\|T(x)\|}{\|x\|} : x \in X\}$. In particular, the infimum of this set cannot be less than $\frac{\|T(x)\|}{\|x\|}$, so $\|T\|\|x\| \geq \|T(x)\|$. □

Continuing with the proof of Proposition 2: For $T, S \in \mathcal{B}(X)$, we have that $\|T\| + \|S\| \in \{M : \|(T + S)(x)\| \leq M\|x\|\}$, since

$$\begin{aligned}
\|(T + S)(x)\| &= \|T(x) + S(x)\| \\
&\leq \|T(x)\| + \|S(x)\| \\
&\leq \|T\|\|x\| + \|S\|\|x\| \\
&= (\|T\| + \|S\|)\,\|x\|.
\end{aligned}$$

Thus the infimum of this set, $\|T + S\|$, is at most $\|T\| + \|S\|$, and we have that $\|T + S\| \leq \|T\| + \|S\|$.

For $\alpha \in \mathbb{F}$ and $T \in \mathcal{B}(X)$,

$$\begin{aligned}
\|\alpha T\| &= \inf\{M : \|\alpha T(x)\| \leq M\|x\|\} \\
&= \inf\{M : |\alpha|\|T(x)\| \leq M\|x\|\} \\
&= \inf\{M : \|T(x)\| \leq \frac{M}{|\alpha|}\|x\|\} \\
&= \inf\{M|\alpha\| : \|T(x)\| \leq M\|x\|\} \qquad \text{(from change of variables } M \mapsto M|\alpha|) \\
&= |\alpha| \inf\{M : \|T(x)\| \leq M\|x\|\} \\
&= |\alpha|\|T\|.
\end{aligned}$$

Next, if $\|T\| = 0$, suppose $T(x_0) \neq 0$ for some x_0. Then we would have that $\|T(x_0)\| > 0$, but $\|T(x)\| \leq \|T\|\|x\| = 0$, so the supposition that $T(x_0) \neq 0$ is a contradiction, and $T(x) = 0$ for all $x \in X$. Thus T is the zero operator.

Finally, we must show that $\mathcal{B}(X)$ is complete when X is complete (with proof from [3], Theorem 5.4.) Suppose $\{T_n\}$ is a Cauchy sequence in $\mathcal{B}(X)$. Then, for some $x \in X$ consider the sequence $\{T_n(x)\}$ in X. For any $\epsilon > 0$, we know that $\frac{\epsilon}{\|x\|} > 0$, and since $\{T_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ so that

$$\|T_n - T_m\| < \frac{\epsilon}{\|x\|}$$

whenever $m, n \geq N$. Then, from Lemma 1, we have

$$\|(T_n - T_m)(x)\| \leq \|T_n - T_m\|\|x\| < \epsilon$$

for $n, m \geq N$, so the sequence $\{T_n(x)\}$ is a Cauchy sequence in X. Then, since we assume X is complete, the sequence $\{T_n(x)\}$ converges to some $T(x) \in X$. That is, for any $\epsilon > 0$, we have that $\frac{\epsilon\|x\|}{2} > 0$, and there is an $N \in \mathbb{N}$ so that

$$\|T(x) - T_n(x)\| < \frac{\epsilon\|x\|}{2}$$

whenever $n \geq N$. Thus $\epsilon/2 \in \{M : \|(T - T_n)(x)\| \leq M\|x\|\}$ and we know that the infimum, $\|T - T_n\|$, of this set is at most $\epsilon/2$. That is, $\|T - T_n| \leq \frac{\epsilon}{2} < \epsilon$, so $\{T_n\}$ converges to T in $\mathcal{B}(X)$, and $\mathcal{B}(X)$ is complete. $\square$

4 Banach Algebras

A linear space A is called an *algebra* if A is also a ring. That is, if A has a second binary operation $(a, b) \mapsto ab$ satisfying

1. $a(bc) = (ab)c$ for $a, b, c \in A$

2. $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for $a, b, c \in A$

then A is an algebra.

A Banach space $\mathcal{A}$ is a **Banach algebra** if $\mathcal{A}$ is an algebra and is also a Banach space relative to norm $\|\cdot\|$ which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

Just as rings may or may not have multiplicative identity, an algebra may or may not have identity. However, if $\mathcal{A}$ is a Banach algebra without multiplicative identity, we can extend $\mathcal{A}$ to another Banach algebra which has multiplicative identity and contains $\mathcal{A}$.

Proposition 3. *(7.1.3 [1]) If $\mathcal{A}$ is a Banach algebra over the field $\mathbb{F}$ which does not have unity, define $\mathcal{A}_1 = \mathcal{A} \times \mathbb{F} = \{(a, \alpha) : a \in \mathcal{A}, \alpha \in \mathbb{F}\}$ with operations and norm given by*

1. *Addition in $\mathcal{A}_1$: $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$ for $(a, \alpha), (b, \beta) \in \mathcal{A}_1$*

2. *Scalar multiplication in $\mathcal{A}_1$: $\beta(a, \alpha) = (\beta a, \beta\alpha)$ for $\beta \in \mathbb{F}, (a, \alpha) \in \mathcal{A}_1$*

3. *(Vector) multiplication in $\mathcal{A}_1$: $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$ for $(a, \alpha), (b, \beta) \in \mathcal{A}_1$*

4. *Norm in $\mathcal{A}_1$: $\|(a, \alpha)\| = \|a\| + |\alpha|$.*

Then $\mathcal{A}_1$ is a Banach algebra with identity $(0, 1)$.

Proof. First, we show that $\mathcal{A}_1$ is a normed space. For (a, α) and $(b, \beta) \in \mathcal{A}_1$,

$$\begin{aligned}
\|(a, \alpha) + (b, \beta)\| &= \|(a + b, \alpha + \beta)\| \\
&= \|a + b\| + |\alpha + \beta| \\
&\leq \|a\| + \|b\| + |\alpha| + |\beta| \\
&= \|(a, \alpha)\| + \|(b, \beta)\|.
\end{aligned}$$

For $\beta \in \mathbb{F}$ and $(a, \alpha) \in \mathcal{A}_1$,

$$\begin{aligned}
\|\beta(a, \alpha)\| &= \|(\beta a, \beta\alpha)\| \\
&= \|\beta a\| + |\beta\alpha| \\
&= |\beta|\|(a, \alpha)\|.
\end{aligned}$$

If $\|(a, \alpha)\| = 0$, then $\|a\| + |\alpha| = 0$, but $\|a\|$ and $|\alpha|$ are both nonnegative, so $\|a\| = 0, |\alpha| = 0$, and $(a, \alpha) = (0, 0)$.

Next, we show that $\mathcal{A}_1$ is complete: Suppose $\{(a_n, \alpha_n)\}$ is a Cauchy sequence in $\mathcal{A}_1$. Then for $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that for $m, n \geq N$,

$$\begin{aligned}
\epsilon &< \|(a_n, \alpha_n) - (a_m, \alpha_m)\| \\
&= \|(a_n - a_m, \alpha_n - \alpha_m)\| \\
&= \|a_n - a_m\| + |\alpha_n - \alpha_m|.
\end{aligned}$$

Thus $\|a_n - a_m\| < \epsilon$, so the sequence $\{a_n\}$ is Cauchy in $\mathcal{A}$, and $|\alpha_n - \alpha_m| < \epsilon$, so $\{\alpha_n\}$ is Cauchy in $\mathbb{F}$ (which is an algebra, since fields are rings and are also linear spaces of dimension 1 over themselves.) Thus $\{a_n\}$ converges to some $a \in \mathcal{A}$, and $\{\alpha_n\}$ converges to some $\alpha \in \mathbb{F}$. In particular, for any $\epsilon > 0$, there is an N_1 so that $\|a - a_n\| < \epsilon/2$ for $n \geq N_1$, and there is an N_2 so that $|\alpha - \alpha_n| < \epsilon/2$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$ so that for $n \geq N$,

$$\begin{aligned}
\|(a, \alpha) - (a_n, \alpha_n)\| &= \|(a - a_n, \alpha - \alpha_n)\| \\
&= \|a - a_n\| + |\alpha - \alpha_n| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Thus $\{(a_n, \alpha_n)\}$ converges in $\mathcal{A}_1$, so $\mathcal{A}_1$ is complete (and thus is a Banach space.)

Furthermore, the direct product of rings is still a ring, and for $(a, \alpha), (b, \beta) \in \mathcal{A}_1$ we have

$$\begin{aligned}
\|(a, \alpha)(b, \beta)\| &= \|(ab + a\beta + \alpha b, \alpha\beta)\| \\
&= \|ab + a\beta + \alpha b\| + |\alpha\beta| \\
&\leq \|a\|\|b + \beta\| + \|\alpha b\| + |\alpha||\beta| \\
&= \|a\|\|b + \beta\| + |\alpha| \left(\|b\| + |\beta|\right) \\
&\leq \|a\| \left(\|b\| + |\beta|\right) + |\alpha| \left(\|b\| + |\beta|\right) \\
&= \|(a, \alpha)\|\|(b, \beta)\|,
\end{aligned}$$

so $\mathcal{A}_1$ is indeed a Banach algebra.

Furthermore, for any $(a, \alpha) \in \mathcal{A}$, we have $(a, \alpha)(0, 1) = (a0 + \alpha 0 + a, \alpha) = (a, \alpha)$ and $(0, 1)(a, \alpha) = (0a + a + 0\alpha, \alpha) = (a, \alpha)$, so $(0, 1)$ is the multiplicative identity for $\mathcal{A}_1$. □

We mentioned in the last section that if X is a Banach space, then the set $\mathcal{B}(X)$ of all bounded linear operators from X to X is also a Banach space. However, we can consider the Banach space $\mathcal{B}(X)$ along with the operation of operator composition, and see that $\mathcal{B}(X)$ will actually be a Banach algebra.

Proposition 4. *If X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra under the operation of operator composition.*

Proof. We already showed that $\mathcal{B}(X)$ is a Banach space when X is a Banach space. Linear operator composition is associative and distributive as ring operations are required to be, so it is sufficient to show that $\|S \circ T\| \leq \|S\|\|T\|$ for $S, T \in \mathcal{B}(X)$. We know that $\|T(x)\| \leq \|T\|\|x\|$, and $\|S(x)\| \leq \|S\|\|x\|$ for all $x \in X$, and since $T(x) \in X$ we have $\|S(T(x))\| \leq \|S\|\|T(x)\| \leq \|S\|\|T\|\|x\|$, so $\|S \circ T\| = \inf\{M : \|S(T(x))\| \leq M\|x\|\} \leq \|S\|\|T\|$. $\square$

An element a in a Banach algebra $\mathcal{A}$ with identity e is **invertible** in $\mathcal{A}$ if there is some $b \in \mathcal{A}$ so that $ab = ba = e$. If $\mathcal{A}$ is a Banach algebra with identity e over $\mathbb{F}$ and $a \in \mathcal{A}$, then the **spectrum** of a as defined in [2], denoted $\mathrm{sp}(a)$, is

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{F} : \lambda e - a \text{ not invertible in } \mathcal{A}\}.$$

If $\mathcal{A}$ is a Banach algebra without identity, then we define the spectrum of $a \in \mathcal{A}$ by

$$\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_{\mathcal{A}_1}(a),$$

where $\mathcal{A}_1$ is the extension of $\mathcal{A}$ to an algebra with identity as constructed in Proposition 3. We note that the spectrum of an element a in a Banach algebra $\mathcal{A}$ over a field $\mathbb{F}$ can also be denoted by $\mathrm{sp}(a) = \{\lambda \in \mathbb{F} : a - \lambda \text{ not invertible in } \mathcal{A}\}$, as in [1].

5 C*-Algebras

A **C*-algebra** is a Banach algebra $\mathcal{A}$ with an involution map $a \mapsto a^*$ for $a \in \mathcal{A}$ such that

1. $(a^*)^* = a$
2. $(ab)^* = b^*a^*$ for $a, b \in \mathcal{A}$
3. $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$ (where $\overline{\alpha}$ denotes the complex conjugate of α)
4. $\|a^*a\| = \|a\|^2$.

Additionally, if $\mathcal{A}$ has multiplicative identity 1, then

$$1^*a = (1^*a)^{**} = (a^*1^{**})^* = (a^*1)^* = (a^*)^* = a,$$

so $1^* = 1$.

At this point, we assume all algebras considered are over the field of complex numbers. Then, we denote by $C(X)$ the set of all continuous functions from a compact set X to the field of complex numbers. For any compact X, this set $C(X)$ is an example of a C^*-algebra. Furthermore, for compact X and $f \in C(X)$, we have the following lemma:

Lemma 2. *(7.3.2 [1]) If X is a compact set so that $C(X)$ is a C^*-algebra, then for $f \in C(X)$, $sp(f) = f(X)$.*

Proof. (7.3.2 [1]) We have that $\text{sp}(f) = \{\lambda \in \mathbb{C} : f - \lambda \text{ not invertible in } C(X)\}$. If $\lambda \in f(X)$, then $\lambda = f(x)$ for some $x \in X$ which implies $(f - \lambda)(x) = 0$ and $f - \lambda$ is not invertible. If $\mu \notin f(X)$, then $f - \mu$ is never zero since $f(x) \neq \mu$ for all $x \in X$, so $f - \mu$ is invertible in $C(X)$, and the spectrum of f is precisely the image of f, $f(X)$. $\square$

A homomorphism ϕ between algebras is an **algebra homomorphism** if ϕ preserves all algebra operations. If $\mathcal{A}$ and $\mathcal{C}$ are C^*-algebras, then an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{C}$ is a ***-homomorphism** if $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$. A *-homomorphism which is bijective is a ***-isomorphism**. The kernel of a homomorphism the set of all elements in the domain of the homomorphism which the homomorphism maps to zero.

We saw that a general Banach algebra $\mathcal{A}$ without identity could be extended to a Banach algebra $\mathcal{A}_1$ with identity so that $\mathcal{A} \subseteq \mathcal{A}_1$. In thte same sense, we can extend C^*- algebras without identity to C^*-algebras with identity. Thus, going forward, we assume every algebra considered has multiplicative identity.

A **subalgebra** is a subset of an algebra which is itself an algebra. If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$, then we denote by $C^*(a)$ the **C*-subalgebra** of $\mathcal{A}$ generated by a and the identity element e. Then $C^*(a)$ is the smallest C^*-subalgebra of $\mathcal{A}$ which contains a.

Proposition 5. *(8.1.14 [1]) If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{C}$, then for $a \in \mathcal{A} \subseteq \mathcal{C}$, $sp_{\mathcal{A}}(a) = sp_{\mathcal{C}}(a)$.*

The proof of this proposition can be found in [1] (Proposition 8.1.14) and is fairly simple but depends on definitions and theorems not covered in this paper, and thus is omitted.

6 Abelian Banach and C*-Algebras

A ring is called commutative if its multiplicative ring operation is commutative. An **abelian algebra** is an algebra which is a commutative ring.

Since algebras are rings, they inherit algebraic properties of rings. First, we define a **division ring**, which is a ring in which every nonzero element is a unit (that is, every nonzero element is invertible.)

Theorem 6.1. ***The Gelfand-Mazur Theorem.*** *If $\mathcal{A}$ is a Banach algebra which is also a division ring, then $\mathcal{A} = \mathbb{C}$. That is, if every nonzero element of $\mathcal{A}$ has an inverse in $\mathcal{A}$, then $\mathcal{A}$ has dimension 1 as a linear space over $\mathbb{C}$.*

Proof. (7.8.1 [1]) Let a be an arbitrary nonzero element of $\mathcal{A}$. Then for $\lambda \in \text{sp}(a)$, $\lambda e - a$ is not invertible by definition of the spectrum, which implies $\lambda e - a = 0$ in the division ring. That is, $a = \lambda e$, and since a was chosen arbitrarily, we have that $\mathcal{A} = \{\lambda e : \lambda \in \mathbb{C}\} = \mathbb{C}$. $\square$

Now we need to consider ring ideals of the abelian Banach and C^*-algebras. An **ideal** of an algebra $\mathcal{A}$ is a subring $I \leq \mathcal{A}$ satisfying $ax \in I$ and $xa \in I$ for all $a \in \mathcal{A}$ and $x \in I$. An ideal I of an algebra $\mathcal{A}$ is a **maximal ideal** if there is no ideal J of $\mathcal{A}$ such that $I \subset J \subset \mathcal{A}$.

Proposition 6. *(7.8.2 [1]) If $\mathcal{A}$ is an abelian Banach algebra, there is a bijective correspondence between maximal ideals of $\mathcal{A}$ and nonzero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. In particular, every maximal ideal of $\mathcal{A}$ is the kernel of a nonzero homomorphism from $\mathcal{A}$ to $\mathbb{C}$, and the kernel of every such homomorphism is a maximal ideal of $\mathcal{A}$.*

The proof of this proposition relies on more algebraic definitions and results than are included in this paper, but can be found in [1] (Proposition 7.8.2).

Definition 1. If $\mathcal{A}$ is an abelian Banach algebra, the **maximal ideal space** of $\mathcal{A}$, denoted by Σ, is the set of all nonzero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. Then Σ is a subset of the dual space of $\mathcal{A}$, and inherits a particular topology, called the weak* topology, from the dual space (for more information on this topology, see Definition 7.8.5 and Example 4.1.8 of [1].)

The name maximal ideal space comes from the bijective correspondence between maximal ideals and elements of Σ. In particular, the set

$$\ker\Sigma = \{\ker\phi : \phi \in \Sigma\}$$

is precisely the set of all maximal ideals of $\mathcal{A}$.

For an abelian Banach algebra $\mathcal{A}$, the maximal ideal space Σ of $\mathcal{A}$ is compact, so $C(\Sigma)$ is a C^*-algebra.

Definition 2. (7.8.8 [1]) If $\mathcal{A}$ is a Banach algebra and $a \in \mathcal{A}$, the **Gelfand transform of a** is the function $\hat{a} : \Sigma \to \mathbb{C}$ defined by $\hat{a}(h) = h(a)$ for homomorphisms $h \in \Sigma$.

The **Gelfand transform** of a Banach algebra $\mathcal{A}$ is the continuous homomorphism $\gamma : \mathcal{A} \to C(\Sigma)$ defined by $\gamma(a) = \hat{a}$.

The Gelfand transform is useful for studying C^*-algebras, as well as general Banach algebras. In particular, if $\mathcal{A}$ is a C^*-algebra with maximal ideal space Σ, then the Gelfand transform $\gamma : \mathcal{A} \to C(\Sigma)$ is a $*-$isomorphism (8.2.1 [1].) For generator a of $\mathcal{A}$ (that is, $\mathcal{A} \cong C^*(a)$) we first restrict the Gelfand transform $\hat{a}$ of a to a function $\tau : \Sigma \to \mathrm{sp}(a) \subseteq \mathbb{C}$, and then extend τ to the function

$$\tau^{\#} : C(\mathrm{sp}(a)) \to C(\Sigma)$$

defined by $\tau^{\#}(f) = f \circ \tau^{\#}$.

Then, if $C^*(a) = \mathcal{A} \subseteq \mathcal{B}$, we know from Proposition 5 that $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_{\mathcal{B}}(a)$, so in particular we can consider τ and $\tau^{\#}$ associated with non-generator elements of C^*-algebras. That is, if $a \in \mathcal{B}$ is not a generator of $\mathcal{B}$, we know that a (along with e) generates the C^*-subalgebra $C^*(a)$, and the function τ defined for $C^*(a)$ is well-defined in $\mathcal{B}$ since the spectrum of a is the same in $\mathcal{B}$ as it is in $C^*(a)$.

Suppose $\mathcal{A}$ is a C^*-algebra with identity and $a \in \mathcal{A}$ is normal (that is, $aa^* = a^*a$.) There is a unique $*$-isomorphism $\rho : C(\mathrm{sp}(a)) \to C^*(a)$ such that the following diagram commutes:

$$\begin{tikzcd}
& C(\mathrm{sp}(a)) \arrow[dl, "\tau^{\#}"'] \arrow[dr, "\rho"] & \\
C(\Sigma) & & C^*(a) \arrow[ll, "\gamma"]
\end{tikzcd}$$

(See 8.2.3-2.5 of [1] and 4.4.5 of [2].) That is, $\gamma(\rho(f)) = \tau^{\#}(f) = f \circ \tau$ for $f : \mathrm{sp}(a) \to \mathbb{C}$. Then for any $h \in \Sigma$, $(f \circ \tau)(h) = (\gamma(\rho(f)))(h)$, which makes sense since $\gamma(\rho(f)) \in C(\Sigma)$ ([1], [2].) Then, we define $f(a) \in C^*(a)$ by

$$f(a) := \rho(f)$$

for $f \in C(\mathrm{sp}(a))$. This map $f \mapsto f(a)$ is called the **functional calculus** for a, and is used for our final theorem.

Theorem 6.2. ***Spectral Mapping Theorem.***

For C^*-algebra $\mathcal{A}$, normal element $a \in \mathcal{A}$, and any $f \in C(sp(a))$, $sp(f(a)) = f(sp(a))$. That is, we have the following set equality:

$$\{\lambda \in \mathbb{C} : \rho(f) - \lambda \textit{ not invertible in } C^*(a)\} = \{f(\lambda) : \lambda \in sp(a)\}.$$

Proof. (8.2.7 [1], 4.4.8 [2]) Since the functional calculus for a is a $*$-isomorphism, we have $\mathrm{sp}(f(a)) = \mathrm{sp}(\rho(f))$ $= \mathrm{sp}(f)$ for $f \in C(\mathrm{sp}(a))$. On the other hand, by Lemma 2, we know that $\mathrm{sp}(f) = f(\mathrm{sp}(a))$, so $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$. $\square$

References

[1] John B. Conway, *A Course in Functional Analysis*, Second Edition. Springer-Verlag New York, Inc. 1990.

[2] Richard V. Kadison and John R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Volume I. Academic Press, Inc. 1983.

[3] Karen Saxe, *Beginning Functional Analysis.* Springer-Verlag New York, Inc. 2002.