CURS 4

Sisteme de ecuații diferențiale

$y_1 = y_1(x)$

$\vdots$

$y_n = y_n(x)$

fct. nec.

$$\begin{cases} y_1'(x) = f_1(x, y_1(x), \dots, y_n(x)) \\ \vdots \\ y_n'(x) = f_n(x, y_1(x), \dots, y_n(x)) \end{cases} \iff$$

$$\underline{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \underline{Y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$\iff \boxed{\underline{Y}' = f(x, \underline{Y}(x))}$$

forma vectorială a sist.

Sisteme liniare de ecuații diferențiale

$\underline{n=2}$

$y_1 = y_1(x)$
$y_2 = y_2(x)$

$$\begin{cases} y_1'(x) = a_{11}(x) \cdot y_1(x) + a_{12}(x) \cdot y_2(x) + b_1(x) \\ y_2'(x) = a_{21}(x)\, y_1(x) + a_{22}(x) \cdot y_2(x) + b_2(x) \end{cases}$$

a_{ij}
b_1, b_2 f.f. cont.

$$\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\boxed{\underline{Y}' = A \cdot \underline{Y} + B}$$

$\underline{Y}' = A\underline{Y} + B$ sistem liniar neomogen

$\underline{Y}' = A\underline{Y}$ sistem liniar omogen.

$$L : C^1([a,b], \mathbb{R}^2) \to C([a,b], \mathbb{R}^2)$$

$$\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto L\underline{Y} = \underline{Y}' - A\underline{Y}$$

sistemul liniar omogen $\Leftrightarrow L\underline{Y} = 0$

sistemul liniar neomogen $\Leftrightarrow L\underline{Y} = B$

operatorul $\underline{L}$ este un <u>operator liniar</u> $\Leftrightarrow$

$\Leftrightarrow \forall \lambda_1, \lambda_2 \in \mathbb{R}, \forall \underline{Y}^1, \underline{Y}^2 \in C^1([a,b], \mathbb{R}^2)$

$$L(\lambda_1 \underline{Y}^1 + \lambda_2 \underline{Y}^2) = \lambda_1 \cdot L\underline{Y}^1 + \lambda_2 \cdot L\underline{Y}^2$$

$L\underline{Y} = 0 \Leftrightarrow S_0 = \{ \underline{Y} \in C^1([a,b], \mathbb{R}^2) \mid L\underline{Y} = 0 \} = \ker L.$

$L\underline{Y} = B \Leftrightarrow S = \{ \underline{Y} \in C^1([a,b], \mathbb{R}^2) \mid L\underline{Y} = B \} =$

$$= \ker L + \{ \underline{Y}^p \}$$

$\underline{Y}^p$ o sol. partic. a sist. $L\underline{Y} = B$.

Teorema 1 (T. de existență și unicitate a sol. probl. Cauchy pt sist. lin.)

Problema Cauchy atașată unui sistem liniar neomogen de ecuații liniare

$$\begin{cases} LY = B \\ Y(x_0) = r \end{cases} \qquad r \in \mathbb{R}^2 \qquad r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

are o unică soluție $Y(\cdot\,; x_0, B, r)$ pt $\forall\, r \in \mathbb{R}^2$.

Cazul omogen

$$LY = 0 \iff \boxed{Y' = AY}$$

Teorema 2

S_0 este un subspațiu vectorial al sp. $C^1([a,b], \mathbb{R}^2)$ cu $\dim S_0 = 2$.

Dem. $\left.\begin{array}{l} S_0 = \ker L \\ L \text{ op. liniar} \end{array}\right\} \Rightarrow S_0$ este subsp. vect. al sp. $C^1([a,b], \mathbb{R}^2)$.

S_0 subsp. liniar $\Leftrightarrow \forall\, Y^1, Y^2 \in S_0$, $\forall\, \lambda_1, \lambda_2 \in \mathbb{R}$
$\lambda_1 Y^1 + \lambda_2 Y^2 \in S_0$.

$\dim S_0 = 2 \Leftrightarrow \exists\, \varphi : \mathbb{R}^2 \to S_0$ izomorf. de spații liniare.

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \longmapsto Y(\cdot\,; a, 0, r).$$

$$\left(\text{sol. probl. Cauchy } \begin{cases} LY = 0 \\ Y(a) = r \end{cases} \right)$$

T1 $\Rightarrow \varphi$ este bijecție

$$\varphi(r^1 + r^2) = \varphi(r^1) + \varphi(r^2), \quad \forall\, r^1, r^2 \in \mathbb{R}^2$$

$$\varphi(\lambda r) = \lambda \cdot \varphi(r) \qquad \forall\, r \in \mathbb{R}^2, \forall\, \lambda \in \mathbb{R}.$$

$\varphi(r^1+r^2) \overset{?}{=} \varphi(r^1)+\varphi(r^2)$

$\varphi(r^1)$ esti sol. probl. Cauchy $\begin{cases} LY=0 \\ Y(a)=r^1 \end{cases}$

$\varphi(r^2)$ esti sol probl. Cauchy $\begin{cases} LY=0 \\ Y(a)=r^2 \end{cases}$

$\Rightarrow$ $\varphi(r^1)+\varphi(r^2)$ esti sol. a probl. Cauchy

$\begin{cases} LY=0 \\ Y(a)=r^1+r^2 \end{cases} \underset{T1}{\Longrightarrow} \varphi(r^1)+\varphi(r^2)=\varphi(r^1+r^2)$

analog $\varphi(\lambda r)=\lambda\cdot\varphi(r)$.

S_0 sp. liniar cu $\dim S_0 = 2 \Longleftrightarrow$

$\Longleftrightarrow \exists\, Y^1, Y^2 \in S_0$ aî $\{Y^1, Y^2\}$ bază în $S_0 \Longleftrightarrow$

$\Longleftrightarrow \exists\, Y^1, Y^2 \in S_0$ aî $\{Y^1, Y^2\}$ esti liniar indep.

$\{Y^1, Y^2\}$ bază în S_0 — sistem. fundam. de soluții.

$$Y^1 = \begin{pmatrix} y_1^1 \\ y_2^1 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}$$

$$U = (Y^1 \; Y^2) = \begin{pmatrix} y_1^1 & y_1^2 \\ y_2^1 & y_2^2 \end{pmatrix}$$

dacă $\{Y^1, Y^2\}$ sist. fundam. de sol $\Leftrightarrow$ U matrice fundam. de soluții.

dacă $\{Y^1, Y^2\}$ sist. fundam. de sol $\Rightarrow$

$\forall\, Y \in S_0 \quad \exists\, c_1, c_2 \in \mathbb{R}$ a.î. $Y = c_1 Y^1 + c_2 Y^2 = U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Soluția generală a sist. liniar omogen.:

$$\boxed{Y^0 = U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \; c_1, c_2 \in \mathbb{R}}$$

Rezolvarea sist. liniar omogen revine la determinarea unei matrici fundam. de soluții (a unui sist. fundam. de soluții).

$$\underline{Y}^1 = \begin{pmatrix} y_1^1 \\ y_2^1 \end{pmatrix}, \quad \underline{Y}^2 = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}$$

$$W(x; \underline{Y}^1, \underline{Y}^2) = \begin{vmatrix} y_1^1(x) & y_1^2(x) \\ y_2^1(x) & y_2^2(x) \end{vmatrix}$$ det. lui Wronski (wronskian).

<u>Teoremă 3</u>

a) $\underline{Y}^1, \underline{Y}^2 \in C([a,b], \mathbb{R}^2)$ sunt liniar dependente $\Rightarrow$

$\Rightarrow W(x; \underline{Y}^1(x), \underline{Y}^2(x)) = 0, \forall x \in [a,b]$

b) $\underline{\underline{Y}^1, \underline{Y}^2 \in S_0}$ sunt liniar indep. $\Rightarrow W(x; \underline{Y}^1, \underline{Y}^2) \neq 0, \forall x \in [a,b]$.

În mulțimea S_0 avem următ. posib:

- $\underline{Y}^1, \underline{Y}^2 \in S_0$ liniar dep. $\rightarrow W(x; \underline{Y}^1, \underline{Y}^2) = 0, \forall x \in [a,b]$
- $\underline{Y}^1, \underline{Y}^2 \in S_0$ liniar indep. $\rightarrow W(x; \underline{Y}^1, \underline{Y}^2) \neq 0, \forall x \in [a,b]$.

Teorema 4 (Criteriul wronskianului)

Fie $\underline{Y}^1, \underline{Y}^2 \in S_0$. Următ. afirm. sunt echiv.:

(i) $\underline{Y}^1, \underline{Y}^2$ formează un sist. fundam. de sol.

(ii) $W(x; \underline{Y}^1, \underline{Y}^2) \neq 0$, $\forall x \in [a,b]$

(iii) $\exists x_0 \in [a,b]$ a.î. $W(x_0; \underline{Y}^1(x_0), \underline{Y}^2(x_0)) \neq 0$.

Cazul neomogen

$$L\underline{Y} = B \iff \underline{Y}' - A\underline{Y} = B \iff \underline{Y}' = A\underline{Y} + B$$

sol gen. a sist. neomogen.

$\underline{Y} = \underline{Y}^0 + \underline{Y}^P$ unde $\underline{Y}^0$ – sol. gen. a sist. omogen $L\underline{Y} = 0$

$\underline{Y}^P$ – o sol. partic. a sist. neomogen $L\underline{Y} = B$.

$\{\underline{Y}^1, \underline{Y}^2\}$ s.f.s $\Rightarrow U = (\underline{Y}^1 \ \underline{Y}^2)$ matr. fundam. de sol.

$$\underline{Y}^0 = U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

$\underline{Y}^p$ se det. prin metoda variației constantelor.

$\Rightarrow$ se caută $\underline{Y}^p = U \cdot \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}$

$c_1(x), c_2(x)$ se det. din cond. ca $\underline{Y}^p$ să fie sol. a sist. neomogen.

$$L\underline{Y}^p = B. \Leftrightarrow (\underline{Y}^p)' - A \cdot \underline{Y}^p = B.$$

$$\left(U \cdot \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}\right)' - A \cdot U \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = B.$$

$$\left(\left(U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)' = U' \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}'\right).$$

$$\Rightarrow \underline{U' \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}} + U \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} - \underline{A \cdot U \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}} = B.$$

$$\underbrace{(U' - AU)}_{=0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + U \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = B. \Rightarrow$$

$(U$ matr. fundam de sol $\Rightarrow U' - AU = 0)$

$$U \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = B. \quad \Rightarrow \quad \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = U^{-1}(x) \cdot B(x) \quad \Rightarrow$$

$\det U(x) = W(x; Y^1, Y^2) \Rightarrow U$ inversab.

$$\Rightarrow \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = \int U^{-1}(x)\, B(x)\, dx \quad \Rightarrow Y^p(x) = U(x) \cdot \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}.$$

Sisteme liniare cu coef. constanți

$$Y' = A \cdot Y + B \quad , \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

coef. const.

$$B = \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix} \in C([a,b], \mathbb{R}^2)$$

Cazul omogen

$$Y' = AY.$$

$$y' = ay.$$

$$y(x) = c \cdot e^{ax}, \; c \in \mathbb{R}.$$

I Metoda matricii exponențiale

$$e^{ax} = 1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + \dots + \frac{(ax)^n}{n!} + \dots$$

$$e^{xA} = I_n + \frac{xA}{1!} + \frac{x^2A^2}{2!} + \dots + \frac{x^m \cdot A^m}{m!} + \dots$$

Propr.

(i) $e^{0 \cdot A} = I_n$

(ii) $e^{(x+y) \cdot A} = e^{xA} \cdot e^{yA}$

(iii) $(e^{xA})^{-1} = e^{-xA}$

(iv) $(e^{xA})' = A \cdot e^{xA}$

$U(x) = e^{xA}$ este o matrice de sol. pt. sist. liniar omogen.

$\det U(0) = \det(I_n) = 1 \neq 0$

$\Rightarrow$ $U(x) = e^{xA}$ este o matr. fundam de sol.

$$\boxed{Y^0 = e^{xA} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}}$$

II Metoda reducerii la ec. cu coef. const.

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{cases}$$

se alege una dintre cele 2 ec. și se deriv. în rap. cu x

$$y_1'' = a_{11} \cdot y_1' + a_{12} y_2' = a_{11}(a_{11}y_1 + a_{12}y_2) + a_{12}(a_{21}y_1 + a_{22}y_2) =$$

$$= \dots =$$

$$\boxed{y_1'' = b_1 y_1 + b_2 y_2}$$

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_1'' = b_1 y_1 + b_2 y_2 \end{cases} \Rightarrow \boxed{y_2 = \frac{1}{a_{12}}(\underline{\underline{y_1'}} - a_{11}\underline{\underline{y_1}})}$$

$$\Rightarrow \dots \Rightarrow \boxed{y_1'' + \beta_1 y_1' + \beta_2 y_1 = 0} \quad \beta_1, \beta_2 \in \mathbb{R}.$$

$$\Rightarrow \dots \Rightarrow \boxed{y_1(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)}$$

III Metoda ecuației caracteristice

se caută soluții $Y = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \alpha_2 e^{\lambda x} \end{pmatrix}$ cu $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a sist.

$Y' = AY. \iff Y' - AY = 0$

$$\begin{pmatrix} \alpha_1 \lambda e^{\lambda x} \\ \alpha_2 \lambda e^{\lambda x} \end{pmatrix} - A \cdot \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \alpha_2 e^{\lambda x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \lambda \\ \alpha_2 \lambda \end{pmatrix} \cdot \cancel{e^{\lambda x}} - A \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \cancel{e^{\lambda x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} - A \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{(\lambda I_2 - A) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{\det(\lambda I_2 - A) = 0}$ ec. caract.

sol. ec. caract. sunt val. proprii ale matricii A.

1. $\exists \lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ $(\Delta > 0)$

$$(\lambda_i I_2 - A)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad i=1,2$$

$\Rightarrow$ alegem $\begin{pmatrix}\alpha_1^i\\ \alpha_2^i\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, $i=1,2$ sol. a sist.

$$\underline{Y}^i = \begin{pmatrix}\alpha_1^i e^{\lambda_i x}\\ \alpha_2^i e^{\lambda_i x}\end{pmatrix}, \quad i=1,2$$

$$U = (\underline{Y}^1 \ \underline{Y}^2) \qquad \underline{Y}^0 = U\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

2. $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$.

$$\underline{Y}^1 = \begin{pmatrix}\alpha_1 e^{\lambda x}\\ \alpha_2 e^{\lambda x}\end{pmatrix}, \quad \underline{Y}^2 = \begin{pmatrix}(ax+b)\, e^{\lambda x}\\ (\alpha x+\beta)\, e^{\lambda x}\end{pmatrix}$$

3. $\lambda_{1,2} = \alpha \pm i\beta$

$$\underline{Y} = \begin{pmatrix}\alpha_1 e^{\lambda x}\\ \alpha_2 e^{\lambda x}\end{pmatrix}$$ unde $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ sol nebanală în $\mathbb{C}$

a sist. $(\lambda I_2 - A)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

fie $\begin{array}{l}\alpha_1 = a_1 + ib_1 \\ \alpha_2 = a_2 + ib_2\end{array}$ o sol. $\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} \neq \begin{pmatrix}0 \\ 0\end{pmatrix}$.

$$\Rightarrow \underline{Y} = \begin{pmatrix}\alpha_1 e^{\lambda x} \\ \alpha_2 e^{\lambda x}\end{pmatrix} = \begin{pmatrix}(a_1 + ib_1) e^{(\alpha + i\beta)x} \\ (a_2 + ib_2) e^{(\alpha + i\beta)x}\end{pmatrix} =$$

$$\boxed{e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)}$$

$$= \begin{pmatrix}(a_1 + ib_1) e^{\alpha x} \cdot (\cos\beta x + i\sin\beta x) \\ (a_2 + ib_2) e^{\alpha x} (\cos\beta x + i\sin\beta x\end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix}e^{\alpha x}(a_1\cos\beta x - b_1\sin\beta x) \\ e^{\alpha x}(a_2\cos\beta x - b_2\sin\beta x)\end{pmatrix}}_{\underline{Y}^1} + i \cdot \underbrace{\begin{pmatrix}e^{\alpha x}(b_1\cos\beta x + a_1\sin\beta x) \\ e^{\alpha x}(b_2\cos\beta x + a_2\sin\beta x)\end{pmatrix}}_{\underline{Y}^2}$$

$$U = (\underline{Y}^1 \ \underline{Y}^2)$$