Seminar 8

1. Evaluati integralele improprii
   a) $\int_0^\infty \frac{\operatorname{arctg} x}{1+x^2}\,\mathrm{d}x$
   b) $\int_{-1}^1 \frac{x+1}{\sqrt{1-x^2}}\,\mathrm{d}x$
   c) $\int_0^\infty x^n \mathrm{e}^{-x}\,\mathrm{d}x, \quad n \in \mathbb{N}$
   d) $\int_1^2 \frac{1}{\sqrt{x(2-x)}}\,\mathrm{d}x$

2. Studiati convergenta integralelor improprii
   a) $\int_0^3 \frac{x^3+1}{\sqrt{9-x^2}}\,\mathrm{d}x$
   b) $\int_0^\infty \frac{\operatorname{arctg} x}{x}\,\mathrm{d}x$
   c) $\int_0^\pi x \ln(\sin x)\,\mathrm{d}x$

3. Studiati convergenta integralei improprii

$$I(\alpha) = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha \mathrm{d}x, \quad \alpha \in \mathbb{R}$$

si calculati valoarea lui $I(\frac{1}{2})$.

4. (**functia Gama**) Consideram integrala improprie

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x, \quad \alpha \in \mathbb{R}$$

Demonstrati urmatoarele proprietati

   a) $\Gamma(\alpha)$ este convergenta, $\forall \alpha > 0$
   b) $\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}$
   c) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \forall \alpha > 0$
   d) $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\,\Gamma(\frac{1}{2}), \quad \forall n \in \mathbb{N}^\star$

5. Exprimati cu ajutorul functiei Γ valoarea urmatoarelor integrale improprii
   a) $\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x$
   b) $\int_{-\infty}^\infty \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x$
   c) $\int_0^1 (\ln x)^{\frac{1}{3}}\,\mathrm{d}x$

Exercitii suplimentare

1. Evaluati integralele improprii
   a) $\int_0^1 \frac{\sqrt{x}+\ln x}{x}\,\mathrm{d}x$
   b) $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$
   c) $\int_0^\infty \mathrm{e}^{-x}\cos x\,\mathrm{d}x$
   d) $\int_0^1 \frac{\ln x}{\sqrt{1-x}}\,\mathrm{d}x$
   e) $\int_1^\infty \frac{\mathrm{d}x}{(x^2+1)\sqrt{x^2-1}}$

2. Studiati convergenta integralelor improprii
   a) $\int_0^1 \frac{1}{\sqrt[4]{1-x^4}}\,\mathrm{d}x$
   b) $\int_0^1 \frac{1}{\sqrt{x(\mathrm{e}^x-\mathrm{e}^{-x})}}\,\mathrm{d}x$
   c) $\int_0^\pi \left(1-\frac{\sin x}{x}\right)^{-1}\,\mathrm{d}x$

3. Determinati valorile lui $\alpha > 0$ pentru care integrala improprie

$$I(\alpha) = \int_1^\infty \frac{x-1}{x^\alpha - 1}\,\mathrm{d}x$$

este convergenta. Calculati valoarea lui $I(3)$.

4. Fie $\alpha > 0$. Studiati convergenta integralei

$$I(\alpha) = \int_1^\infty \left[\frac{1}{x^\alpha} - \frac{1}{(x+1)^\alpha}\right]\mathrm{d}x$$

si calculati valoarea lui $I(\frac{1}{2})$.

5. Fie $f : [1,+\infty) \to [0,+\infty)$ o functie continua, pozitiva si descrescatoare. Aratati ca

   i) $f(n+1) \leq \int_n^{n+1} f(x)\,\mathrm{d}x \leq f(n), \quad \forall n \in \mathbb{N}^\star$

   ii) $f(2)+f(3)+\ldots+f(n) \leq \int_1^n f(x)\,\mathrm{d}x \leq f(1)+f(2)+\ldots+f(n), \quad \forall n \in \mathbb{N}, n \geq 2$

   iii) (**criteriul integral**)
   Seria $\sum\limits_{n=1}^{\infty} f(n)$ este convergenta $\iff$ integrala $\int_1^\infty f(x)\,\mathrm{d}x$ este convergenta

   iv) Sirul $c_n = f(1)+f(2)+\ldots+f(n) - \int_1^n f(x)\,\mathrm{d}x$ este convergent.