3.1.41.

1) $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$$

$$f(\alpha x + \beta y) = f(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3))$$

$$= f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$= (\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, \alpha x_2 + \beta y_2 - \alpha x_3 - \beta y_3, \alpha x_3 + \beta y_3 - \alpha x_1 - \beta y_1)$$

$$= (\alpha x_1 - \alpha x_2, \alpha x_2 - \alpha x_3, \alpha x_3 - \alpha x_1) + (\beta y_1 - \beta y_2, \beta y_2 - \beta y_3, \beta y_3 - \beta y_1)$$

$$= \alpha f(x) + \beta f(y), \forall x, y \in \mathbb{R}^3, \forall \alpha, \beta \in \mathbb{R}.$$

$$\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0\}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \\ x_3 - x_1 = 0 \end{cases} \iff x_1 = x_2 = x_3$$

$$\Rightarrow \operatorname{Ker} f = \langle (1,1,1) \rangle$$

$$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3 : f(x) = y\}$$

$$= \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid \exists (x_1, x_2, x_3) \in \mathbb{R}^3 : \overset{(S)}{\begin{cases} x_1 - x_2 = y_1 \\ x_2 - x_3 = y_2 \\ x_3 - x_1 = y_3 \end{cases}}\}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}; \quad \bar{A} = \left(\begin{array}{ccc|c} 1 & -1 & 0 & y_1 \\ 0 & 1 & -1 & y_2 \\ -1 & 0 & 1 & y_3 \end{array}\right)$$

$y \in \operatorname{Im} f \iff (S)$ este compatibil $\iff \operatorname{rg} A = \operatorname{rg} \bar{A}$

minor principal

$$\det A = 0; \quad \begin{matrix} \boxed{\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ -1 \end{matrix} \\ \begin{matrix} -1 & 0 \end{matrix} & 1 \end{matrix}$$

$$\Downarrow$$

$$\operatorname{rg} A = 2;$$

$$\operatorname{rg} \bar{A} = 2 \iff \det \begin{pmatrix} 1 & -1 & y_1 \\ 0 & 1 & y_2 \\ -1 & 0 & y_3 \end{pmatrix} = 0 \iff \begin{vmatrix} 1 & -1 & y_1 \\ 0 & 1 & y_2 \\ 0 & 0 & y_1 + y_2 + y_3 \end{vmatrix} = 0$$

$\iff y_1+y_2+y_3=0$

$\operatorname{Im} f=\{y\in\mathbb{R}^3 \mid y_1+y_2+y_3=0\}$

2) $f:\mathbb{R}^3\to\mathbb{R}^3$

$f(x_1,x_2,x_3)=(x_1-1,\ x_2+2,\ x_3+1)$

Observăm că $f(0)\neq 0 \Rightarrow f$ nu este liniară

3) $f:\mathbb{R}^3\to\mathbb{R}^3$

$f(x_1,x_2,x_3)=(2x_1-3x_2+x_3,\ -x_1+x_2+3x_3,\ x_1+x_2+x_3)$

$$f(\alpha x+\beta y)=(2\alpha x_1+2\beta y_1-3\alpha x_2-3\beta y_2+\alpha x_3+\beta y_3,$$
$$-\alpha x_1-\beta y_1+\alpha x_2+\beta y_2+3\alpha x_3+3\beta y_3,$$
$$\alpha x_1+\beta y_1+\alpha x_2+\beta y_2+\alpha x_3+\beta y_3)=$$

$$=\alpha(2x_1-3x_2+x_3,\ -x_1+x_2+3x_3,\ x_1+x_2+x_3)+$$
$$\beta(2y_1-3y_2+y_3,\ -y_1+y_2+3y_3,\ y_1+y_2+y_3)=$$
$$=\alpha f(x)+\beta f(y)$$

$\operatorname{Ker} f=\{x\in\mathbb{R}^3 \mid f(x)=0\}$

$$\begin{cases}2x_1-3x_2+x_3=0\\-x_1+x_2+3x_3=0\\x_1+x_2+x_3=0\end{cases};\quad A=\begin{pmatrix}2&-3&1\\-1&1&3\\1&1&1\end{pmatrix}$$

$$\det(A)=\begin{vmatrix}2&-3&1\\-1&1&3\\1&1&1\end{vmatrix}=2-9-1-1-3-6=-18$$

$\Rightarrow$ sistem compatibil determinat $\Rightarrow$ sol. unică

sistem omogen ↳ admite sol. banală $\Bigg\} \Rightarrow$

$\Rightarrow x_1=x_2=x_3=0$; $\operatorname{Ker} f=\mathbb{R}^3$; $\operatorname{Im} f=\mathbb{R}^3$.

4) $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f(x_1, x_2) = (x_1 + x_2,\ x_1 - x_2,\ 2x_1 + x_2)$

$f(\alpha x + \beta y) = (\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2,\ \alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2,\ 2\alpha x_1 + 2\beta y_1 + \alpha x_2 + \beta y_2)$

$= \alpha(x_1 + x_2,\ x_1 - x_2,\ 2x_1 + x_2) +$

$\beta(y_1 + y_2,\ y_1 - y_2,\ 2y_1 + y_2)$

$= \alpha f(x) + \beta f(y)$

$\operatorname{Ker} f = \{x \in \mathbb{R}^2 : f(x) = 0\}$

$\left.\begin{matrix} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 = x_2 = 0$

$\operatorname{Ker} f = \{(0,0)\}$

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^2 : f(x) = y\}$

$\left\{\begin{matrix} x_1 + x_2 = y_1 \\ x_1 - x_2 = y_2 \\ 2x_1 + x_2 = y_3 \end{matrix}\right.$

$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$; $\bar{A} = \left(\begin{array}{cc:c} 1 & 1 & y_1 \\ 1 & -1 & y_2 \\ 2 & 1 & y_3 \end{array}\right)$

$\operatorname{rg} A = 2 \quad \left(\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3 \right)$

$\Rightarrow \begin{vmatrix} 1 & 1 & y_1 \\ 1 & -1 & y_2 \\ 2 & 1 & y_3 \end{vmatrix} = 0 \iff 3y_1 + y_2 - 2y_3 = 0$

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid 3y_1 + y_2 - 2y_3 = 0\}$

6) $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x_1, x_2) = (a_{11}x_1 + a_{21}x_2, a_{12}x_1 + a_{22}x_2)$

f este liniară, pentru că putem identifica

o matrice $A = \begin{pmatrix} a_{11} & a_{22} \\ a_{21} & a_{22} \end{pmatrix}$; $f(x) = x \cdot A = (x_1, x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

- $\mathrm{Ker} f = \{x \in \mathbb{R}^2 \mid f(x) = 0\}$

$$\begin{cases} a_{11}x_1 + a_{21}x_2 = 0 \\ a_{12}x_1 + a_{22}x_2 = 0 \end{cases}$$

I. $\det A \neq 0 \Rightarrow$ sistem comp. det. $\Rightarrow$ sol. unică

$\mathrm{Ker} f = \{(0,0)\}$

II. $\mathrm{rg} A = 1$, adică unul dintre a_{11}, a_{12}, a_{21} sau a_{22} este nenul

a_{11} nenul: $a_{11}x_1 + a_{21}x_2 = 0 \Rightarrow x_1 = \frac{-a_{21}}{a_{11}}\alpha$; $\alpha = x_2$

→

→

→

$\mathrm{Ker} f = \{\alpha(-\frac{a_{21}}{a_{11}}, 1) \mid \alpha \in \mathbb{R}\}$

III. $\mathrm{rg} A = 0 \Rightarrow \mathrm{Ker} f = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\} = \mathbb{R}^2$

- $\mathrm{Im} f = \{y \in \mathbb{R}^2 \mid \exists x \in \mathbb{R}^2 : f(x) = y\}$

$a_{11}x_1 + a_{21}x_2 = y_1$

$a_{12}x_1 + a_{22}x_2 = y_2$

IV. $a_{ij} = 0$, $i,j = \overline{1,2}$ $\Rightarrow \mathrm{Im} f = \{(0,0)\}$

$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$;

I. $\det A \neq 0 \Rightarrow \mathrm{Im} f = \mathbb{R}^2$

II. a_{11} și a_{21} nule; a_{12} și a_{22} nenule $\Rightarrow \mathrm{Im} f = \{(0, a)\}$

III. a_{12} și a_{22} nule; a_{11} și a_{21} nenule $\Rightarrow \mathrm{Im} f = \{(a, 0)\}$

Obs.: $f: \mathbb{R}^m \to \mathbb{R}^n$

$$f(x) = x \cdot A, \quad A \in M_{m\times n}(\mathbb{R})$$

$$(x_1, x_2, \dots, x_m)\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \left(\sum_{i=1}^{m} x_i a_{i1}, \sum_{i=1}^{m} x_i a_{i2}, \dots, \sum_{i=1}^{m} x_i a_{in}\right)$$

$$\left[(1, m) \cdot (m, n) = (1, n)\right]$$

$$(S_0): f(x) = 0 \iff x \cdot A = 0 \iff A^T \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(S)\ f(x) = y \iff x \cdot A = 0 \iff A^T \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$\operatorname{Ker} f$ = mulț. soluțiilor sist. (S_0)

$\operatorname{Im} f = \{ y \in \mathbb{R}^n \mid$ sist. (S) e compatibil$\}$

3.2.33. $[x, y] \in V^{2\times 1}$; $\alpha_1 x + \alpha_2 y = 0$

- Dacă $\alpha_1 \neq 0 \Rightarrow \alpha_1 x + \alpha_2 y = 0 \iff x = \underbrace{\frac{-\alpha_2}{\alpha_1}}_{=:\alpha} y$

$x = \alpha y \Rightarrow x + (-\alpha) y = 0 \to$ liniar dependentă

- Dacă $\alpha_2 \neq 0 \Rightarrow \alpha_1 x + \alpha_2 y = 0 \iff y = \underbrace{\frac{-\alpha_1}{\alpha_2}}_{=:\alpha} x$

$y = \alpha x \Rightarrow (-\alpha) x + y = 0 \to$ liniar dependentă

- $K = \mathbb{R}$

$V = \mathbb{R}^3$

$[x, y, z]^t \in (\mathbb{R}^3)^{3\times 1}$

$\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$,

I. $\alpha_1 \neq 0 \Rightarrow x = \underbrace{\frac{-\alpha_2}{\alpha_1}}_{=:\alpha} y + \underbrace{\frac{-\alpha_3}{\alpha_1}}_{=:\beta} z$

$x = \alpha y + \beta z$

II și III. Analog, $\alpha_2 \neq 0 \Rightarrow y = \alpha x + \beta z$

$\alpha_3 \neq 0 \Rightarrow z = \alpha x + \beta y$

3.2.35.

$f : V \to W$ apl. lin., $X \subseteq V$.

Ar. că $f(\langle X \rangle) = \langle f(X) \rangle$

$y \in f(\langle X \rangle) \Leftrightarrow \exists t \in \langle X \rangle : f(t) = y$

$t \in \langle X \rangle \Leftrightarrow t = \alpha_1 x_1 + \ldots + \alpha_m x_m$, $m \in \mathbb{N}$, $\alpha_i \in \mathbb{N}$, $x_i \in X$
$i \in \{1, \ldots, m\}$

$y = f(t) \Leftrightarrow y = f(\alpha_1 x_1 + \ldots + \alpha_n x_n) \Leftrightarrow$

$\Leftrightarrow y = \alpha_1 f(x_1) + \ldots + \alpha_m f(x_n) \Leftrightarrow y \in \langle f(X) \rangle$

3.2.36.

$\mathbb{Q} + \mathbb{Q}\sqrt{2} = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

Temă: $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ este $\mathbb{Q}$-spațiu vectorial bază : $[1, \sqrt{2}]$

- $\langle 1, \sqrt{2} \rangle = \{ a \cdot 1 + b \cdot \sqrt{2} \mid a, b \in \mathbb{Q} \} = \mathbb{Q} + \mathbb{Q}\sqrt{2}$
- $a \cdot 1 + b \cdot \sqrt{2} = 0 \iff a = -b \cdot \sqrt{2}$, $a, b \in \mathbb{Q}$

  $\Rightarrow a = b = 0 \Rightarrow \langle 1, \sqrt{2} \rangle$ liniar independentă

$\dim(\mathbb{Q} + \mathbb{Q}\sqrt{2}) = 2$ (baza are două elem.)

3.2.37.

Temă: $\mathbb{Q} + \mathbb{Q}\sqrt[3]{p} + \mathbb{Q}\sqrt[3]{p^2} = \{ a + b\sqrt[3]{p} + c\sqrt[3]{p^2} \mid a, b, c \in \mathbb{Q} \}$

este un $\mathbb{Q}$-sp. vect.

bază : $[1, 1, 1]$

- $\langle 1, 1, 1 \rangle = \{ a + b\sqrt[3]{p} + c\sqrt[3]{p^2} \mid a, b, c \in \mathbb{Q} \} = \mathbb{Q} + \mathbb{Q}\sqrt[3]{p} + \mathbb{Q}\sqrt[3]{p^2}$
- $a \cdot 1 + b \cdot \sqrt[3]{p} + c \cdot \sqrt[3]{p^2} = 0 \iff \underset{\mathbb{Q}}{\underset{\cap}{a}} = \underset{\mathbb{Q}\sqrt[3]{p}}{\underset{\cap}{\underbrace{-b\sqrt[3]{p}}}} + \underset{\mathbb{Q}\sqrt[3]{p^2}}{\underset{\cap}{\underbrace{(-c)\sqrt[3]{p^2}}}} \iff$

$\iff a = 0 = \underbrace{-b\sqrt[3]{p} + (-c)\sqrt[3]{p^2}}$

$\iff \underset{\mathbb{Q}\sqrt[3]{p}}{\underset{\cap}{\underbrace{b\sqrt[3]{p}}}} = \underset{\mathbb{Q}\sqrt[3]{p^2}}{\underset{\cap}{\underbrace{(-c)\sqrt[3]{p^2}}}} \iff b = c = 0.$

$a = b = c = 0$

$\dim(\mathbb{Q} + \mathbb{Q}\sqrt[3]{p} + \mathbb{Q}\sqrt[3]{p^2}) = 3.$

3.2.38.

(a) f inj. $+ v=[v_1,\dots,v_m]^t \in V^{m\times 1}$ lin. indep. $\Rightarrow$

$\Rightarrow f(v)$ lin. indep.

$f(v)=[f(v_1),\dots,f(v_m)]$

v. lin. indep. $\Rightarrow a_1v_1+a_2v_2+\dots+a_mv_m=0 \Rightarrow a_1=\dots=a_m=0$

$b_1f(v_1)+\dots+b_mf(v_m)=0 \Leftrightarrow f(b_1v_1+\dots+b_mv_m)=0$

f injectivă $\Rightarrow f(x)=0$ sol. unică $\Big\}$ $x=0$
avem soluţia banală

$\Rightarrow b_1v_1+\dots b_mv_m=0 \Rightarrow b_1=\dots=b_m=0 \Rightarrow f(v)$ lin. indep.

(b) $v\subseteq V \Rightarrow f(\langle v\rangle)=\langle f(v)\rangle = f(V) \overset{f \text{ surj.}}{=} W$

(c) f inj. $+ v$ lin. indep. $\Rightarrow f(v)$ lin. indep. $\Big\}$ $\Rightarrow$
f surj. $+ \langle v\rangle = V \Rightarrow \langle f(v)\rangle = W$

$\Rightarrow f(v)$ bază pt. W.