Suppose there are N keys in the table at the start of this program, and let

$$\alpha = N/M = \text{load factor of the table.} \tag{14}$$

Then the average value of A in an unsuccessful search is obviously α, if the hash function is random; and exercise 39 proves that the average value of C in an unsuccessful search is

$$C'_N = 1 + \frac{1}{4}\left(\left(1+\frac{2}{M}\right)^N - 1 - \frac{2N}{M}\right) \approx 1 + \frac{e^{2\alpha}-1-2\alpha}{4}. \tag{15}$$

Thus when the table is half full, the average number of probes made in an unsuccessful search is about $\frac{1}{4}(e+2) \approx 1.18$; and even when the table gets completely full, the average number of probes made just before inserting the final item will be only about $\frac{1}{4}(e^2+1) \approx 2.10$. The standard deviation is also small, as shown in exercise 40. These statistics prove that *the lists stay short even though the algorithm occasionally allows them to coalesce*, when the hash function is random. Of course C can be as high as N, if the hash function is bad or if we are extremely unlucky.

In a successful search, we always have $A = 1$. The average number of probes during a successful search may be computed by summing the quantity $C + A$ over the first N unsuccessful searches and dividing by N, if we assume that each key is equally likely. Thus we obtain

$$\begin{aligned} C_N = \frac{1}{N}\sum_{0\le k<N}\left(C'_k + \frac{k}{M}\right) &= 1 + \frac{1}{8}\frac{M}{N}\left(\left(1+\frac{2}{M}\right)^N - 1 - \frac{2N}{M}\right) + \frac{1}{4}\frac{N-1}{M} \\ &\approx 1 + \frac{e^{2\alpha}-1-2\alpha}{8\alpha} + \frac{\alpha}{4} \end{aligned} \tag{16}$$

as the average number of probes in a random successful search. Even a full table will require only about 1.80 probes, on the average, to find an item! Similarly