Seminar 3

Criteriul Stolz-Cesaro. Fie $(a_n)_{n\in\mathbb{N}}$ un sir oarecare de numere reale si $(b_n)_{n\in\mathbb{N}}$ un sir strict monoton si divergent.

$$\text{Daca} \quad \exists \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \overline{\mathbb{R}} \quad \text{atunci} \quad \exists \lim_{n\to\infty} \frac{a_n}{b_n} = l$$

1. Fie $(x_n)_{n\in\mathbb{N}}$ un sir cu termeni strict pozitivi. Daca exista $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = l$ atunci $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = l$. Reciproca este adevarata?

2. Calculati limita sirurilor

a) $y_n = \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n}$
b) $y_n = \sqrt[n]{n!}$
c) $y_n = \frac{\sqrt[n]{n!}}{n}$

3. Pentru sirurile de mai jos

$$a_n = \sum_{k=1}^{n} \frac{1+(-1)^k}{2}, \; b_n = n, \quad \forall n \in \mathbb{N}^\star$$

calculati valoarea limitelor $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ si $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$. Contrazice acest lucru criteriul Stolz-Cesaro?

4. Scrieti urmatoarele serii cu ajutorul simbolului suma
a) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$
b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$
c) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \ldots$

5. Calculati suma urmatoarelor serii
a) $\sum\limits_{n=0}^{\infty} \frac{1}{n!}$
b) $\sum\limits_{n=1}^{\infty} \frac{1}{5^n}$
c) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$
d) $\sum\limits_{n=1}^{\infty} \frac{1}{4n^2-1}$
e) $\sum\limits_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$
f) $\sum\limits_{n=1}^{\infty} \frac{n2^n}{(n+2)!}$

Exercitii suplimentare

1. Calculati limita sirurilor
   a) $y_n = \frac{1+\sqrt{2}+...+\sqrt{n}}{n\sqrt{n}}$
   b) $y_n = \frac{1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}}{\ln n}$
   c) $y_n = \frac{\sqrt[n]{(n+1)(n+2)\cdot...\cdot(n+n)}}{n}$
   d) $y_n = \frac{\ln n!}{n \ln n}$

2. Fie $(x_n)_{n\in\mathbb{N}}$ un sir cu termeni strict pozitivi. Daca exista $\lim\limits_{n\to\infty} x_n = l$ atunci

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \ldots + x_n}{n} = \lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdot \ldots \cdot x_n} = l$$

3. Calculati suma urmatoarelor serii
   a) $\sum\limits_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$
   b) $\sum\limits_{n=2}^{\infty} \frac{1}{\mathrm{C}_n^2}$
   c) $\sum\limits_{n=1}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1}$
   d) $\sum\limits_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n}$, $a > 0$

Criteriul Stolz-Cesaro (cazul 0/0). Fie $(a_n)_{n\in\mathbb{N}}$ un sir de numere reale si $(b_n)_{n\in\mathbb{N}}$ un sir strict monoton, avand proprietatile $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

$$\text{Daca} \quad \exists \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l \in \overline{\mathbb{R}} \quad \text{atunci} \quad \exists \lim_{n\to\infty} \frac{a_n}{b_n} = l$$