Seminar 9

3.1.31. Arătați că $\mathbb{R}_+^*$ este un $\mathbb{R}$-spațiu vectorial în raport cu adunarea vectorilor

$$\boxplus : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$$

$$x \boxplus y = xy, \forall x, y \in \mathbb{R}_+^*$$

și cu înmulțirea cu scalari

$$\boxdot : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}_+^*$$

$$\alpha \boxdot x = x^\alpha, \forall x \in \mathbb{R}_+^*, \alpha \in \mathbb{R}.$$

1) parte stabilă: $x, y \in \mathbb{R}_+^*$, $x \boxplus y = xy \in \mathbb{R}_+^*$ ✓

2) asociativitatea: $x, y, z \in \mathbb{R}_+^*$,

$$(x \boxplus y) \boxplus z = (xy) \boxplus z = (xy)z =$$

$$= x(yz) = x \boxplus (yz) = x \boxplus (y \boxplus z)$$ ✓

3) comutativitatea: $x, y \in \mathbb{R}_+^*$,

$$x \boxplus y = xy = yx = y \boxplus x$$ ✓

4) element neutru: $x \in \mathbb{R}_+^*$,

$$x \boxplus e = x \iff x \cdot e = x \iff x = 1 \in \mathbb{R}_+^*$$ ✓

5) element simetrizabil: $x \in \mathbb{R}_+^*$,

$$x \boxplus x' = e \iff x \cdot x' = 1 \iff x' = \frac{1}{x} \in \mathbb{R}_+^*$$ ✓

Din 1), 2), ..., 5) $\Rightarrow (\mathbb{R}_+^*, \boxplus)$ grup Abelian

6) $\alpha \boxdot (x \boxplus y) = \alpha \boxdot (xy) = (xy)^{\alpha} =$

$= x^{\alpha} y^{\alpha} = x^{\alpha} \boxplus y^{\alpha} = (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$ ✓

7) $(\alpha + \beta) \boxdot x = x^{\alpha+\beta} = x^{\alpha} \cdot x^{\beta} = x^{\alpha} \boxplus x^{\beta} =$

$= (\alpha \boxdot x) \boxplus (\beta \boxdot x)$ ✓

8) $\alpha \boxdot (\beta \boxdot x) = \alpha \boxdot (x^{\beta}) = (x^{\beta})^{\alpha} = x^{\alpha\beta} = (\alpha\beta) \boxdot x$ ✓

9) $1 \boxdot x = x^{1} = x$ ✓

Din toate cele de mai sus, avem că $\mathbb{R}_+^*$ este un $\mathbb{R}$ - spațiu vectorial împreună cu $\boxplus$ și operația externă $\boxdot$.

3.1.32. Să se verifice că operațiile:

$\boxplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $x \boxplus y = \sqrt[5]{x^5 + y^5}$, $\forall x, y \in \mathbb{R}$

$\boxdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $\alpha \boxdot x = \sqrt[5]{\alpha} x$, $\forall x, \alpha \in \mathbb{R}$

definesc o structură de $\mathbb{R}$- spațiu vectorial pe $\mathbb{R}$.

am modificat din $\boxed{\alpha \sqrt[5]{\alpha}\, x}$, pentru că nu obținem proprietățile operației externe

1) parte stabilă: $x, y \in \mathbb{R}$, $x \boxplus y = \sqrt[5]{x^5 + y^5} \in \mathbb{R}$ ✓

2) asociativitatea: $x, y, z \in \mathbb{R}$

$(x \boxplus y) \boxplus z = \sqrt[5]{x^5 + y^5} \boxplus z = \sqrt[5]{\left(\sqrt[5]{x^5 + y^5}\right)^5 + z^5} = \sqrt[5]{x^5 + y^5 + z^5}$

$$x \boxplus (y \boxplus z) = x \boxplus \sqrt[5]{y^5+z^5} = \sqrt[5]{x^5+\left(\sqrt[5]{y^5+z^5}\right)^5} =$$

$$= \sqrt[5]{x^5+y^5+z^5} = (x \boxplus y) \boxplus z \quad \checkmark$$

3) comutativitate: $x, y \in \mathbb{R}$,

$$x \boxplus y = \sqrt[5]{x^5+y^5} = \sqrt[5]{y^5+x^5} = y \boxplus x \quad \checkmark$$

4) element neutru: $x \in \mathbb{R}$,

$$x \boxplus e = x \iff \sqrt[5]{x^5+e^5} = \sqrt[5]{x^5} \iff$$

$$\iff x^5+e^5 = x^5 \iff e = 0 \in \mathbb{R} \quad \checkmark$$

5) element simetrizabil: $x \in \mathbb{R}$,

$$x \boxplus x' = e \iff \sqrt[5]{x^5+x'^5} = 0 \iff$$

$$\iff x'^5 = -x^5 \iff x' = -x \quad \checkmark$$

Din 1) - 5), avem că $(\mathbb{R}, \boxplus)$ este grup Abelian.

6) $$\alpha \boxdot (x \boxplus y) = \alpha \boxdot \sqrt[5]{x^5+y^5} = \sqrt[5]{\alpha} \cdot \sqrt[5]{x^5+y^5} =$$

$$= \sqrt[5]{\alpha(x^5+y^5)}$$

$$(\alpha \boxdot x) \boxplus (\alpha \boxdot y) = \sqrt[5]{\alpha}\, x \boxplus \sqrt[5]{\alpha}\, y =$$

$$= \sqrt[5]{(\sqrt[5]{\alpha}\, x)^5 + (\sqrt[5]{\alpha}\, y)^5} = \sqrt[5]{\alpha x^5 + \alpha y^5} =$$

$$= \sqrt[5]{\alpha(x^5+y^5)} = \alpha \boxdot (x \boxplus y) \quad \checkmark$$

7) $$\alpha \boxdot (\beta \boxdot x) = \alpha \boxdot \sqrt[5]{\beta}\, x = \sqrt[5]{\alpha}\sqrt[5]{\beta}\, x = \sqrt[5]{\alpha\beta}\, x =$$

$$= (\alpha\beta) \boxdot x \quad \checkmark$$

8)

$$(\alpha+\beta)\boxdot x = \sqrt[5]{\alpha+\beta}\,x = \sqrt[5]{\alpha+\beta}\cdot\sqrt[5]{x^5} =$$

$$= \sqrt[5]{x^5(\alpha+\beta)} = \sqrt[5]{\alpha x^5+\beta x^5} = \sqrt[5]{(\sqrt[5]{\alpha})^5 x^5 + (\sqrt[5]{\beta})^5 x^5} =$$

$$= \sqrt[5]{(\sqrt[5]{\alpha}\,x)^5 + (\sqrt[5]{\beta}\,x)^5} = \sqrt[5]{\alpha}\,x \boxplus \sqrt[5]{\beta}\,x =$$

$$= (\alpha \boxdot x) \boxplus (\beta \boxdot x) \ \checkmark$$

9) $1 \boxdot x = \sqrt[5]{1}\,x = x$ ✓

$\Rightarrow$ $\mathbb{R}$ este un $\mathbb{R}$-spațiu vectorial împreună cu operația internă $\boxplus$ și cea externă $\boxdot$.

3.1.33. Care dintre urm. submulțimi ale lui $\mathbb{R}^3$ sunt $\mathbb{R}$-subspații:

- $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0\}$

$$\Updownarrow$$

$$x_1 = \frac{x_3 - x_2}{2}$$

(a) $0 = (0,0,0) \in A$ ✓

(b) $x = \left(\frac{x_3-x_2}{2}, x_2, x_3\right)$, $y = \left(\frac{y_3-y_2}{2}, y_2, y_3\right) \in A$,

$$x+y = \left(\frac{(x_3+y_3)-(x_2+y_2)}{2}, x_2+y_2, x_3+y_3\right) \in A \ \checkmark$$

(c) $x \in A$, $\alpha \in \mathbb{R}$

$$\alpha x = \left(\alpha\frac{x_3-x_2}{2}, \alpha x_2, \alpha x_3\right)$$

Verif.: $2\left(\alpha\frac{x_3-x_2}{2}\right) + \alpha x_2 - \alpha x_3 = 0$

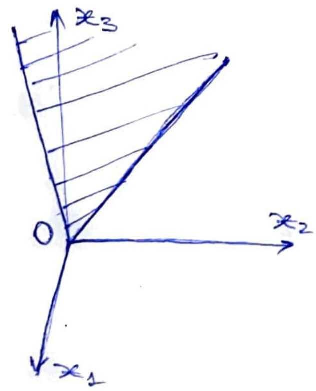
z 5 2

$\alpha x_3 - \alpha x_2 + \alpha x_2 - \alpha x_3 = 0$ ✓

$\Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$.

Reprezentare geometrică

Im planul x_2Ox_3 $(x_1 = 0)$



$x_1 = \frac{x_3 - x_2}{2} = 0 \Leftrightarrow x_3 = x_2$

(ecuația primei bisectoare)

im planul x_1Ox_3 $(x_2 = 0)$

$x_1 = \frac{x_3}{2} \Rightarrow x_3 = 2x_1$

→ A este planul comun al celor două drepte

- $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 1\}$

(a) $0 = (0,0,0) \notin B$

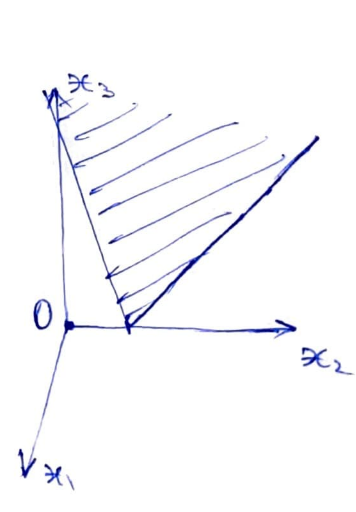
$2 \cdot 0 + 0 - 0 = 1 \Leftrightarrow 0 = 1$ (X)

$\Rightarrow B \nleq_{\mathbb{R}} \mathbb{R}^3$

Repr. geom.

→ la fel ca A, deplasat cu +1 unitate pe axa Ox_2

$0 = (0,0,0) \notin B$



- $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$

(a) $0 = (0,0,0) \in C$ ✓

(b) $x = (x_1, x_1, x_1) \in C$, $y = (y_1, y_1, y_1) \in C$

$x + y = (x_1 + y_1, x_1 + y_1, x_1 + y_1) \in C$ ✓

(c) $\alpha \in \mathbb{R}$, $x = (x_1, x_1, x_1) \in C$

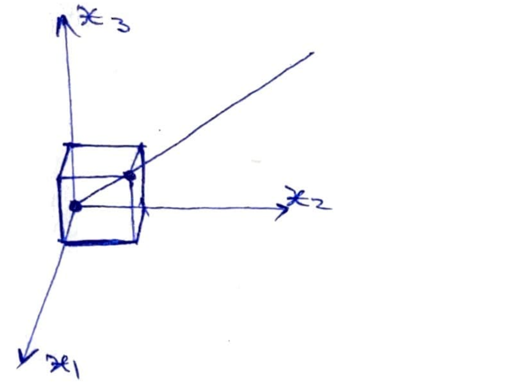
$\alpha x = (\alpha x_1, \alpha x_1, \alpha x_1) \in C$ ✓

$\Rightarrow C \leq_{\mathbb{R}} \mathbb{R}^3$.

Repr. geom.

→ C este diagonala cubului cu latura de 1 unitate;

→ trece prin $(0,0,0)$ și $(1,1,1)$;



- $D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2 = 0\}$

(a) $0 = (0,0,0) \in D$ ✓

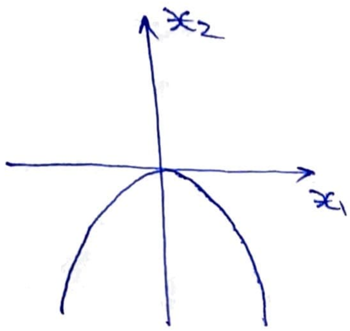
(b) $x, y \in D$

$$(x_1, -x_1^2, x_3) + (y_1, -y_1^2, y_3) = (x_1 + y_1, -(x_1^2 + y_1^2), x_3 + y_3)$$

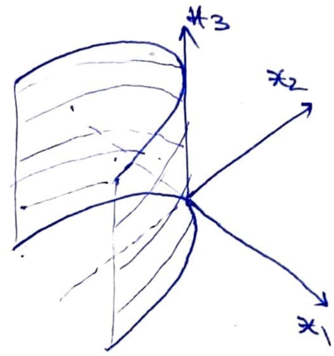
$$\neq$$

$$-(x_1 + y_1)^2$$

Repr. geom.:

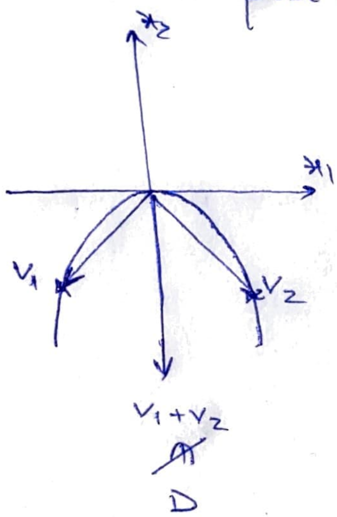


$\boxed{\{D \cap (x_1 O x_2)\}}$



$\boxed{\{x_1 O x_2 O x_3\}}$

contraexemplu:



$v_1 = (-1,-1,0)$

$v_2 = (1,-1,0)$

$v_1+v_2 = (0,-2,0) \notin D$

- $E = \mathbb{R}^3 \setminus A \Rightarrow$

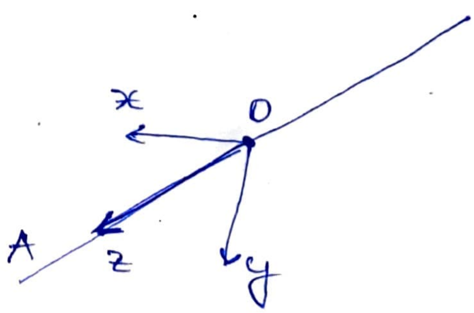
$\Rightarrow 0 \notin E \Rightarrow E \not\leq_{\mathbb{R}} \mathbb{R}^3$

- $F = E \cup \{0\} = (\mathbb{R}^3 \setminus A) \cup \{0\}$

(a) $0 \in F$ ✓

(b) $x, y \in F$, $z = x+y \in A \Rightarrow$

$\Rightarrow z \in F$



$\Rightarrow F \not\leq_{\mathbb{R}} \mathbb{R}^3$

Obs.: $S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid A \cdot x^T = 0\}$, $x^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{m \times n}(\mathbb{R})$$

$$A \cdot x^T = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in M_{m \times 1}(\mathbb{R})$$

$$A \cdot x^T \Leftrightarrow \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{cases}$$

Atunci $S \leq_{\mathbb{R}} \mathbb{R}^n$:

- $0 \in S$ ✓
- $x, y \in S$; $\alpha, \beta \in \mathbb{R}$:

$$A \cdot (\alpha x + \beta y)^T = A \cdot (\alpha x^T + \beta y^T) = \alpha \underbrace{A \cdot x^T}_{=0} + \beta \underbrace{A \cdot y^T}_{=0} = 0 \ ✓$$

3.1.35.

- $S = \langle (1,2,-1) \rangle$ (subgrupul generat de un vector)

$$\| $$

$$\{\alpha(1,2,-1) \mid \alpha \in \mathbb{R}\} = \{(\alpha, 2\alpha, -\alpha) \mid \alpha \in \mathbb{R}\}$$

$$\begin{cases} x_1 = \alpha \\ x_2 = 2\alpha \quad (\alpha \in \mathbb{R}) \\ x_3 = -\alpha \end{cases} \iff \begin{cases} 2x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$$ (una din reprezentări, nu e unică)

ecuații parametrice — ecuații generale

$$\iff S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 = x_1 + x_3 = 0\}$$

- $T = \langle (1,2,1), (-2,1,-3) \rangle = \{(\alpha, 2\alpha, \alpha) + (-2\beta, \beta, -3\beta) \mid \alpha, \beta \in \mathbb{R}\}$

$$= \{(\alpha - 2\beta, 2\alpha + \beta, \alpha - 3\beta) \mid \alpha, \beta \in \mathbb{R}\}$$

$$\begin{cases} x_1 = \alpha - 2\beta \\ x_2 = 2\alpha + \beta \quad (\alpha, \beta \in \mathbb{R}) \\ x_3 = \alpha - 3\beta \end{cases}$$

Vom avea o ecuație generală de forma:

$$a x_1 + b x_2 + c x_3 = 0$$

$$a(\alpha - 2\beta) + b(2\alpha + \beta) + c(\alpha - 3\beta) = 0 \iff$$

$$\iff \begin{cases} a + 2b + c = 0 \mid \cdot 3 & \text{(corespunzător coeficientului lui } \alpha) \\ -2a + b - 3c = 0 & (\text{—— " —— " —— lui } \beta) \end{cases}$$

$(+)$

$$a + 7b = 0 \iff a = -7b \quad \text{(înlocuim în prima ec.)}$$

$-7b+2b+c=0 \Rightarrow c=5b$

Alegem $b=1 \Rightarrow a=-7,\ c=5$

$\Rightarrow -7x_1+x_2+5x_3=0$

$T=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid -7x_1+x_2+5x_3=0\}$

3.1.36.

- $S=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1-x_2-x_3=0\}$

$x_1-x_2-x_3=0$ (1 singură ecuație $\Rightarrow$ 2 parametri)

$\left.\begin{array}{l} x_2=\alpha \\ x_3=\beta \end{array}\right\} \Rightarrow x_1=\alpha+\beta \Leftrightarrow$

$\Leftrightarrow S=\{(\alpha+\beta,\alpha,\beta) \mid \alpha,\beta\in\mathbb{R}\}$

$S=\{\alpha(1,1,0)+\beta(1,0,1) \mid \alpha,\beta\in\mathbb{R}\}$

$S=\langle(1,1,0),(1,0,1)\rangle$ $\rightarrow$ nu e soluție unică

$\rightarrow$ Temă: T.

3.1.37.

$S=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1+x_2+x_3=0\}$

$T=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1=x_2=x_3\}$

$\rightarrow$ Temă: S, T subspații în $\mathbb{R}^3$

- $S\oplus T=\mathbb{R}^3$ (Prop. 3.1.18 pt. detalii)

1) $S\cap T=0 \Leftrightarrow \left\{\begin{array}{l} x+x_2+x_3=0 \\ x_1=x_2+x_3 \end{array}\right.$ $\Rightarrow x_1=x_2=x_3=0$

2) $S+T = \{(a,b,c) \in \mathbb{R}^3 \mid \exists x \in S, \exists y \in T : x+y=(a,b,c)\}$

ecuațiile generale pentru S+T

$$\begin{cases} x_1+y_1=a \\ x_2+y_2=b \\ x_3+y_3=c \\ x_1=x_2=x_3 \\ y_1+y_2+y_3=0 \end{cases}$$

• adunăm primele trei ecuații:

$$x_1+x_2+x_3+\underbrace{y_1+y_2+y_3}_{=0}=a+b+c$$

$$\Rightarrow x_1=x_2=x_3=\frac{a+b+c}{3}$$

• din prima ecuație și $x_1=\frac{a+b+c}{3}$:

$$y_1=a-x_1=\frac{2a-b-c}{3}$$

Analog, $y_2=\frac{-a+2b-c}{3}$

$$y_3=\frac{-a-b+3c}{3}$$

$\Rightarrow \forall a,b,c \in \mathbb{R}^3$, $(a,b,c)\in S+T$

$$(a,b,c)=\left(\frac{a+b+c}{3},\frac{a+b+c}{3},\frac{a+b+c}{3}\right)+$$

$$+\left(\frac{2a-b-c}{3},\frac{-a+2b-c}{3},\frac{-a-b+2c}{3}\right)$$

scriere unică, $\forall (a,b,c)$

$\Rightarrow S\oplus T=\mathbb{R}^3$.