2.1.45

$* : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$x*y = xy-5x-5y+30 = (x-5)(y-5)+5$

$(\mathbb{R}, *)$ grup (?)

(i) $\forall x,y \in \mathbb{R},\ x*y \in \mathbb{R}$:

$$x*y = \underbrace{(x-5)}_{\in\mathbb{R}}\underbrace{(y-5)}_{\in\mathbb{R}} + \underset{\mathbb{R}}{\underset{\cap}{5}} \in \mathbb{R} \ \checkmark$$

(ii) $\forall x,y,z \in \mathbb{R},\ (x*y)*z = x*(y*z)$:

- $(x*y)*z = [(x-5)(y-5)+5]*z =$

$= [(x-5)(y-5)+5-5](z-5)+5 = (x-5)(y-5)(z-5)+5$

- $x*(y*z) = x*[(y-5)(z-5)+5] = (x-5)[(y-5)(z-5)+5-5]+5 =$

$= (x-5)(y-5)(z-5)+5$

(iii) $\exists e \in \mathbb{R}$ a.î. $e*x = x*e = x,\ \forall x\in\mathbb{R}$:

$x*e = x \iff (x-5)(e-5)+5 = x$

$\cancel{\pm}$ $(x-5)(e-5)-(x-5)=0$

$(x-5)(e-6)=0,\ \forall x\in\mathbb{R} \Rightarrow$

$\Rightarrow e-6=0 \Rightarrow \underline{\underline{e=6}} \in \mathbb{R}$

$e*x = (6-5)(x-5)+5 =$

$= x-5+5 = x \ \checkmark$

(iv) $\forall x\in\mathbb{R},\ \exists x'\in\mathbb{R}$ a.î. $x*x' = x'*x = 6$

$x*x' = e \iff (x-5)(x'-5)+5 = 6 \iff$

$\iff (x-5)(x'-5) = 1 \iff x' = \frac{1}{x-5}+5,\ \forall x\in\mathbb{R}\setminus\{5\}$

$\Rightarrow$ pt. $x=5$ nu avem element simetrizabil

$\Rightarrow (\mathbb{R}, *)$ nu este grup

- $(\mathbb{R}\setminus\{5\}, *)$ grup(?)

(i) $(x-5)(y-5)+5 \neq 5$, $\forall x,y \in \mathbb{R}\setminus\{5\}$

(ii) ✓

(iii) $e=6 \in \mathbb{R}\setminus\{5\}$

(iv) ✓ $x' = \frac{1}{x-5} + 5$, $\forall x \in \mathbb{R}\setminus\{5\}$

$\Rightarrow (\mathbb{R}\setminus\{5\}, *)$ grup

- $((5,\infty), *)$ grup (?)

(i) $\underbrace{(x-5)}_{>0}\underbrace{(y-5)}_{>0}+5 > 5 \Rightarrow$ ~~xxxxxxxx~~.

$(5,\infty)$ p.s. a lui $\mathbb{R}\setminus\{5\}$ în raport cu $*$ $\Rightarrow$ (ii) ✓

(iii) $e=6>5$

(iv) $x' = \underbrace{\frac{1}{x-5}}_{>0} + 5 > 5$ ✓, $\forall x \in$ ~~xxxx~~ $(5,\infty)$

$\Rightarrow ((5,\infty), *)$

- $((-\infty,5), *)$ grup(?)

(i) $\underbrace{(x-5)}_{<0}\underbrace{(y-5)}_{<0}+5 < 5$ ✓ $\forall x \in (-\infty,5)$

(iii) $e=6 \notin (-\infty,5)$

$\Rightarrow ((-\infty,5), *)$ nu este grup

2.1.52

$H = \{\pm 1, \pm i, \pm j, \pm k\}$

Pp. că știm $-(-x) = x$

$(-x)y = x(-y) = -xy$

- $i^2 = j^2 = k^2 = -1$
- $ij = k = -ji$

  $jk = i = -kj$

  $ki = j = -ik$

$\cdot$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

→ nu e comutativă

→ $\forall x, y \in H$, $x \cdot y \in H$

→ asociativitatea $(*)$

→ 1 este element neutru

→ $1' = 1$; $j' = -j$;

$-1' = -1$; $-j' = j$;

$i^{-1} = -i$; $k' = -k$;

$-i' = i$; $-k' = k$;

$\Rightarrow (H, \cdot)$ grup

Met. 1 : computațional (8^3 posib.)

Met. 2 : Încercăm să găsim o operație despre care știm că e asociativă și 8 elemente care operate între ele dau tabla prezentată :

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ;$$

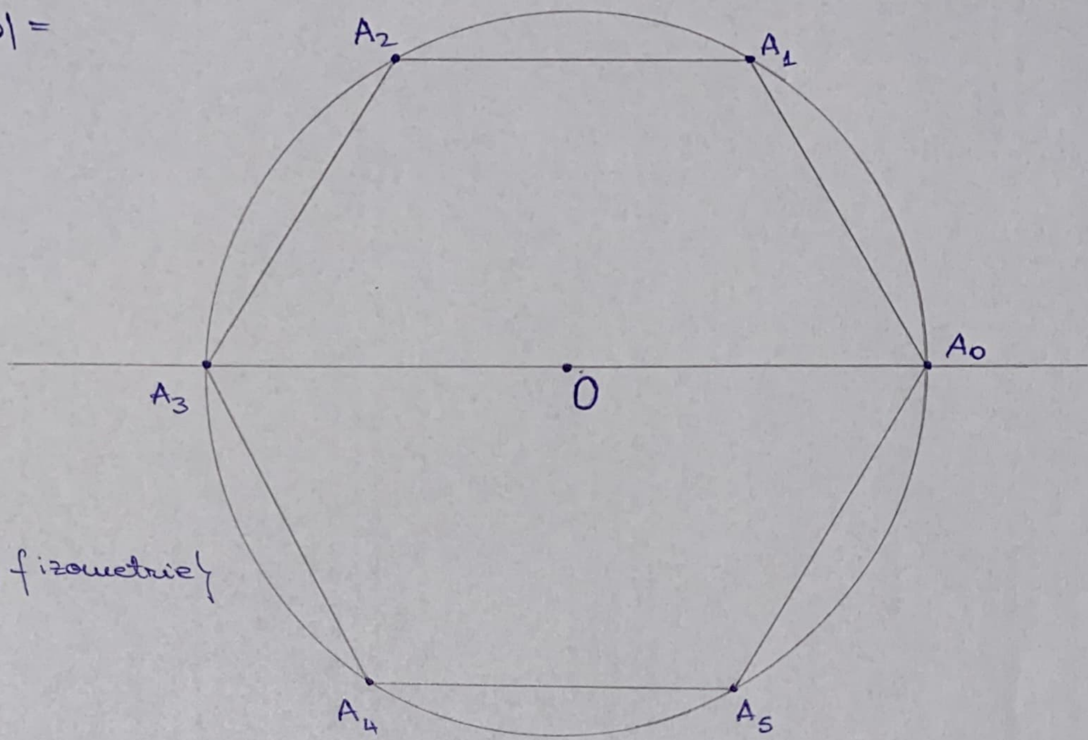
$$j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ; \quad \mathbb{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Met. 3 : matrice aparținând $M_{2,2}(\mathbb{R})$

2.1.51

izometrie:

$f: \alpha \to \alpha : |f(x), f(y)| = $
$= |xy|, \forall x, y \in \alpha$



1) $Izom(\alpha) = \{ f: \alpha \to \alpha \mid f$ izometrie$\}$

Presupunem adevărat că orice izometrie este bijectivă (fără demonstrație).

$\hookrightarrow S(\alpha) = \{ f: \alpha \to \alpha \mid f$ bijectivă$\}$

a) Vom arăta că $Izom(\alpha) \leq S(\alpha)$

• $1_\alpha : \alpha \to \alpha \in Izom(\alpha)$:

$x, y \in \alpha : \quad |1_\alpha(x) 1_\alpha(y)| = |xy|;$ ✓

• $f, g \in Izom(\alpha) \Rightarrow f \circ g \in Izom(\alpha)$

$x, y \in \alpha : |(f \circ g)(x)(f \circ g)(y)| = |f(g(x)) f(g(y))|$

$\xlongequal{f \in Izom(\alpha)} |g(x) g(y)| \xlongequal{g \in Izom(\alpha)} |xy|$

$\Rightarrow f \circ g \in Izom(\alpha);$ ✓

- $f \in Izom(\alpha) \Rightarrow f$ bij. $\Rightarrow \exists f^{-1}: \alpha \to \alpha$ (a.î. $f \circ f^{-1} = 1_\alpha = f^{-1} \circ f$)

$x, y \in \alpha$:

$$|\underbrace{f^{-1}(x)}_{\in \alpha}\underbrace{f^{-1}(y)}_{\in \alpha}| \overset{f \in Izom(\alpha)}{=\!=} |f(f^{-1}(x))f(f^{-1}(y))| =$$

$$= |(f \circ f^{-1})(x)(f \circ f^{-1})(y)| = |1_\alpha(x) 1_\alpha(y)| = |xy|; \checkmark$$

b) $D_m = \{f \in Izom(\alpha) \mid f(A_0A_1 \dots A_{m-1}) = A_0A_1 \dots A_{m-1}\}$

Acum vom arăta $\underline{\underline{D_m}} \leq \underline{\underline{Izom(\alpha)}}$ și $|D_m| = 2m$ elemente

- $1_\alpha(A_0A_1 \dots A_{m-1}) = A_0A_1 \dots A_{m-1} \Rightarrow 1_\alpha \in D_m$

- $f, g \in D_m \Rightarrow (f \circ g)(A_0A_1 \dots A_{m-1}) = f(g(A_0A_1 \dots A_{m-1})) \overset{g \in D_m}{=\!=}$

$$= f(A_0A_1 \dots A_{m-1}) \overset{f \in D_m}{=\!=} A_0A_1 \dots A_m \Rightarrow f \circ g \in D_m$$

- $f \in D_m \Rightarrow f \in Izom(\alpha) \Rightarrow f$ bij. $\Rightarrow$

$\Rightarrow \exists f^{-1}: \alpha \to \alpha \Rightarrow f^{-1} \in Izom(\alpha)$

$$f^{-1}(A_0A_1 \dots A_{m-1}) = f^{-1}(f(A_0A_1 \dots A_{m-1})) = 1_\alpha(A_0A_1 \dots A_{m-1}) =$$

$$= A_0A_1 \dots A_{m-1}$$

$\Rightarrow D_m \leq Izom(\alpha)$

c) Fie $f \in D_m \Rightarrow f(A_0) \in \{A_0, A_1, \dots, A_{m-1}\}$

Notăm $A_k = f(A_0)$, $0 \le k \le m-1$

$\Rightarrow f(A_1) \in \{A_{k+1}, A_{k-1}\}$

I). Dacă $f(A_1) = A_{k+1} \Rightarrow f(A_2) = A_{k+2}$

$\Rightarrow f(A_i) = A_{k+i}$ (suma indicelui este modulo m)

II) Dacă $f(A_1) = A_{k-1} \Rightarrow f(A_2) = A_{k-2}$

Notăm :

s - rotația cu $\frac{2\pi}{m}$ în jurul lui O

t - simetria axială față de axa OA_0

$\Rightarrow f(A_i) = A_{k-i}$

- în cazul II $f = s^k t$
- Conform modelului, $s^m = 1 = t^2$

$\Rightarrow D_m = \{1, s, s^2, \dots, s^{m-1}, t, st, \dots, s^{m-1}t\}$

$= \langle s, t \mid s^m = 1 = t^2, ts = s^{m-1}t \rangle$

$D_3 = \{1, s, s^2, t, st, s^2t\}$

	1	s	s^2	t	st	s^2t
1	1	s	s^2	t	st	s^2t
s	s	s^2	1	st	s^2t	t
s^2	s^2	1	s	s^2t	t	st
t	t	s^2t	st	1	s^2	s
st	st	t	s^2t	s	1	s^2
s^2t	s^2t	st	t	s^2	s	1

Temă: tabla pt. D_4