# CURS 11

## Stabilitatea punctelor de echilibru prin funcții Lyapunov

$$(1)\ \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

$X^*(x^*, y^*)$ pct. de echilibru pt (1)

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \qquad f = (f_1, f_2)$$

$$J_f(x^*, y^*)$$

$D \subseteq \mathbb{R}^2$ domeniu, $X^* \in D$

Pt o funcție $V = V(x,y) \in C^1(D)$ numim derivata lui $V$ de-a lungul traiectoriilor sist. (1) și o notăm prin

$$\dot{V}(x,y) = \frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial V}{\partial y}(x(t), y(t)) \cdot y'(t)$$

$$\Rightarrow \boxed{\dot{V}(x,y) = \frac{\partial V}{\partial x}(x,y)\cdot f_1(x,y) + \frac{\partial V}{\partial y}(x,y)\cdot f_2(x,y)}$$

## Teorema lui Lyapunov

Fie $V \in C^1(D)$ a.î.

(i) $V(X^*) = V(x^*,y^*) = 0$ și $V(x,y) > 0$, $\forall (x,y) \in D \setminus \{(x^*,y^*)\}$

Atunci:

(ii) Dacă $\dot{V}(x,y) \le 0$, $\forall (x,y) \in D \Rightarrow X^*(x^*,y^*)$ este local stabil

(iii) Dacă $\dot{V}(x,y) < 0$, $\forall (x,y) \in D \setminus \{(x^*,y^*)\} \Rightarrow$

$\Rightarrow X^*(x^*,y^*)$ este local asimptotic stabil

(iv) Dacă $\dot{V}(x,y) > 0$, $\forall (x,y) \in D \setminus \{(x^*,y^*)\} \Rightarrow$

$\Rightarrow X^*(x^*,y^*)$ este instabil.

Obs: Dacă $V$ satisf. (i)+(ii) $\Rightarrow V$ fct Lyapunov

Dacă $V$ satisf. (i)+(iii) $\Rightarrow V$ fct. Lyapunov strictă

Exemple

1) $\begin{cases} x' = -y^3 \\ y' = x^3 \end{cases}$ $\quad X^*(0,0)$ este pct de echilibru

$f_1(x,y) = -y^3$

$f_2(x,y) = x^3$

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -3y^2 \\ 3x^2 & 0 \end{pmatrix}$$

$$J_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0 \text{ val. proprii} \Rightarrow$$

$$\operatorname{Re} \lambda_{1,2} = 0$$

$\Rightarrow$ nu se poate aplica T. stab. in prima aproximație

$V(x,y) = x^4 + y^4 \quad , D = \mathbb{R}^2$

$V(0,0) = 0 \qquad V(x,y) > 0 \ , \ \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\dot{V}(x,y) = \frac{\partial V}{\partial x} \cdot f_1 + \frac{\partial V}{\partial y} \cdot f_2 = 4x^3 \cdot (-y^3) + 4y^3 \cdot x^3 =$$

$$= -4x^3y^3 + 4x^3y^3 = 0 .$$

$\dot{V}(x,y) \leq 0 , \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow X^*(0,0)$ este local stabil.

2) $\begin{cases} x' = -x + y^3 \\ y' = -x - y \end{cases}$ $\underline{X^*(0,0)}$ pct. de echilibru

$f_1(x,y) = -x + y^3$

$f_2(x,y) = -x - y$

$V(x,y) = \frac{1}{2}x^2 + \frac{1}{4}y^4 , D = \mathbb{R}^2$

$V(0,0) = 0$ și $V(x,y) > 0 , \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}. \Rightarrow$ (i) satisf.

$$\dot{V}(x,y) = \frac{\partial V}{\partial x} \cdot f_1 + \frac{\partial V}{\partial y} \cdot f_2 = x \cdot (-x + y^3) + y^3 \cdot (-x - y) =$$

$$= -x^2 \cancel{+ xy^3} \cancel{- xy^3} - y^4 = -x^2 - y^4$$

$\Rightarrow \dot{V}(x,y) < 0,\ \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \Rightarrow$ (iii) satisf.

$\Rightarrow X^*(0,0)$ este local asimptotic stabil

3) $\begin{cases} x' = x - y^3 \\ y' = x + y \end{cases}$ $X^*(0,0)$ pct de echil.

$f_1(x,y) = x - y^3$

$f_2(x,y) = x + y$

$V(x,y) = \frac{1}{2}x^2 + \frac{1}{4}y^4,\ D = \mathbb{R}^2$

(i) satisf (vezi ex. 2)

$\dot{V}(x,y) = \frac{\partial V}{\partial x} \cdot f_1 + \frac{\partial V}{\partial y} \cdot f_2 = x \cdot (x - y^3) + y^3(x+y) =$

$= x^2 - \cancel{xy^3} + \cancel{xy^3} + y^4 = x^2 + y^4$

$\dot{V}(x,y) > 0,\ \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \Rightarrow$ (iv) satisf.

$\Rightarrow X^*(0,0)$ este instabil

4) Sistemul pradă- prădător.

$$\begin{cases} x' = ax - bxy \\ y' = -cy + d\cdot xy \end{cases} \qquad a, b, c, d > 0$$

$f_1(x,y) = ax - bxy$

$f_2(x,y) = -c + d\cdot xy$

$X_1^*(0,0)$ și $X_2^*\left(\frac{c}{d}, \frac{a}{b}\right)$

$X_1^*$: instabil (T. stab. în primă aprox.)

$X_2^*$: nu se poate aplica T. stab. în prima aprox.

$J_f\left(\frac{c}{d}, \frac{a}{b}\right)$ are val. proprii $\lambda_{1,2} = \pm i\sqrt{ac}$

$\underline{Re\, \lambda_{1,2} = 0}$.

$D = \mathbb{R}_+^2 = [0,+\infty) \times [0,+\infty)$

$$V(x,y) = \left(d\cdot x - c\ln x - c + c\ln\frac{c}{d}\right) + \left(by - a\ln y - a + a\ln\frac{a}{b}\right)$$

$$= d\left(x - \frac{c}{d} - \frac{c}{d}\ln\frac{xd}{c}\right) + b\left(y - \frac{a}{b} - \frac{a}{b}\ln\frac{yb}{a}\right)$$

$$V\left(\frac{c}{d}, \frac{a}{b}\right) = 0$$

se știe că $\quad \ln t + 1 \le t \quad \forall t > 0$

egalitatea are loc pt $t = 1$.

$$V(x,y) = d\left[x - \frac{c}{d}\left(1 + \ln\frac{x\cdot d}{c}\right)\right] + b\left[y - \frac{a}{b}\left(1 + \ln\frac{by}{a}\right)\right]$$

$$= c\Bigg[\underbrace{x\cdot\frac{d}{c} - \left(1 + \ln\frac{x\cdot d}{c}\right)}_{>0}\Bigg] + a\Bigg[\underbrace{\frac{yb}{a} - \left(1 + \ln\frac{by}{a}\right)}_{>0}\Bigg]$$

$t = \frac{x\cdot d}{c} \qquad \ln\frac{xd}{c} + 1 < \frac{xd}{c}$

$\frac{xd}{c} \neq 1$

$\overline{x \neq \frac{c}{d}}$

$t = \frac{yb}{a} \Rightarrow 1 + \ln\frac{by}{a} < \frac{by}{a}$

$\frac{by}{a} \neq 1$

$\Updownarrow$

$y \neq \frac{a}{b}$

$\Rightarrow \underline{V(x,y) > 0}, \ \forall (x,y) \in \mathbb{R}^2 \setminus \{X_2^*\}$

$\Rightarrow$ (ii) satisf.

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\cdot f_1 + \frac{\partial V}{\partial y}\cdot f_2 =$$

$$= \left(d - \frac{c}{x}\right)\left(ax - bxy\right) + \left(b - \frac{a}{y}\right)\left(-cy + d\cdot xy\right) =$$

$$= \cancel{adx} - \cancel{bdxy} - \cancel{ac} + \cancel{cby} - \cancel{bcy} + \cancel{bdxy} + \cancel{ac} - \cancel{adx}$$

$$= 0$$

$\Rightarrow \dot{V}(x,y) = 0 \leq 0, \ \forall (x,y) \in \mathbb{R}_+^2 \ \Rightarrow$ (ii) satisf.

$\Rightarrow X_2^*\left(\frac{c}{d}, \frac{a}{b}\right)$ este local stabil

$$\begin{cases} x' = ax - bxy \\ y' = -cy + dxy \end{cases}$$

$$\frac{dx}{dy} = \frac{ax - bxy}{-cy + dxy}$$ ec. dif. a orbitelor

$$\frac{dx}{dy} = \frac{a - by}{y} \cdot \frac{x}{-c + d \cdot x}$$

$$\frac{a - by}{y} \cdot dy = \frac{-c + d \cdot x}{x} \cdot dx$$

$$\int \left(\frac{a}{y} - b\right) dy = \int \left(-\frac{c}{x} + d\right) dx$$

$$a \ln y - by = -c \ln x + dx + \mathcal{C}$$

$$\boxed{d \cdot x - c \ln x + by - a \ln y + \mathcal{C} = 0}$$