2.1.53

$(\mathbb{R},+) \simeq (\mathbb{R}_+^*,\cdot)$

$f:(\mathbb{R},+) \to (\mathbb{R}_+^*,\cdot)$

$f(x) = a^x$ , $a>0$ , $a \neq 1$

1) morfism: $\forall x,y \in \mathbb{R}$, $f(x+y)=f(x)f(y)$

$f(x+y) = a^{x+y} = a^x \cdot a^y = f(x)f(y)$

$f(0)=1$

2) $f$ injectivă:

$x,y \in \mathbb{R}$: $f(x)=f(y) \Leftrightarrow a^x = a^y \mid \log_a \Rightarrow$

$\Rightarrow x=y$ ✓

3) $f$ surjectivă:

$y \in \mathbb{R}_+^*$, $\exists x = \log_a y$ a.î. $f(x)=y$

$a \neq 1$

$\Rightarrow x \in \mathbb{R}$

$f$ bijectivă $\Rightarrow \exists f^{-1}: (\mathbb{R}_+^*,\cdot) \to (\mathbb{R},+)$

$f^{-1}(x) = \log_a x$ , $a \neq 1$, $a>0$

2.1.54

$f : (\mathbb{C}^*, \cdot) \to (\mathbb{R}, +)$

$f(x) = \arg(x)$

Greşeala :

pt. $x = y = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$xy = \cos(3\pi) + i\sin(3\pi)$

$f(xy) = \arg(\cos 3\pi + i\sin 3\pi) = \pi$

$f(x) + f(y) = \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi \neq \pi$

Dacă

$(\mathbb{R}, \mathbb{R}, \equiv) : x \equiv y \iff \arg(x) - \arg(y) = 2k\pi, k \in \mathbb{Z}$

atunci: • echivalență pe $\mathbb{R}$

• $(\mathbb{R}/_{\equiv}, +)$ grup,

$f : \mathbb{C}^* \to \mathbb{R}/_{\equiv}$

$f(x) = [\arg(x)]_{\equiv}$ | funcție bine definită
- morfisme de grupuri
-

Temă:

$\operatorname{Ker} f = ?$

$\operatorname{Im} f = ?$

2.1.57

$(\mathbb{Z}, +)$ grup

$H \leq \mathbb{Z}$

$\underbrace{Sub(\mathbb{Z},+) = \{n\mathbb{Z} \mid n \in \mathbb{N}\}}_{\text{de arătat}}$, unde $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$

$(H,+) \leq (\mathbb{Z},+)$ ~~...~~

„$\subseteq$" $H \in Sub(\mathbb{Z})$:

I. $H = \{0\} \Rightarrow H = 0 \cdot \mathbb{Z} \in \{n\mathbb{Z} \mid n \in \mathbb{N}\}$

II. $H \neq 0 \Rightarrow \exists m \in H, m \neq 0 \Rightarrow -m \in H$

$\Rightarrow H \cap \mathbb{N}^* \neq \emptyset$

$(\mathbb{N}, \leq)$ lanț $\Rightarrow$ orice submulțime nevidă are un cel mai mic element $\Rightarrow$

$\Rightarrow \underline{\exists n \in H \cap \mathbb{N}^*} : \forall m \in H, n \leq m$

Fie $x \in H \Rightarrow x = nq + r \Rightarrow r = x - nq$, $q \in \mathbb{Z}$

$r \in \mathbb{N}, r < n$

$\Rightarrow r = \underset{\substack{\cap \\ H}}{x} - (\overbrace{\underbrace{n + n + \ldots + n}_{\text{de } q \text{ ori}}}^{\in H}) \Rightarrow r \in H, r < n \underset{\substack{\uparrow \\ \text{cel mai mic element din } H}}{\overset{=}{=}} \Rightarrow$

$\Rightarrow r = 0 \Rightarrow x = nq \Rightarrow$

$\Rightarrow H \subseteq n\mathbb{Z}$

„$\supseteq$" $x \in m\mathbb{Z} \Rightarrow \exists y \in \mathbb{Z}$:

$$x = my = \underbrace{m + \dots + m}_{\text{de } y \text{ ori}} , m \in H \Rightarrow m\mathbb{Z} \subseteq H$$

Arătăm că $(m\mathbb{Z}, +) \leq (\mathbb{Z}, +)$:

$$m \in \mathbb{N} \; ; \; m\mathbb{Z} = \{my \mid y \in \mathbb{Z}\} =$$

$$= \{\dots, -m, 0, m, 2m, \dots\}$$

- $0 \in m\mathbb{Z}$
- $x_1, x_2 \in m\mathbb{Z} \Rightarrow \begin{cases} x_1 = my_1 \\ x_2 = my_2 \end{cases} \Rightarrow x_1 - x_2 = m(y_1 - y_2) \in m\mathbb{Z}$

<u>Obs.:</u> $H \leq \mathbb{Z}$, $m \in H \Rightarrow m\mathbb{Z} \subseteq H$

$$\Rightarrow m\mathbb{Z} = \langle m \rangle$$

## 2.1.58

Exemplu: $2\mathbb{Z} \cup 3\mathbb{Z}$

$(2\mathbb{Z}, +)$ grup

$(3\mathbb{Z}, +)$ grup

$2+3=5 \notin 2\mathbb{Z}+3\mathbb{Z}$ → nu respectă partea stabilă

## 2.1.59

$(G, +)$ grup abelian

$H, K \leq G$

$H+K = \{x+y \mid x \in H, y \in K\} \subseteq G$

$H+K = \langle H \cup G \rangle \iff$ 1) $H+K \leq G$ → subgrup

2) $H \cup K \subseteq H+K$ → conține reuniunea

3) $L \leq G, H \cup K \subseteq L \Rightarrow H+K \subseteq L$ → cel mai mic

1) • $0 = \underset{H}{\underset{\cap}{0}} + \underset{K}{\underset{\cap}{0}} \in H+K$

• $t_1, t_2 \in H+K$

$t_1 = x_1 + y_1$, $x_1, x_2 \in H$

$t_2 = x_2 + y_2$, $y_1, y_2 \in K$

$t_1 - t_2 = x_1 + y_1 - (x_2 + y_2) \overset{\text{comutativitate}}{=}$

$= \underbrace{x_1 - x_2}_{\in H} + \underbrace{y_1 - y_2}_{\in K} \in H+K$

$\Rightarrow H+K \leq G$ ✓

2) $t \in H \cup K \Rightarrow \begin{cases} t \in H \\ \text{sau} \\ t \in K \end{cases}$

I. $t \in H \Rightarrow t = t + 0 \in H + K$

II. $t \in K \Rightarrow t = 0 + t \in H + K$

$\Rightarrow t \in H + K, \forall t \in H \cup K$

$\Rightarrow H \cup K \subseteq H + K$ ✓

3) $L \leq G$ a.î. $H \cup K \subseteq L$

Fie $t \in H + K$, $t = \underset{\substack{\in \\ H}}{x} + \underset{\substack{\in \\ K}}{y}$

$$\left.\begin{array}{l} x \in H \subseteq H \cup K \subseteq L \Rightarrow x \in L \\ y \in K \subseteq H \cup K \subseteq L \Rightarrow y \in L \end{array}\right\} \xrightarrow{L\ p.s.} x + y \in L$$

$\Rightarrow H + K \subseteq L$ ✓

$\Rightarrow \langle H \cup K \rangle = H + K$

2.1.61

(a) $n\mathbb{Z} \subseteq m\mathbb{Z} \iff m \mid n$

" $\Rightarrow$ " $n = n \cdot 1 \in n\mathbb{Z}$

$n\mathbb{Z} \subseteq m\mathbb{Z} \Rightarrow n \in m\mathbb{Z} \Rightarrow n = mq, \ q \in \mathbb{Z}$

$\Rightarrow m \mid n$ ✓

" $\Leftarrow$ " $m \mid n \iff n = mq_1, \ q_1 \in \mathbb{Z}$

$x \in n\mathbb{Z} \Rightarrow x = n \cdot q_2, \ q_2 \in \mathbb{Z}$ $\Big\} \Rightarrow$

$\Rightarrow x = m \cdot \underbrace{(q_1 q_2)}_{\in \mathbb{Z}} \Rightarrow x \in m\mathbb{Z}$ ✓

(b) $n, m \in \mathbb{N} \Rightarrow n\mathbb{Z}, m\mathbb{Z} \leq \mathbb{Z} \Rightarrow$ $\cap$ a oricărei familii de subgrupuri este subgrup

$\Rightarrow \exists k \in \mathbb{N} : n\mathbb{Z} \cap m\mathbb{Z} =: k\mathbb{Z}$

Dar $n\mathbb{Z} \cap m\mathbb{Z} \subseteq n\mathbb{Z} \Rightarrow k\mathbb{Z} \subseteq n\mathbb{Z} = n \mid k$

$n\mathbb{Z} \cap m\mathbb{Z} \subseteq m\mathbb{Z} \Rightarrow k\mathbb{Z} \subseteq m\mathbb{Z} = m \mid k$

Fie $k' \in \mathbb{N} : n \mid k', m \mid k' \Rightarrow k'\mathbb{Z} \subseteq n\mathbb{Z}$

$k'\mathbb{Z} \subseteq m\mathbb{Z}$

$\Rightarrow k'\mathbb{Z} \subseteq n\mathbb{Z} \cap m\mathbb{Z} \Rightarrow k \mid k'$

$\Rightarrow k = \operatorname{lcm}(n, m)$

(c) $n\mathbb{Z}, m\mathbb{Z} \leq \mathbb{Z} \Rightarrow n\mathbb{Z} + m\mathbb{Z} = \langle n\mathbb{Z} \cup m\mathbb{Z} \rangle \leq \mathbb{Z}$

$\Rightarrow n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}, \ d \in \mathbb{N}$

Dar $n\mathbb{Z} \subseteq n\mathbb{Z} \cup m\mathbb{Z} \subseteq n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z} \overset{(a)}{\Longrightarrow} d \mid n$

$m\mathbb{Z} \subseteq n\mathbb{Z} \cup m\mathbb{Z} \subseteq n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z} \Rightarrow d \mid m$

Fie $d' \in \mathbb{N} : d' \mid n$ şi $d' \mid m$

$\Rightarrow n\mathbb{Z} \subseteq d'\mathbb{Z}, \ m\mathbb{Z} \subseteq d'\mathbb{Z} \Rightarrow$

$\Rightarrow n\mathbb{Z} \cup m\mathbb{Z} \subseteq d'\mathbb{Z}$, $d'\mathbb{Z}$ subgrup

$\Rightarrow n\mathbb{Z} + m\mathbb{Z} \subseteq d'\mathbb{Z}$

$\Rightarrow d\mathbb{Z} \subseteq d'\mathbb{Z} \Leftrightarrow d' \mid d$

$\Rightarrow d = \gcd(n, m)$

Diagrama Hasse a laticii $(Sub(\mathbb{Z}), \subseteq)$