8.17 O hiperbolă trece prin punctul $M(\sqrt{6}, 3)$ şi este tangentă dreptei $9x+2y-15=0$.
Stabiliţi ecuaţia hiperbolei, ştiind că axele sale coincid cu axele coordonate.

$M(\sqrt{6}, 3)$

$d: 9x+2y-15=0$

Ecuaţia hiperbolei:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \iff x^2b^2-y^2a^2=a^2b^2$$

Considerăm cazul general:

$$Ax+By+C=0, \text{ unde } \begin{cases} A=9 \\ B=2 \\ C=-15 \end{cases}$$

$$\Rightarrow y=\frac{-C-Ax}{B}=-\frac{Ax+C}{B}$$

$$-y^2a^2+x^2b^2-a^2b^2=0 \mid \cdot(-1)$$

$$y^2a^2-x^2b^2+a^2b^2=0$$

$$\left(-\frac{Ax+C}{B}\right)^2a^2-x^2b^2+a^2b^2=0$$

$$\frac{(Ax+C)^2}{B^2}a^2-x^2b^2+a^2b^2=0 \mid \cdot B^2$$

$$(A^2x^2+2ACx+C^2)a^2-b^2B^2x^2+a^2b^2B^2=0$$

$$a^2A^2x^2+2a^2ACx+a^2C^2-b^2B^2x^2+a^2b^2B^2=0$$

$$(a^2A^2-b^2B^2)x^2+2a^2ACx+(b^2B^2+C^2)a^2=0$$

$\Delta = b^2 - 4ac =$

$= (2a^2AC)^2 - 4(a^2A^2 - b^2B^2)(b^2B^2 + C^2)a^2 =$

$= 4a^4A^2C^2 - 4a^2(a^2b^2A^2B^2 + a^2A^2C^2 - b^4B^4 - b^2B^2C^2) =$

$= 4a^2(\cancel{a^2A^2C^2} - a^2b^2A^2B^2 - \cancel{a^2A^2C^2} + b^4B^4 + b^2B^2C^2) =$

$= 4a^2(-a^2b^2A^2B^2 + b^2B^2 \cdot b^2B^2 + b^2B^2C^2) =$

$= 4a^2b^2B^2(-a^2A^2 + b^2B^2 + C^2)$

Pentru ca dreapta să fie tangentă la hiperbolă, trebuie să avem $\Delta = 0$ $\Rightarrow$

~~(a, b)~~

$$\left.\begin{array}{l} \Rightarrow 4a^2b^2B^2(-a^2A^2 + b^2B^2 + C^2) = 0 \\ a, b > 0 \\ B = 12 \end{array}\right\} \Leftrightarrow$$

$\Leftrightarrow -a^2A^2 + b^2B^2 + C^2 = 0 \mid \cdot (-1)$

$a^2A^2 - b^2B^2 - C^2 = 0 \Leftrightarrow 81a^2 - 4b^2 - 225 = 0$

$M(\sqrt{6}, 3) \in$ hiperbolei $\Rightarrow 9a^2 - 6b^2 + a^2b^2 = 0$

$$\begin{cases} 81a^2 - 4b^2 - 225 = 0 \\ 9a^2 - 6b^2 + a^2b^2 = 0 \end{cases}$$

Notăm $\begin{cases} x = a^2 > 0 \\ y = b^2 > 0 \end{cases}$

$$\begin{cases} 81x - 4y - 225 = 0 \Rightarrow -4y = 225 - 81x \Leftrightarrow y = \dfrac{225 - 81x}{-4} = -\dfrac{81x - 225}{4} \\ 9x - 6y + xy = 0 \end{cases}$$

$$9 \cdot x - 6\left(-\frac{-81x+225}{4}\right) + x\left(-\frac{-81x+225}{4}\right) = 0$$

$$9 \cdot x + 6 \cdot \frac{-81x+225}{4} - x \cdot \frac{-81x+225}{4} = 0$$

$$9 \cdot x + \frac{-486x+1350}{4} - \frac{-81x^2+225x}{4} = 0 \quad | \cdot 4$$

$$36x - 486x + 1350 + 81x^2 - 225x = 0$$

$$81x^2 - 675x + 1350 = 0$$

$$\Delta = b^2 - 4ac =$$

$$= (-675)^2 - 4 \cdot 81 \cdot 1350 =$$

$$= 455.625 - 437.400 =$$

$$= 18.225 > 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{675 \pm 135}{162} \Rightarrow$$

$$\Rightarrow x_1 = 5 \Rightarrow y_1 = -\frac{-81 \cdot 5 + 225}{4} = -\frac{-405+225}{4} = -\frac{-180}{4} =$$

$$= -(-45) = 45$$

$$\Rightarrow x_2 = \frac{10}{3} \Rightarrow y_2 = -\frac{-81 \cdot \frac{10}{3} + 225}{4} = -\frac{\frac{-810+675}{3}}{4} =$$

$$= -\frac{-135^{(3}}{12} = \frac{45}{4}$$

$$\Rightarrow \left\{ \begin{array}{c} a^2 = 5 \text{ și } b^2 = 45 \\ \text{sau} \\ a^2 = \frac{10}{3} \text{ și } b^2 = \frac{45}{4} \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{c} \frac{x^2}{5} - \frac{y^2}{45} = 1 \\ \text{sau} \\ \frac{3x^2}{10} - \frac{4y^2}{45} = 1 \end{array} \right.$$