ON $\mathbb{Z}^d$-ODOMETERS ASSOCIATED TO INTEGER MATRICES

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Abstract. We extend the results of [GPS19] on characterization of conjugacy, isomorphism, and continuous orbit equivalence of $\mathbb{Z}^d$-odometers to dimensions $d > 2$. We then apply these exensions to the case of odometers defined by matrices with integer coefficients.

1. Introduction

In this note we first supply an argument that extends the main result of the paper titled "$\mathbb{Z}^d$-odometers and cohomology" by T. Giordano, I. F. Putnam, and C. F. Skau [GPS19, Theorem 1.5 (1), (2), (3)] to dimensions greater than 2. More precisely, the proofs of conjugacy, isomorphism, and continuous orbit equivalence characterizations of [GPS19, Theorem 1.5] are based on [GPS19, Theorem 4.4], the only result in that paper where the dimension restriction $d = 1, 2$ is important. Our Proposition 2.3 below lifts this restriction and therefore leads to the characterizations in arbitrary dimension $d \geq 1$. In turn, our proof of Proposition 2.3 uses a simple Lemma 2.1 that made the relevant computations of the image of the first cohomology group under a natural map τ_μ^1 below simple and possible in higher dimensions. In the last section we apply these results along with the earlier results by the second author [S22a] in the setting of odometers defined by matrices with integer coefficients. Below we use the notations and terminology from [GPS19], [S22], and [S22a] and refer to these papers for more details.

If $d \in \mathbb{N}$ and G is a subgroup of $\mathbb{Z}^d$, the group $\mathbb{Z}^d$ acts on $\mathbb{Z}^d/G$ by

$$\phi_G^k(l + G) = k + l + G, \quad k, l \in \mathbb{Z}^d.$$

Let $\mathcal{G} = \{G_1, G_2, \dots\}$ be a decreasing sequence of subgroups of $\mathbb{Z}^d$ of finite index, i.e.,

$$\mathbb{Z}^d = G_1 \supseteq G_2 \supseteq \dots, \quad [\mathbb{Z}^d \colon G_n] < \infty,\ n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ and $\mathbb{Z}^d \supseteq G_n \supseteq G_{n+1}$, let $q_n \colon \mathbb{Z}^d/G_{n+1} \to \mathbb{Z}^d/G_n$ denote the quotient map $q_n(k + G_{n+1}) = k + G_n,\ k \in \mathbb{Z}^d$.

Definition 1.1. *If $\mathcal{G} = \{G_1, G_2, \dots\}$ is as above, a* $\mathbb{Z}^d$-odometer *is the inverse limit system $(X_\mathcal{G}, \phi_\mathcal{G})$ of*

$$(\mathbb{Z}^d/G_1, \phi_{G_1}) \xleftarrow{q_1} (\mathbb{Z}^d/G_2, \phi_{G_2}) \xleftarrow{q_2} \dots.$$

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For $n \in \mathbb{N}$, the natural projection $(X_{\mathcal{G}}, \phi_{\mathcal{G}})$ to $(\mathbb{Z}^d/G_n, \phi_{G_n})$ is denoted by π_n.

If $G_n \neq G_{n+1}$ for infinitely many $n \in \mathbb{N}$, then $X_{\mathcal{G}}$ is a Cantor set and $\phi_{\mathcal{G}}$ is a minimal action of $\mathbb{Z}^d$ on it that is free if $\cap_{n=1}^{\infty} G_n = \{0\}$. The action $\phi_{\mathcal{G}}$ is also isometric in the metric $d_{\mathcal{G}}$ given by

$$d_{\mathcal{G}}(x, y) = \sup\{0, 1/n \colon \pi_n(x) \neq \pi_n(y)\}, \quad x, y \in X_{\mathcal{G}}.$$

Moreover, $X_{\mathcal{G}}$ supports a unique $\phi_{\mathcal{G}}$-invariant probability measure $\mu_{\mathcal{G}}$ such that

$$\mu_{\mathcal{G}}\left(\pi_n^{-1}\{k + G_n\}\right) = \frac{1}{[\mathbb{Z}^d \colon G_n]}, \quad n \in \mathbb{N},\ k \in \mathbb{Z}^d.$$

In [GPS19] the authors proposed an equivalent system to $(X_{\mathcal{G}}, \phi_{\mathcal{G}})$ in the sense of conjugacy defined below, using Pontryagin duality. Namely, let H be a subgroup of $\mathbb{Q}^d$ so that H contains $\mathbb{Z}^d$. Let $Y_H = \widehat{H/\mathbb{Z}^d}$ be the Pontryagin dual of the quotient. Here the groups H and $H/\mathbb{Z}^d$ are endowed with the discrete topology, and hence Y_H is compact. Identifying the Pontryagin dual $\widehat{\mathbb{T}^d}$ of d-torus $\mathbb{T}^d$ with $\mathbb{Z}^d$, we obtain an action ψ_H of $\mathbb{Z}^d$ on Y_H given by

$$\psi_H^k(x) = x + \hat{\rho}(k), \quad k \in \mathbb{Z}^d,\ x \in Y_H,$$

where $\rho\left((r_1, \ldots, r_d) + \mathbb{Z}^d\right) = (e^{2\pi i r_1}, \ldots, e^{2\pi i r_d})$. The action (Y_H, ψ_H) is free if and only if H is dense in $\mathbb{Q}^d$.

Definition 1.2. *Two actions (X, ϕ) and (Y, ψ) of the same group G are said to be* conjugate *if there exists a homeomorphism $h \colon X \to Y$ such that*

$$h \circ \phi^g = \psi^g \circ h,$$

for all $g \in G$. In this case we refer to h as a conjugacy *between the two actions.*

If H is an increasing union of finite index extensions H_n, $n \in \mathbb{N}$, of $\mathbb{Z}^d$, then, up to conjugacy, (Y_H, ϕ_H) is given as the inverse limit of

$$(Y_{H_1}, \phi_{H_1}) \xleftarrow{\widehat{i_1}} (Y_{H_2}, \phi_{H_2}) \xleftarrow{\widehat{i_2}} \ldots,$$

where $i_n \colon H_n/\mathbb{Z}^d \to H_{n+1}/\mathbb{Z}^d$, $n \in \mathbb{N}$, is the inclusion. The correspondence between $\mathbb{Z}^d$-odometers and systems (Y_H, ϕ_H), up to conjugacy, is established by realizing (Y_H, ϕ_H) as an inverse limit as above passing to dual lattices: if K is a lattice in $\mathbb{R}^d$, its dual lattice is

$$K^* = \{x \in \mathbb{R}^d \colon \langle k, x\rangle \in \mathbb{Z}, \text{ for all } k \in K\}.$$

We now consider odometers associated to integer matrices. For a non-singular $d \times d$-matrix A with integer coefficients, $A \in \mathrm{M}_d(\mathbb{Z})$, define

$$G_A = \left\{A^{-k}\mathbf{x} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d,\, k \in \mathbb{Z}\right\}, \quad \mathbb{Z}^d \subseteq G_A \subseteq \mathbb{Q}^d.$$

One can readily check that G_A is a subgroup of $\mathbb{Q}^d$. Applying the process described above to the group $H = G_A$, we get an associated $\mathbb{Z}^d$-odometer Y_{G_A}, defined up to conjugacy. In [S22], the second author classified groups G_A in the 2-dimensional case and applied the results to 2-dimensional odometers Y_{G_A} using [GPS19, Theorem 1.5]. In [S22a], it was studied when G_A, G_B are isomorphic as abstract groups for non-singular $A, B \in \mathrm{M}_d(\mathbb{Z})$ for an arbitrary d. We combine the results from [S22a] with the results of this paper to analyze when Y_{G_A}, Y_{G_B} are equivalent with respect to conjugacy, isomorphism, continuously orbit equivalence, and orbit equivalence.

2. Conjugacy and Isomorphism

Proposition 2.3 below extends [GPS19, Theorem 4.4] to higher dimensions and its proof requires the following lemma.

Lemma 2.1. *Let $d \in \mathbb{N}$ and let G be a subgroup of $\mathbb{Z}^d$ of finite index. Then for each $i = 1, \ldots, d$, there exists a basis $\mathcal{B} = \{f_1, \ldots, f_d\}$ of G such that $f_i = ae_i$, where $a \in \mathbb{N}$ and e_i is the i-th vector of the standard basis of $\mathbb{Z}^d$. Equivalently, for any non-singular $N \in \mathrm{M}_d(\mathbb{Z})$, a $d \times d$-matrix with integer coefficients and non-zero determinant, and each $i = 1, \ldots, d$, there exists $P = P(i) \in \mathrm{GL}_d(\mathbb{Z})$ such that the i-th column of NP is ae_i for some $a \in \mathbb{N}$.*

Proof. Since G is a subgroup of $\mathbb{Z}^d$ of finite index, G is a free group of rank d. Thus, the columns of N form a basis of G. We first prove the following:

Lemma 2.2. *There exists a basis $g_1, \ldots, g_d$ of G such that the j-th component $[g_l]_j$ of g_l is zero for any $l = 2, \ldots, d$ and $[g_1]_j = a$, $a \in \mathbb{N}$. Equivalently, the j-th row of the matrix $\begin{pmatrix} g_1 & g_2 & \ldots & g_d \end{pmatrix}$ (we write coordinates of each g_i as a column) is $\begin{pmatrix} a & 0 & \ldots & 0 \end{pmatrix}$.*

Proof of Lemma 2.2. Let $g_1, \ldots, g_d$ be an arbitrary basis of G and let

$$M = \begin{pmatrix} g_1 & g_2 & \ldots & g_d \end{pmatrix} = (g_{ij}),$$

$M \in \mathrm{M}_d(\mathbb{Z})$ is non-singular. Note that there exists at least one non-zero element in the j-th row of M, since $\det M \neq 0$. Assume there are two non-zero elements in the j-th row of M. Without loss of generality (we can interchange columns of M), $g_{j1} \neq 0$, $g_{j2} \neq 0$. By multiplying g_1, g_2 by -1 if necessary, we can assume $g_{j1} > 0$, $g_{j2} > 0$. If $g_{j1} = g_{j2}$, then $g'_1, \ldots, g'_d$ is a new basis of G with $g'_2 = g_2 - g_1$, $g'_l = g_l$, $l \neq 2$, and $[g'_2]_j = 0$. If $g_{j1} \neq g_{j2}$, then without loss of generality, we can assume $g_{j1} > g_{j2}$. Let $a_1 = g_{j1}$, $a_2 = g_{j2}$, and $a_1 = a_2 k + a_3$, for some $k, a_3 \in \mathbb{Z}$, $0 \leq a_3 < a_2$. Then $g'_1, \ldots, g'_d$ is a new basis of G with $g'_1 = g_2$, $g'_2 = g_1 - kg_2$, $g'_l = g_l$, $l \neq 1, 2$, and hence $[g'_1]_j = a_2$, $[g'_2]_j = a_3$, $0 \leq a_3 < a_2 < a_1$. Continuing this way, we get a sequence $0 \leq \cdots < a_h < \cdots < a_3 < a_2 < a_1$ of natural numbers, which in finitely many steps has to reach zero (this is essentially the Euclidean algorithm). This shows that there exists a basis $g'_1, \ldots, g'_d$ of G such that $[g'_2]_j = 0$. Repeating the process for any other g'_s, g'_t with $s \neq t$, $[g'_s]_j \neq 0$, $[g'_t]_j \neq 0$, we conclude that there exists a basis $g''_1, \ldots, g''_d$ of G such that $[g''_1]_j \in \mathbb{N}$, $[g'_l]_j = 0$ for any $l \neq 1$. □

We now use induction on d to prove Lemma 2.1. Let $d = 1$. Then $e_1 = 1$, $G = a\mathbb{Z}$ for some $a > 0$, $f_1 = ae_1$, and the claim follows. Let $d > 1$. We consider two cases: $i > 1$ and $i = 1$. Let $i > 1$. By Lemma 2.2 applied to $j = 1$, there exists a basis $g_1, \ldots, g_d$ of G

such that

$$M = \begin{pmatrix} g_1 & \dots & g_d \end{pmatrix} = \begin{pmatrix} a & 0 \\ * & M' \end{pmatrix}, \qquad a \in \mathbb{N},\ M' \in \mathrm{GL}_{d-1}(\mathbb{Z}).$$

By induction on d, there exists $S \in \mathrm{GL}_{d-1}(\mathbb{Z})$ such that the $(i-1)$-st column of $M'S$ is be'_{i-1}, where $b \in \mathbb{N}$ and e'_{i-1} is the $(i-1)$-st vector of the standard basis of $\mathbb{Z}^{d-1}$. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \in \mathrm{GL}_d(\mathbb{Z}).$$

Then the i-th column of MP is be_i and the claim follows for $i > 1$.

Let $i = 1$. By Lemma 2.2 applied to $j = d$, there exists a basis $g_1, \dots, g_d$ of G such that

$$M = \begin{pmatrix} g_1 & \dots & g_d \end{pmatrix} = \begin{pmatrix} M'' & * \\ 0 & a \end{pmatrix}, \qquad a \in \mathbb{N},\ M'' \in \mathrm{GL}_{d-1}(\mathbb{Z}).$$

As in the case $i > 1$, we apply the induction on d to M''. Thus, there exists $S' \in \mathrm{GL}_{d-1}(\mathbb{Z})$ such that the 1-st column of $M''S'$ is $b'e'_1$, where $b' \in \mathbb{N}$ and e'_1 is the 1-st vector of the standard basis of $\mathbb{Z}^{d-1}$. Let

$$P' = \begin{pmatrix} S' & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_d(\mathbb{Z}).$$

Then the 1-st column of MP' is $b'e_1$ and the claim follows in the case $i = 1$ as well. □

For a system (X, ϕ), where ϕ is a $\mathbb{Z}^d$-action on X, the first cohomology group $H^1(X, \phi)$ is defined as follows. A 1-cocycle is any continuous function $\theta\colon X \times \mathbb{Z}^d \to \mathbb{Z}$ such that

$$\theta(x, m+n) = \theta(x, m) + \theta(\phi^m(x), n), \quad x \in X,\ m, n \in \mathbb{Z}^d.$$

A 1-cocycle θ is a coboundary if there exists a continuous function $h\colon X \to \mathbb{Z}$ such that

$$\theta(x, n) = h\left(\phi^n(x)\right) - h(x), \quad x \in X,\ n \in \mathbb{Z}^d.$$

Let $Z^1(X, \phi)$ and $B^1(X, \phi)$ denote the groups of 1-cocycles and coboundaries, respectively, and let $H^1(X, \phi) = Z^1(X, \phi)/B^1(X, \phi)$ be the first cohomology group.

For any Cantor $\mathbb{Z}^d$-system (X, ϕ) with an invariant probability measure μ, the map τ^1_μ is defined as follows. If θ is a 1-cocycle, $\tau^1_\mu(\theta) \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ is given by

$$\tau^1_\mu(\theta)(n) = \int_X \theta(x, n)\, d\mu(x), \quad n \in \mathbb{Z}^d.$$

Since $\tau^1_\mu(\theta) = 0$ if θ is a coboundary, τ^1_μ passes to a well-defined group homomorphism

$$\tau^1_\mu\colon H^1(X, \phi) \to \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R}).$$

The space $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ is identified with $\mathbb{R}^d$ via the map that takes $\alpha \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ to $(\alpha(e_1), \dots, \alpha(e_d)) \in \mathbb{R}^d$.

Proposition 2.3. *Let $d \in \mathbb{N}$ and H be a group such that $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. Let μ be the unique invariant probability measure for the system (Y_H, ψ_H). Then the map*

$$\tau_\mu^1 \colon H^1(Y_H, \psi_H) \to H$$

is an isomorphism.

Proof. The proof here follows the lines of that of [GPS19, Theorem 4.4], with the exception that we use Lemma 2.1 above to simplify computations, that also work in any dimension.

We write $H = \cup_{n\in\mathbb{N}} H_n$, the union of an increasing sequence of finite index extensions H_n of $\mathbb{Z}^d$. Let $G_n = H_n^*$ be the dual lattice. Then $\mathcal{G} = \{G_1, G_2, \dots\}$ is a decresing sequence of finite index subgroups of $\mathbb{Z}^d$ such that (Y_H, ψ_H) and $(X_\mathcal{G}, \phi_\mathcal{G})$ are conjugate.

The first cohomology group $H^1(Y_H, \psi_H)$ is the direct limit of the sequence $H^1(\mathbb{Z}^d/G_n, \phi_{G_n})$. Therefore, if $[\theta] \in H^1(Y_H, \psi_H)$, then θ is a cocycle in $Z^1(\mathbb{Z}^d/G_n, \phi_{G_n})$. We can write

$$\tau_\mu^1(\theta) = [\mathbb{Z}^d \colon G_n]^{-1} \sum_{k\in F} \left(\theta(k + G_n, e_1), \dots, \theta(k + G_n, e_d)\right), \tag{2.1}$$

where F is any fundamental domain for G_n. Since each $\theta(k + G_n, e_i)$, $i = 1, \dots, d$, is integer, $\tau_\mu^1(\theta)$ is in H_n, and hence in H.

Conversely, let $h \in H$, and hence $h \in H_n$ for some $n \in \mathbb{N}$. We show that there exists $[\theta] \in H^1(Y_H, \psi_H)$ such that $\tau_\mu^1(\theta) = h$. According to [GPS19, Lemma 4.2], for each $\theta \in Z^1(\mathbb{Z}^d/G_n, \phi_{G_n})$, the map $\alpha(\theta)$ given by

$$\alpha(\theta)(g) = \theta(G_n, g), \quad g \in G_n,$$

induces an isomorphism $\alpha \colon H^1(\mathbb{Z}^d/G_n, \phi_{G_n}) \to \mathrm{Hom}(G_n, \mathbb{Z})$. The space $\mathrm{Hom}(G_n, \mathbb{Z})$ is identified with H_n via the inner product, and thus there exists $\theta \in Z^1(\mathbb{Z}^d/G_n, \phi_{G_n})$ with $\alpha(\theta) = \langle \cdot, h\rangle$.

It remains to show that $\tau_\mu^1(\theta) = h$, i.e.,

$$\sum_{k\in F} \theta(k + G_n, e_i) = [\mathbb{Z}^d \colon G_n]\langle e_i, h\rangle, \quad \text{for each } i = 1, \dots, d. \tag{2.2}$$

For each fixed $i = 1, \dots, d$, we apply Lemma 2.1 to find a basis $\mathcal{B} = \{f_1, \dots, f_d\}$ for G_n such that $f_i = a e_i$, where $a \in \mathbb{N}$. We choose a fundamental domain F for G_n in Equation (2.2) to consist of $\mathbb{Z}^d$ elements in a parallelepiped in $\mathbb{R}^d$ determined by the basis $\mathcal{B}$. Namely,

$$F = \{t_1 f_1 + \cdots + t_d f_d \colon 0 \le t_1, \dots, t_d < 1\} \cap \mathbb{Z}^d.$$

The map θ is defined by

$$\theta(k_1 + G_n, k_2 + g) = \langle g', h\rangle,$$

for $k_1, k_2 \in F, g \in G_n$, where $g' \in G_n$ is the unique element such that $k_1 + k_2 + g = k' + g'$ and $k' \in F$. To compute the left-hand side of (2.2) we choose $k_1 = k \in F$, $k_2 = e_i$, and $g = 0$. Note that $k_2 \in F$ because of our choice of F. If $k \in F$ is such that $k + e_i$ is also in F, then $g' = 0$, and thus the term $\theta(k + G_n, e_i)$ does not contribute anything to the sum in (2.2).

Now assume that $k \in F$ is such that $k + e_i$ is not in F. Let F_i denote the set of all such k. In this case $g' = f_i$ because $k' = k + e_i - f_i \in F$. Indeed, let $P_{k,i}$ consist of all elements $n \in F$ such that for each $j = 1, \dots, d,\ j \neq i$, the j'th component of n coincides with the j'th component of k. Since F is a parallelepiped with side $f_i = ae_i,\ a \in \mathbb{N}$, the i'th coordinates of elements $n \in P_{k,i}$ form a sequence of a consecutive integers $m, m+1, \dots, m+a-1$. Now, the choice of k gives that its i'th component is $m + a - 1$. For $j = 1, \dots, d,\ j \neq i$, the j'th component of k' equals the j'th component of k. The i'th component of k' equals m, and thus $k' \in P_{k,i} \subseteq F$.

We conclude that $\theta(k + G_n, e_i) = \langle f_i, h\rangle = a\langle e_i, h\rangle$ for each $k \in F_i$. The number of elements k in F_i equals the number of elements in the projection P_i of F to $\mathbb{Z}^{d-1}$ obtained from $\mathbb{Z}^d$ by omitting the i'th coordinate. Indeed, if $k \in F_i \subseteq F$, omitting its i'th coordinate we get an element in P_i. Conversely, for any $k_i \in P_i$, again, since F is a parallelepiped with side $f_i = ae_i,\ a \in \mathbb{N}$, the i'th coordinates of all elements in F that project to p_i form a sequence $m, m+1, \dots, m+a-1$. The element k that projects to k_i and whose i'th coordinate is $m+a-1$ is in F_i. The projection P_i is itself a parallelepiped in $\mathbb{Z}^{d-1}$. Let b denote the number of elements in P_i, which is the same as the number of elements in F_i. We therefore conclude

$$\sum_{k \in F} \theta(k + G_n, e_i) = \sum_{k \in F_i} \theta(k + G_n, e_i) = ab\langle e_i, h\rangle.$$

But ab is the number of elements in F, which is the same as $[\mathbb{Z}^d \colon G_n]$, and we are done. $\square$

A characterization of a Cantor $\mathbb{Z}^d$-system (Y, ψ) up to conjugacy now follows from Proposition 2.3 and the following elementary lemma.

Lemma 2.4. *If (Y_1, ψ_1) and (Y_2, ψ_2) are conjugate, then*

$$\tau^1_{\mu_1}\left(H^1(Y_1, \psi_1)\right) = \tau^1_{\mu_2}\left(H^1(Y_2, \psi_2)\right),$$

where μ_1, μ_2 are the unique probability measures for $(Y_1, \psi_1), (Y_2, \psi_2)$, respectively.

Proof. Let h be a conjugacy between (Y_1, ψ_1) and (Y_2, ψ_2). The conjugacy h induces an isomorphism of cohomologies via the map

$$h^* \colon H^1(Y_2, \psi_2) \to H^1(Y_1, \psi_1)$$

given by $h^*([\theta]) = [h^*(\theta)]$, where $h^*(\theta(x, n)) = \theta(h(x), n)$. The invariance of μ_1 and uniqueness of μ_2 imply that $h_*(\mu_1) = \mu_2$, where $h_*(\mu_1)$ denotes the push-forward measure of μ_1 under h. Therefore, for $[\theta(y, n)] \in H^1(Y_2, \psi_2)$, one has

$$\begin{aligned} \tau^1_{\mu_2}(\theta)(n) &= \int_{Y_2} \theta(y, n)\, d\mu_2 = \int_{Y_2} \theta(y, n)\, dh_*(\mu_1) \\ &= \int_{Y_1} \theta(h(x), n)\, d\mu_1 = \tau^1_{\mu_1}(h^*(\theta))(n). \end{aligned}$$

$\square$

Corollary 2.5. *Two $\mathbb{Z}^d$-actions (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are conjugate if and only if $H_1 = H_2$.*

Proof. If $H_1 = H_2$, the claim conjugacy is trivial. Assume now that (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are conjugate. Proposition 2.3 gives

$$\tau^1_{\mu_1}\left(H^1(Y_{H_1}, \psi_{H_1})\right) = H_1, \quad \tau^1_{\mu_2}\left(H^1(Y_{H_2}, \psi_{H_2})\right) = H_2,$$

where μ_1 and μ_2 are the unique invariant probability measures for (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}), respectively. Now Lemma 2.4 gives $H_1 = H_2$. □

Definition 2.6. *Let (X, ϕ) be an action of a group G and (Y, ψ) an action of a group H. An* isomorphism *between the actions is a pair (h, α), where $h\colon X \to Y$ is a homeomorphism and $\alpha\colon G \to H$ is a group isomorphism, such that*

$$h \circ \phi^g = \psi^{\alpha(g)} \circ h.$$

If such a pair exists, (X, ϕ) and (Y, ψ) are said to be isomorphic.

Proposition 2.7. *Let H_1 (resp., H_2) be a subgroup of $\mathbb{Q}^{d_1}$ (resp., $\mathbb{Q}^{d_2}$) that contains $\mathbb{Z}^{d_1}$ (resp., $\mathbb{Z}^{d_2}$). A $\mathbb{Z}^{d_1}$-action (Y_{H_1}, ψ_{H_1}) is isomorphic to a $\mathbb{Z}^{d_2}$-action (Y_{H_2}, ψ_{H_2}) if and only if $d_1 = d_2$, the common value being denoted by d, and there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $AH_1 = H_2$.*

Proof. If $d = d_1 = d_2$ and $AH_1 = H_2$ for some $A \in \mathrm{GL}_d(\mathbb{Z})$, then (Y_{AH_1}, ψ_{AH_1}) is trivially conjugate to (Y_{H_2}, ψ_{H_2}). Moreover, the actions (Y_{AH_1}, ψ_{AH_1}) and (Y_{H_1}, ψ_{H_1}) are isomorphic via the automorphism of $\mathbb{Z}^d$ sending $n \in \mathbb{Z}^d$ to $An \in \mathbb{Z}^d$; see [GPS19, Proposition 2.8].

Conversely, assume that (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are isomorphic. The existence of a group isomorphism $\alpha\colon H_1 \to H_2$ implies $d_1 = d_2$. Indeed, d_i is the rank of H_i, $i = 1, 2$, and a group isomorphism preserves the rank. We denote the common value of d_1, d_2 by d. Then [GPS19, Proposition 2.8] gives $A \in \mathrm{GL}_d(\mathbb{Z})$ such that (Y_{AH_1}, ψ_{AH_1}) is conjugate to (Y_{H_2}, ψ_{H_2}). We now apply Corollary 2.5 to conclude $AH_1 = H_2$. □

3. Continuous Orbit Equivalence

Definition 3.1. *An action (X, ϕ) of a group G and an action (Y, ψ) of another group G' are said to be* orbit equivalent *if there exists a homeomorphism $h\colon X \to Y$ such that for each $x \in X$ one has*

$$h\left(\{\phi^g(x)\colon g \in G\}\right) = \{\psi^{g'}(h(x))\colon g' \in G'\}.$$

In [GPS19] the authors characterize orbit equivalence of systems (Y_H, ψ_H), where H is a dense subgroup of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$, using superindex $[[H\colon \mathbb{Z}^d]]$. The *superindex* is defined as

$$[[H\colon \mathbb{Z}^d]] = \left\{[H'\colon \mathbb{Z}^d]\colon \mathbb{Z}^d \subseteq H' \subseteq H,\ [H'\colon \mathbb{Z}^d] < \infty\right\}.$$

Theorem 3.2. [GPS19, Corollary 5.5] *Let H be a dense subgroup of $\mathbb{Q}^d$ that contains $\mathbb{Z}^d$ and H' be a dense subgroup of $\mathbb{Q}^{d'}$ that contains $\mathbb{Z}^{d'}$. The systems (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are orbit equivalent if and only if $[[H\colon \mathbb{Z}^d]] = [[H'\colon \mathbb{Z}^{d'}]]$.*

For free orbit equivalent actions (X, ϕ) and (Y, ψ) via a homeomorphism h there exist unique *orbit cocycles* $\alpha\colon X \times G \to G'$ and $\beta\colon Y \times G' \to G$ such that

$$h(\phi^g(x)) = \psi^{\alpha(x,g)}(h(x)),$$

for all $x \in X, g \in G$, and also

$$h^{-1}(\psi^{g'}(y) = \phi^{\beta(y,g')}(h^{-1}(y)),$$

for all $y \in Y, g' \in G'$.

Definition 3.3. *Free orbit equivalent actions (X, ϕ) and (Y, ψ) of groups G and G', respectively, are* continuously orbit equivalent *if there exists a homeomorphism $h\colon X \to Y$, such that the associated cocycles α and β are continuous in the corresponding product topologies.*

Proposition 3.4. *Let (Y_{H_1}, ψ_{H_1}) be a $\mathbb{Z}^{d_1}$-action and (Y_{H_2}, ψ_{H_2}) a $\mathbb{Z}^{d_2}$-action. Then (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are continuously orbit equivalent if and only if $d_1 = d_2$ and there exists $A \in \mathrm{GL}_d(\mathbb{Q})$ with $\det A = \pm 1$ and $AH_1 = H_2$.*

Proof. The proof follows the lines of the proof of [GPS19, Theorem 5.7] where one uses Proposition 2.7 in place of [GPS19, Corollary 5.1]. □

4. Odometers defined by matrices

In this section we generalize the results in [S22] on $\mathbb{Z}^2$-odometers defined by matrices with integer coefficients to the d-dimensional case, $d > 2$. For convenience, we first put together the results from previous sections in one theorem.

Theorem 4.1. *Let H_1, H_2 be dense subgroups of $\mathbb{Q}^d$ such that $\mathbb{Z}^d \subseteq H_1$, $\mathbb{Z}^d \subseteq H_2$. Then*

(1) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are conjugate if and only if $H_1 = H_2$.*
(2) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are isomorphic if and only if there is $T \in \mathrm{GL}_d(\mathbb{Z})$ such that $TH_1 = H_2$.*
(3) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are continuously orbit equivalent if and only if there is $T \in \mathrm{GL}_d(\mathbb{Q})$ such that $\det T = \pm 1$ and $TH_1 = H_2$.*
(4) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are orbit equivalent if and only if $[[H_1 : \mathbb{Z}^d]] = [[H_2 : \mathbb{Z}^d]]$.*

Proof. This is the content of Corollary 2.5, Proposition 2.7, Theorem 3.2, and Proposition 3.4 above. □

Recall that for a non-singular $d \times d$-matrix A with integer coefficients, $A \in \mathrm{M}_d(\mathbb{Z})$,

$$G_A = \left\{ A^{-k}\mathbf{x} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d,\, k \in \mathbb{Z} \right\}, \quad \mathbb{Z}^d \subseteq G_A \subseteq \mathbb{Q}^d,$$

a subgroup of $\mathbb{Q}^d$. For simplicity, we denote Y_{G_A} by Y_A.

We start with a characterization of dense groups G_A in $\mathbb{Q}^d$. Let $h_A \in \mathbb{Z}[t]$ be the characteristic polynomial of A and let $h_A = h_1 h_2 \cdots h_s$, where $h_1, h_2, \ldots, h_s \in \mathbb{Z}[t]$ are non-constant and irreducible.

Lemma 4.2. [S22a, Lemma 8.1] *The group G_A is dense in $\mathbb{Q}^d$ if and only if $h_i(0) \neq \pm 1$ for all $i \in \{1, 2, \ldots, s\}$.*

Next, we describe orbit equivalent odometers.

Lemma 4.3. [S22a, Lemma 8.2] *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp., G_B) is dense in $\mathbb{Q}^d$. Then $\mathbb{Z}^d$-actions Y_A, Y_B are orbit equivalent if and only if* $\det A, \det B$ *have the same prime divisors (in $\mathbb{Z}$).*

The next lemma is a special (simple) case when all the equivalences hold at the same time.

Lemma 4.4. *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp., G_B) is dense in $\mathbb{Q}^d$. Assume that for any prime $p \in \mathbb{N}$ that divides* $\det A$ *we have*

$$h_A \equiv t^n \,(mod\ p).$$

Then the following are equivalent:

(1) *$\mathbb{Z}^d$-actions Y_A, Y_B are conjugate;*
(2) *$\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic;*
(3) *$\mathbb{Z}^d$-actions Y_A, Y_B are continuously orbit equivalent;*
(4) $\det A$, $\det B$ *have the same prime divisors and for any prime $p \in \mathbb{N}$ that divides* $\det B$ *we have*

$$h_B \equiv t^n \,(mod\ p).$$

Proof. Follows from Theorem 4.1 and [S22, Lemma 3.10]. □

Let

$$\det A = a p_1^{s_1} p_2^{s_2} \cdots p_l^{s_l}$$

be the prime-power factorization of $\det A$, where $p_1, p_2, \ldots, p_l \in \mathbb{N}$ are distinct primes, $a = \pm 1$, and $s_1, s_2, \ldots, s_l \in \mathbb{N}$. Let

$$\mathcal{P} = \mathcal{P}(A) = \{p_1, p_2, \ldots, p_l\}.$$

In what follows we assume $\mathcal{P} \neq \emptyset$, since otherwise G_A is not dense in $\mathbb{Q}^d$ by Lemma 4.2. Denote

$$\mathcal{P}' = \mathcal{P}'(A) = \{p \in \mathcal{P},\ h_A \not\equiv t^n \,(\mathrm{mod}\ p)\}\,,$$

where $h_A \in \mathbb{Z}[t]$ denotes the characteristic polynomial of A. The case $\mathcal{P}' = \emptyset$ is settled in Lemma 4.4, so in what follows we assume $\mathcal{P}' \neq \emptyset$. Finally, for a prime $p \in \mathbb{N}$ let $t_p = t_p(A)$ denote the multiplicity of zero in the reduction of the characteristic polynomial of A modulo p, $0 \leq t_p \leq n$. Thus, in our notation $t_p \neq n$ if and only if $p \in \mathcal{P}'$.

Even though the results in [S22a] apply to arbitrary non-singular $A, B \in \mathrm{M}_d(\mathbb{Q})$, not to make the paper too technical, we only consider a generic case when the characteristic polynomials of A, B are irreducible. An interested reader could use [S22a] together with Theorem 4.1 to treat other cases. Another additional assumption we make is the condition that there exists $p \in \mathcal{P}'$ such that the greatest common divisor (t_p, d) of t_p and d is 1. It seems to be a right generalization of the 2-dimensional case. In particular, in the

2-dimensional case, if h_A is irreducible and $G_A \cong G_B$, then h_B is irreducible and an isomorphism $T \in \mathrm{GL}_d(\mathbb{Q})$ from G_A to G_B takes an eigenvector of A to an eigenvector of B. It turns out these facts remain true under the assumption $(t_p, d) = 1$ and not true in general, *e.g.*, all $t_p = 2$ and $d = 4$ (see [S22a] for specific examples).

Let $\overline{\mathbb{Q}}$ denote a fixed algebraic closure of $\mathbb{Q}$. Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular and let $\lambda_1, \dots, \lambda_d \in \overline{\mathbb{Q}}$ (resp., $\mu_1, \dots, \mu_d \in \overline{\mathbb{Q}}$) denote eigenvalues of A (resp., B). Assume there exists $T \in \mathrm{GL}_d(\mathbb{Q})$ that satisfies $T(G_A) = G_B$. Suppose further the characteristic polynomial $h_A \in \mathbb{Z}[t]$ of A is irreducible and there exists a prime $p \in \mathbb{N}$ such that $(t_p(A), d) = 1$. Then $h_B \in \mathbb{Z}[t]$ is irreducible. Moreover,

$$\begin{aligned}&\mathbb{Q}(\lambda_1) = \mathbb{Q}(\mu_1) \text{ and } \lambda_1,\ \mu_1 \text{ have the same}\\ &\text{prime divisors in the ring of integers of } K_1 = \mathbb{Q}(\lambda_1) = \mathbb{Q}(\mu_1).\end{aligned} \tag{4.1}$$

In particular, the splitting fields of h_A, h_B coincide. Let $K \subset \overline{\mathbb{Q}}$ denote the common splitting field. Furthermore, both A and B are diagonalizable over $\overline{\mathbb{Q}}$ and there exist $M, N \in \mathrm{GL}_d(K)$ such that

$$A = M \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_d \end{pmatrix} M^{-1},\ B = N \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_d \end{pmatrix} N^{-1}, \tag{4.2}$$

and $T = NM^{-1}$ [S22a, Proposition 5.7]. Assume G_A (resp., G_B) is dense in $\mathbb{Q}^d$. If $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate (or isomorphic, or continuously orbit equivalent), then by Theorem 4.1, we have $T = I_d$, the identity matrix, (or $T \in \mathrm{GL}_d(\mathbb{Z})$, or $\det T = \pm 1$, respectively) and, hence, $M = N$ (or $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$, or $\det NM^{-1} = \pm 1$, respectively). This proves the following lemma.

Lemma 4.5. *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp., G_B) is dense in $\mathbb{Q}^d$. Assume the characteristic polynomial $h_A \in \mathbb{Z}[t]$ of A is irreducible and $(t_p, d) = 1$ for some prime $p \in \mathbb{N}$. If $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate (or isomorphic, or continuously orbit equivalent), then (4.1) holds, there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that (4.2) holds, and $M = N$ (or $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$, or $\det NM^{-1} = \pm 1$, respectively).*

As in the 2-dimensional case, the conditions in Lemma 4.5 are also sufficient in the cases of conjugacy and isomorphism.

Lemma 4.6. *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp., G_B) is dense in $\mathbb{Q}^d$. Assume $h_A \in \mathbb{Z}[t]$ is irreducible and $(t_p, d) = 1$ for some prime $p \in \mathbb{N}$.*

- *(i) If (4.1) holds and there exists $M \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that (4.2) holds for $N = M$, then $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate.*
- *(ii) If (4.1) holds, there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$ and (4.2) holds, then $\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic.*

Proof. First, (ii) follows easily from (i) as in the proof of [S22, Lemma 8.10]. We repeat the argument for the sake of completeness. Let $X = NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$ and assume (4.1) and (4.2) hold. Then $XM = N$ and

$$X(G_A) = G_{XAX^{-1}} = G_{N\Lambda N^{-1}} = G_B, \quad \Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_d \end{pmatrix}.$$

Here, $G_{XAX^{-1}} = X(G_A)$, $XAX^{-1} = N\Lambda N^{-1} \in \mathrm{M}_d(\mathbb{Z})$, since $X \in \mathrm{GL}_d(\mathbb{Z})$. Also, $G_{N\Lambda N^{-1}} = G_B$ by (i) and Theorem 4.1 (clearly, $G_{N\Lambda N^{-1}}$ is dense, since G_A is dense by assumption). Thus, $\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic by Theorem 4.1.

We now prove (i). Assume (4.1), (4.2) hold and $N = M$. One can verify that the identity matrix $T = I_d$ gives an isomorphism between G_A and G_B by [S22a, Lemma 6.1]. $\square$

Remark 4.7. Assume the characteristic polynomial of A is irreducible. Then the condition $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$ is equivalent to $I_{\mathbb{Z}}(A, \lambda_1) = xI_{\mathbb{Z}}(B, \mu_1)$ for some $x \in K$, where $I_{\mathbb{Z}}(A, \lambda_1)$ (resp., $I_{\mathbb{Z}}(B, \mu_1)$) denotes the ideal class of A (resp., B) corresponding to λ_1 (resp., μ_1). Moreover, if A, B share the same characteristic polynomial, equivalently, A, B are conjugated by a matrix in $\mathrm{GL}_d(\mathbb{Q})$, then $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$ is equivalent to

$$I_{\mathbb{Z}}(A, \lambda_1) = xI_{\mathbb{Z}}(B, \lambda_1) \text{ for some } x \in K.$$

Thus, the classical theory of ideal classes can be used in this case (see, *e.g.*, [N99]).

Remark 4.8. Note that $M = N$ implies A, B commute. However, $AB = BA$ does not imply conjugacy, since the ordering of eigenvalues matters. For example, $G_A \neq G_B$ for $A = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$.

Remark 4.9. We know from the 2-dimensional case that for odometers Y_A defined by $A \in \mathrm{M}_d(\mathbb{Z})$, continuous orbit equivalence is more subtle than conjugacy and isomorphism. Even in the 2-dimensional case, general sufficient conditions for $\mathbb{Z}^2$-actions Y_A, Y_B to be continuously orbit equivalent under the conditions of Lemma 4.5 become rather technical. However, in each particular example, it is possible to resolve the question based on Theorem 4.1 and the results in [S22] and [S22a].

Example 4.10. In this example, $d = 4$, $A, B \in \mathrm{M}_4(\mathbb{Z})$ have the same irreducible characteristic polynomial $h(t) = t^4 + t^2 + 9$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 9 & 0 & 2 & 1 \\ 9 & 0 & 1 & 1 \\ -18 & -9 & 7 & -1 \end{pmatrix}.$$

One can show that $G_A = G_B$ [S22a, Example 11], hence $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate.

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