The dynamics of birational maps over finite fields and their signatures

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Abstract

In this talk, I will review various aspects of our research into algebraic and arithmetic properties of birational difference equations or birational maps.

Birational maps with suitable coefficients (e.g. rational coefficients) can be reduced to birational maps over infinitely many finite fields. Now the phase space is finite and the dynamics is encoded in the proportion of the finite phase space occupied by cycles and by aperiodic orbits and the length distributions of such orbits. We have studied the reductions of *integrable* birational maps over finite fields and the reductions of *reversible* birational maps. Integrable birational maps of the plane are measure-preserving birational maps $L : (x, y) \mapsto (x', y')$ that preserve a rational integral of motion $I(x, y) = n(x, y)/d(x, y)$. A reversible map L can be written as the composition of 2 involutions, a type of time-reversal symmetry as it implies the existence of an involution G conjugating the map to its inverse: $G \circ L \circ G^{-1} = L^{-1}$.

We find that the dynamics of these low-complexity highly deterministic maps has some *universal* (i.e. map-independent) aspects, expressible in terms of the distribution functions of orbit lengths. Integrable maps over finite fields have quantised periods and singular distributions [4, 1]. Reversible maps over finite fields have smooth distributions based around the gamma distribution $R(x) = 1 - e^{-x}(1 + x)$ [5, 2, 3]. These observations can be used to make efficient detectors for integrability and reversibility. Models to explain the integrable distribution use algebraic geometry and the Hasse-Weil bound, whereas the reversible distribution can be explained by a composition of two random involutions and the related statistical averages [6, 3].

References

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