Homework 2
Due Wednesday, September 21st

1. Prove that S_3 is not the direct product of any family of its proper subgroups. The same is true for $\mathbb{Z}_{p^n}$ (p is a prime, $m \geq 1$).

2. Let H, K, N be nontrivial normal subgroups of a group G and suppose $G = H \times K$. Prove that N is in the center of G or N intersects one of H, K nontrivially. Give examples to show that both possibilities can actually occur when G is non-abelian.

3. A normal subgroup H of a group G is said to be a *direct factor* (*direct summand* if G is abelian) if there exists a (normal) subgroup K of G such that $G = H \times K$. Show

(a) if H is a direct factor of K and K is a direct factor of G, then H is normal in G;

(b) if H is a direct factor of G, then every homomorphism $H \longrightarrow G$ may be extended to an endomorphism $G \longrightarrow G$. However, a monomorphism need not be extendible to an automorphism $G \longrightarrow G$.

4. Let E and E_i, $i \in \{1, 2, \ldots, m\}$, be (left) modules over a ring R. Let $\phi_i : E_i \longrightarrow E$ and $\psi_i : E \longrightarrow E_i$ be R-module homomorphisms such that

$$\psi_i \circ \phi_i = \mathrm{id}, \quad \psi_i \circ \phi_j = 0,\ i \neq j, \quad \sum_{i=1}^{m} \phi_i \circ \psi_i = \mathrm{id}.$$

Show that the map $x \mapsto (\psi_1(x), \ldots, \psi_m(x))$ is an isomorphism of E onto the direct product of the E_i, $i \in \{1, 2, \ldots, m\}$, and that the map $(x_1, \ldots, x_m) \mapsto \phi_1(x_1) + \cdots + \phi_m(x_m)$ is an isomorphism of this direct product onto E.

Converesely, if E is equal to a direct product (or direct sum) of submodules E_i, $i \in \{1, 2, \ldots, m\}$, if we let ϕ_i be the inclusion of E_i in E and ψ_i the projection of E on E_i, then these maps satisfy the above-mentioned properties.

5. A topological space X is said to be *Noethertian* if the open subsets of X satisfy the ascending chain condition, i.e., for every increasing sequence $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ of open subsets $U_1, U_2, U_3, \ldots$ of X there exists N such that $U_N = U_{N+1} = \cdots$.

i) Show that X is Noetherian if and only if the closed subsets of X satisfy the descending chain condition.

ii) A topological space X is said to be *quasi-compact* if any open cover of X has a finite subcover. Show that if X is Noetherian, then every subspace of X is Noetherian, and that X is quasi-compact.

iii) Prove that the following are equivalent:

1) X is Noetherian.

2) Every open subspace of X is quasi-compact.

3) Every subspace of X is quasi-compact.