Homework 6
Due Wednesday, October 26th

1. Suppose the vector space V is the direct sum of cyclic $F[x]$-modules whose annihilators are $(x+1)^2$, $(x-1)(x^2+1)^2$, (x^4-1), and $(x+1)(x^2-1)$. Determine the invariant factors and elementary divisors for V.
2. Read about eigenvalues, eigenvectors, and diagonalizable matrices. Prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the $n \times n$ matrix A then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k for any $k \geq 0$.
3. Read section 12.2 from Dummit and Foote, 3rd edition. Determine which of the following matrices are similar:

$$\begin{pmatrix} -1 & 4 & -4 \\ 2 & -1 & 3 \\ 0 & -4 & 3 \end{pmatrix}, \quad \begin{pmatrix} -3 & -4 & 0 \\ 2 & 3 & 0 \\ 8 & 8 & 1 \end{pmatrix}, \quad \begin{pmatrix} -3 & 2 & -4 \\ 2 & 1 & 0 \\ 3 & -1 & 3 \end{pmatrix}.$$

4. Prove that any matrix A is similar to its transpose A^t.
5. Prove that if the minimal polynomial m of a matrix $A \in Mat_n(\mathbb{C})$ has distinct roots, i.e., each root of m has multiplicity one, then A is diagonalizable.
6. Show that a matrix $A \in Mat_n(\mathbb{C})$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over any field F? Prove or give a counterexample.
7. Let λ be an eigenvalue of the linear transformation T on the finite-dimensional vector space V over the field F. Let $r_k = \dim_F (T-\lambda)^k V$ be the rank of the linear transformation $(T-\lambda)^k$ on V. For any $k \geq 1$ prove that $r_{k-1} - 2r_k + r_{k+1}$ is the number of Jordan blocks of T corresponding to λ of size k. (This gives an efficient method for determining the Jordan canonical form for T by computing the ranks of the matrices $(A - \lambda I)^k$ for a matrix A representing T.)