Homework 8
Due Wednesday, November 16th

1. Let R be a commutative ring with 1. Prove that $R[x]$ is a flat R-module.

2. Let R and S be commutative rings with identities and let M be an R-module and let N be an (R,S)-bimodule. Prove that if M if flat over R and N is flat over S, then $M \otimes_R N$ is flat over S.

3. Let R be a commutative ring and let M, N be flat R-modules. Prove that $M \otimes_R N$ is a flat R-module.

4. Let R be a commutative ring, let $S = M_{n\times n}(R)$, let $M = M_{1\times n}(R)$ be the set of all $1 \times n$ vectors (rows), and let $N = M_{n\times 1}(R)$ be the set of all $n \times 1$ vectors (columns). Show that M is a right S-module under matrix multiplication and N is a left S-module under matrix multiplication. Prove that the map $M \times N \longrightarrow R$ given by matrix multiplication induces an isomorphism $M \otimes_S N \cong R$.

5. Let A_1 and A_2 be torsion-free abelian groups and let $B_1 \subseteq A_1$, $B_2 \subseteq A_2$ be subgroups. Show that the natural map $B_1 \otimes_{\mathbb{Z}} B_2 \longrightarrow A_1 \otimes_{\mathbb{Z}} A_2$ is injective. Deduce that $A_1 \otimes_{\mathbb{Z}} A_2 \neq 0$ if $A_1 \neq 0$ and $A_2 \neq 0$.

6. Let R be a ring and let the following diagram of R-modules and R-module homomorphisms be commutative with exact rows

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C'. &
\end{tikzcd}$$

Assume that f and h are isomorphisms. Show that α is injective, β' is surjective, and g is an isomorphism.

7. Let R be a commutative ring with 1. Consider the functors $M \otimes_R -$ and $Hom_{R\,mod}(M, -)$ from *R-mod* to *R-mod*. Show that $M \otimes_R -$ is a left adjoint for $Hom_{R\,mod}(M, -)$. Prove that $Hom_{R\,mod}(M, -)$ is left exact. Deduce that a covariant functor F : *R-mod* $\longrightarrow$ *R-mod* with a left adjoint must be left exact. Prove that the functor $Hom_{R\,mod}(-, M)$ is left exact (i.e., if $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ is an exact sequence of R-modules and R-module homomorphisms, then the induced sequence $0 \longrightarrow Hom_{R\,mod}(N_3, M) \longrightarrow Hom_{R\,mod}(N_2, M) \longrightarrow Hom_{R\,mod}(N_1, M)$ is exact) and deduce that a covariant functor G : *R-mod* $\longrightarrow$ *R-mod* with a right adjoint must be right exact. (This is another proof that $M \otimes_R -$ is right exact.)

8. (The snake lemma.) Let R be a ring and let the following diagram of R-modules and R-module homomorphisms be commutative with exact rows

$$\begin{tikzcd}
& A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C'. &
\end{tikzcd}$$

(a) Prove that there exists an exact sequence

$$\ker f \xrightarrow{\bar{\alpha}} \ker g \xrightarrow{\bar{\beta}} \ker h \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\bar{\alpha}'} \operatorname{coker} g \xrightarrow{\bar{\beta}'} \operatorname{coker} h,$$

where $\bar{\alpha}$, $\bar{\beta}$ are restrictions of α, β, and $\bar{\alpha}'$, $\bar{\beta}'$ are induced by α', β'. Also, the *boundary homomorphism* δ is defined as follows: let $x \in \ker h$, then $x = \beta(y)$ for some $y \in B$ and $g(y) = \alpha'(z)$ for some $z \in A'$. Define $\delta(x)$ to be the image of z in $\operatorname{coker} f$. (Show that δ is a well-defined R-module homomorphism.)

(b) Show that if α is injective and β' is surjective, then $\bar{\alpha}$ is injective and $\bar{\beta}'$ is surjective.