**Homework 7**
**Due Wednesday, November 2nd**

**1**. Let $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ be categories and let $F, G$ be functors of two variables (both contravariant in the first variable and covariant in the second variable) from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$. Give a definition of a natural transformation from $F$ to $G$.

**2**. Show that the map assigning to a ring its group of (multiplicative) units defines a functor $\mathbb{G}_m : \mathcal{R}ing \longrightarrow \mathcal{G}rp$ from the category of (non-trivial) rings with 1 to the category of groups. Show by an explicit example that this functor is not faithful.

**3**. If $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor, let $\mathcal{I}m\, F$ consist of the objects $\{F(A) \,|\, A \in Obj(\mathcal{C})\}$ and morphisms $\{F(f) \,|\, f : A \to B \in Mor(\mathcal{C})\}$. Then show that $\mathcal{I}m\, F$ need not be a category. If the object function of $F$ is injective or $F$ is a full functor, then show that $\mathcal{I}m\, F$ is a category.

**4**. Construct functors as follows:
(a) A covariant functor $\mathcal{G}rp \to \mathcal{S}et$ that assigns to each group the set of all its subgroups.
(b) A covariant functor $\mathcal{R}ing \to \mathcal{R}ing$ that assigns to each ring $R$ the polynomial ring $R[x]$.
(c) A functor covariant in both variables $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that $(A, B) \mapsto A \oplus B$.

**5**. Let $\mathcal{V}$ be the category whose objects are all finite dimensional vector spaces over a field $F$ (of characteristic $\neq 2, 3$) and whose morphisms are all vector-space *isomorphisms*.
(a) Construct a covariant functor $F : \mathcal{V} \to \mathcal{V}$ such that $F(V) = V^*$, where $V \in Obj(\mathcal{V})$ and $V^* = Hom_F(V, F)$ is the dual of $V$.
(b) For each $V \in Obj(\mathcal{V})$ choose a basis $\{e_1, \ldots, e_n\}$ and let $\{\delta_1, \ldots, \delta_n\}$ be the dual basis of $V^*$. Define a map $\alpha_V : V \to V^*$ via $e_i \mapsto \delta_i$. Show that the assignment $V \mapsto \alpha_V$ is not a natural isomorphism from the identity functor $\mathcal{I}_{\mathcal{V}}$ to $F$ (Hint: consider a one-dimensional vector space with basis $\{e\}$ and let $f(e) = ce$ with $c \neq 0, \pm 1$).

**6**. Prove that covariant representable functors $\mathcal{S}et \to \mathcal{S}et$ preserve surjective maps.

**7**. Show that the forgetful functor $\mathcal{R}ing \to \mathcal{S}et$ is representable.

**8**. Show that the forgetful functor $\mathcal{T}op \to \mathcal{S}et$ is representable.

**9**. Show that if $F : \mathcal{C} \to \mathcal{S}et$ is a covariant functor that has a left adjoint, then $F$ is representable.

**10**. Let $X$ be a fixed set and define a functor $F_X : \mathcal{S}et \to \mathcal{S}et$ by $Y \mapsto X \times Y$. Show that $F_X$ is a left adjoint of the covariant hom functor $h_X = Hom_{\mathcal{S}et}(X, -)$.