Homework 9
Due Wednesday, November 23rd

1. *Direct limits commute with tensor products.* Let R be a commutative ring with 1 and let $(M_i, \mu_{ij})_{i\in I}$ be a direct system in the category *R-mod* over a directed set I and let N be any R-module. Show that $(M_i \otimes_R N, \mu_{ij} \otimes 1)$ is a direct system and denote $P = \varinjlim (M_i \otimes_R N)$. Let (M, μ_i) be a direct limit of (M_i, μ_{ij}). Then for each $i \in I$ we have an R-module homomorphism $\mu_i \otimes 1 : M_i \otimes_R N \longrightarrow M \otimes_R N$. Show that $\mu_i \otimes 1$, $i \in I$, induce an R-module isomorphism from P to $M \otimes N$.

2. *Direct limits exist in the category of rings.* Let $(R_i, \alpha_{ij})_{i\in I}$ be a direct system in the category of rings (not necessarily commutative with identities) over a directed set I. Regarding each R_i as a $\mathbb{Z}$-module we can form an abelian group $R = \varinjlim R_i$ with group homomorphisms $\alpha_i : R_i \longrightarrow R$. Show that R inherits a ring structure from the R_i's so that each α_i is a ring homomorphism. Assume that each R_i has an identity. Show that if $R = 0$ then $R_i = 0$ for some $i \in I$.

3. Let R be an integral domain and let M be a flat R-module. Prove that M is $R-$torsion-free.

4. Prove that the inverse limit of simple groups G_n, $n \in \mathbb{N}$, in which the homomorphisms are surjective is either the trivial group or a simple group.

5.
(i) Let p be a prime. Show that $\mathbb{Z}_p$ maps surjectively on each $\mathbb{Z}/p^n\mathbb{Z}$ under the projection maps.
(ii) Do problem 11 (p. 270, Chap. 7) from Dummit and Foote (2nd edition).
(iii) Consider all non-zero ideals of $\mathbb{Z}$ as forming an inverse system by divisibility, i.e., for each $n, m \in \mathbb{Z}$ with $n|m$ define $\lambda_{mn} : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ induced by the identity map on $\mathbb{Z}$. Prove that

$$\varprojlim (\mathbb{Z}/n\mathbb{Z}) \cong \prod_p \mathbb{Z}_p,$$

where the limit on the left is taken over all $n \in \mathbb{N}$ and the product on the left is taken over all primes p.

6. Let $\{M_i\}$ be a family of modules over a ring R. For any R-module N show that there are R-module isomorphisms:

$$\mathrm{Hom}_R(\oplus_i M_i, N) \cong \prod_i \mathrm{Hom}_R(M_i, N),$$

$$\operatorname{Hom}_R(N, \prod_i M_i) \cong \prod_i \operatorname{Hom}_R(N, M_i).$$

7. Let (L_i, λ_{ij}) be an inverse system of modules over a ring R. For any R-module N show that there is an R-module isomorphism

$$\operatorname{Hom}_R(N, \varprojlim M_i) \cong \varprojlim \operatorname{Hom}_R(N, M_i).$$

8. Show that any module is a direct limit of finitely generated submodules.