**Homework 4**
**Due Wednesday, October 5th**

**1**. Let $\Gamma$ be a subgroup of $\mathbb{R}^n$ such that in each ball there are finitely-many elements of $\Gamma$. Show that $\Gamma$ is a free $\mathbb{Z}$-module generated by at most $n$ elements.

**2**. Prove that a non-trivial discrete subgroup of $\mathbb{R}$ is a free $\mathbb{Z}$-module of rank 1.

**3**. Read sections 10.1–10.3 (Module theory), do problem 27 on p. 358, problems 10, 11 on p. 344 from Dummit and Foote, 3rd edition.

**4**. Read sections 8.1–8.3, do problems 1, 3, 5, 8 on p. 292 from Dummit and Foote, 3rd edition.

**5**. A non-empty subset $Y$ of a topological space $X$ is called irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in $Y$. The empty space is not considered to be irreducible.

(i) Give an example of an irreducible space and an example of a space which is not irreducible.

(ii) Show that any non-empty open subset of an irreducible space is irreducible and dense.

(iii) Show that in an irreducible space any two non-empty open subsets have a non-empty intersection.

(iv) If $Y$ is a subset of a topological space $X$, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.