ROOT NUMBERS OF HYPERELLIPTIC CURVES

ARMAND BRUMER, KENNETH KRAMER, AND MARIA SABITOVA

1. Introduction

The root number $W(A)$ associated to an abelian variety A over a number field L is the sign in the (conjectural) functional equation of its L-function, and so the conjectures of Birch and Swinnerton-Dyer imply that the rank of the Mordell–Weil group $A(L)$ satisfies

$$W(A) = (-1)^{\operatorname{rank} A(L)}.$$

For each place v of L let L_v denote the completion of L with respect to v. Let $W(A_v)$ be the local root number associated to $A_v = A \times_L L_v$. By definition,

$$W(A) = \prod_v W(A_v),$$

where v runs through all the places of L. If $v|\infty$, then

$$W(A_v) = (-1)^{\dim A}$$

(see Proposition 1 in [Ro93] or by Lemma 2.1 on p. 4272 in [S07]). If v is finite, then $W(A_v)$ is defined as follows. Let $\overline{L_v}$ be a fixed algebraic closure of L_v. For a rational prime l different from the residual characteristic of L_v let $T_l(A_v)$ be the l-adic Tate module of A_v and let $V_l(A_v) = T_l(A_v) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Let $\sigma'_v = \sigma'_{v,A}$ denote a (complex) representation of the Weil–Deligne group $\mathcal{W}'(\overline{L_v}/L_v)$ of L_v associated to $V_l(A_v)$ via the Deligne–Grothendieck construction (see e.g., [Ro94]). Then

$$W(A_v) = W(\sigma'_v).$$

We assume from now on a non-archimedean local field of definition K of A. Let $\mathcal{A}_{\mathcal{O}_K}$ and $\mathcal{A}_{\mathcal{O}_F}$ be the Néron models over the rings of integers of K and a (possibly trivial) Galois extension F/K such that A has semistable reduction over F. We analyze $W(A)$ in terms of contributions from the toroidal and abelian variety components of the closed fibers of these Néron models. In particular, our results can be used to calculate local root numbers of abelian varieties in two main instances: first, for abelian varieties with additive reduction over K and totally toroidal reduction over F, provided that the residual characteristic of K is odd and the wild part of the inertia group $I(F/K)$ of F/K is abelian, and second, for the Jacobian $J(X)$ of a stable curve X over K. More precisely, in the first case

$$W(A) = \chi(-1)^n,$$

Date: March 19, 2015.

where χ is a ramified quadratic character of $K^\times$ and n is the number of irreducible components of the complex representation of $I(F/K)$ associated to (the toric part of) the closed fiber of $\mathcal{A}_{\mathcal{O}_F}$ (see (2.7), Proposition 2.2, and Proposition 3.2 below). In the second case,

$$W(J(X)) = (-1)^{n_k+s_k+1} \prod_{[z]\in\Sigma_k} \tau_{[z]},$$

where k is the residue field of K, Σ_k is the set of the singular k-points of the reduction $\mathcal{X}$ of X over k, $n_k = |\Sigma_k|$, s_k is the number of irreducible k-components of $\mathcal{X}$ and $\tau_{[z]} = \pm 1$ is defined in Proposition 4.2 below. We also give examples of hyperelliptic curves of genus 2 over $\mathbb{Q}$, for which the obtained results are enough to calculate the global root numbers of their Jacobians.

For a somewhat different viewpoint on epsilon factors associated to abelian varieties which acquire semistable reduction over a tamely ramified extension of a local field, see Lemma 1 in [Sa93].

The paper is organized as follows. Section 2 contains general facts and notation concerning root numbers and abelian varieties that are used in the paper. In Section 3 we calculate the contribution to the local root number $W(A)$ from the toric parts of the closed fibers of $\mathcal{A}_{\mathcal{O}_K}$ and $\mathcal{A}_{\mathcal{O}_F}$ in the case when the residual characteristic of K is odd and the wild part of the inertia group of F/K is abelian. In Section 4 we calculate the root number of the Jacobian of a stable curve X over K in terms of the reduction of X over K. Finally, Section 5 provides several examples of hyperelliptic curves of genus 2 over $\mathbb{Q}$, for which the obtained results allow one to calculate global root numbers of their Jacobians.

2. General facts and notation

2.1. **Base field and representations of the Weil–Deligne group.** Let K be a local non-archimedean field with ring of integers $\mathcal{O}_K$ and residue field k of characteristic p. Let $\overline{K}$ and $\overline{k}$ be fixed algebraic closures of K and k, respectively. Let φ be the inverse of the Frobenius automorphism of the absolute Galois group of k and let $\Phi \in \mathrm{Gal}(\overline{K}/K)$ denote a preimage of φ under the decomposition map. We call a continuous homomorphism $\mu : K^\times \longrightarrow \mathbb{C}^\times$ a *(multiplicative) character of* $K^\times$. By definition, the Weil group $\mathcal{W}(\overline{K}/K)$ (also denoted by $\mathcal{W}_K$) of K is a subgroup of $\mathrm{Gal}(\overline{K}/K)$ equal to $\mathrm{Gal}(\overline{K}/K^{unr}) \rtimes \langle\Phi\rangle$, where $K^{unr} \subset \overline{K}$ denotes the maximal unramified extension of K contained in $\overline{K}$, $\langle\Phi\rangle$ denotes the infinite cyclic group generated by Φ, and $I_K = \mathrm{Gal}(\overline{K}/K^{unr})$ is the inertia group of K. The group $\mathcal{W}(\overline{K}/K)$ is a topological group (see e.g., [Ro94]). Throughout the paper we will identify one-dimensional complex continuous representations of $\mathcal{W}(\overline{K}/K)$ with characters of $K^\times$ via local class field theory assuming that a uniformizer of K corresponds to a geometric Frobenius $\Phi \in \mathrm{Gal}(\overline{K}/K)$. Let $\omega : \mathcal{W}(\overline{K}/K) \longrightarrow \mathbb{C}^\times$ be the one-dimensional representation of $\mathcal{W}(\overline{K}/K)$ given by

$$\omega|_{I_K} = 1, \quad \omega(\Phi) = q^{-1},$$

where $q = \mathrm{card}(k)$.

Let $\mathcal{W}'(\overline{K}/K)$ denote the Weil–Deligne group of K. By a representation σ' of $\mathcal{W}'(\overline{K}/K)$ we mean a continuous homomorphism

$$\sigma' : \mathcal{W}'(\overline{K}/K) \longrightarrow \mathrm{GL}(U),$$

where U is a finite-dimensional complex vector space and the restriction of σ' to the subgroup $\mathbb{C}$ of $\mathcal{W}'(\overline{K}/K)$ is complex analytic. It is known that there is a bijection between representations of $\mathcal{W}'(\overline{K}/K)$ and pairs (σ, N), where $\sigma : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}(U)$ is a continuous complex representation of $\mathcal{W}(\overline{K}/K)$ and N is a nilpotent endomorphism on U such that

$$\sigma(g)N\sigma(g)^{-1} = \omega(g)N, \quad g \in \mathcal{W}(\overline{K}/K).$$

In what follows we identify σ' with the corresponding pair (σ, N) and write $\sigma' = (\sigma, N)$. Also, a representation σ of $\mathcal{W}(\overline{K}/K)$ is identified with the representation $(\sigma, 0)$ of $\mathcal{W}'(\overline{K}/K)$ (see e.g., §§1–3 in [Ro94]).

2.2. **Abelian varieties.** Let A be an abelian variety over K. For a rational prime l different from $p = \mathrm{char}(k)$ let $T_l(A)$ be the l-adic Tate module of A. It is a free $\mathbb{Z}_l$-module of rank $2g$, where $g = \dim A$. Put $V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and let

$$\rho_l : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(V_l(A))$$

denote the natural l-adic representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_l(A)$. The Weil pairing together with an isogeny from A to the dual abelian variety of A induces a nondegenerate, skew-symmetric, $\mathrm{Gal}(\overline{K}/K)$-equivariant pairing

$$\langle -, - \rangle : V_l(A) \times V_l(A) \longrightarrow \mathbb{Q}_l \otimes \omega_l, \tag{2.1}$$

where ω_l is the l-adic cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$ (see e.g., [S07] for more detail).

Let $F/K \subset \overline{K}/K$ be a finite Galois extension over which A acquires semistable reduction ([G67], p. 21, Theorem 6.1). Denote by k_F the residue field of F. Let $\mathcal{A}$ denote the Néron model of A. Let $\mathcal{A}_{v_K} = \mathcal{A} \times_{\mathcal{O}_K} k$ and $\mathcal{A}_{v_F} = \mathcal{A} \times_{\mathcal{O}_K} k_F$ be the closed fibers of $\mathcal{A}$ over k and k_F, respectively, with corresponding connected components $\mathcal{A}^0_{v_K}$ and $\mathcal{A}^0_{v_F}$. Let $\mathcal{T}_{v_K}$ and $\mathcal{T}_{v_F}$ be the maximal subtori of $\mathcal{A}^0_{v_K}$ and $\mathcal{A}^0_{v_F}$, respectively. As an algebraic group, $\mathcal{A}^0_{v_K}$ has a natural decomposition

$$0 \to \mathcal{U} \to \mathcal{A}^0_{v_K}/\mathcal{T}_{v_K} \to \mathcal{B}_{v_K} \to 0, \tag{2.2}$$

in which $\mathcal{U}$ is a unipotent algebraic group over k and $\mathcal{B}_{v_K}$ is an abelian variety over k. Since A acquires semistable reduction over F,

$$\mathcal{A}^0_{v_F}/\mathcal{T}_{v_F} \cong \mathcal{B}_{v_F}, \tag{2.3}$$

where $\mathcal{B}_{v_F}$ is an abelian variety over k_F.

We will use the following notation:

$$\begin{aligned} V &= V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \\ V_1 &= V^{I_K}, \\ V_2 &= V^{I_F}, \end{aligned}$$

so that $V_1 \subseteq V_2 \subseteq V$ as $\mathrm{Gal}(\overline{K}/K)$-modules over $\mathbb{Q}_l$. The reduction map $\pi : \mathcal{O}_K \twoheadrightarrow k$ induces the following isomorphisms:

$$\begin{aligned} V_1 &\cong T_l(\mathcal{A}^0_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}, \\ V_2 &\cong T_l(\mathcal{A}^0_{v_F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \end{aligned} \tag{2.4}$$

(as follows easily from [ST68], p. 495, Lem. 2). We endow $T_l(\mathcal{A}^0_{v_K})$ and $T_l(\mathcal{A}^0_{v_F})$ with the action of $\mathrm{Gal}(\overline{K}/K)$ via isomorphisms (2.4). It follows from the orthogonality theorem ([G67], p. 338, Theorem 2.4) that the reduction map π induces the following isomorphism of $\mathrm{Gal}(\overline{K}/K)$-modules:

$$V_1 \cap V_1^{\perp} \cong \ T_l(\mathcal{T}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \tag{2.5}$$

where $V_1^{\perp}$ is an orthogonal complement to V_1 in V with respect to the pairing (2.1).

Proposition 2.1. *We have*

- $V_2^{\perp} \subseteq V_2$,
- $V_1 \cap V_1^{\perp} \subseteq V_2^{\perp}$.

Proof. This follows from Proposition 3.5 on p. 350 in [G67] and Equation (2.7) on p. 32 in [E95]. □

Note that (2.2) and (2.3) induce exact sequences

$$\begin{aligned} 0 \longrightarrow T_l(\mathcal{T}_{v_K}) \longrightarrow T_l(\mathcal{A}^0_{v_K}) \longrightarrow T_l(\mathcal{B}_{v_K}) \longrightarrow 0, \\ 0 \longrightarrow T_l(\mathcal{T}_{v_F}) \longrightarrow T_l(\mathcal{A}^0_{v_F}) \longrightarrow T_l(\mathcal{B}_{v_F}) \longrightarrow 0, \end{aligned}$$

since $\mathcal{T}(\overline{k})$ is a divisible group for a torus $\mathcal{T}$ over k, so that $\varprojlim$ is an exact functor, and $T_l(\mathcal{U})$ is trivial. Thus from (2.5) and Proposition 2.1 we have the following diagrams corresponding to each other via the reduction map

$$\begin{tikzcd} V_1 \arrow[r, hook] & V_2 \\ V_1 \cap V_1^{\perp} \arrow[u, hook] \arrow[r, hook] & V_2^{\perp} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} T_l(\mathcal{A}^0_{v_K}) \arrow[r, hook] & T_l(\mathcal{A}^0_{v_F}) \\ T_l(\mathcal{T}_{v_K}) \arrow[u, hook] \arrow[r, hook] & T_l(\mathcal{T}_{v_F}) \arrow[u, hook] \end{tikzcd}$$

Since $V_1 \cap V_2^{\perp} = V_1 \cap V_1^{\perp}$ by Proposition 2.1, this gives identification of $\mathrm{Gal}(\overline{K}/K)$-modules

$$V_1/(V_1 \cap V_1^{\perp}) \cong T_l(\mathcal{B}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \quad V_2/V_2^{\perp} \cong T_l(\mathcal{B}_{v_F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

as well as an injection of $\mathrm{Gal}(\overline{K}/K)$-modules

$$T_l(\mathcal{B}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong (T_l(\mathcal{A}^0_{v_K})/T_l(\mathcal{T}_{v_K})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \hookrightarrow T_l(\mathcal{B}_{v_F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong (T_l(\mathcal{A}^0_{v_F})/T_l(\mathcal{T}_{v_F})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Let $\bar{\rho}^{tor}$ denote the representation of $\mathcal{W}_K$ on the quotient $U = (T_l(\mathcal{T}_{v_F})/T_l(\mathcal{T}_{v_K})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ considered as a representation over $\mathbb{C}$ via a fixed embedding $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$:

$$\bar{\rho}^{tor} : \mathcal{W}_K \longrightarrow \mathrm{GL}((T_l(\mathcal{T}_{v_F})/T_l(\mathcal{T}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \otimes_\imath \mathbb{C}).$$

Let

$$H = I_K/I_F = \mathrm{Gal}(F^{unr}/K^{unr}).$$

Clearly, by (2.5) the action of I_K on U factors through H and $U^H = \{0\}$. Let $\bar{\rho}^{av}$ denote the representation of $\mathcal{W}(\overline{K}/K)$ on the quotient of $T_l(\mathcal{B}_{v_F})$ by $T_l(\mathcal{B}_{v_K})$ considered as a representation over $\mathbb{C}$ via $\imath$:

$$\bar{\rho}^{av} : \mathcal{W}_K \longrightarrow \mathrm{GL}(((T_l(\mathcal{B}_{v_F})/T_l(\mathcal{B}_{v_K})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \otimes_\imath \mathbb{C}).$$

We are interested in the representation $\rho' = (\rho, N)$ of $\mathcal{W}'(\overline{K}/K)$ associated to ρ_l by the standard procedure (see e.g., [Ro94], §4). More precisely, we fix a non-trivial homomorphism $t_l : I_K \longrightarrow \mathbb{Z}_l$, e.g., the one that factors through the natural projection $\mathrm{Gal}(K^l/K^{unr}) \longrightarrow \mathbb{Z}_l$ corresponding to the maximal pro-l extension $K^l \subset \overline{K}$ of K^{unr}. Then there exists a unique nilpotent endomorphism N_l of V such that

$$\rho_l(h) = \exp(t_l(h)N_l), \quad h \in I_F.$$

We have $\ker N_l = V_2$ and the image of N_l is contained in $V_2^\perp$, thanks to the perfect pairing (2.1). Indeed,

$$\langle (\rho_l(h) - \mathrm{id})w, v\rangle = 0, \quad h \in I_F,\ v \in V_2,\ w \in V.$$

Let $\imath : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ be a field embedding and let $N = N_l \otimes_\imath \mathbb{C}$. Define a representation ρ of $\mathcal{W}(\overline{K}/K)$ on $V \otimes_\imath \mathbb{C}$ as

$$\rho(g) = \rho_l(g)\exp(-t_l(h)N_l), \quad g = \Phi^m h \in \mathcal{W}(\overline{K}/K),\ m \in \mathbb{Z},\ h \in I_K. \tag{2.6}$$

Then $\rho' = (\rho, N)$ is a representation of $\mathcal{W}'(\overline{K}/K)$ on $V \otimes_\imath \mathbb{C}$. It is known that in our context ρ' does not depend on the choice of l and $\imath$ (see e.g., [S07]). By definition,

$$W(A) = W(\rho') = W(\rho) \cdot \delta(\rho'), \tag{2.7}$$

where

$$\delta(\rho') = \det\left(-\Phi|_{(V\otimes_\imath \mathbb{C})^{I_K}/(\ker N)^{I_K}}\right)$$

(see e.g., [Ro94], §§11, 12, for the definition and properties of $W(\rho)$).

Proposition 2.2. *We have*

$$W(\rho) = W(\bar{\rho}^{av}) \cdot (\det \bar{\rho}^{tor})(-1).$$

In particular, if A already is semistable over K, then $W(\rho) = 1$.

Proof. By additivity, $W(\rho)$ is the product of the root numbers of the representations of $\mathcal{W}(\overline{K}/K)$ (over $\mathbb{C}$ via $\imath$) afforded by the successive quotients in

$$0 \subseteq V_2^\perp \subseteq V_2 \subseteq V.$$

Since $N_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ acts as zero on each of these quotients, $\rho(\mathcal{W}_K)$ and $\rho_l(\mathcal{W}_K)$ agree on each quotient.

By Proposition 2.1, the first constituent $V_2^\perp$ affords the representation

$$\rho^{tor} : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}((T_l(\mathcal{T}_{v_F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \otimes_\imath \mathbb{C})$$

and the second constituent $V_2/V_2^\perp$ affords the representation

$$\rho^{av} : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}((T_l(\mathcal{B}_{v_F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \otimes_\imath \mathbb{C}).$$

The perfect pairing

$$V_2^\perp \times (V/V_2) \to \mathbb{Q}_l \otimes \omega_l$$

shows that the representation ν of $\mathcal{W}(\overline{K}/K)$ afforded by the third constituent V/V_2 satisfies $\nu \cong (\rho^{tor})^* \otimes \omega$. Hence, $W(\rho^{tor} \oplus \nu) = (\det \rho^{tor})(-1)$ (see e.g., [Ro94], §§11, 12). Any lift $\tau \in \mathcal{W}(\overline{K}/K)$ of $-1 \in \overline{K}^\times \cong \mathcal{W}(\overline{K}/K)^{ab}$ is in I_K and therefore acts trivially on the submodule V_1 of V_2. Therefore, $(\det \rho^{tor})(-1) = (\det \bar{\rho}^{tor})(-1)$.

Finally, the subrepresentation $\rho_0 : \mathcal{W}(\overline{K}/K) \longrightarrow \mathrm{GL}((T_l(\mathcal{B}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \otimes_\imath \mathbb{C})$ of ρ^{av} is unramified, so $W(\rho_0) = 1$ by the standard properties of root numbers (see e.g., [Ro94], §§11, 12). Hence, $W(\rho^{av}) = W(\bar{\rho}^{av})$. □

3. Calculation of $\det \bar{\rho}^{tor}(-1)$

Lemma 3.1. *Let B be a finite abelian group of odd order, let $C = \langle c \rangle$ be a finite cyclic group of order h, and let $G = B \rtimes C$ be a semi-direct product with C acting on B. Assume that there exists a non-trivial orthogonal irreducible representation σ of G over $\mathbb{C}$. Then h is even. Moreover, if σ is not trivial on B, then there exists a subgroup $\Gamma = B \rtimes \langle c^x \rangle$ of G and a one-dimensional representation ψ of Γ such that*

$$\sigma \cong \mathrm{Ind}_\Gamma^G \psi,$$

where x is even and $\psi(c^x) = 1$.

Proof. Clearly, if σ is a non-trivial orthogonal irreducible representation and σ is trivial on B, then σ has order 2 and h is even. Thus, for the rest of the proof we assume that σ is not trivial on B.

Using Mackey method of small subgroups, an irreducible representation σ of G can be constructed from an irreducible (one-dimensional) representation ψ of B in the following way. Let Γ denote the stabilizer of ψ in G, i.e.,

$$\Gamma = \{g \in G \,|\, \psi(g^{-1}bg) = \psi(b),\ \forall b \in B\}.$$

Then $\Gamma = B \rtimes \langle c^x \rangle$ for some positive integer $0 < x \leq h$. We fix a choice of x such that x divides h. It turns out that ψ can be extended to a representation of Γ. Namely, since c^x belongs to the stabilizer of ψ, we have $\psi(c^{-x}bc^x) = \psi(b)$ for any $b \in B$. For a (h/x)-th root of unity ξ we can extend ψ to a representation of Γ via

$$\psi(c^x) = \xi.$$

By abuse of notation we will denote the extension of ψ to Γ also by ψ, so that

$$\psi(bc^{xv}) \quad = \quad \psi(b)\cdot\xi^v, \quad b\in B,\ v\in\{0,1,\dots,\frac{h}{x}-1\}.$$

Furthermore, $\sigma = \mathrm{Ind}_\Gamma^G\psi$ ([S77], p. 62, Prop. 25). Since by assumption, σ is not trivial on B, ψ is not trivial.

We are interested in the case when σ is orthogonal. Then, in particular, $\mathrm{Res}_B^G\,\sigma$ is orthogonal and hence

$$\mathrm{Res}_B^G\,\sigma \cong 1^n \oplus \mu \oplus \overline{\mu},$$

where μ is a representation of B and $\overline{\mu}$ denotes its contragredient (recall that by assumption B is abelian of odd order). On the other hand,

$$\mathrm{Res}_B^G\,\sigma \cong \psi \oplus \psi_c \oplus \cdots \oplus \psi_{c^{x-1}},$$

where ψ_{c^i} denotes the representation of B given by $\psi_{c^i}(b) = \psi(c^{-i}bc^i)$, $b\in B$. Since ψ is not trivial on B, we have $n=0$ and $\overline{\psi} = \psi_{c^j}$ as representations of B for some j. Furthermore, one can easily check that

- x is even,
- $\overline{\psi} = \psi_{c^{x/2}}$,
- the character of ψ is not real-valued, i.e., ψ is neither trivial nor quadratic.

Recall that for an irreducible representation σ of a finite group G we have

$$\frac{1}{|G|}\cdot\sum_{y\in G}\mathrm{tr}\,\sigma(y^2) = \begin{cases} 1, & \text{if and only if } \sigma \text{ is orthogonal,}\\ -1, & \text{if and only if } \sigma \text{ is symplectic,}\\ 0, & \text{if and only if the character of } \sigma \text{ is not real-valued}\end{cases} \tag{3.1}$$

([S77], p. 109, Prop. 39). We now apply (3.1) to $G = B\rtimes\langle c\rangle$ and $\sigma = \mathrm{Ind}_\Gamma^G\psi$. We first note that $\mathrm{tr}\,\sigma(y^2) = 0$ unless $y^2\in\Gamma$. Let $y = bc^v$ for $b\in B$, $v\in\{0,1,\dots,h-1\}$, so that $y^2\in\Gamma$ if and only if x divides $2v$ and, since x is even, if and only if $\frac{x}{2}$ divides v. Let $v = \frac{x}{2}n$, then we have

$$S = \sum_{y\in G}\mathrm{tr}\,\sigma(y^2) = \sum_{b,\,n}\mathrm{tr}\,\sigma(bc^vbc^v) = S_1 + S_2, \tag{3.2}$$

where

$$S_1 = \sum_{\substack{b\in B\\ n=even}}\mathrm{tr}\,\sigma(bc^vbc^v), \qquad S_2 = \sum_{\substack{b\in B\\ n=odd}}\mathrm{tr}\,\sigma(bc^vbc^v).$$

We first show that $S_1 = 0$. Let W be a representation space of G corresponding to σ and let $V\subseteq W$ be a one-dimensional subrepresentation of $\mathrm{Res}_\Gamma^G\sigma$ isomorphic to ψ. Then

$W = V \oplus cV \oplus c^2V \oplus \cdots \oplus c^{x-1}V$ and in a suitable basis σ has the following form

$$\sigma(b) = \begin{pmatrix} \psi(b) & 0 & \dots & 0 \\ 0 & \psi_c(b) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_{c^{x-1}}(b) \end{pmatrix}, \quad \sigma(c) = \begin{pmatrix} 0 & 0 & \dots & \psi(c^x) \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\psi_{c^{x/2}} = \overline{\psi}$, $\psi_{c^{x/2+1}} = \overline{\psi}_c$, $\dots$, $\psi_{c^{x-1}} = \overline{\psi}_{c^{x/2-1}}$. Let $v = \frac{x}{2}n$, where $n = 2l$ is even, so that $v = xl$, $l \in \{1, 2, \dots, h/x\}$, and $\sigma(c^x) = \psi(c^x)I$, where I is the $x \times x$-identity matrix. Hence,

$$\begin{aligned} S_1 = \sum_{b\in B,\, l} \operatorname{tr} \sigma((bc^{xl})^2) &= \sum_{\gamma\in\Gamma} \operatorname{tr} \psi(\gamma^2) + \sum_{\gamma\in\Gamma} \operatorname{tr} \psi_c(\gamma^2) + \cdots + \\ &\quad + \sum_{\gamma\in\Gamma} \operatorname{tr} \psi_{c^{x-1}}(\gamma^2) = x \sum_{\gamma\in\Gamma} \operatorname{tr} \psi(\gamma^2). \end{aligned} \tag{3.3}$$

Since by assumption the character of ψ considered as a representation of Γ is not real-valued, it follows from (3.1) that $S_1 = 0$.

We now calculate S_2. Let $n = 2l + 1$, $v = \frac{x}{2}n$. Using the matrix representations of $\sigma(b)$ and $\sigma(c)$ above and taking into account that $\psi_{c^{x/2}}(b)\psi(b) = 1$ for any $b \in B$, one can easily check that $\sigma((bc^v)^2) = \xi^{2l+1}I$. Finally,

$$S_2 = x \cdot |B| \cdot \sum_{l=0}^{\frac{h}{x}-1} \xi^{2l+1} = x \cdot |B| \cdot \xi \cdot \begin{cases} 0, & \xi^2 \neq 1, \\ \frac{h}{x}, & \xi^2 = 1, \end{cases}$$

since $\xi^{h/x} = 1$. Recall that $S_1 + S_2 = S_2 = |G| = h \cdot |B|$, which implies $\xi = 1$.

Analogously, σ is symplectic if and only if x is even and $\xi = -1$. These generalize the results of Proposition 1.5 in [S07]. $\square$

For the rest of the section we will use the notation of Section 2.2.

Proposition 3.2. *If $p \neq 2$ and the p-Sylow subgroup of $H = \mathrm{Gal}(F^{unr}/K^{unr})$ is abelian, then*

$$(\det \bar{\rho}^{tor})(-1) = \chi(-1)^n, \tag{3.4}$$

where χ is a ramified quadratic character of $K^\times$ and n is the number of irreducible components of $\bar{\rho}^{tor}$ considered as a representation of H (in what follows denoted by $\bar{\rho}^{tor}|_H$).

Proof. Let $H = B \rtimes C$, where $C = \langle c\rangle$ is a finite cyclic group of order t not divisible by p and B is a p-group, i.e., B is the wild subgroup of H and C is the tame quotient. Let $\alpha \in I_K$ correspond to -1 under the local class field theory homomorphism $\mathcal{W}_K \longrightarrow K^\times$ and let $\bar{\alpha} = \beta\gamma \in H$, $\beta \in B$, $\gamma \in C$, be the image of α under the quotient map $I_K \longrightarrow H$, so that

$$(\det \bar{\rho}^{tor})(-1) = (\det \bar{\rho}^{tor})(\alpha) = (\det \bar{\rho}^{tor})(\bar{\alpha}).$$

If the order of H is odd (equivalently, the order of B is odd), then clearly $\bar{\alpha} = 1$ and the left side of (3.4) is 1. On the other hand, using $(\bar{\rho}^{tor})^H = \{0\}$ and Lemma 3.1, as an orthogonal representation of H, $\bar{\rho}^{tor}|_H \cong \mu \oplus \mu^*$, where μ is a representation of H and μ^* denotes its contragredient. Thus, n is even and the right side of (3.4) is also 1.

Assume for the rest of the proof that the order of H is even. Since $p \neq 2$, this means the order of B is even. Since $W^H = \{0\}$, as an orthogonal representation of H,

$$\bar{\rho}^{tor}|_H \cong \mu \oplus \mu^* \oplus \lambda_1 \oplus \cdots \oplus \lambda_m \oplus \nu^q,$$

where μ is a representation of H, μ^* denotes its contragredient, each λ_i is an irreducible orthogonal representation of H with $\dim \lambda_i \geq 2$, and ν is the one-dimensional representation of H of order 2. Using Lemma 3.1 it is easy to see that $\det \lambda_i(c) = -1$.

Since $\bar{\rho}^{tor}$ is orthogonal and by assumption the order of B is odd, $(\det \bar{\rho}^{tor})(\beta) = 1$ for any $\beta \in B$ and hence

$$(\det \bar{\rho}^{tor})(-1) = (\det \bar{\rho}^{tor})(\alpha) = (\det \bar{\rho}^{tor})(\gamma) = \begin{cases} 1, & \gamma \in C^2, \\ (-1)^{m+q}, & \text{otherwise.} \end{cases} \tag{3.5}$$

Note that every two ramified quadratic characters of $K^\times$ differ by multiplication by the unramified quadratic character of $K^\times$, which makes $\chi(-1)$ well-defined. Let ϖ_K be a uniformizer of K and let $\sqrt[t]{\varpi_K} \in \overline{K}$ be a t-th root of ϖ_K. Then $K_1 = K(\sqrt[t]{\varpi_K})$ is a totally ramified tame extension of K. We can take a quadratic character χ of $K^\times$ given by

$$\chi : \mathcal{W}_K \hookrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(K_1/K) \cong \mathrm{Gal}(K_1^{unr}/K^{unr}) = C \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z},$$

where $\phi(c) = -1$ (recall that the order of C is even) and the rest are the natural maps. Then $\chi(-1) = \chi(\alpha) = \phi(\gamma)$ and the proposition follows from (3.5). $\square$

4. Calculation of $\delta(\rho')$

In this section we keep the notation of Section 2.2.

Proposition 4.1. *Denote*

$$\begin{aligned} V_N^{I_K} &= (\ker(N_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l))^{I_K}, \\ V_l(\mathcal{T}_{v_K}) &= T_l(\mathcal{T}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \end{aligned}$$

There is a perfect pairing

$$\left(V^{I_K}/V_N^{I_K}\right) \times V_l(\mathcal{T}_{v_K}) \to \mathbb{Q}_l \otimes \omega_l.$$

Proof. Recall the notation $V_1 = V^{I_K}$, $V_2 = V^{I_F}$. To determine V^{I_K}, consider the filtration $V \supseteq V_1^\perp + V_2 \supseteq V_2 \supseteq 0$. Since $\rho(I_F) = 1$ (by the definition of ρ), the action of $\rho(I_K)$ on V factors through the finite group $I_K/I_F \cong \mathrm{Gal}(F^{nr}/K^{nr})$ and therefore is semisimple. Hence, we find the $\rho(I_K)$-invariants on the successive quotients, i.e., on

$$V/(V_1^\perp + V_2) \;\oplus\; (V_1^\perp + V_2)/V_2 \;\oplus\; V_2. \tag{4.1}$$

On each piece $N_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ acts trivially, so $\rho(I_K)$ and $\rho_l(I_K)$ give the same action.

The action of $\rho_l(I_K)$ is trivial on the first quotient $V/(V_1^{\perp}+V_2)$, because

$$(\rho_l(i)-\mathrm{id})(V)\subseteq V_1^{\perp},\quad i\in I_K.$$

By abuse of language we will say that the action of a group on a vector space is *fixed point free* if the only fixed point is 0. We claim that the action of $\rho_l(I_K)$ on $(V_1^{\perp}+V_2)/V_2$ is fixed point free. It suffices to show that the action of $\rho_l(I_K)$ on the dual space is fixed point free. Note that via the pairing (2.1), we have

$$\left((V_1^{\perp}+V_2)/V_2\right)^*\cong V_2^{\perp}/(V_2^{\perp}\cap V_1).$$

But the latter injects into V_2/V_1, where 0 clearly is the only invariant point.

As for the third term in (4.1), we have $V_2^{I_K}=V_1$ and hence

$$V^{I_K}\cong V/(V_1^{\perp}+V_2)\oplus V_1.$$

Finally, $\ker(N_l\otimes_{\mathbb{Z}_l}\mathbb{Q}_l)=V_2$ and so $V_N^{I_K}=V_1$. It follows that

$$V^{I_K}/V_N^{I_K}\cong V/(V_1^{\perp}+V_2).$$

The latter pairs perfectly with $V_2^{\perp}\cap V_1=V_1^{\perp}\cap V_1$, which is isomorphic to $V_l(\mathcal{T}_{v_K})$ by (2.5). Moreover, the action of $\Phi\in\mathrm{Gal}(\overline{K}/K)$ on $V^{I_K}/V_N^{I_K}$ corresponds to that of $\varphi\in\mathrm{Gal}(\overline{k}/k)$ on $V_l(\mathcal{T}_{v_K})$. □

Let

$$\mathcal{H}(\mathcal{T}_{v_K})=\mathrm{Hom}_{\mathbb{Z}}\left(\mathcal{T}_{v_K}(\overline{k}),\overline{k}^{\times}\right)$$

be the character group of the torus $\mathcal{T}_{v_K}$ and recall that $\varphi\in\mathrm{Gal}(\overline{k}/k)$ denotes the inverse of the Frobenius automorphism of $\overline{k}$. The natural $\mathrm{Gal}(\overline{k}/k)$-equivariant pairing

$$\mathcal{T}_{v_K}\left(\overline{k}\right)\times\mathcal{H}(\mathcal{T}_{v_K})\to\overline{k}^{\times}$$

induces a perfect pairing

$$V_l(\mathcal{T}_{v_K})\times\mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K})\to\mathbb{Q}_l\otimes\omega_l,$$

where $\mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K})=\mathcal{H}(\mathcal{T}_{v_K})\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$. Proposition 4.1 therefore gives

$$\delta(\rho')=\det\left(-\Phi|_{(V\otimes_{\iota}\mathbb{C})^{I_K}/(\ker N)^{I_K}}\right)=\det\left(-\varphi|_{\mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K})\otimes_{\iota}\mathbb{C}}\right).$$

4.1. **Root numbers of Jacobians of curves.** In this subsection we assume that A is the Jacobian $J(X)$ of a smooth geometrically irreducible proper curve X over K such that the reduction $\mathcal{X}_{v_K}$ of X is a stable curve over k in the sense of Definition 1.1 on p. 76 in [DM69]. Thus $A=J(X)$ has semistable reduction over K (by Theorem 2.4 on p. 89 in [DM69]), i.e., in our notation above $F=K$. Under these conditions $W(\rho)=1$ by Proposition 2.2 and hence

$$W(J(X))=\delta(\rho').$$

It is known that the connected component of the reduction of $J(X)$ over k is the Jacobian of the reduction of X, i.e., $\mathcal{A}^0_{v_K}=J(\mathcal{X}_{v_K})=\mathrm{Pic}^0(\mathcal{X}_{v_K})$ (by the results in [R70]). To simplify the notation throughout this subsection we will write $\mathcal{X}$ and $\mathcal{T}$ instead of

$\mathcal{X}_{v_K}$ and $\mathcal{T}_{v_K}$, respectively. Let $\eta : \tilde{\mathcal{X}} \to \mathcal{X}$ be the normalization of $\mathcal{X}$, let $\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_s$ be irreducible components of $\tilde{\mathcal{X}}$, and let Σ be the set of singular points of $\mathcal{X}$ over $\overline{k}$ with $n = |\Sigma|$. There is the following long exact sequence

$$1 \to \overline{k}^\times \longrightarrow \oplus_{i=1}^s \overline{k}^\times \longrightarrow \oplus_{i=1}^n \overline{k}^\times \to \mathcal{T}(\overline{k}) \to 1 \tag{4.2}$$

([BLR90], p. 247).

To describe the character group $\mathcal{H}(\mathcal{T}) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(\overline{k}), \overline{k}^\times)$, we construct a directed graph Γ, as follows. The set of vertices $\mathcal{V}$ is the set of irreducible components of $\tilde{\mathcal{X}}$. For each node of $\mathcal{X}$, say $z \in \Sigma$, fix an ordering, say (x, y) for the pair of points over z on $\tilde{\mathcal{X}}$. Suppose that x and y lie on the components $\tilde{\mathcal{X}}_i$ and $\tilde{\mathcal{X}}_j$, respectively. Let there be a directed edge e_z of Γ from the vertex $\tilde{\mathcal{X}}_i$ to the vertex $\tilde{\mathcal{X}}_j$. We allow the possibility of a loop if $i = j$. Also, we define the boundary map $\gamma(e_z) = \tilde{\mathcal{X}}_j - \tilde{\mathcal{X}}_i$. Denote the set of edges of Γ by $\mathcal{E}$. Let $C_0(\Gamma, \mathbb{Z})$ and $C_1(\Gamma, \mathbb{Z})$ be the free abelian groups on $\mathcal{V}$ and $\mathcal{E}$, respectively. It follows from [BLR90], p. 247, that dualizing (4.2), we obtain the following exact sequence of $\mathrm{Gal}(\overline{k}/k)$-modules:

$$0 \to \mathcal{H}(\mathcal{T}) \to C_1(\Gamma, \mathbb{Z}) \xrightarrow{\gamma} C_0(\Gamma, \mathbb{Z}) \xrightarrow{a} \mathbb{Z} \to 0, \tag{4.3}$$

in which a is the augmentation map induced by $a(\tilde{\mathcal{X}}_i) = 1$ for all i. Here $\mathrm{Gal}(\overline{k}/k)$ acts trivially on $\mathbb{Z}$, the Galois action on $C_0(\Gamma, \mathbb{Z})$ is induced from the Galois action on the set of irreducible components $\mathcal{C} = \{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_s\}$, the Galois action on $C_1(\Gamma, \mathbb{Z})$ is induced from the action on ordered pairs $(x, y) \mapsto (g(x), g(y))$ for $g \in \mathrm{Gal}(\overline{k}/k)$, and $\mathrm{Gal}(\overline{k}/k)$ acts canonically on $\mathcal{H}(\mathcal{T})$.

It is convenient to consider the set of singular k-points Σ_k of $\mathcal{X}$. Thus each element of Σ_k is an orbit, denoted $[z]$, of a node $z \in \Sigma$ under the action of $\mathrm{Gal}(\overline{k}/k)$. Similarly, define $\mathcal{C}_k$, the set of irreducible k-components of $\tilde{\mathcal{X}}$. Let z_1 and z_2 be the points over z in $\tilde{\mathcal{X}}$ and let d be the size of the orbit $[z]$. Define a sign $\tau_{[z]}$ to be $+1$ if φ^d fixes z_1 and z_2, while $\tau_{[z]} = -1$ if φ^d switches z_1 and z_2.

Proposition 4.2. *Let $n_k = |\Sigma_k|$ be the number of singular k-points of $\mathcal{X}$ and let $s_k = |\mathcal{C}_k|$ be the number of irreducible k-components of $\tilde{\mathcal{X}}$. Then*

$$\det\left(-\varphi|_{\mathcal{H}(\mathcal{T})\otimes_{\mathbb{Z}}\mathbb{Q}}\right) = (-1)^{n_k+s_k+1} \prod_{[z]\in\Sigma_k} \tau_{[z]}.$$

Proof. We tensor (4.3) with $\mathbb{Q}$ over $\mathbb{Z}$ and compute $\det(-\varphi)$ on each term in the exact sequence, beginning on the right side, where φ acts trivially on $\mathbb{Z}$. Hence $\det\left(-\varphi|_{\mathbb{Q}}\right) = -1$.

Consider the orbit $\mathfrak{O}$ of a vertex of Γ under φ. Let S be the $\mathrm{Gal}(\overline{k}/k)$-invariant submodule of $C_0(\Gamma, \mathbb{Z})$ generated by the elements of $\mathfrak{O}$ and let $d = |\mathfrak{O}|$. Then φ acts on $\mathfrak{O}$ as a cyclic permutation of length d and sign $(-1)^{d-1}$, so we have

$$\det(-\varphi|_{S\otimes_{\mathbb{Z}}\mathbb{Q}}) = (-1)^d \text{ sign } \varphi = (-1)^d\, (-1)^{d-1} = -1.$$

Note that each such orbit corresponds to a component of $\tilde{\mathcal{X}}$ defined over k. Hence the number of distinct orbits is s_k. By decomposing $C_0(\Gamma, \mathbb{Z})$ according to these distinct orbits we find that

$$\det\left(-\varphi|_{C_0(\Gamma,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}}\right) = (-1)^{s_k}.$$

Consider the orbit $\mathfrak{O}_1 = [z]$ of a node z of $\mathcal{X}$, $z \in \Sigma$, under the action of φ. The Galois conjugates of the associated edge e_z on the graph Γ generate a $\mathrm{Gal}(\overline{k}/k)$-invariant submodule S_1 of $C_1(\Gamma, \mathbb{Z})$ whose rank is $d_1 = |\mathfrak{O}_1|$ and on which φ^{d_1} acts as multiplication by $\tau_{[z]}$. It follows as above that

$$\det\left(-\varphi|_{S_1\otimes_{\mathbb{Z}}\mathbb{Q}}\right) = -\tau_{[z]}.$$

Now each such orbit $[z]$ corresponds to an element of Σ_k. Since there are n_k distinct orbits, we have

$$\det\left(-\varphi|_{C_1(\Gamma,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}}\right) = (-1)^{n_k} \prod_{[z]\in\Sigma_k} \tau_{[z]}.$$

□

Corollary 4.3. *Let $k' \subset \overline{k}$ be the quadratic extension of k and let $n_{k'} = |\Sigma_{k'}|$ be the number of singular k'-points of $\mathcal{X}$. Then*

$$\det\left(-\varphi|_{\mathcal{H}(\mathcal{T})}\right) = (-1)^{n_{k'}+n_k+s_k+1}.$$

Proof. For each singular point $[z]$ of $\mathcal{X}$ defined over k, there are one or two singular points defined over k' according as $\tau_{[z]} = -1$ or $+1$. Hence

$$\prod_{[z]\in\Sigma_k} \tau_{[z]} = (-1)^{n_{k'}}.$$

□

5. Examples

In this section we give examples of hyperelliptic curves C of genus 2 over $\mathbb{Q}$ for which we can compute root numbers W of their Jacobians $J(C)$ using our results in the previous sections. More precisely, each Jacobian in the examples has additive reduction at $p = 3$ (i.e., in the above notation $K = \mathbb{Q}_3$ and in (2.2) the connected component $\mathcal{A}^0_{v_K}$ of the reduction of $J(C)$ at $p = 3$ equals $\mathcal{U}$) and has totally toroidal semistable reduction over a wildly ramified extension of $\mathbb{Q}_3$ satisfying conditions of Proposition 3.2 (i.e., in the above notation $\mathcal{A}^0_{v_F}$ is a torus). Thus in the notation of Section 2.2 above for $K = \mathbb{Q}_3$ we have $\overline{\rho}^{av}$ is trivial, $W(\overline{\rho}^{av}) = 1$, $V^{I_K} \cong (T_l(\mathcal{A}^0_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \otimes_\imath \mathbb{C}$ is trivial (since $\mathcal{A}^0_{v_K}$ is unipotent), so that $\delta(\rho') = 1$, and hence by (2.7) and Proposition 2.2

$$W_3 = W(J(C)_3) = (\det \overline{\rho}^{tor})(-1), \tag{5.1}$$

which can be computed using Proposition 3.2. At other primes $q \neq 3$ each C (and hence $J(C)$ by Example 8 on p. 246 in [BLR90]) has stable reduction over $\mathbb{Q}_q$, so that

$$W_q = W(J(C)_q) = \delta(\rho'),$$

which can be computed using Proposition 4.2.

Let

$$f(x) = x^3 + 3ax^2 + 3bx + 3c, \quad a, b, c \in \mathbb{Z},\ 3 \nmid c. \tag{5.2}$$

Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$. Let $L \subset \overline{\mathbb{Q}}$ be the splitting field of f over $\mathbb{Q}$ and factor $f(x) = (x - r_1)(x - r_2)(x - r_3)$, $r_1, r_2, r_3 \in L$. Since f is Eisenstein at 3, the inertia subgroup $\mathcal{I}_v(L/\mathbb{Q})$ of $\mathrm{Gal}(L/\mathbb{Q})$ at a place v over 3 in L contains an element σ of order 3. Furthermore, $\mathcal{I}_v(L/\mathbb{Q})$ is isomorphic to the cyclic group C_3 if n is even and $\mathcal{I}_v(L/\mathbb{Q})$ is isomorphic to the symmetric group S_3 if n is odd.

Thanks to the action of σ, the various differences of roots of f have the same v-adic valuation, say $m = v(r_i - r_j)$ for all $i \neq j$. But $\prod_{1 \le i < j \le 3} (r_i - r_j)^2 = \mathrm{disc}(f)$, so

$$m = \frac{1}{6}\, v(\mathrm{disc}(f)) = \frac{1}{6}\, v(3)\, n = \begin{cases} n, & n \text{ is odd}, \\ n/2, & n \text{ is even}. \end{cases}$$

Fix an element t in the ring of integers $\mathcal{O}_L$ such that $v(t) = 1$ and write

$$r_2 - r_1 = t^m u \quad \text{and} \quad r_3 - r_1 = t^m u'.$$

Then u and u' embed as v-adic units in the completion L_v. Furthermore, u and u' are distinct modulo t, since $t^m(u' - u) = r_3 - r_2$ also has v-adic valuation m. By stretching and translation, we have

$$f(t^m X + r_1) = t^m X(t^m X - t^m u)(t^m X - t^m u') = t^{3m} X(X - u)(X - u'). \tag{5.3}$$

Note that

$$\mathrm{disc}(f) = -27(4b^3 + 12a^3 c + 9c^2 - 18cab - 3a^2 b^2).$$

Hence, we have

$$n = \begin{cases} 3, & \text{if } 3 \nmid b, \\ 4, & \text{if } 3 \mid b \text{ and } 3 \nmid a, \\ 5, & \text{if } 3 \mid b \text{ and } 3 \mid a. \end{cases}$$

Proposition 5.1. *Let $g(x) = \sum_{j=0}^{5} c_j x^j \in \mathbb{Z}[x]$ with*

$$9 \mid c_0 \quad \textit{and} \quad 3 \mid c_1, c_2, c_3.$$

Given $f(x)$ as in (5.2), assume that the polynomial $f(x)^2 + 3^{n-1} g(x)$ has distinct roots in $\overline{\mathbb{Q}}$. Then the Jacobian of a hyperelliptic genus 2 curve C over $\mathbb{Q}$ defined via

$$C\colon\ y^2 = f(x)^2 + 3^{n-1} g(x)$$

has additive reduction over $\mathbb{Q}_3$, acquires totally toroidal reduction over L_v, and L_v is the minimal Galois extension of $\mathbb{Q}_3$ over which it acquires semistable reduction.

Proof. By (5.3), the change of variables $x = t^m X + r_1$, $y = t^{3m} Y$ leads to a model C' of C over L_v of the form

$$C' : Y^2 = X^2(X-u)^2(X-u')^2 + tG(X), \tag{5.4}$$

in which the coefficients of $G(X) = 3^{n-1} g(t^m X + r_1)/t^{6m+1}$ are v-integral because of the congruences imposed on the coefficients of $g(x)$. Indeed, let

$$e = \begin{cases} 2, & n \text{ is odd}, \\ 1, & n \text{ is even}. \end{cases}$$

Then $v(3) = 3e$ and $v(t^m) = m > v(r_1) = e$. It is enough to choose $\alpha_j = \mathrm{ord}_3(c_j)$, so that $t^{6m+1} \mid 3^{n-1} c_j (r_1)^j$ for all j. Equivalently,

$$\alpha_j \geq \frac{6m+1-je}{3e} - n + 1,$$

which leads to the congruences stated in the proposition.

Clearly, C' defines a stable curve (see Definition 1.1 on p. 76 in [DM69]) over the ring of integers $\mathcal{O}_{L_v}$ of L_v, whose reduction $\tilde{C}$ over the residue field k of $\mathcal{O}_{L_v}$ consists of two projective lines intersecting transversally at three points. Let J denote the Jacobian variety of C considered as an abelian variety over L_v. Then the connected component $\mathcal{Y}$ of the reduction of J over k (considered over an algebraic closure $\overline{k}$ of k by extension of scalars) is isomorphic to $\mathrm{Pic}^0(\tilde{C}/\overline{k})$ (by the results in [R70]) and hence J has totally toroidal reduction over k. Indeed, let α denote the dimension of the abelian variety part of $\mathcal{Y}$ and let β denote the dimension of the toric part of $\mathcal{Y}$. Then α equals the sum of genera of the irreducible components of $\tilde{C}$ and β equals the first Betti number of $\tilde{C}$ (see Example 8 on p. 246 and Corollary 12 on p. 250 in [BLR90]).

Let v denote the 3-adic valuation on $\mathbb{Q}$ with $v(3) = 1$. One can easily check that C has a model over $\mathbb{Z}$ of the form

$$y^2 = (x^3 + e_1 x^2 + e_2 x + e_3)^2 + e_4 x^2 + e_5 x + e_6, \quad \forall e_i \in \mathbb{Z},$$

where $v(e_3) = 1$, $v(e_i) \geq 1$, $i = 1, 2$, $v(e_i) \geq 2$, $i = 4, 5, 6$. It follows from the results on p. 151–152 in [L94] that 3 divides the degree of the minimal Galois extension of $\mathbb{Q}_3$ over which C has stable reduction. Furthermore, the closed fiber $\mathcal{F}$ of a minimal model of C over $\mathbb{Z}_3$ has type $[\mathrm{III}_N]$ for a natural number N (see [NU73]). In particular, $\mathcal{F}$ has no cycles and its irreducible components are projective lines. As was explained at the end of the previous paragraph, this implies that the reduction of $J(C)$ over $\mathbb{Q}_3$ must be additive. Finally, by the Deligne-Mumford theorem on stable curves and their Jacobians (Theorem 2.4 on p. 89 in [DM69]), L_v is the minimal Galois extension over which the Jacobian $J(C)$ of C has semistable reduction. $\square$

We now give a hyperelliptic curve C of genus 2

$$C\colon y^2 + Q(x)y = P(x), \quad P, Q \in \mathbb{Z}[x],$$

where $Q(x)$ is monic of degree 3 and $\deg P(x) \le 5$. The discriminant Δ of C is calculated as follows

$$\Delta = 2^{-12}\,\mathrm{disc}(F), \text{ where } F = Q^2 + 4P$$

([L96], p. 4581). We give a cubic f, Eisenstein at 3 as in (5.2), such that $F = f^2 + 3^{n-1}g$ with g as in Proposition 5.1. By Proposition 5.1 this is enough to guarantee that the Jacobian $J = J(C)$ of C acquires totally toroidal semistable reduction over the completion L_v of the splitting field L of f at each place v over 3. Moreover, using Proposition 5.1, Equation (5.1), and Proposition 3.2, one could easily check that $W_3 = 1$ if $\mathcal{I}_v(L/\mathbb{Q}) \cong C_3$ and $W_3 = -1$ if $\mathcal{I}_v(L/\mathbb{Q}) \cong S_3$.

In the examples, for a prime $q \neq 3$ dividing Δ, we have $\mathrm{ord}_q(\Delta) = 1$. Hence the equation $y^2 = F(x)$ defines a minimal Weierstrass model of C over $\mathbb{Z}_q$ (see [L96]) and the reduction $\mathcal{C}$ of C at q has the form

$$\mathcal{C} : y^2 \equiv (x-r)^2 h(x) \bmod q, \quad h(x) \in \mathbb{Z}[x], \quad h(r) \not\equiv 0 \bmod q, \quad \deg h = 4.$$

There is one ordinary double point corresponding to $x = r$ and the blow-up of $\mathcal{C}$ at $x = r$ is a curve of genus 1. This implies that J has semistable reduction at $q = 3$ whose abelian variety and toric parts are both of dimension one. Since $\mathcal{C}$ is irreducible, we get $s_q = 1$ and $n_q = 1$ as well. The root number therefore is

$$W_q = -\tau_{[z]} = -\mathrm{legendre}(h(r), q),$$

so that $W_q = -1$ if the slopes above are rational and $W_q = 1$ otherwise.

Case 1: $\mathcal{I}_v(L/\mathbb{Q}) \cong C_3$, $f = x^3 + 3x^2 - 3$, $n = 4$

Q	$x^3 + x^2 + 3$
P	$-80x^5 + 2x^4 - 3x^3 - 6x^2$
Δ	$3^{25} \cdot 13 \cdot 109 \cdot 4507$
g	$-12x^5$
$W = 1$	$W_3 = W_{109} = 1$ $W_{13} = W_{4507} = -1$

Case 2: $\mathcal{I}_v(L/\mathbb{Q}) \cong C_3$, $f = x^3 + 3x^2 - 3$, $n = 4$

Q	$x^3 + x^2 + 3$
P	$82x^5 + 2x^4 - 3x^3 - 6x^2$
Δ	$3^{25} \cdot 13 \cdot 561359$
g	$12x^5$
$W = -1$	$W_3 = W_{13} = 1$ $W_{561359} = -1$

Case 3: $\mathcal{I}_v(L/\mathbb{Q}) \cong C_3$, $f = x^3 + 3x^2 - 3$, $n = 4$

Q	$(x+1)^3$
P	$12x^4 + 34x^3 + 12x^2 + 39x + 2$
Δ	$3^{21} \cdot 852263$
g	$2x^4 + 6x^3 + 3x^2 + 6x$
$W = -1$	$W_3 = 1$ $W_{852263} = -1$

Case 4: $\mathcal{I}_v(L/\mathbb{Q}) \cong S_3$, $f = x^3 - 3x - 3$, $n = 3$

Q	$x^3 + x + 1$
P	$-2x^4 - 2x^3 + 2x^2 - 77x + 2$
Δ	$3^{19} \cdot 383 \cdot 1607$
g	$-36x$
$W = 1$	$W_{1607} = 1$ $W_3 = W_{383} = -1$

Case 5: $\mathcal{I}_v(L/\mathbb{Q}) \cong S_3$, $f = x^3 - 3x - 3$, $n = 3$

Q	x^3
P	$-6x^4 - 15x^3 - 18x^2 - 9x - 18$
Δ	$3^{16} \cdot 23 \cdot 37 \cdot 167$
g	$-2x^4 - 6x^3 - 9x^2 - 6x - 9$
$W = 1$	$W_{23} = W_{167} = 1$ $W_3 = W_{37} = -1$

References

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 21. Springer-Verlag, Berlin, 1990.

[DM69] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math., no. 36 (1969), 75–109.

[E95] B. Edixhoven, *On the prime to p-part of the groups of connected components of Néron models*, Compositio Math. 97, 1995, 29–49.

[G67] A. Grothendieck, Séminaire de Géométrie Algébrique du Bois Marie, 1967-69, *Groupes de monodromie en géométrie algébrique*, (SGA 7), vol. 1.

[L93] Q. Liu, *Courbes stables de genre* 2 *et leur schéma de modules*, Math. Ann. 295 (1993), no. 2, 201–222.

[L94] Q. Liu, *Modèles minimaux des courbes de genre deux*, J. Reine Angew. Math. 453 (1994), 137–164.

[L96] Q. Liu, *Modèles entiers des courbes hyperelliptiques sur un corps de valuation discrète*, Trans. Amer. Math. Soc. 348 (1996), no. 11, 4577–4610.

[NU73] Y. Namikawa, K. Ueno, *The complete classification of fibres in pencils of curves of genus two*, Manuscripta Math. 9 (1973), 143–186.

[R70] M. Raynaud, *Spécialisation du foncteur de Picard*, Inst. Hautes Études Sci. Publ. Math., no. 38, 1970, 27–76.

[Ro94] D. E. Rohrlich, *Elliptic curves and the Weil-Deligne group*, Elliptic Curves and Related Topics (CRM Proc. Lecture Notes, 4, Amer. Math. Soc., Providence, RI, 1994), 125–157.

[Ro93] D. E. Rohrlich, *Variation of the root number in families of elliptic curves*, Compositio Math. 87 (1993), no. 2, 119–151.

[S07] M. Sabitova, *Root numbers of abelian varieties*, Trans. Amer. Math. Soc. 359 (2007), no. 9, 4259–4284.

[Sa93] T. Saito, *Epsilon-factor of a tamely ramified sheaf on a variety*, Invent. Math. 113, (1993), 389–417.

[ST68] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. 88 (1968), 492–517.

[S77] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977.