STATISTICAL COMPUTATIONAL METHODS

Seminar Nr. 5, Counting Processes

1. On the average, 6 airplanes per minute land at a certain international airport. Assume the number of landings is modeled by a Binomial counting process.
a) What frame length should be used to guarantee that the probability of a landing does not exceed 0.1?
b) Using the chosen frames, compute the probability of no landings during the next half a minute;
c) Using the chosen frames, compute the probability of more than 170 landed airplanes during the next 30 minutes.

Solution:
a) We have $\lambda = 6/\ \mathrm{min}$. So, if we want $p \le 0.1$, then

$$\Delta \;=\; \frac{p}{\lambda} \le \frac{0.1}{6}\,\mathrm{min} \;=\; 1\,\mathrm{sec}.$$

b) Let $\Delta = 1$ sec. In $t = 1/2$ min. $= 30$ sec., there are $n = \dfrac{t}{\Delta} = \dfrac{30}{1} = 30$ frames. The number of landings $X(30)$ during 30 frames has Binomial distribution with parameters $n = 30$ and $p = 0.1$. We want

$$P(X(30) = 0) \;=\; \mathit{binopdf}(0, 30, 0.1) \;=\; 0.0424.$$

c) Similarly, in 30 minutes $= 1800$ sec., there are $n = \dfrac{1800}{1} = 1800$ frames. Thus, the number of landings $X(1800)$ during the next half hour has Binomial distribution with parameters $n = 1800$ and $p = 0.1$. We want to compute

$$P(X(1800) > 170) \;=\; 1 - P(X(1800) \le 170) \;=\; 1 - \mathit{binocdf}(170, 1800, 0.1) \;=\; 0.7709.$$

2. Messages arrive at a communications center according to a Binomial counting process with 30 frames per minute. The average arrival rate is 40 messages per hour. How many messages can be expected to arrive between 10 a.m. and 10:30 a.m.? What is the standard deviation of that number of messages?

Solution:

The arrival rate is $\lambda = 40/\text{hr} = \frac{2}{3}/$ min. and the frame length is $\Delta = \frac{1}{30}$ min. Then, the probability of a new message arriving during any given frame is

$$p = \lambda\Delta = \frac{2}{3} \cdot \frac{1}{30} = \frac{1}{45}.$$

Between 10 and 10:30 a.m., there are $t = 30$ minutes, so there are $n = \frac{t}{\Delta} = \frac{30}{1/30} = 900$ frames, hence, the number of new messages $X(900)$ arriving during this time has Binomial distribution with $n = 900$ and $p = 1/45$, so

$$\begin{aligned} E(X) &= np = 20 \text{ messages}, \\ \sigma(X) &= \sqrt{np(1-p)} = 4.4222 \text{ messages}. \end{aligned}$$

3. An internet service provider offers special discounts to every third connecting customer. Its customers connect to the internet according to a Poisson process with the rate of 5 customers per minute. Compute
a) the probability that no offer is made during the first 2 minutes;
b) the probability that no customers connect for 20 seconds;
c) expectation and standard deviation of the time of first offer.

Solution:
a) We have $\lambda = 5/\text{min}$. In $t = 2$ minutes, the number of connections X has Poisson distribution with parameter $\lambda t = 5 \cdot 2 = 10$. No offer is made if there are *fewer* than three connections during that time. So, we want

$$P(\text{no offer}) = P(X < 3) = P(X \le 2) = \mathit{poisscdf}(2, 10) = 0.0028.$$

b) The time between connections (interarrival time) T has Exponential(λ) distribution. No customers connect for 20 sec. $= 1/3$ minutes, if the interarrival time exceeds that. So, we want

$$P(T > 1/3) = 1 - P(T \le 1/3) = 1 - \mathit{expcdf}(1/3, 1/5) = 0.1889.$$

Or we can express λ in seconds, $\lambda = 5/60 = 1/12\,/\text{sec}$. Then we compute

$$P(T > 20) = 1 - P(T \le 20) = 1 - \mathit{expcdf}(20, 12) = 0.1889.$$

Alternatively, we want 0 connections (arrivals) in $t = 20$ seconds $= 1/3$ minutes. Just like in part a), the number of connections X has Poisson distribution with parameter $\lambda t = 5 \cdot 1/3 = 5/3$. Then we compute

$$P(X = 0) \;=\; poisspdf(0, 5/3) \;=\; 0.1889.$$

c) The time T_3 of the third connection (arrival) (and therefore, the first offer) is the sum of 3 independent $Exp(5)$ times, so it has Gamma distribution with parameters $\alpha = 3$ and $\lambda = 1/5$. Then

$$\begin{aligned} E(T_3) &= \alpha\lambda &= 3/5 &= 0.6\text{ min}, \\ \sigma(T_3) &= \sqrt{V(T_3)} &= \sqrt{\alpha\lambda^2} &= 0.3464\text{ min}. \end{aligned}$$

4. On the average, Mr. X drinks and drives once in 4 years. He knows that

- every time he drinks and drives, he is caught by the police;
- according to the law of his state, the third time he is caught drinking and driving, he loses his driver's license;
- a Poisson counting process models such "rare events" as drinking and driving.

What is the probability that Mr. X will keep his driver's license for at least 10 years?

Solution:

The arrival rate of drinking and driving is $\lambda = 1/4$/year. Let X be the number of times Mr. X is caught drinking and driving during 10 years. Then X has Poisson distribution with parameter $\lambda t = (1/4)(10) = 2.5$. Keeping his driver's license is equivalent to being caught *less* than three times in 10 years. Then,

$$\begin{aligned} P(\text{Mr. X keeps his driver's license}) &= P(X < 3) \;=\; P(X \le 2) \\ &= poisscdf(2, 2.5) \;=\; 0.5438. \end{aligned}$$