EUCLID'S ALGORITHM

Problem 1.

a) Compute the gcd of $(13, 8)$ using Euclid's Algorithm.
b) Find the continued fraction (CF) expansion for $\frac{13}{8}$. Write down both the full form and the list form.
c) Find the convergents $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\frac{p_3}{q_3}$, $\frac{p_4}{q_4}$.

Problem 2. Do some simple algebra to write each of the numbers $[4; 8, 1]$, $[6; 2, 3]$, and $[1; 8, 2, 2]$ in the form $\frac{a}{b}$, for some integers a, b (this means that all these numbers are rational numbers). After you are done, find the decimal expansion of each of them.

Problem 3. Find an easy way to compute the reciprocal of a number x if you only know its CF expansion $[a_0; a_1, a_2, \ldots, a_n]$, without computing its decimal expansion.
Hint

Problem 4. (Challenge problem) In this problem we will prove that:
All the rationals have finite CF expansion, and rationals are the **only** numbers with **finite** CF expansion.

In parts a) and b) you will prove that the rationals are the only numbers with finite CF expansion.

a) Let x have the finite CF expansion $x = [a_0; a_1, \ldots a_n]$. Write this expansion in full form.
b) Stare at your solution to Problem 2 to explain why x can be written in the form $\frac{a}{b}$, with a, b integers.

In parts c) and d) you will prove that all rational numbers have finite CF expansion.

c) Let $x = \frac{a}{b}$ be a rational number. How would you use the Euclidean Algorithm to find its CF expansion? Write out the first few iteration.
d) Will the Euclidean Algorithm in part c) go on forever, or will it stop at some point? Explain why your answer to this question means that the CF expansion of $x = \frac{a}{b}$ is finite.

Let's compare the CF expansions with the decimal expansions:

e) You proved that the finite CF expansions are precisely the rational numbers. Is this the same with the decimal expansions? (Reminder: we found what numbers have finite decimal expansions in a previous worksheet)