Leaf Growth Pattern in Plants

(or: “All pigs are equal, but some are more equal than others.” – Orwell, Animal Farm.)

• Google images of the following: “sunflower leaves from above”, and then “teal cactus leaves from above”, and also “aloe leaves from above”. Do you see a pattern in the pictures you found?

Question 1: Is this leaf pattern random?

Question 2: Are all irrational numbers equally irrational?

Theorem 1. (Hurwitz's theorem) Let α be an irrational number, with continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots]$. Let $\frac{p_n}{q_n}$ be its n-th convergent. Then

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_n q_n^2}.$$

Imagine you are a plant. You can sprout off leaves from your stem in various directions. You want to maximize the amount of sunlight that hits your leaves. In other words, you want the **angular distance** between your leaves to be as wide as possible.

However, you can only grow in patterns with a **constant angle** between any two successive leaves—so if your first leaf points north and your second points east, the third one must point south and the fourth one west (make a quick picture of this to make sure you understand it).

Fortunately, millions of years of evolution make sure that your solution to the problem of maximizing the sunlight you get is the **best possible**. Let's find this solution:

Call the angle between successive leaves θ. For simplicity, we will consider θ in units of fractions of a circle. For example, if $\theta = \frac{1}{4}$, this corresponds to an angle of $\frac{\pi}{2}$ radians, or 90 degrees.

a) Why is $\theta = \frac{1}{2}$ a disastrous choice for your leaf growth pattern?
b) Is $\theta = \frac{1}{2+\epsilon}$ any better? ϵ is any small quantity.
c) What about other small denominator fractions like $\theta = \frac{1}{4}$? $\frac{1}{3}$? $\frac{2}{3}$? $\frac{4}{5}$?
d) Generalize what you have found so far. What types of growth angles θ would be disastrous?
e) What about choosing a big denominator, like for example if the angle is $\theta = \frac{4999}{10000}$?
f) What if you choose an irrational θ? Is this enough to guarantee that you will have efficient leaf growth?
g) Taking all the above into consideration, what is it that we are really looking for?

h) Stare at the Hurwitz Theorem for a few minutes to figure out what the Continued Fraction expansion of θ should look like. Then find its decimal expansion.

A simple google search reveals that, from the plants that grow their leaves in various angles in order to maximize the sunlight they receive, **92**% of them have Fibonacci phyllotaxis (this is the fancy name for the kind of leaf growth patterns you found in part (h)).

So what the Hurwitz theorem predicts is observable in nature!

Solution: The optimal angle is the angle which is as far away as possible from all rational numbers with small denominators. In other words: what number is least well-approximated by its low-denominator rational approximations? Looking at the Hurwitz theorem, we find that the CF expansion of that number (the "most irrational" number) should be $[1, 1, 1, \ldots]$. The decimal expansion of this number is $\frac{\sqrt{5}-1}{2}$, i.e. the small golden mean. The students had already been exposed to all the necessary material to solve this problem (i.e. relevant terminology, approximation properties of irrational numbers, how to think of math jargon like absolute values in plain English, etc) on previous days of the course.

Credit for this problem: **https://www.benkuhn.net/cf-plants/fnref:3**