Inverted Pendulum

$x_1 = \theta \quad x_2 = \dot{\theta} \quad x_3 = x \quad x_4 = \dot{x}$

$$\frac{dx}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \quad \dot{x} = \begin{pmatrix} x_2 \\ \dfrac{u\cos x_1 - (M+m)g\sin x_1 + m l \cos x_1 \sin x_1 \, x_2^2}{m l \cos^2 x_1 - (M+m) l} \\ x_4 \\ \dfrac{u + m l \sin x_1 \, x_2^2 - m g \cos x_1 \sin x_1}{M+m - m\cos^2 x_1} \end{pmatrix}$$

$$\cos x_1 \sin x_1 = \frac{\sin 2x_1}{2}$$

$$y = \begin{bmatrix} 1000 \\ 0010 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

Linearized form of inverted pendulum:

$$\frac{d}{dt}\delta x = J_x(x_0, u_0)\,\delta x + J_u(x_0, u_0)\,\delta u$$

J_x - ?

(1) $= -u\sin x_1 - (M+m)g\cos x_1 - m l \cos 2x_1 \, x_2^2$

(2) $-m l 2\cos x_1 \sin x_1$

$\Rightarrow$ (3) $= (-u\sin x_1 - (M+m)g\cos x_1 - m l \cos 2x_1 \, x_2^2)(m l \cos^2 x_1 - (M+m) l) +$

$+ m l 2\cos x_1 \sin x_1 (u\cos x_1 - (M+m) g \sin x_1 + m l \cos x_1 \sin x_1 \, x_2^2) =$

$= u\sin x_1 (M+m) l - (M+m) g \cos^3 x_1 \, m l + (M+m)^2 g l \cos x_1 - m^2 l^2 (\cos^2 x_1 - \sin^2 x_1) x_2^2 \cdot \cancel{\cos^2 x_1}$

$+ m l^2 (\cos^2 x_1 - \sin^2 x_1) x_2^2 (M+m) - 2(M+m) g \sin^2 x_1 \cos x_1 \, m l + u\cos^2 x_1 \sin x_1 \, m l$

$+ 2m^2 l^2 \cos^2 x_1 \sin^2 x_1 x_2^2 = u\sin x_1 (M+m) l + u\cos^2 x_1 \sin x_1 \, m l +$

$+ (M+m) g [-\cos^3 x_1 \cdot m l + (M+m) l \cos x_1 - 2\sin^2 x_1 \cos x_1 \, m l] +$

$+ 3m^2 l^2 \cos^2 x_1 \sin^2 x_1 \cdot x_2^2 - m^2 l^2 \cos^4 x_1 \cdot x_2^2 + x_2^2 (M+m) m l^2 (\cos^2 x_1 - \sin^2 x_1)$

(4) $[m l \cos^2 x_1 - (M+m) l]^2 = m^2 l^2 \cos^4 x_1 + (M+m)^2 l^2 -$

$- 2 m l^2 \cos^2 x_1 (M+m) = m^2 l^2 \cos^4 x_1 + (M+m)^2 l^2 - 2m^2 l^2 \cos^2 x_1 -$

$- 2mM l^2 \cos^2 x_1 = m^2 l^2 \cos^2 x_1 (\cos^2 x_1 - 2) + (M+m)^2 l^2 - 2mM l^2 \cos^2 x_1$

$$\frac{(3)}{(4)}\Bigg|_{\substack{x_0=0 \\ u_0=0}} = \frac{-(M+m) g \cdot m l + (M+m)^2 g l}{m^2 l^2 + (M+m)^2 l^2 - 2m^2 l^2 - 2mM l^2} = \frac{(M+m) g l (M+m-m)}{M^2 l^2} = \boxed{\frac{(M+m) g}{M l}}$$

$$\| \quad \boxed{f'_{2\,x_1}}$$

(5) $m\ell \cos x_1 x_2^2 + mg \sin^2 x_1 - mg\cos^2 x_1 \Big|_{x_0,u_0} = -mg$

(6) $+2m \cos x_1 \cdot \sin x_1 \Big|_{x_0,u_0} = 0$

(7) $(M+m-m\cos^2 x_1)^2\Big|_{x_0,u_0} = M^2$

$$\Rightarrow f'_{4_{x_1}} = \frac{(-mg)(M+m-m) - 0}{M^2} = -\frac{mgM}{M^2} = -\frac{mg}{M}$$

The others are set to 0 in x_0, u_0 (except (1,2) and (3,4))

That's why $\boxed{J_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{M\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix}} = \tilde{A}$

(8) $\cos x_1$

(9) $$\frac{\cos x_1}{m\ell\cos^2 x_1 - (M+m)\ell}\Bigg|_{x_0,u_0} = \frac{1}{m\ell - M\ell - m\ell} = -\frac{1}{M\ell}$$

(10) $\frac{1}{M+m-m} = \frac{1}{M}$

$$\Rightarrow \boxed{J_\theta = \begin{bmatrix} 0 \\ -\frac{1}{M\ell} \\ 0 \\ \frac{1}{M} \end{bmatrix}} = \tilde{B}$$

$\Rightarrow A = \tilde{A}$

$B = \tilde{B}$

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ //

* in the files → useful paper