INTRODUCTION TO THE HOT SPOTS CONJECTURE

MARICHI GUPTA

Abstract. The "hot spots" conjecture, posed by Jeff Rauch at a conference in 1974, claims that on any domain the second Neumann eigenfunction of the Laplacian attains its maxima (and minima) on the boundary. The conjecture has been proven generally false, but affirmative examples have been found for a handful of classes of domains, and interest remains in repairing the conjecture and seeing for what types of domains the claim holds. In this write-up we describe Rauch's original conjecture, summarize current progress on the problem (both affirmative and counter examples), and describe some of the machinery being used to work on the problem.

1. Introduction

1.1. **Problem Statement.** Let $D \subset \mathbb{R}^n$ be an bounded domain, and consider the Helmholtz equation on D with Neumann boundary conditions:

$$\begin{cases} \Delta u + \mu u = 0 \text{ in } D \\ \partial_\nu u = 0 \text{ on } \partial D, \end{cases} \tag{1.1}$$

where $\partial_\nu u$ denotes the normal derivative of u. Denote solutions to the above as $\{u_j\}_{j=1}^{\infty}$, which form an orthonormal (when scaled appropriately) basis for $L^2(D)$, with corresponding eigenvalues $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow \infty$. The question we study here is whether the extrema of the second eigenfunction, u_2, lie on the boundary of D. Jeffrey Rauch, at a conference at Tulane in 1974, conjectured that they did [13][1]

Conjecture 1.1. (Hot Spots): If ϕ is any non-zero eigenfunction with eigenvalue μ_2, then for all $y \in \operatorname{int}(D)$,

$$\inf_{x \in \partial D} \phi(x) < \phi(y) < \sup_{x \in \partial D} \phi(x).$$

The conjecture has several stated forms - the above is taken from [1]. Rauch's original conjecture was stated in the context of solutions to the heat equation with zero flux boundary conditions, and claimed that the "hot spots" of the domain tended to the boundary as time goes on. More formally, consider instead the heat equation on D:

$$\begin{cases} \partial_t \varphi = \Delta \varphi \text{ in } D, t > 0 \\ \varphi(x, 0) = f(x) \in L^2(D) \\ \partial_\nu \varphi(x,t) = 0 \text{ on } \partial D, t > 0. \end{cases} \tag{1.2}$$

[1]According to [1], the conjecture has never appeared in print under Rauch's name. The article [13] contains related discussion on the heat equation with Neumann boundary conditions.

Solving (1.2) by way of eigenfunction expansion using (1.1), we write the solution φ as

$$\varphi(x,t) = \sum_{j=1}^{\infty} c_j e^{-\mu_j t} u_j(x), \tag{1.3}$$

where $c_j = (f, u_j)_{L^2(D)}$. The first eigenfunction u_1 will be constant (and can be taken as $u_1(x) = |D|^{-1/2}$ when assumed to have unit mass). Suppose that μ_2 is simple (i.e., $\mu_2 \neq \mu_3$): we may then rewrite the above as

$$\varphi(x,t) = C_1 + c_2 e^{-\mu_2 t} u_2(x) + R(t,x), \tag{1.4}$$

for some new constant C_1 and where $R(t,x) \to 0$ faster than $e^{-\mu_2 t}$ as $t \to \infty$ uniformly for $x \in D$. The above implies that the position of the hot spots at t goes to infinity will be determined precisely by the extrema of $u_2(x)$. When μ_2 is degenerate, a stronger form of the conjecture claims that each eigenfunction in the multidimensional subspace of $L^2(D)$ corresponding to μ_2 has its extrema on the boundary.

It is worth noting (as pointed out by [1]) that even for suitably nice domains, the conjecture cannot hold for all initial conditions $f(x)$: for instance, if $\varphi(x,0)$ is initialized to be a higher-energy mode $u_k(x)$ with an interior maximum, then clearly from (1.3) that mode (along with its interior maximum) will persist for all time. The conjecture therefore seeks to describe behavior for "most" initial conditions [1].

Despite the simplicity of the statement and the physical intuition that supports it, hot spots conjecture has proven to be a tricky problem to study and has seen somewhat sparse progress over the last 40 or so years. While it has been shown that the conjecture does not hold for all domains in $\mathbb{R}^n$ (see Figure 2 shown here from [15] and discussion further below), there has been a desire to repair the conjecture by stating additional assumptions on the domains. Of course, the problem with Dirichlet boundary conditions is better understood (see Ch. 6, Prop. 1.1 of [16]).

1.2. **History of Progress on the Problem.** The first result on the problem came from Kawohl in 1985, who proved that the conjecture held on cylindrical domains $D = D_1 \times [0,1] \subset \mathbb{R}^n$ (where $D_1 \subset \mathbb{R}^{n-1}$); his work guided subsequent research by posing the need to confirm the conjecture on simple classes of domains [4].[2] Beyond this, progress was quiet on the conjecture until burst of activity occurred around 2000, on the hinge of development of both analytic and probabilistic approaches. In 1999, Bañuelos and Burdzy [1] posed a probabilistic framework for studying the problem in terms of Brownian motion, and proved the conjecture on obtuse triangles and symmetric domains (with additional assumptions). Later that year, using the same probabilistic techniques, Burdzy and Werner [5] offered the first counter-example in a domain with two holes. In 2000, Jerison and Nadirashvili [2] used analytic approaches, primarily by way of studying the nodal set of the eigenfunction u_2 to prove the conjecture on domains with two axes of symmetry, and in 2002 Pascu [18] used probabilistic techniques to prove it on domains with one axis of symmetry.

In 2004, Atar and Burdzy [3] proved the conjecture on planar *lip-domains*, such as the one shown in Figure 1. The name comes from the specification that such lip domains

[2]This cites to a summary paper by Burdzy on the problem where he describes Kawohl's work, not Kawohl's original work itself.

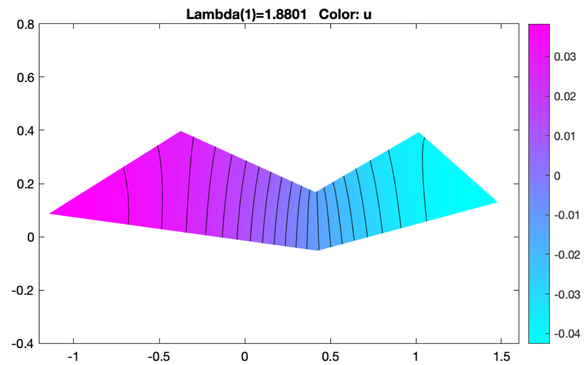

FIGURE 1. Example of a lip-domain studied by [3]. My own numerical example using MATLAB's PDE Solver.

be of form $D = \{(x, y) : f_1(x) \le y \le f_2(x)\}$ where f_1, f_2 are Lipschitz functions with constant less than or equal to 1. (The name is slightly tongue-in-cheek for obvious reason.) A simpler counterexample, compared to the two-hole counterexample, was found by Burdzy in 2005 [10]; Figure 2 shows the numeric rendering of u_2 on this domain as done by Kleefeld [15] earlier this year.

Much more recently, Judge and Mondal [11] proved the conjecture on all triangles using a somewhat similar approach as [2], and Steinerberger proved a related result using probabilistic methods that the hot spots for convex planar domains are close, within $O(\mathrm{inrad}(D))$ to the tips (i.e. points $x, y \in D$ such that $d(x, y) = \mathrm{diam}(D)$). In a Zoom conference talk, Jerison [19] outlined some ideas for further progress for the problem on convex domains in $\mathbb{R}^n$ and suggested performing quantitative analysis of the level sets of u_2 using Harnack inequalities to give the conjecture. Further, he suggested using a method of continuity by deforming the domain continuously and preserving the structure of the nodal sets - this is very akin to the approach taken by Judge and Mondal in [11].

In the following sections, we detail the probabilistic and analytic approaches being used by way of example: namely, we offer sketches of the proofs for the conjecture on lip domains and on Euclidean triangles.

2. Some Results on Laplacian Eigenfunctions

It is worth building up some basic results on Laplacian eigenfunctions that pop up in some of the papers. First off, there is a useful variational formulation for the second Neumann eigenvalue, using what is called the *Rayleigh quotient*:

Proposition 2.1. *Let $D \subset \mathbb{R}^n$, and consider the functional $R : H^1(D) \to [0, \infty)$ (the Rayleigh quotient) given by*

$$R[w] = \frac{\int_D |\nabla w(x)|^2 dx}{\int_D |w(x)|^2 dx} = \frac{\|\nabla w\|_{L^2(D)}}{\|w\|_{L^2(D)}}. \tag{2.1}$$

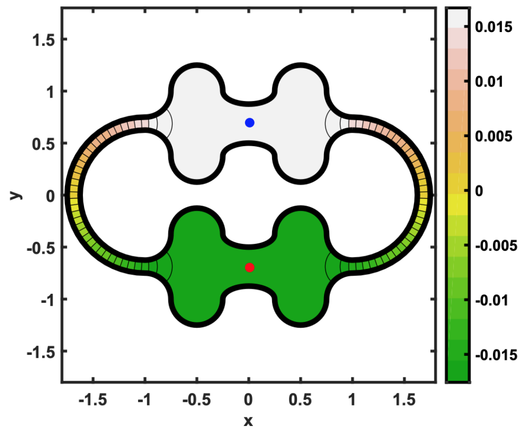

FIGURE 2. Figure 16(a) in [15], numerically validating counterexample of a domain with one hole described by [10]. Blue and red dots denote the extrema.

Then the second Neumann eigenvalue of (1.1) can be written as

$$\mu_2 = \min\left\{R[w] \,:\, w \perp u_1\right\}. \tag{2.2}$$

This variational formulation is used in [1] and [11]. The proof is simple, and was taught to me by Tom Beck a few years back:

Proof. First, consider $R[u_2]$, where $\Delta u_2 + \mu_2 u_2 = 0$. Since $\|u_2\|_{L^2(D)} = 1$, we can integrate by parts and apply $\partial_\nu u_2 = 0$ to write

$$R[u_2] = \int_D |\nabla u_2|^2 \, dx = -\int_D (\Delta u_2)u_2 \, dx = \mu_2 \int_D u_2^2 \, dx = \mu_2. \tag{2.3}$$

Now, let $w \in H_1(D)$ and write $w(x) = \sum_{j=1}^\infty a_j u_j$, where $a_j = (w, u_j)_{L^2(D)}$. Integration by parts again gives that $\|\nabla w\|_{L^2(D)} = \sum_{j=1}^\infty \mu_j a_j^2$. Further, since $\{u_j\}$ is an orthonormal basis, we have that $\|w\|_{L^2(D)} = \sum a_j^2$. In other words,

$$R[w] = \frac{\sum_{j=1}^\infty \mu_j a_j^2}{\sum_{j=1}^\infty a_j^2}. \tag{2.4}$$

Let's now consider the term $\min\{R[w] \,:\, w \perp u_1\}$. Since $w \perp u_1$, then $a_1 = 0$, and therefore we seek to minimize

$$\frac{\sum_{j=2}^\infty \mu_j a_j^2}{\sum_{j=2}^\infty a_j^2}.$$

The minimum value will be attained when $a_j = 0$ for all j where $\mu_j > \mu_2$. (If μ_2 is simple, this holds for $j \geq 3$). Let $J = \{j \,:\, \mu_j = \mu_2\}$. We then have that the minimum possible value of $R[w]$ is

$$\frac{\sum_{j \in J} \mu_2 a_j^2}{\sum_{j \in J} a_j^2} = \mu_2,$$

as desired. □

The useful thing about the Rayleigh quotient is that it allows you to get estimates (or bounds) on the eigenvalue by passing appropriate test functions to $R[w]$. It has use in proving the following theorem, which is useful in its own right for working on the hot spots conjecture:

Proposition 2.2. *(Courant's Nodal Domain Theorem) Let u_k be an eigenfunction of (1.1), ordering by energy counting multiplicity. Let $\#(k)$ be the number of nodal domains of u_k on D. (That is, $\#(k)$ = number of connected components of $D \backslash u_k^{-1}(0)$). Then $\#(k) \leq k$.*

The theorem is stated in [9]; their proof assumes that u_k has $k+1$ domains and uses the variational formulation of μ_k to obtain a contradiction. It's worth commenting on a related problem to the hot spots conjecture: whenever u_k has exactly k nodal domains, we call such an eigenfunction *Courant Sharp*. Similar to wondering what types of domains have the hot spots property, there is currently much research being conducted into seeing for a given domain which of its eigenfunctions are Courant Sharp (see, for instance, [7], [6], [8]). One early paper discussing this problem is due to Pleijel [14], which states that for domains in $\mathbb{R}^n$, $n \geq 2$, there can be only finitely many Courant Sharp eigenfunctions. Such analysis of Courant Sharp eigenfunctions, like some of the analytic approaches to the hot spots conjecture performed by [2] and [11], involves delicate analysis of the nodal sets.

As for the usefulness of Courant's Nodal Domain Theorem, it implies that the second eigenfunction u_2 has exactly two nodal domains. We also have the result due to Pólya that for any planar domain D, we have $\mu_2(D) < \lambda_1(D)$ where μ_2 is the second Neumann eigenvalue and λ_1 is the first Dirichlet eigenvalue on D [21]. Consequently, these imply that the nodal set of u_2 does not form a closed loop ([11], see Section 5). For if it did and enclosed some region $\Omega \subset D$, observe then that $\lambda_1(\Omega) \leq \mu_2(D)$ by using u_2 as a test function in the Rayleigh quotient against $\lambda_1(\Omega)$. By domain monotonicity[3] for Dirichlet eigenvalues, we also have that $\lambda_1(\Omega) \geq \lambda_1(D)$. But this contradicts Pólya's inequality. Therefore, the nodal line of u_2 must "cut" the domain D in some way, and intersects the boundary. This fact has use both in the probabilistic and analytic approaches.

3. Probabilistic Approaches:

The probabilistic approaches used to studying the hot spots conjecture, used in [1], [3] as well as in the counterexamples [5] and [10], involve the use of Brownian motion to study solutions to the heat equation. There is a strong motivation for this approach (ref. [17], pg. 14): Recall that solutions to (1.2) can be written using the heat kernel for $x \in D \subset \mathbb{R}^n$:

$$p(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}. \tag{3.1}$$

A key observation is to interpret $p(x,t)$ as a probability distribution - i.e., a Gaussian obeying $\int_{\mathbb{R}^n} p(x,t)\, dx = 1$ for all $t > 0$. Given initial data $f \in C(\overline{D})$, we may write the

[3] If $D_1 \subset D_2$, then their corresponding first Dirichlet values satisfy $\lambda_1 \geq \lambda_2$. I learned this from Tom Beck and Jeremy Marzuola in a reading course I took with them a few years back, and it is a generally known result.

solution u as

$$u(x,t) = \int_D p(x-y,t) f(y)\, dy. \tag{3.2}$$

But the above describes a convolution of the probability density $p(x-y,t)$ with f; in other words, it has the form of the expected value of f under p. Letting $\{X_t^x\}_{t>0}$ be a family of random variables with mean x and variance $2t$, we can rewrite (3.2) as the expectation

$$u(x,t) = \mathbb{E}\left[f(X_t^x)\right]. \tag{3.3}$$

To give an example of applying this approach, we sketch the proof of the hot spots conjecture for lip-domains.

3.1. **Sketch of Proof of Conj. 1.1 for lip domains.** The following proof sketch comes from [4]. Let $D \subset \mathbb{R}^2$ be a lip-domain as described above. Let $x, y \in D$, and suppose that "$x \leq y$" in the sense that the angle between $y - x$ and the horizontal $[1,0]^T$ lies in $[-\pi/4, \pi/4]$ (where the choice of $\pi/4$ is determined by the fact that the boundary functions of the lip domains are Lipschitz with constant ≤ 1). The goal will be to show that $u_2(x) \leq u_2(y)$ for all $x \leq y$, since this implies Conj. 1.1.

Let X_t, Y_t be Brownian motions in D starting from x and y respectively and that reflect upon hitting the boundary ∂D. Using (3.2), the solution to (1.2) for $\varphi(\cdot,t)$ at x and y can be written as $\varphi(x,t) = \mathbb{E}\left[f(X_t)\right]$ and $\varphi(y,t) = \mathbb{E}\left[f(Y_t)\right]$. Therefore, from (1.4) we have

$$\begin{aligned} u_2(x) - u_2(y) &= Ce^{-\mu_2 t}\left(\varphi(x,t) - \varphi(y,t)\right) + R(t,x,y) \\ &= Ce^{-\mu_2 t}\left(\mathbb{E}\left[f(X_t)\right] - \mathbb{E}\left[f(Y_t)\right]\right) + R(t,x,y), \end{aligned} \tag{3.4}$$

where the remainder $R(t,x,y) \to 0$ as $t \to \infty$ and the constant C can be assumed positive without loss (since if u_2 is an eigenfunction, then so is $-u_2$). To show that $u_2(x) \leq u_2(y)$, it suffices to show $\mathbb{E}\left[f(X_t)\right] - \mathbb{E}\left[f(Y_t)\right] \leq 0$.

To this end, suppose that X_t and Y_t are driven by the same Brownian motion and reflect off of ∂D; that is, suppose that

$$\begin{aligned} X_t &= x + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X \\ Y_t &= y + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y, \end{aligned}$$

where $\mathbf{n}(\cdot)$ is the unit inward normal vector on ∂D, and L_s^X is the local time of X on ∂D. The integral on the right can be understood as a term that kicks the Brownian motion back inside D (along the unit normal) whenever it touches ∂D. As noted, since D is a lip-domain with Lipschitz constant ≤ 1, the normal vector $\mathbf{n}(z)$ forms an angle less than $\pi/4$ with the vertical for any $z \in \partial D$. This is enough to guarantee that $X_t \leq Y_t$ for all $t \geq 0$ provided $x \leq y$ in the geometric sense stated above.

To conclude, slice through D with a vertical cut, and denote the right-most region as A. Take the initial data $f(x) = \chi_A(x)$. Then since $X_t \leq Y_t$ for all $t \geq 0$, it can never happen that $X_t \in A$ and $Y_t \in D \backslash A$. This then implies that $\mathbb{E}[\chi_A(X_t)] \leq \mathbb{E}[\chi_A(Y_t)]$, which implies $u_2(x) \leq u_2(y)$ by (3.4). Since this holds for all $x \leq y$ (in the sense assumed here), this forces the extrema to be at the left-most and right-most points of D, as suggested by Figure 1. This completes the proof sketch.

There are extensions of this method that can be applied to other types of domains. For instance, with domains with an axis of symmetry (such as those partially considered in [1]), [4] describes a "coupling" of the reflected Brownian motions in the following sense: Suppose D is symmetric on some vertical line K. The eigenfunction u_2 must be odd about this line, and therefore $u_2(K) = \{0\}$. Label the component of D lying to the right of K as D_1. Then, u_2 corresponds to the first eigenmode on D_1 with mixed boundary conditions (Dirichlet on K and Neumann on $\partial D_1 \cap \partial D$). Our Brownian motion will then be reflected on $\partial D_1 \cap \partial D$ and killed on K. Again pick $x, y \in D$ (though [4] emphasizes that one must carefully factor in the geometry of D in this step), and let X_t, Y_t be Brownian motions starting from them. Denote $T_K^X = \inf\{t \geq 0 : X_t \in K\}$ be the hitting time of K by X_t and define T_K^X respectively. Then, showing (3.4) amounts to showing that $T_K^X \leq T_K^Y$; i.e., that K kills X_t before it does Y_t.

4. Analytic Approaches: Judge and Mondal 2020

We conclude by giving a brief overview of the approach taken in [11], following exposition given by Mondal in a Zoom conference talk [20]. Denote the triangle as T. Judge and Mondal's strategy is to show that u_2 has finitely many critical points on T, and then show that these critical points must lie on ∂T. For this latter part, they use a method of continuity by beginning on the right triangle T_1 vertices at $(0,0), (0,1)$, and $(1,0)$, and define a one-parameter family of triangles T_t passing towards some target triangle T_0. Eventually, they show that there is at most one critical point of u_2 in T_0 that must lie on the boundary. In their approach, they make use of the fact that the Laplacian commutes with constant and rotational vector fields. Namely if L is a constant vector field, and if R_p is a rotational vector field about $p = (p_1, p_2)$ of form

$$R_p = -(y - p_2) \cdot \partial_x + (x - p_1) \cdot \partial_y,$$

then whenever u is an eigenfunction of $-\Delta$, so are Lu and $R_p u$ with the same eigenvalue. It is worth noting that by way of defining eigenfunctions on sectors, much of the analysis becomes more concrete thanks to Bessel function expansions of the eigenfunction (see Sections 4, 9 of [11]).

Let u_2 be the second Neumann eigenfunction on T. We seek to show the following:

Lemma 4.1. *(Corr. 5.7 in* [11]*) The set crit*(u_2) *is finite collection of points.*

Proof. We follow [20], and take for granted that $\mathrm{crit}(u_2)$ has finitely many connected components to begin with (Prop. 5.6 of [11]). To start, we recall from discussion much earlier that $u_2^{-1}(0)$ cannot contain a closed loop. Since $u_2 \in C^\infty(T)$ is analytic, the connected components of $u_2^{-1}(0)$ are either points or real-analytic arcs that end on ∂T.

Suppose $\gamma \subset \mathrm{crit}(u_2)$ was such an arc. Then for any vector field X, the zero set $(Xu)^{-1}(0) \supset \gamma$. Pick the vertex v that γ encloses on T. (That is, if γ intersects T on sides e and e', pick v adjacent to e and e'). Choose the rotational vector field R_v and apply it to u_2. Due to the Neumann boundary conditions, this forces $R_v u_2(x) = 0$ for $x \in e \cup e' \subset \partial T$. However, this means that γ, e, and e' form a closed loop and $R_v u_2 \equiv 0$ on this loop, a contradiction. This gives the claim. $\square$

Now that we know there are just finitely many critical points, the task becomes to study their position, and the [11] then seeks to rule out the possibility internal critical points. To this end, Judge and Mondal first consider the right triangle T_1 described earlier, for which the second Neumann eigenfunction can be written as $u_2^{(1)}(x,y) = \cos(\pi x) - \cos(\pi y)$ and where $\mu_2(T_1)$ is simple. Further, $\mathrm{crit}(u_2^{(1)}(x,y)) \subset \partial T_1$. Then, given triangle T_0 (that we want to confirm has no interior critical point), consider the continuous deformation[4] of triangles T_t from T_0 to T_t with their corresponding eigenfunctions $u_2^{(t)}(x,y)$. From here, one seeks a contradiction by assuming that $u_2^{(0)}(x,y)$ on T_0 had an interior critical point. Some subtleties arise in performing this continuous deformation in ensuring that no point in $\mathrm{crit}(u_2)$ becomes degenerate (i.e., has singular Hessian).

References

[1] Rodrigo Bañuelos and Krzysztof Burdzy. On the "hot spots" conjecture of J. Rauch. *Journal of Functional Analysis*, 1999.

[2] David Jerison and Nikolai Nadirashvili. The "hot spots" conjecture for domains with two axes of symmetry. *Journal of the American Mathematical Society*, 2000.

[3] Rami Atar and Krzysztof Burdzy. On Neumann eigenfunctions in lip domains. *Journal of the American Mathematical Society*, 2004.

[4] Krzysztof Burdzy. Neumann eigenfunctions and Brownian couplings. *Advanced Studies in Pure Mathematics*, 2006.

[5] Krzysztof Burdzy and Wendelin Werner. A counterexample to the "hot spots" conjecture. *Annals of Mathematics*, 1999.

[6] Pierre Bérard and Bernard Helffer, Dirichlet eigenfunctions of the square membrane: Courant's property, and A. Stern's and Å. Pleijel's analyses. *Springer Proceedings in Mathematics and Statistics*, 2015.

[7] Bernard Helffer and Mikael Persson Sundqvist. Nodal domains in the square – the Neumann case. *Moscow Mathematical Journal*, 2014.

[8] Pierre Bérard, Bernard Helffer and Rola Kiwan. Courant-sharp eigenvalues of compact flat surfaces: Klein bottles and cylinders. Accepted to *Proceedings of the American Mathematical Society*, 2021.

[9] Richard Courant and David Hilbert. Methods of Mathematical Physics: Vol. 1. *New York: Wiley*, 2nd Edition, 1989 (1st Edition, 1952). See p. 452 for Courant's nodal domain theorem statement and proof.

[10] Krzysztof Burdzy. "The hot spots problem in planar domains with one hole." *Duke Mathematical Journal*, 2005.

[11] Chris Judge and Sugata Mondal. Euclidean triangles have no hot spots. *Annals of Mathematics*, 2020.

[12] Stefan Steinerberger. Hot spots in convex domains are in the tips (up to an inradius). *Communications in Partial Differential Equations*, 2020.

[13] Jeffrey Rauch. Lecture #1. Five problems: An introduction to the qualitative theory of partial differential equations. In: Goldstein J.A. (eds) Partial Differential Equations and Related Topics. Lecture Notes in Mathematics, vol 446. Springer, Berlin, Heidelberg, 1975.

[14] Åke Pleijel. Remarks on Courant's nodal line theorem. *Commun. Pure Appl. Math.*, 1956.

[15] Andreas Kleefeld. "The hot spots conjecture can be false: Some numerical examples." Preprint, 2021. https://arxiv.org/pdf/2101.01210.pdf.

[16] Michael Taylor. "Partial Differential Equations, Volume I." *Springer*, 2011.

[4]This is straightforward to formulate for triangles by way of defining functions $v_i(t)$ for $i = 1, 2, 3$ connecting the ith vertex of T_0 to T_1 and setting $T_t = (v_1(t), v_2(t), v_3(t)$.

[17] James Nolen. “Partial Differential Equations and Diffusion Processes” course notes. http://math.stanford.edu/ ryzhik/STANFORD/STANF227-10/notes227-09.pdf.

[18] Mihai Pascu. “Scaling coupling of reflecting brownian motions and the hot spots problem.” *Transactions of the American Mathematical Society*, 2002.

[19] David Jerison. “Optimal Shapes of Level Sets.” Presentation given as part of Simons Collaboration on Localization of Waves, 2020. https://www.youtube.com/watch?v=Lvn032d9iMk.

[20] Sugata Mondal. “Hot spots conjecture for polygonal domains.” Presentation given as part of Spectral geometry in the clouds seminar, 2020. https://www.youtube.com/watch?v=P24nSF12Dv0.

[21] George Pólya. “Remarks on the Foregoing Paper.” *Journal of Mathematics and Physics*, 1952.

Email address: `marichi@live.unc.edu`