Lecture 4: Integration Strategies

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Substitution Rule (Change of Variables)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \text{ where } u = g(x)$$

Examples:

(a) $\int (2x^3 + 1)^7 x^2\,dx$

(b) $\int x\sqrt{7 - 6x^2}\,dx$

(c) $\int \sin 5x\,dx$

(d) $\int_0^4 \sqrt{3x + 4}\,dx$

(e) $\int_1^2 e^{-\frac{2x}{t}}\,dx$

(f) $\int_{e^2}^{e^6} \frac{(\ln x)^4}{x}\,dx$

Integration by Parts the Hard Way

Let $u = f(x)$ and $v = g(x)$ be two functions. Then

$$\int f(x)g'(x)\,dx = \int u\,dv = uv - \int v\,du$$

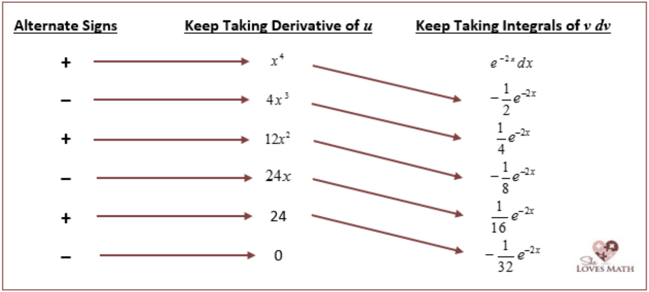
Rule of thumb: Select the most complicated part of the integral that can easily be integrated for dv.

Example:

(a) $\int xe^{2x}\,dx$

Integration by Parts the Easy Way

The problem with the preceding formula is that you may need to apply it successively (eg: x^4e^{-2x}) which is very time consuming and hard to keep track of. A shortcut is to use the line method or Tanzalin Method (?).



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Integration by Parts the Easy Way: Examples

Integrate the following:

(a) $\int xe^{2x}\,dx$

(b) $\int \ln x\,dx$

(c) $\int_0^{\pi} x^2\cos(4x)\,dx$

(d) $\int_0^1 (4x^3 - 9x^2 + 7x + 3)e^{-x}\,dx$

(e) $\int x^2 \ln 4x\,dx$

Improper Integrals

An integral over an open interval or half open interval (eg: $(a, b]$) is called an improper integral. There are two types of improper integral. For instance consider

$$\int_{\infty}^{\infty} e^{-x^2}\,dx \text{ or } \int_0^5 \frac{1}{x^2}\,dx$$

Type I: One or both of the endpoints are infinite
Type II: The interval contains a point of discontinuity

An integral is *divergent* if the evaluated integral is not a finite number or does not exist and *convergent* if the evaluated integral is a finite number.

Type I Improper Integrals

(i) If $\int_a^t f(x)\,dx$ exists $\forall t \geqslant a$ then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(ii) If $\int_t^b f(x)\,dx$ exists $\forall t \leqslant b$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

(iii) If $\int_a^t f(x)\,dx$ and $\int_t^a f(x)\,dx$ exist $\forall a \in \mathbb{R}$ then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Examples:

(a) $\int_1^\infty \frac{1}{x}\,dx$

(b) For what values of p is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Type II Improper Integrals

(i) If f is continuous on $[a, b)$ but discontinuous at b then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

(ii) If f is continuous on $(a, b]$ but discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

(iii) If f has a discontinuity at $c \in [a, b]$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Examples:

(a) $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

(b) $\int_{-2}^7 \frac{1}{(x+1)^{2/3}}\,dx$