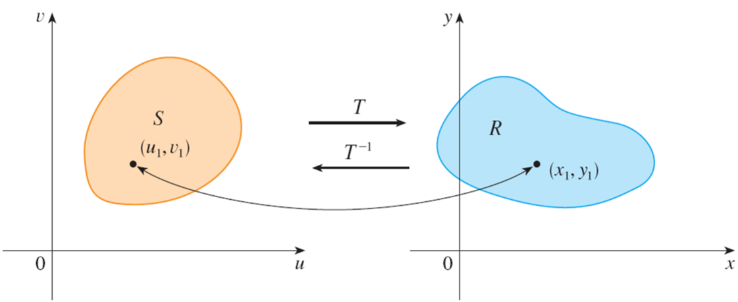
Lecture 12: Change of Variables II

October 3, 2019

Review: Change of Variables Formula



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If $J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0$,

$$\iint_R f(x, y)\,dy\,dx = \iint_S f[g(u, v), h(u, v)]\,|J|\,dv\,du$$

A Bunch of Examples

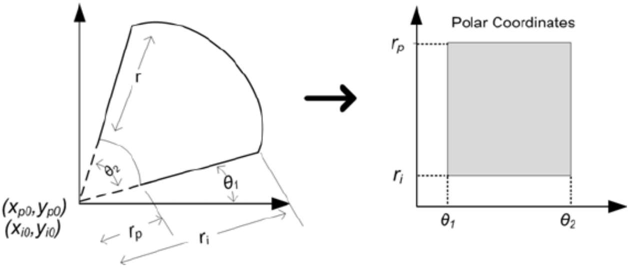
(a) $T^{-1}: u = xy, v = xy^2$. Evaluate $\iint\limits_R y^2 dA$ where R is bounded by the curves $xy = 1, xy = 2, xy^2 = 1, xy^2 = 2$.

(b) $T: x = u/v, y = v$. Evaluate $\iint\limits_R xy\, dA$ where R is the region in the first quadrant bounded by $y = x, y = 3x, xy = 1, xy = 3$.

(c) $T^{-1}: u = 2x - y, v = x + y$. Evaluate $\iint\limits_R 6x - 3y dA$ where R is the region bounded by $2x - y = 1, x + y = 1, 2x - y = 3, x + y = 3$.

(d) $T^{-1}: u = y - x, v = y + x$. Evaluate $\iint\limits_R xy dA$ where R is a square with verticies at $(0, 1), (1, 1), (2, 0), (1, -1)$

Polar Coordinates



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Polar coordinates is just a specific type of transformation where

$$T : x = r\cos\theta, y = r\sin\theta$$

Then the Jacobean is found by

$$M = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \text{ giving } J = |M| = r\cos^2\theta + r\sin^2\theta = r$$

More Examples

Examples:

(a) Evaluate $\iint\limits_R e^{x^2+y^2}\,dy\,dx$ where R is the unit circle.

(b) Evaluate $\iint_R 4xy - 7\,dA$ where R is the region bounded by $x^2 + y^2 = 2$ in the first quadrant.

(c) Evaluate $\int_{-\infty}^{\infty} e^{-x^2/2}dx$.

Triple Integrals & Change of Variables

T now maps region S in the (u, v, w) space onto R in the (x, y, z) space via

$$T : x = g(u, v, w), y = g(u, v, w), z = k(u, v, w)$$

This gives a Jacobean of

$$J = |M| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

And the follow change of variables formula

$$\iiint\limits_R f(x, y, z) dV = \iiint\limits_S f(g(u, v, w), h(u, v, w), k(u, v, w)) |J| du\, dv\, dw$$