BIOS 2081 Sample Midterm Exam

SOLUTIONS 2018 Fall

For problems 1-2, evaluate the following integrals:

1. $\int x\, e^{2x}\, dx$

use integration by parts

$u = x \qquad dv = e^{2x} dx$

$du = dx \qquad v = \frac{1}{2} e^{2x}$

$= \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx$

$=$

$= \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + c$

2. $\int x\sqrt{(1 - x^2)}\, dx$

use the substitution

$u = 1 - x^2$

$du = -2x dx$

$= -\frac{1}{2} \int u^{1/2} du$

$= -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} + c$

$= -\frac{1}{3} \sqrt{(1 - x^2)^3} + c$

3. Determine whether the series is convergent or divergent. If convergent, find the sum.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^{n-1}}$$

$= \frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$ geometric series with a = 1 and r = -3/4

Geometric series converges to $\frac{a}{1-r}$

$$=\frac{1}{4}\left(\frac{1}{1-\left(-\frac{3}{4}\right)}\right) = \frac{1}{4} * \frac{4}{7} = \frac{1}{7}$$

Series is convergent, and converges to 1/7

4. Determine whether the integral is convergent or divergent. Evaluate if convergent.

$$\int_{4}^{20} \frac{1}{\sqrt{x-4}} dx$$

Discontinuous at x = 4

$$\lim_{t \to 4^+} \int_{t}^{20} (x-4)^{-\frac{1}{2}} dx$$

$$= \lim_{t \to 4^+} 2(x-4)^{\frac{1}{2}} \Big|_{t}^{20}$$

$$= \lim_{t \to 4^+} \left(2 * \sqrt{16}\right) - \left(2 * \sqrt{t-4}\right) = 2 * 4 = 8$$

Integral is convergent, and converges to 8

5. Find all second partial derivatives

$$f(x, y) = \frac{x}{x + y}$$

$Z_{xx} = \frac{-2y}{(x+y)^2}$

$Z_{xy} = Z_{yx} = \frac{(x-y)}{(x+y)^3}$

$Z_{yy} = \frac{2x}{(x+y)^3}$

6. Find the critical points and determine whether they are local minima or maxima.

$$f(x,y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x$$

$$\frac{\partial f}{\partial x} = 3x^2 + 12x - 12y + 9 = 0$$

$$\frac{\partial f}{\partial y} = 6y - 12x = 0 \quad \Rightarrow \quad y = 2x \quad \text{substitute this into } \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial x} = 3x^2 + 12x - 12(2x) + 9 = 0$$

$$= 3x^2 - 12x + 9 = 0$$

$$= x^2 - 4x + 3 = (x-1)(x-3) = 0$$

$$x = 1, x = 3$$

$$x = 1 \Rightarrow y = 2$$

$$x = 3 \Rightarrow y = 6$$

critical points (1,2), (3,6)

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using the 2nd derivative test we get

$$\frac{\partial^2 f}{\partial x^2} = 6x + 12$$

$$\frac{\partial^2 f}{\partial y^2} = 6$$

$$\frac{\partial^2 f}{\partial x \partial y} = -12$$

$D(1,2) = 18 \cdot 6 - 144 < 0$, so (1,2) is a saddle pt.

$D(3,6) = 30 \cdot 6 - 144 > 0$, $\frac{\partial^2 f}{\partial x^2}(3,6) = 30 > 0$, so (3,6) is a local minimum

7. Find the extreme values on the set D, such that

$D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}$ and $f(x,y) = x^2 - 5xy + 2y^2$

$$\frac{\partial f}{\partial x} = 2x - 5y = 0 \quad \Rightarrow \quad x = \frac{5}{2}y, \text{ substitute into } \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = -5x + 4y = 0 \qquad -5\left(\frac{5}{2}y\right) + 4y = 0$$

$$-\frac{25}{2}y + 4y = 0$$

$$-\frac{17}{2}y = 0$$

y=0 $\Rightarrow$ x=0

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critical pt (0,0), f(0,0)=0

What about the boundaries?

$y = 0 \quad \Rightarrow \quad f(x,0) = x^2 \qquad 0 \le x \le 2$

$f(0,0) = 0,\ f(2,0) = 4$

$x = 2 \quad \Rightarrow \quad f(2,y) = 4 - 10y + 2y^2 \qquad 0 \le y \le 3$

$f(2,4) = 0,\ f(2,3) = -8$

$y = 3 \quad \Rightarrow \quad f(x,3) = x^2 - 15y + 18 \qquad 0 \le x \le 2$

$f(0,3) = 18,\ f(2,3) = -8$

$x = 0 \quad \Rightarrow \quad f(0,y) = 2y^2 \qquad 0 \le y \le 3$

$f(0,0) = 0,\ f(0,3) = 18$

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absolute max f(0,3)=18

absolute min f(2,3)=-8

8. Use the given transformation to evaluate the given integral

$$\iint_R (3x + 6y)^2$$

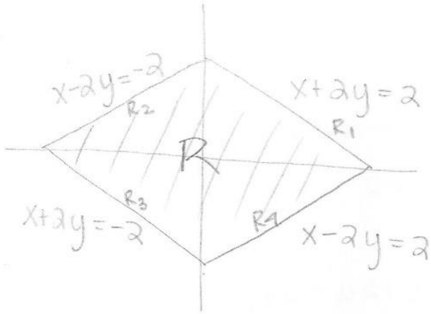
Where R is the region bounded by:

x-2y = 2, x+2y = 2, x+2y = -2, and x-2y = -2

and T^{-1}: $\begin{bmatrix} U = x + 2y \\ V = x - 2y \end{bmatrix}$

8. $\iint\limits_R (3x+6y)^2$ R is region bounded by

$x-2y=2,\ x+2y=2,\ x+2y=-2,\ x-2y=-2$



$T^{-1}:\quad u=x+2y$
$\qquad\quad v=x-2y$

$x=v+2y$
$u=(v+2y)+2y$
$u=v+4y$
$4y=u-v$
$y=\frac{1}{4}(u-v)$
$x=v+2(\frac{1}{4}(u-v))$
$x=\frac{1}{2}(u+v)$

$T:\quad x=\frac{1}{2}(u+v)$
$\qquad y=\frac{1}{4}(u-v)$

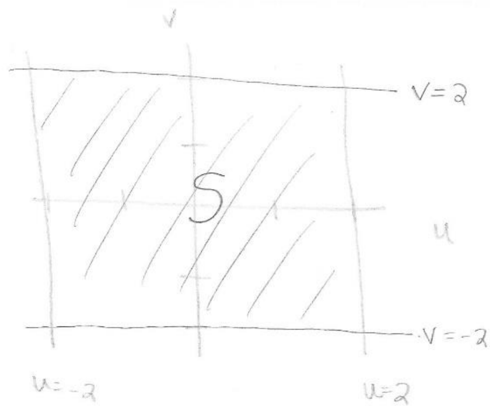
$$J=\begin{vmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{vmatrix} = -\frac{1}{8}-\left(\frac{1}{8}\right)=-\frac{2}{8}=-\frac{1}{4}$$

R_1: $x+2y=2 \Rightarrow x=2-2y$
$u=2-2y+2y \Rightarrow \boxed{u=2}$

R_2: $x-2y=-2 \Rightarrow x=2y-2$
$v=2y-2-2y \Rightarrow \boxed{v=-2}$

R_3: $x+2y=-2 \Rightarrow x=-2y-2$
$u=-2y-2+2y \Rightarrow \boxed{u=-2}$

R_4: $x-2y=2 \Rightarrow x=2y+2$
$v=2y+2-2y \Rightarrow \boxed{v=2}$



$$-2 \le u \le 2$$
$$-2 \le v \le 2$$

$$\int_{-2}^{2}\int_{-2}^{2} \left(3\left(\tfrac{1}{2}u+\tfrac{1}{2}v\right)+6\left(\tfrac{1}{4}u-\tfrac{1}{4}v\right)\right)^2 \left|-\tfrac{1}{4}\right| \, du\,dv$$

$$= \frac{1}{4}\int_{-2}^{2}\int_{-2}^{2} \left(\tfrac{3}{2}u+\tfrac{3}{2}v+\tfrac{3}{2}u-\tfrac{3}{2}v\right)^2 du\,dv$$

$$= \frac{1}{4}\int_{-2}^{2}\int_{-2}^{2} 9u^2 \, du\,dv$$

$$= \frac{1}{4}\int_{-2}^{2} 3u^3 \Big|_{-2}^{2} dv$$

$$= \frac{1}{4}\int_{-2}^{2} 24-(-24) \, dv$$

$$= \frac{1}{4}\int_{-2}^{2} 48 \, dv$$

$$= \frac{1}{4}\left(48v \Big|_{-2}^{2}\right)$$

$$= \frac{1}{4}\left(96-(-96)\right)$$

$$= \frac{1}{4}(196) = \boxed{48}$$