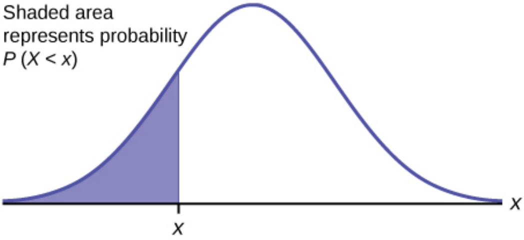
Lecture 3: Integration Fundamentals

September 3, 2019

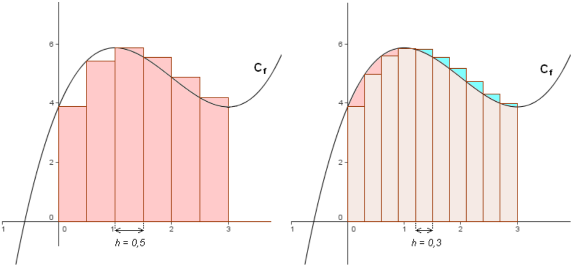
Intro to Integration



Lumen Learning

Integration is a technique to find the area under a curve $f(x)$ over some interval $[a, b]$. In statistics, we use this often to find probabilities like $P(X < x)$ for some random variable X.

Riemann Integration: Basic Idea



Science Direct

Riemann integration involves using smaller and smaller rectangles to find the area under $f(x)$. As the width of these rectangles goes to zero, the approximation of the area becomes more and more accurate.

Approximating the Area Under $F(x)$

Goal: find the area S under $f(x)$ over some interval $I = [a, b]$.

1. Divide I into n smaller subintervals by choosing points $a = x_0 < x_1 < ... < x_n = b$
2. The n subintervals are then $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$. This is called the partition of I and is denoted by $\mathbb{P}$
3. Define $\Delta x_i = x_i - x_{i-1}$.
4. The length of the longest subinterval is given by the norm $||\mathbb{P}|| = \max\{\Delta x_1, ..., \Delta x_n\}$
5. Divide S into strips $s_1, ..., s_n$ by drawing the vertical lines at $x_0, x_1, ..., x_n$
6. Choose an x_i^* in each subinterval and construct a rectangle R_i with base Δx_i and height $f(x_i^*)$.
7. Define $A_i = f(x_i^*)\Delta x_i$ as the area of R_i.
8. Then $\sum_{i=1}^{n} A_i =$ area of S
9. $A = \lim\limits_{||\mathbb{P}|| \to 0} \sum_{i=1}^{n} A_i$

Definite Integral

If f is a function defined on a closed interval $[a,b]$, let $\mathbb{P}$ be a partition of $[a,b]$ with points $x_0, ..., x_1$ where $a = x_0 < x_1 < ... < x_n = b$. Choose points x_i^* in $[x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $||\mathbb{P}|| = \max\{\Delta x_i\}$. Then the definite integral of f from a to b is given by

$$\int_a^b f(x)dx = \lim_{||\mathbb{P}|| \to 0} \sum_{i=1}^{n} \Delta x_i f(x_i^*)$$

if this limit exists. If it does exists f is integrable on $[a,b]$.

Note: f is integrable on $[a,b]$ if it is continuous on (a,b).

Example:

(a) Estimate the area of the region between the x-axis and $f(x) = x^3 - 2x^2 + 4$ over the interval $[0, 10]$ using the left, right, and midpoint of subintervals for the height of $n = 5$ rectangles.

Properties of the Definite Integral

Let $c \in \mathbb{R}$ and let $f(x), g(x)$ be continuous functions on the closed interval $[a, b]$. Then the following properties hold:

1. $\int_a^b c\, dx = c(b-a)$
2. $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
3. $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$
4. $\int_a^b f(x)\, dx \leqslant \int_a^b g(x)\, dx$ if $f(x) \leqslant g(x)\ \forall x \in [a, b]$
5. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ for $c \in [a, b]$
6. $\int_a^a f(x)\, dx = 0$
7. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
8. $\left| \int_a^b f(x)\, dx \right| \leqslant \int_a^b |f(x)|\, dx$

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

If f is continuous on $[a,b]$ and the function F is defined by

$$F(x) = \int_a^x f(t)\,dt$$

then F is an anti-derivative of f on $[a,b]$ and $F'(x) = f(x)$. Furthermore, if F is any anti-derivative of f then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

A list of common anti-derivatives can be found ▸ Here.

Some Examples Using Definite Integrals

(a) Determine the value of $\int_2^9 f(x)\,dx$ given that $\int_5^2 f(x)\,dx = 3$ and $\int_5^9 f(x)\,dx = 8$.

(b) $\int_a^b 3x^4 + 6x^2 + 2\,dx$

(c) $\int_a^b 7e^x + \frac{2}{x}\,dx$

(d) $\int_0^4 f(x)\,dx$ where $f(x) = \begin{cases} 2x & x > 1 \\ 1 - 3x^2 & x \leqslant 1 \end{cases}$

(e) $\int_3^6 |2x - 10|\,dx$

Indefinite Integrals

A definite integral has a specified interval $[a, b]$ while an indefinite integral has unspecified interval.

$$\int f(x)\, dx = F(x) + C$$

Examples:

(a) $\int \sqrt[3]{x} + 10\sqrt[5]{x^3}\, dx$

(b) $\int \frac{2}{\theta} e^{-\frac{2x}{\theta}}\, dx$

(c) Determine $f(x)$ given that $f'(x) = 12x^2 - 4x$ and $f(-3) = 17$