Lecture 6: Sequences & Series II

September 12, 2019

Series With Positive Terms

(i) If $a_i \geqslant 0$ for all i, then the partial sums S_n must be non-decreasing

(ii) If the S_n's are to approach a limit they cannot become arbitrarily large.

Suppose there exists some $B \geqslant S_n$ for all n, then B is called an upper bound of the sequence $\{S_n\}$. The sequence is bounded. If $B' < B$ such that $S_n > B'$. B is then called the least upper bound and

$$\sum_{n=1}^{\infty} a_n = B$$

Can characterize the sum of a infinte series as

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \text{ or } \sup_{n=1}^{\infty}\{S_n\}$$

Tests of Convergence: Integral Test

If f is a continuous, positive, decreasing function on $[k, \infty]$ and $a_n = f(n)$, then the series $\sum_{n=k}^{\infty} a_n$ is convergent if and only if the improper integral $\int_k^{\infty} f(x)\,dx$ is convergent.

Examples:

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$

(b) $\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

(c) For what values of p does the series $\sum_{n=1}^{\infty} \frac{1}{x^p}$ converge?

Tests of Convergence: Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. Then

(i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or diverge.

(ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

Examples:

(a) Test $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence.

(b) Test $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

(c) Test $\sum_{n=3}^{\infty} \frac{e^{-n}}{n^2 + 2n}$ for convergence.

Tests of Convergence: Ratio Test

Let a_n be any sequence

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then the series $\sum a_n$ is convergent.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test fails.

Examples:

(a) Test $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ for convergence

(b) Test $\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$ for convergence

(c) Test $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n+7}$ for convergence

Absolute Convergence

Consider any sequence $a_1, a_2, \ldots$. A series $\sum_{n=1}^{\infty} a_n$ convergences absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Examples:

(a) Test $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$ for absolute convergence

(b) Test $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n}$ for absolute convergence

Strategies for Testing Series

Classify according to the form of the series

(1) P-series $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$

(2) Geometric series $\sum ar^{n-1}$ or $\sum ar^n$ is convergent if $|r| < 1$ and divergent if $|r| \geqslant 1$

(3) If the series has the form that is similar to (1) or (2), then use the limit comparison test

(4) If you can notice that $\lim\limits_{n\to\infty} a_n \neq 0$ then use the test for divergence theorem

(5) For series involving factorials or other products, use the ratio test

(6) If $a_n = f(n)$ and $\int_k^\infty f(x)\,dx$ is easy to evaluate, use the integral test

Power Series

Let x be any number and $\{a_n\}$ by a sequence of numbers. A power series is any series of the form

$$\sum_{n=1}^{\infty} a_n x^n$$

with partial sums $S_n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

Examples:

(a) For what values of x does the series $\sum_{n=0}^{\infty} r x^n$ converge?

(b) For what values of x does the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Taylor Series

Most common use is to approximate the value of functions at particular points.

Theorem (Taylor Series)

Let f be a function defined on a neighborhood of $c \in \mathbb{R}$ with n continuous derivatives. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor series of f around c.

Examples:

(a) Find the Taylor Series expansion of $f(x) = e^{-x}$ about $x = 0$

(b) Find the Taylor Series expansion of $f(x) = \ln x$ about $x = 2$

(c) Find the Taylor Series expansion of $f(x) = (1+x)^n$ about $x = 0$