Lecture 5: Sequences & Series I

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What is a Sequence?

A sequence is a set of numbers written in some order

$$a_1, a_2, a_3, ..., a_{n-1}, a_n$$

An infinite sequence can be thought of as a function whose domain is the set of positive integers $i \in \mathbb{I}$ such that

$$a_i = f(i)$$

Examples:

Give the first four terms of the following sequences

(a) $a_n = \frac{n}{n+1}$

(b) $a_n = 2 + (.1)^n$

(c) $a_n = (-1)^{n+1} \frac{n^2}{3n-1}$

(d) $a_n = 4$

Limits of Sequences

A sequence $\{a_n\}$ has the limit

$$\lim_{n\to\infty}\{a_n\} = L$$

if for all $\epsilon > 0$ there exits a positive N such that $|a_n - L| < \epsilon$ whenever $n > N$. If the limit exists then the sequence converges, otherwise the sequence diverges.

Examples:

If possible, find the limit of the following sequences:

(a) $\left\{\frac{n}{2n+1}\right\}$

(b) $\left\{\frac{(-1)^n n^2}{1+n^3}\right\}$

(c) $\left\{(1+\frac{c}{n})^n\right\}$

What is a Series?

If we add the terms of a sequence $\{a_n\}$ then we get an infinite series.

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{i=1}^{\infty} a_i$$

If we add the first n terms of a sequence, we get the n^{th} partial sum.

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ \vdots &= \vdots \\ S_n &= a_1 +, \ldots, + a_n \end{aligned}$$

If the sequence of partial sums $\{S_n\}$ converges and $\lim_{n \to \infty} S_n = S$ then the series $\sum_{i=1}^{\infty} a_i$ is convergent. Otherwise $\sum_{i=1}^{\infty} a_i$ is divergent.

Geometric Series

The geometric series has the form

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1} \text{ for } a \neq 0$$

For $|r| < 1$ the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Examples:

(a) Find the sum of $\sum_{n=1}^{\infty} 2(\frac{3}{4})^{n-1}$

(b) Is this convergent and find its sum: $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \ldots$.

(c) Find the sum of $\sum_{x=0}^{\infty} p(1-p)^x$ for $p \in (0, 1)$

(d) Does $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ converge?

Limits and Convergence

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then*

$$\lim_{n\to\infty} a_n = 0$$

If the $\lim\limits_{n\to\infty} a_n$ *does not exist or if* $\lim\limits_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Note: If $\lim\limits_{n\to\infty} a_n = 0$, this does not mean that $\sum_{n=1}^{\infty} a_n$ is convergent.

Examples:

(a) Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

(b) Does $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge?

Convergence Rules

Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B and let c be a constant. Then

(i) $\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$

(ii) $\sum_{n=1}^{\infty} ca_n = cA$

(iii) $(\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n) = AB$

Examples:

(a) Does $\sum_{n=1}^{\infty}(9^{-n+2}4^{n+1} + \frac{1}{2^n})$ converge?

(b) Coin flipping.

Series With Positive Terms

(i) If $a_i \geqslant 0$ for all i, then the partial sums S_n must be non-decreasing

(ii) If the S_n's are to approach a limit they cannot become arbitrarily large.

Suppose there exists some $B \geqslant S_n$ for all n, then B is called an upper bound of the sequence $\{S_n\}$. The sequence is bounded. If $B' < B$ such that $S_n > B'$. B is then called the least upper bound and

$$\sum_{n=1}^{\infty} a_n = B$$

Can characterize the sum of a infinte series as

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \text{ or } \sup_{n=1}^{\infty}\{S_n\}$$