Lecture 1: Limits and Continuity

August 27, 2019

What is a Limit?

We are often interested in values of $f(x)$ of a function f when x is very close to a number a, but not necessarily equal to a. In many cases $f(a)$ is not defined.

$$\lim_{x \to a} f(x) = L$$

In words: The limit of $f(x)$ as x approaches a is equal to L.

Example:

(a) $\lim_{x \to 4} \frac{1}{2}(3x - 1)$

(b) $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$

(c) $\lim_{x \to 2} x^2 - x + 2$

Formal Definition of Limit

Let $f(x)$ be a function of x defined on some open interval $\mathbb{R}$ that contains a point a except possibly a itself. Then

$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Some notes:

1. $f(x)$ does not have to equal L for the limit to exist.
2. Examine the behavior of $f(x)$ as $x \to \infty$.
3. The limit does not have to exist.

Example:

(a) Show $\lim_{x \to 0} x^2 = 0$

One-sided Limits

We can examine the limit as x approaches from the left or right. This is called a one-sided limit.

$$\text{Left-handed limit}: \quad \lim_{x \to a^-} f(x) = L^-$$

$$\text{Right-handed limit}: \quad \lim_{x \to a^+} f(x) = L^+$$

For the limit to exist $L^- = L^+ = L$

Example:

(a) $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}$

Limit Laws

Suppose c is a constant, n is a positive integer, and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{G(X)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

Limit Laws

7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

Examples:

(a) Find $\lim_{x \to 2} \frac{3x+4}{5x+7}$

(b) Show that for every real number a, $\lim_{x \to a} x^3 = a^3$

(c) Find $\lim_{x \to 2} (3x + 4)^5$

Continuity

A function is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$. This definition requires that:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

A continuous function has the property that a small change in x produces only a small change in $f(x)$.

Example:

(a) Is $f(x) = \frac{x^2 - x + 2}{x - 2}$ continuous at 8 and 4?

Types of Continuity

Left & Right Continuity

We can examine the continuity of a function from either the left or the right.

$$\text{Left continuous}: \quad \lim_{x \to a^-} f(x) = f(a)$$

$$\text{Right continuous}: \quad \lim_{x \to a^+} f(x) = f(a)$$

f is continuous if and only if f is continuous from the left and right at every point in its domain.

Interval Continuity

f is continuous on $[a, b]$ if it is continuous on every point on the interval

Example:

(a) Show that $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$