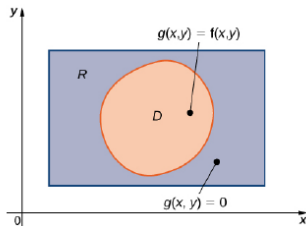


# Lecture 12: Multiple Integration Part 2

October 4, 2018

# Double Integration with General Region



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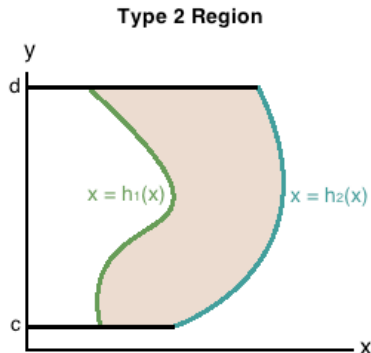
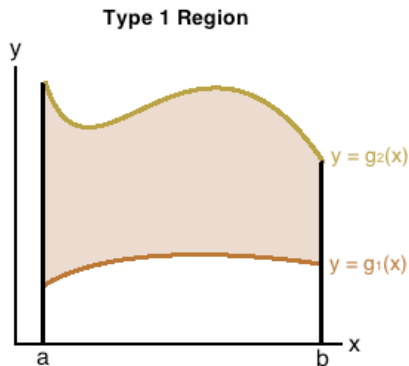
If we want to integrate  $f(x, y)$  over  $D$ , define a new function with domain  $R$

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

Then if  $F$  is integrable over  $R$ ,  $f$  is integrable over  $D$ .

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA$$

# Type I vs Type II Regions



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# Type I Region

A Type I region is defined as:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where  $g_1$  and  $g_2$  are continuous on  $[a, b]$ . Then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

**Example:**

(a) Integrate  $f(x, y) = x + 2y$  on  $D = \{(x, y) \mid -1 \leq x \leq 2, 2x^2 \leq y \leq 1 + x^2\}$

## Type II Region

A Type II region is defined as:

$$D = \{(x, y) \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

where  $h_1$  and  $h_2$  are continuous on  $[c, d]$ . Then

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

### Example:

(a) Integrate  $f(x, y) = xe^y$  on  $D = \{(x, y) \mid \sqrt{y} \leq x \leq 1/2y, 0 \leq y \leq 1\}$ .

# The Approach I Use (An Aside)

In practice, you'll rarely have  $D$  presented to you in a nice way that makes it obvious what type of region you're looking at. More often it looks like:

Integrate  $f(x, y) = e^{x+y}$  over  $y, x > 0$  and  $x > y$ .

So you need to be able to tell from a graph how to set up your bounds. I typically use something called the "Rectangle and Line" method.

## Example:

- (a) Integrate  $f(x, y) = 4xy$  on the trapezoid with corners at  $(0, 0)$ ,  $(4, 0)$ ,  $(2, 2)$ , and  $(4, 2)$ .

# Several Examples

- (a) Find the volume of the solid that lies under  $f(x, y) = x^2 + y^2$  and above region  $D$  that is bounded by  $x = y/2$  and  $x = \sqrt{y}$
- (b) Integrate  $f(x, y) = xy$  over the region bounded by  $y = x - 1$  and  $y^2 = 2x + 6$ .
- (c) Integrate  $f(x, y) = e^{y^2}$  where  $y \leq 1$ ,  $y \geq x$ , and  $x \geq 0$ .
- (d) Integrate  $f(x, y) = x^2 + y^3$  over the region in the first quadrant bounded by  $y = x^2$  and  $x = y^4$ .