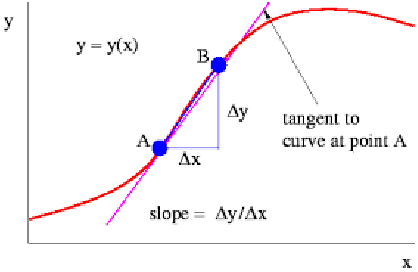
# Lecture 2: Derivatives

August 29, 2019

# Formal Definition of a Derivative




*labman.phys.uk.edu*

$f'(a)$ is the derivative of a function $f$ at $a$ if

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

**Example:**

(a) Given $f(x) = \frac{x}{2x-1}$ find $f'(a)$.

# Derivative Laws

1. $f(x) = c \implies f'(x) = 0$
2. $f(x) = x^n \implies f'(x) = nx^{n-1}$
3. $F(x) = f(x)g(x) \implies F'(x) = f(x)g'(x) + g(x)f'(x)$
4. $F(x) = \frac{f(x)}{g(x)} \implies F'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$
5. $F(x) = f(x) + g(x) \implies f'(x) + g'(x)$
6. $F(x) = f(x) - g(x) \implies f'(x) - g'(x)$
7. $f(x) = e^x \implies f'(x) = e^x$
8. $f(x) = \log(x) \implies f'(x) = 1/x$

**Examples:**

Find the derivative of the following functions

(a) $f(t) = \frac{6}{\sqrt[3]{t^5}}$

(b) $f(x) = \frac{x}{x+c/x}$

# Using Derivatives to Find Max & Min

Let $F$ be a function of $x$ and $A$ a set of numbers contained on the domain of $F$.

(i) A point $x$ in $A$ is a maximum for $F$ on $A$ if $F(x) \geqslant F(y)$ for every $y$ on $A$. $F(x)$ is called the maximum of $F$ on $A$.

(ii) A point $x$ in $A$ is a minimum for $F$ on $A$ if $F(x) \leqslant F(y)$ for every $y$ on $A$. $F(x)$ is called the minimum of $F$ on $A$.

**Theorem**

*Let F be any function defined on $(a, b)$. If $x$ is a max or min point for F on $(a, b)$ and F is differentiable, then $F'(x) = 0$*

The second derivative tells us which type of point we are dealing with.

# Using Derivatives to Find Max & Min

A critical or stationary point of a function $F$ is a number $x$ such that $F'(x) = 0$. $F(x)$ itself is called the critical value of $F$.

**Finding a Max or Min**

Consider a function on a closed interval $[a, b]$. In order to locate the max and/or min three kinds of points must be considered.

1. The critical points of $F$ in $[a, b]$.
2. The end points of $[a, b]$.
3. The points in $[a, b]$ such that $F$ is not differentiable at $x$.

**Examples:**

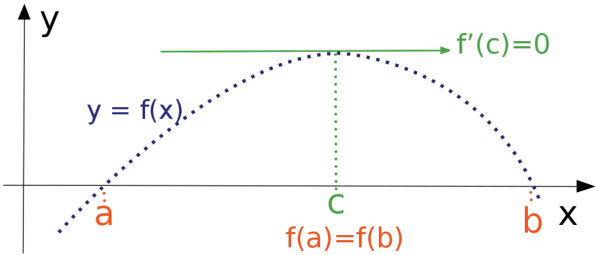
Find the max and min of each of the following

(a) $f(x) = 2x^3 + 3x^2 - 12x + 4$ on $[-4, 2]$

(b) $f(x) = 2x^3 + 3x^2 - 12x + 4$ on $[0, 2]$

(c) $f(x) = \frac{(x-1)^2}{x}$ on $[1/2, 2]$

# Rolle's Theorem

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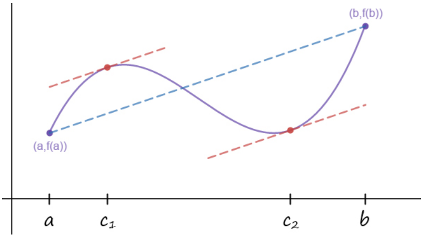
**Theorem (Rolle's Theorem)**

*If $F$ is continuous on $[a, b]$ and differentiate on $(a, b)$ and $F(a) = F(b)$ then there is a number $c$ in $(a, b)$ such that $F'(c) = 0$.*

# Mean Value Theorem

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**Theorem (Mean Value Theorem)**

*If $F$ is continuous on $[a, b]$ and differentiable on $(a, b)$ then there is a number $c$ in $(a, b)$ such that*

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

# Cauchy Mean Value Theorem

Theorem (Cauchy Mean Value Theorem)

*If $F$ and $g$ are continuous on $[a,b]$ and differentiable on $(a,b)$ then there is a number $c$ in $(a,b)$ such that*

$$\frac{F'(c)}{g'(c)} = \frac{F(b)-F(a)}{g(b)-g(a)} \implies [F(b)-F(a)]g'(c) = [g(b)-g(a)]F'(c)$$

# L'Hopitals Rule

**Theorem (L'Hopitals Rule)**

*Suppose that* $\lim_{x \to a} F(x) = 0$ *and* $\lim_{x \to a} g(x) = 0$ *and*

*(i)* $F'$ *and* $g'$ *exist for each* $x$ *of an interval about* $x = a$ *except possibly* $a$ *itself.*

*(ii)* $g' \neq 0$ *for* $x \neq a$ *in this interval.*

*(iii)* $\lim_{x \to a} \frac{F'(x)}{g'(x)} = A$

*Then* $\lim_{x \to a} \frac{F(x)}{g(x)} = A$.

**Example:**

(a) $\lim_{x \to 0} \frac{e^x - 1}{\ln(1+x)}$