Lecture 7: Limits, Continuity, and Derivatives in Several Variables

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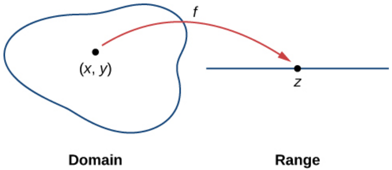
Functions of Two Variables



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A function of two variables is a rule that assigns to each ordered pair (x, y) in D a unique real nuber denote by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on.

Examples:

(a) Find the domain of f and g and evaluate $f(2, 5), f(1, 2), g(3, 2)$

$$f(x, y) = \frac{xy - 5}{2\sqrt{y - x^2}}, \; g(x, y)\frac{\sqrt{x + y + 1}}{x - 1}$$

Limits in Two Variables

Let f be a function of a two variables defined on ad isk with center at (a, b) expect possibly at (a, b) itself. The limit of $f(x, y)$ as (x, y) approaches (a, b) is given by

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x, y) - L| < \epsilon$ whenever $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Examples:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Continuity in Two Variables

Let f be a function of two variables defined on a disk with center (a, b). Then f is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

If f is continuous at (a, b) and g is a function of one variable that is continuous at $f(a, b)$ then g of $f(x, y)$, $g(f(x, y))$ is also continuous at (a, b).

Examples:

(a) Is the function $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ continuous at $(1, 2)$?

(b) Where is the function $h(x, y) = \ln(x^2 + y^2 - 1)$ continuous?

Intro to Partial Derivatives

Suppose f is a function of two variables x and y, then

$$\begin{aligned}\frac{\partial f}{\partial x}(a,b) &= \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}\\ \frac{\partial f}{\partial y}(a,b) &= \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}\end{aligned}$$

To find $\partial f/\partial x$ regard y as a constant and differentiate with respect to x. To find $\partial f/\partial y$ regard x as a constant and differentiate with respect to y.

Examples:

(a) Find all first order partials of $g(x,y,z) = \frac{x\sin(y)}{z^2}$

(b) Find all first and second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

Mixed Partial Derivatives

Theorem (Clairaut's Theorem)

If f is defined on a disk D containing (a,b) and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous, then

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

An example of a function which does not have equal mixed partials can be found ▸ HERE

Chain Rule

The chain rule can be extended to functions of two variables.

(i) If x and y are each functions of a single variable t such that $x = g(t)$ and $y = h(t)$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

(ii) If x and y are each functions of two variables s, t such that $x = g(s,t)$ and $y = h(s,t)$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \text{ and } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Examples:

(a) Find $\frac{\partial z}{\partial t}$ where $z = x^2y + 3xy^4$, $x = e^t$, and $y = \sin t$.

(b) Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ where $z = e^x \sin y$, $x = st^2$, and $y = s^2 t$.