Lecture 12: Change of Variables I

October 3, 2019

Review of Single Variable Case

In the univariate case we have:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Where

(i) $g'(x)$ is continuous on $[a, b]$.

(ii) $f(x)$ is continuous on a range of g.

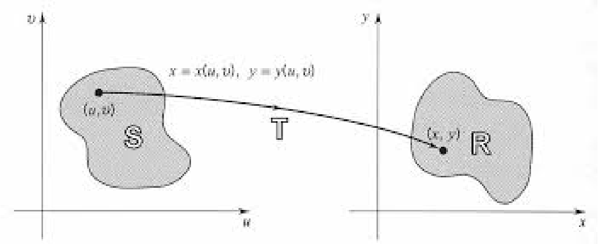
(iii) $u = g(x)$ is the transformation

This process is also called u-substitution.

Example:

(a) $\int_0^4 \sqrt{3x + 4}\, dx$

Transformations in Two Variables



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We consider a change of variables given by the transformation T from the (u, v) plane to the (x, y) plane.

$$\iint_R f(x, y)\,dx\,dy = \iint_S g(u, v)\,dv\,du$$

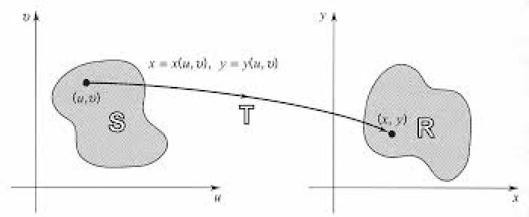
Transformations

- A function F that maps $R^2 \to R^2$ is called "one-to-one" if $\forall (u, v)$ in the domain of F on the (u, v) plane, $\exists$ only one (x, y) in the (x, y) plane.
- The transformation T is given by

$$T(u, v) = (x, y) \text{ or } x = g(u, v), y = h(u, v)$$

- If $T(u, v) = (x, y)$, then (x_1, y_1) is the image of (u_1, v_1) under the transformation T. If no two points have the same image, T is one-to-one.
- T is a C^1 transformation which means that g and h have continuous 1st order partials.

Transformations



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- T transforms S into a region R in the (x, y) plane called the image of S.
- T is said to be ”onto” R if each point in R is the image of some point in S.
- A transformation T that is one-to-one and onto is invertable, denoted by T^{-1}.

$$T^{-1}(x, y) = (u, v) \text{ or } u = G(x, y), v = H(x, y)$$

Is T Invertible?

How do you tell if T is invertible

(1) Given that $x = g(u, v)$ and $y = h(u, v)$, take all the partial derivatives of x and y with respect to u and v.

(2) Form the Jacobian, $J = |M|$, of the transformation which is the determinate of the matrix

$$M = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \text{ giving } J = |M| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

(3) If $J \neq 0$ then T^{-1} exists.

Example:

(a) T is given by $x = u^2 - v^2$ and $y = 2uv$. If $S = \{(u, v) | 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$. Is T invertible?

Finding the Image of S Under T

How do we find R?

- Need to find new limits of integration
- Process is to work along the boundaries of S and determine what happens on the (x, y) plane.

Example:

(a) T is given by $x = u^2 - v^2$ and $y = 2uv$. If $S = \{(u, v) | 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$. Find region R.

(b) T is given by $u = x + y$ and $v = x - y$. If $R = \{(x, y) | 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. Find region S.

Change of Variables Formula

If the following applies:

(i) T is invertible. (Namely $J \neq 0$.)

(ii) T is C^1.

(iii) T maps region S in (u, v) plane onto region R in (x, y) plane.

(iv) f is continuous in R.

(v) R and S are Type I or Type II regions

Then we can apply the change of variables formula:

$$\iint_R f(x, y)\,dy\,dx = \iint_S f[g(u, v), h(u, v)]\,|J|\,dv\,du$$

Example:

(a) T is given by $x = u^2 - v^2$ and $y = 2uv$. If $S = \{(u, v) | 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$. Express and evaluate the integral $\iint_R y\,dy\,dx$ in terms of (u, v).