Lecture 4: Integration Strategies

September 5, 2019

Substitution Rule (Change of Variables)

If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I then

$$\int f(g(x))g'(x) dx = \int f(u) du, \text{ where } u = g(x)$$

Examples:

- (a) $\int (2x^3 + 1)^7 x^2 dx$
- (b) $\int x \sqrt{7 6x^2} \, dx$
- (c) $\int \sin 5x \, dx$
- (d) $\int_0^4 \sqrt{3x+4} \, dx$
- (e) $\int_{1}^{2} e^{-\frac{2x}{t}} dx$
- (f) $\int_{e^2}^{e^6} \frac{(\ln x)^4}{x} dx$



Integration by Parts the Hard Way

Let u = f(x) and v = g(x) be two functions. Then

$$\int f(x)g'(x) dx = \int u dv = uv - \int v du$$

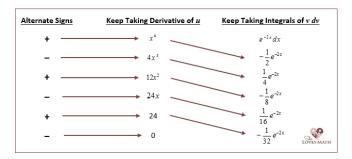
Rule of thumb: Select the most complicated part of the integral that can easily be integrated for dv.

Example:

(a)
$$\int xe^{2x} dx$$

Integration by Parts the Easy Way

The problem with the preceding formula is that you may need to apply it successively (eg: x^4e^{-2x}) which is very time consuming and hard to keep track of. A shortcut is to use the line method or Tanzalin Method (?).



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Integration by Parts the Easy Way: Examples

Integrate the following:

- (a) $\int xe^{2x} dx$
- (b) $\int_0^1 x^3 e^{2x} dx$
- (c) $\int_0^\pi x^2 \cos(4x) \, dx$
- (d) $\int x^2 \ln 4x \, dx$
- (e) $\int \ln x \, dx$
- (f) $\int e^x \cos(x) dx$

Improper Integrals

An integral over an open interval or half open interval (eg: (a, b]) is called an improper integral. There are two types of improper integral. For instance consider

$$\int_{\infty}^{\infty} e^{-x^2} dx \text{ or } \int_{0}^{5} \frac{1}{x^2} dx$$

Type I: One or both of the endpoints are infinite

Type II: The interval contains a point of discontinuity

An integral is *divergent* if the evaluated integral is not a finite number or does not exist and *convergent* if the evaluated integral is a finite number.

Type I Improper Integrals

(i) If $\int_a^t f(x) dx$ exists $\forall t \ge a$ then

$$\int_{a}^{\infty} f(x) dx = \lim_{t \to \infty} \int_{a}^{t} f(x) dx$$

(ii) If $\int_t^b f(x) dx$ exists $\forall t \leq b$ then

$$\int_{-\infty}^{b} f(x) dx = \lim_{t \to -\infty} \int_{t}^{b} f(x) dx$$

(iii) If $\int_a^t f(x) dx$ and $\int_t^a f(x) dx$ exist $\forall a \in \mathbb{R}$ then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{a} f(x) dx + \int_{a}^{\infty} f(x) dx$$

Examples:

- (a) $\int_1^\infty \frac{1}{x} dx$
- (b) For what values of p is $\int_1^\infty \frac{1}{x^p} dx$ convergent? (Assume $p \neq 1$.)

Type II Improper Integrals

(i) If f is continuous on [a, b) but discontinuous at b then

$$\int_{a}^{b} f(x) dx = \lim_{t \to b^{-}} \int_{a}^{t} f(x) dx$$

(ii) If f is continuous on (a, b] but discontinuous at a then

$$\int_{a}^{b} f(x) dx = \lim_{t \to a^{+}} \int_{t}^{b} f(x) dx$$

(iii) If f has a discontinuity at $c \in [a, b]$ and both $\int_a^c f(x) dx$ and $\int_a^b f(x) dx$ converge then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Examples: (a)
$$\int_2^5 \frac{1}{\sqrt{x-2}} dx$$

(b)
$$\int_{-2}^{7} \frac{1}{(x+1)^{2/3}} dx$$

