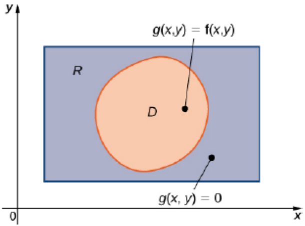
Lecture 11: Multiple Integration II

September 26, 2019

Double Integration with General Region



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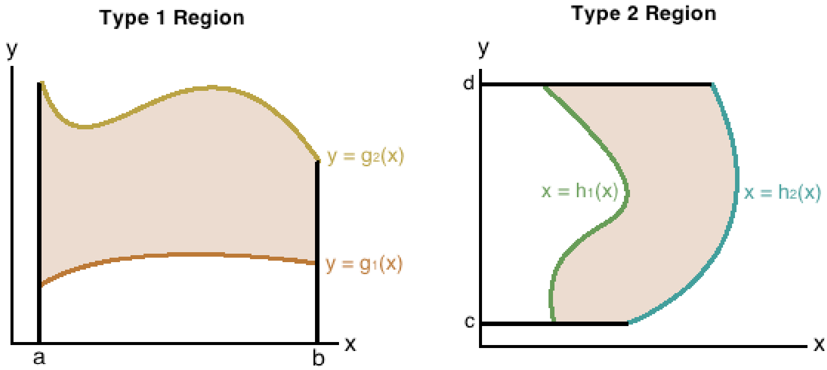
If we want to integrate $f(x, y)$ over D, define a new function with domain R

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

Then if F is integrable over R, f is integrable over D.

$$\iint_D f(x, y)dA = \iint_D F(x, y)dA$$

Type I vs Type II Regions



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Type I Region

A Type I region is defined as:

$$D = \{(x, y) | \, a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy \, dx$$

Example:

(a) Integrate $f(x, y) = x + 2y$ on $D = \{(x, y) | -1 \leqslant x \leqslant, 2x^2 \leqslant y \leqslant 1 + x^2\}$

Type II Region

A Type II region is defined as:

$$D = \{(x, y) \,|\, h_1(y) \leqslant x \leqslant h_2(y), c \leqslant y \leqslant d\}$$

where h_1 and h_2 are continuous on $[c, d]$. Then

$$\iint\limits_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dx\,dy$$

Example:

(a) Integrate $f(x, y) = xe^y$ on $D = \{(x, y)|\sqrt{y} \leqslant x \leqslant 1/2y, 0 \leqslant y \leqslant 1\}$.

Which Type of Region Do You Have?

In practice, you'll rarely have D presented to you in a nice way that makes it obvious what type of region you're looking at. More often it looks like:

$$\text{Integrate } f(x,y) = e^{x+y} \text{ over } y, x > 0 \text{ and } x > y.$$

So you need to be able to tell from a graph how to set up your bounds. I typically use something called the "Rectangle and Line" method.

Example:

(a) Integrate $f(x,y) = 4xy$ on the trapezoid with corners at $(0,0)$, $(4,0)$, $(2,2)$, and $(4,2)$.

Several Examples

(a) Find the volume of the solid that lies under $f(x, y) = x^2 + y^2$ and above region D that is bounded by $x = y/2$ and $x = \sqrt{y}$

(b) Integrate $f(x, y) = xy$ over the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

(c) Integrate $f(x, y) = e^{y^2}$ where $y \leqslant 1$, $y \geqslant x$, and $x \geqslant 0$.

(d) Integrate $f(x, y) = x^2 + y^3$ over the region in the first quadrant bounded by $y = x^2$ and $x = y^4$.

(e) Integrate $f(x, y) = 5x^3 \cos(y^3)$ over the region in the first quadrant bounded by $y = 2$ and $y = \frac{1}{4}x^2$.