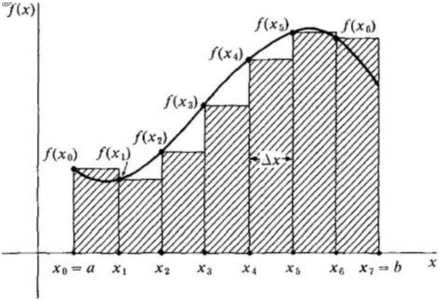
Lecture 9: Multiple Integration Part I

September 24, 2019

Single to Double Integration



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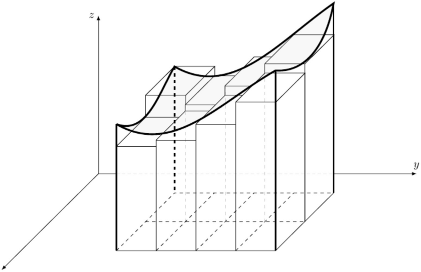
(a) Single integration



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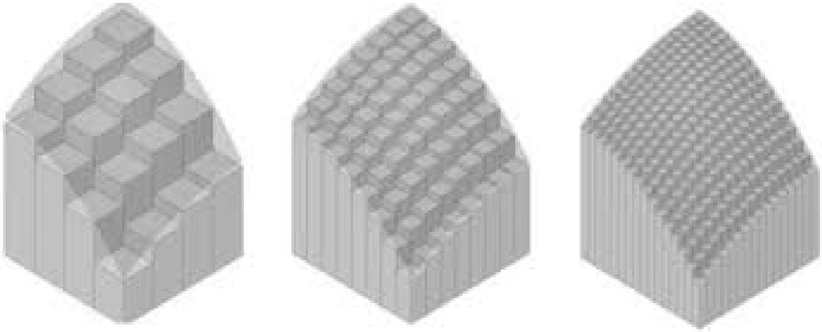
(b) Double integration.

Riemann Double Integral



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$$\iint\limits_{\mathbb{R}} f(x,y)\,dA = \lim_{||P||\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{i,j}$$

Properties

(a) $\iint\limits_{\mathbb{R}} cf(x,y)\,dx\,dy = c\iint\limits_{\mathbb{R}} f(x,y)\,dx\,dy$ if c is a real number

(b) $\iint\limits_{\mathbb{R}} [f(x,y) + g(x,y)]\,dx\,dy = \iint\limits_{\mathbb{R}} f(x,y)\,dx\,dy + \iint\limits_{\mathbb{R}} g(x,y)\,dx\,dy$

(c) If $\mathbb{R}$ is the union of two nonoverlapping regions $\mathbb{R}_1 + \mathbb{R}_2$,

$$\iint\limits_{\mathbb{R}} f(x,y)\,dx\,dy = \iint\limits_{\mathbb{R}_1} f(x,y)\,dx\,dy + \iint\limits_{\mathbb{R}_2} f(x,y)\,dx\,dy$$

(d) If $f(x,y) \geqslant 0$ throughout $\mathbb{R}$, then $\iint\limits_{\mathbb{R}} f(x,y)\,dx\,dy \geqslant 0$

Evaluation of the Double Integral: Iterated Integral

It is generally impossible to evaluate the double integral using the definition. Instead it is typical to express the double integral as an iterated integral and evaluate as two single integrals.

$$\int_a^b \int_c^d f(x,y)dx\,dy = \int_a^b \left[\int_c^d f(x,y)dx \right] dy$$

Examples:

(a) $\int_1^4 \int_{-1}^2 (2x + 6x^2y)dy\,dx$

(b) $\int_{-1}^2 \int_1^4 (2x + 6x^2y)dx\,dy$

Evaluation of the Double Integral: Fubini's Theorem

Theorem (Fubini's Theorem)

If f is a continuous function on rectangle $\mathbb{R} = \{(x, y)|a \leqslant b, c \leqslant d\}$ then

$$\iint\limits_{\mathbb{R}} f(x, y) dA = \int_a^b \int_c^d f(x, y) dx\, dy = \int_c^d \int_a^b f(x, y) dx\, dy$$

Sometimes one way is easier to evaluate than the other.

Example:

(a) Integrate $f(x, y) = 4xy$ on the trapezoid with corners at $(0, 0)$, $(4, 0)$, $(2, 2)$, and $(4, 2)$.

Evaluation of the Double Integral: Factorizable f

If $f(x, y) = g(x)h(y)$ on $\mathbb{R} = [a, b] \times [c, d]$ then

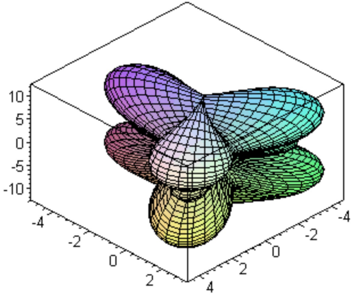
$$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Note: This only works if the limits are constant. (Don't depend on x or y.)

Example:

(a) $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x) \cos(y)\,dx\,dy$

Triple Integral



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$$\iiint\limits_{\mathbb{R}} F(x,y,z)\,dV = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \int_{z=z_1(x,y)}^{z=z_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

Example:

(a) $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz\,dz\,dy\,dx$