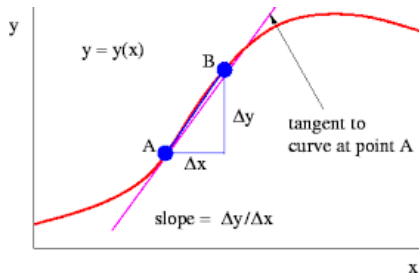


## Lecture 2: Derivatives

August 29, 2019

# Formal Definition of a Derivative



*labman.phys.uk.edu*

$f'(a)$  is the derivative of a function  $f$  at  $a$  if

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

**Example:**

(a) Given  $f(x) = \frac{x}{2x-1}$  find  $f'(a)$ .

# Derivative Laws

1.  $f(x) = c \implies f'(x) = 0$
2.  $f(x) = x^n \implies f'(x) = nx^{n-1}$
3.  $F(x) = f(x)g(x) \implies F'(x) = f(x)g'(x) + g(x)f'(x)$
4.  $F(x) = \frac{f(x)}{g(x)} \implies F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$  if  $g(x) \neq 0$
5.  $F(x) = f(x) + g(x) \implies f'(x) + g'(x)$
6.  $F(x) = f(x) - g(x) \implies f'(x) - g'(x)$
7.  $f(x) = e^x \implies f'(x) = e^x$
8.  $f(x) = \log(x) \implies f'(x) = 1/x$

## Examples:

Find the derivative of the following functions

- (a)  $f(t) = \frac{6}{\sqrt[3]{t^5}}$
- (b)  $f(x) = \frac{x}{x+c/x}$

# Using Derivatives to Find Max & Min

Let  $F$  be a function of  $x$  and  $A$  a set of numbers contained on the domain of  $F$ .

- (i) A point  $x$  in  $A$  is a maximum for  $F$  on  $A$  if  $F(x) \geq F(y)$  for every  $y$  on  $A$ .  
 $F(x)$  is called the maximum of  $F$  on  $A$ .
- (ii) A point  $x$  in  $A$  is a minimum for  $F$  on  $A$  if  $F(x) \leq F(y)$  for every  $y$  on  $A$ .  
 $F(x)$  is called the minimum of  $F$  on  $A$ .

## Theorem

*Let  $F$  be any function defined on  $(a, b)$ . If  $x$  is a max or min point for  $F$  on  $(a, b)$  and  $F$  is differentiable, then  $F'(x) = 0$*

The second derivative tells us which type of point we are dealing with. If  $f''(x) > 0$  then  $x$  is a minimum. If  $f''(x) < 0$  then  $x$  is a maximum.

# Using Derivatives to Find Max & Min

A critical or stationary point of a function  $F$  is a number  $x$  such that  $F'(x) = 0$ .  $F(x)$  itself is called the critical value of  $F$ .

## Finding a Max or Min

Consider a function on a closed interval  $[a, b]$ . In order to locate the max and/or min three kinds of points must be considered.

1. The critical points of  $F$  in  $[a, b]$ .
2. The end points of  $[a, b]$ .
3. The points in  $[a, b]$  such that  $F$  is not differentiable at  $x$ .

## Examples:

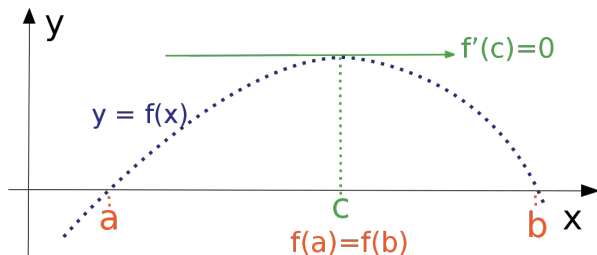
Find the max and min of each of the following

(a)  $f(x) = 2x^3 + 3x^2 - 12x + 4$  on  $[-4, 2]$

(b)  $f(x) = 2x^3 + 3x^2 - 12x + 4$  on  $[0, 2]$

(c)  $f(x) = \frac{(x-1)^2}{x}$  on  $[1/2, 2]$

# Rolle's Theorem



wikipedia

## Theorem (Rolle's Theorem)

*If  $F$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $F(a) = F(b)$  then there is a number  $c$  in  $(a, b)$  such that  $F'(c) = 0$ .*

# Mean Value Theorems

## Theorem (Mean Value Theorem)

*If  $F$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then there is a number  $c$  in  $(a, b)$  such that*

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

## Theorem (Cauchy Mean Value Theorem)

*If  $F$  and  $g$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$  then there is a number  $c$  in  $(a, b)$  such that*

$$\frac{F'(c)}{g'(c)} = \frac{F(b) - F(a)}{g(b) - g(a)} \implies [F(b) - F(a)]g'(c) = [g(b) - g(a)]F'(c)$$

# L'Hopitals Rule

## Theorem (L'Hopitals Rule)

*Suppose that*

$$\lim_{x \rightarrow a} \frac{F(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x \rightarrow a} \frac{F(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

*where  $a$  can be any real number, infinity, or negative infinity. Then*

$$\lim_{x \rightarrow a} \frac{F(x)}{g(x)} = \lim_{x \rightarrow a} \frac{F'(x)}{g'(x)}$$

## Examples:

- (a)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\ln(1+x)}$
- (b)  $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$