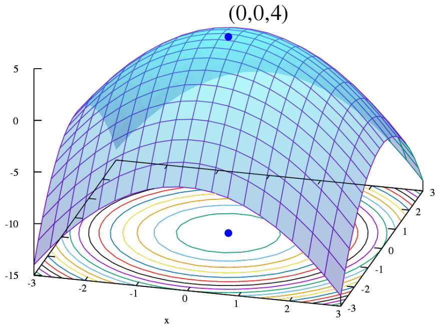
Lecture 8: Optimization

September 19, 2019

Optimization



Wikipedia

One of the main application of derivatives is finding a maximum or minimum.
We discussed this using one variable, and similar concepts apply.
Optimization deals with finding max or min points subject to some constraint.

Finding Local Extrema

A function of two variables has a local maximum at (a, b) if $f(x, y) \leqslant f(a, b)$ for all (x, y) in some disk with center (a, b). $f(a, b)$ is then the local maximum. If $f(x, y) \geqslant f(a, b)$ on such a disk, $f(a, b)$ is the local minimum. *Note: If the inequalities hold on the entire domain of $f(x, y)$ then we have a global max or min.*

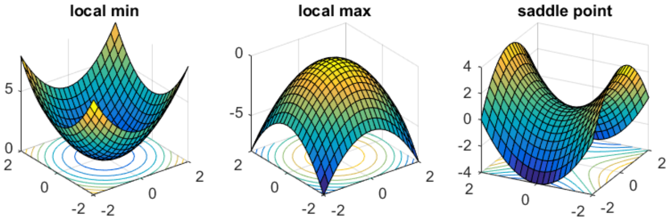
(i) If f has a local extrema at (a, b) and the 1st partial derivative exists, then

$$\frac{\partial f}{\partial x}(a, b) = 0, \text{ and } \frac{\partial f}{\partial y}(a, b) = 0$$

(ii) A point (a, b) where all existing partial derivatives equals zero is called a critical or stationary point.

(iii) A critical point is not necessarily an extrema of f. A critical point can be a max, min, or saddle point.

Finding Local Extrema



Off the convex path

Examples:

(a) $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

(b) $f(x, y) = y^2 - x^2$

Second Derivative Test for Extrema

To test whether a critical point is a max, min, or saddle calculate D where

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

-If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local min.

-If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local max.

-If $D < 0$ then $f(a,b)$ is a saddle point.

-If $D = 0$ then the test fails.

Examples:

(a) $f(x,y) = x^2 - y^2$

(b) $f(x,y) = x^2 - xy + y^2$

(c) $f(x,y) = 4 + x^3 + y^3 - 3xy$

Finding Absolute Extreme Values

Theorem (Extreme Value Theorem)

If $f(x,y)$ is continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute min value $f(a_1,b_1)$ and also a max value $f(a_2,b_2)$ at some points (a_1,b_1) and (a_2,b_2) in D such that

$$f(a_1,b_1) \leqslant f(x,y) \leqslant f(a_2,b_2)$$

To find the absolute extrema of a continuous function on a closed, bounded D:

(i) Find the values of f at the critical points of f in D.

(ii) Find the extreme values of f on the boundary of D.

Examples:

(a) $f(x,y) = 1 + x^2 + y^2$ where $0 \leqslant x^2 + y^2 \leqslant 4$

(b) $f(x,y) = 5 - 3x + 4y$ on closed triangle with vertices $(0,0)$, $(4,0)$, $(4,5)$.

Lagrange Multipliers

Suppose we want to minimize $f(x,y)$ subject to constraint $g(x,y) = 0$. Then we use $F = f(x,y) + \lambda g(x,y)$ and solve the following system of equations:

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} &= 0 \\ \frac{\partial F}{\partial \lambda} &= 0 \end{aligned}$$

Examples:

(a) Minimize $f(x,y) = (x-1)^2 + y^2$ subject to $y^2 = 4x$

(b) Minimize $f(x,y) = x^2 - y^2$ subject to $x^2 + y^2 = 1$

(c) Minimize $f(x,y) = x^2 + y$ subject to $x^2 + y^2 = 1$