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DS 98

AMATYC Spring 2019

1. C **A store sells rope by the whole foot. If a landscaper needs a rope at least 16,800 mm long, what is the least number of feet she must purchase. Use 1 inch = 2.54 cm.**
   Since 1 foot = 12 inches = 30.48cm = 304.8mm, $\frac{16800}{304.8}$ feet $\approx$ 55.11 feet. Therefore, store needs to buy at 56 feet.

2. C **Let $A = \{1, 2, 3, 4\}$. Let M = the number of distinct proper subsets of A. Let N = the number of distinct differences that can be found by subtracting two distinct elements of A (for example, 1 would be one such difference since $3 - 1 = 2$). Find $M + N$.**
   The proper sets are 4 sets containing one element, 6 containing two elements, 4 containing three elements and empty set. Therefore $M = 15$.
   There are 6 distinct differences $(-3, -2, -1, 1, 2, 3)$ that can be found by subtracting two distinct elements. Therefore $N = 6$.
   Then $M + N = 21$.

3. D **How many different ways can a cashier break (return an equivalent dollar amount in small denominations) a \$50 bill if there are an unlimited number of \$20, \$10, \$5, and \$1 bills available to the cashier? Assume bills of the same denomination are indistinguishable.**
   Consider ways how to get \$50:

   - use two \$20 bills $\rightarrow$ 4 ways
     Since there are 4 ways how to get \$10 ($1 \cdot \$10$, $2 \cdot \$5$, $1 \cdot \$5 + 5 \cdot \$1$, $10 \cdot \$1$), there are 4 ways how to get this combination.
   - use one \$20 bill $\rightarrow$ 16 ways
     - use three \$10 bills $\rightarrow$ 1 way
     - use two \$10 bills $\rightarrow$ 3 ways
       Since there are three ways how to get \$10 using \$5 and \$1 bills, there are three ways.
     - use one \$10 bill $\rightarrow$ 5 ways
       Since there are five ways how to get \$20 using \$5 and \$1 bills, there are five ways.
     - don't use any \$10 bill $\rightarrow$ 7 ways
       Since there are seven ways how to get \$30 using \$5 and \$1 bills, there are seven ways.
   - don't use any \$20 bill $\rightarrow$ 36 ways
     - use five \$10 bills $\rightarrow$ 1 way
     - use four \$10 bills $\rightarrow$ 3 ways
     - use three \$10 bills $\rightarrow$ 5 ways
     - use two \$10 bills $\rightarrow$ 7 ways

- use one \$10 bill $\rightarrow$ 9 ways
- don't use any \$10 bills $\rightarrow$ 11 ways

Therefore there are $4 + 16 + 36 = 56$ ways.

4. E **An isosceles triangle has two sides of length 40 and a base of length 48. A circle circumscribes the triangle. What is the radius of the circle?**

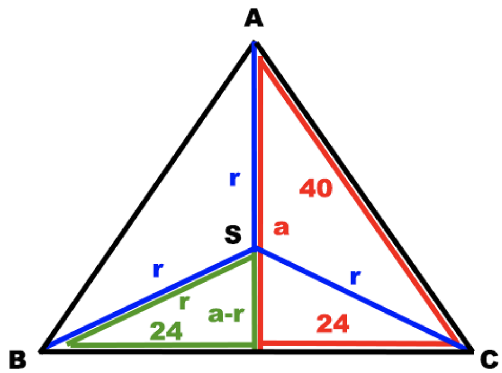

Figure 1: Isosceles triangle

We know that center S of the circle is going to be on the height to the base because $|SB| = |SC| = r$.
We can find height a from the red right triangle, $40^2 = a^2 + 24^2$. That is $a = 32$.
Then we can find radius r from the green right triangle, $r^2 = 24^2 + (32 - r)^2$. That is $r = 25$.

5. A **Let M be the number of digits $\{0, 1\}$ required to express the largest prime factor of 2019 in base 2. Let N be the number of hex digits $\{0, 1, 2, ...E, F\}$ required to express 2019 in base 16. Find $M - N$.**
Note that $2019 = 3 \cdot 673$. Therefore, largest prime factor is 673. Numbers of binary digits required to express number can be expressed as $2^M - 1$. Therefore, $2^M - 1 = 673$. That is $M = \log_2(672) \approx 9.4$, so we need 10 binary digits.
Number of he digits can be found similarly. Therefore, $N = log_{16}672 = 2.3$, so we need 3 hex digits.
Then $M - N = 10 - 3 = 7$.

6. A **Triangle ABC has vertices at $(8, 8), (6, 4)$ and $(10, 7)$. Find the sum of the lengths of the three altitudes of this triangle, rounded to the nearest tenth.**
Side AB of triangle has direction vector $v_{AB} = \langle 8 - 6, 8 - 4\rangle = \langle 2, 4\rangle$. That is normal vector $n_{AB} = \langle 4, -2\rangle$. Equation of the line is therefore $4x - 2y - 16 = 0$. Distance from C is $d_{AB-C} = \frac{|4 \cdot 10 - 2 \cdot 7 - 16|}{\sqrt{4^2 + (-2)^2}} = \sqrt{5}$. After doing the same for sides AC and BC, we get

$v_{AC-B} = 2\sqrt{5}$ and $v_{BC-A} = 2$.
Therefore sum of altitudes is $\sqrt{5} + 2\sqrt{5} + 2 \approx 8.7$.

7. D **The polynomials $2x^3 + x^2 + cx + d$ is divisible by $x + 1$. If d and c are integers with $d + c = 29$, find the sum of the two non-real roots of this polynomial.**
   $(2x^3 + x^2 + cx + d) \div (x + 1) = 2x^2 - x + (c + 1) + \frac{d-c-1}{x+1}$. Since polynomial is divisible by $x + 1$, $d - c - 1 = 0$. Since $d - c - 1 = 0$ and $d + c = 29$, $c = 14$ and $d = 15$.
   We need to find sum of two non-real roots of $2x^2 - x + 15$. Since the complex roots are complex conjugates, we can ignore complex part of the roots and find just sum of real parts of the roots. Therefore $x_1 + x_2 = 2(\frac{1}{4}) = \frac{1}{2}$.

8. D **Let N be the smallest integer greater than 2 such that N^{N-1} is not the square of an integer. Find the product of all rational numbers that could be roots of $5x^4 + bx^3 + cx^2 + dx + N$, where b, c, and d can be any integers. Round your answer to the nearest hundredth.**
   Since 3^2, $4^3 = 2^6$, 5^4 are all squares of integers and 6^5 is not square, the smallest $N = 6$.
   Therefore, we have $5x^4 + bx^3 + cx^2 + dx + 6$.
   According to Rational Root Theorem, the rational roots of the polynomial are
   $$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{6}{5}, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{6}{5}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{6}{1}, -\frac{1}{1}, -\frac{2}{1}, -\frac{3}{1}, -\frac{6}{1}$$
   Their product is $\frac{1}{5^2} \cdot \frac{2^2}{5^2} \cdot \frac{3^2}{5^2} \cdot \frac{6^2}{5^2} \cdot 2^2 \cdot 3^2 \cdot 6^2 = 4.3998 \approx 4.3$.

9. E **Three people (X, Y, Z) are in a room with you. One is a knight (knights always tell the truth), one is knave (knaves always lie), and the other is a spy (spies may either lie or tell the truth). X says "I am a spy." Y says "X is telling the truth." Z says "I am not a spy." Which of the following correctly identifies all three people?**
   Since X said "I am a spy.", X is either spy or knave. Since Z said "I am not a spy.", Z is either knight or spy.
   We have two options for X:
   - X is spy. Since Y said "X is telling the truth." and X was indeed telling the truth, Y has to be knight. Then Z has to be knave since there is no other role left. That is not possible because Z tells the truth. This option is not valid
   - X is knave. Since Y said "X is telling the truth." and Y was lying, Y is spy. Then Z is knigth. This option is valid.

10. A **Morse code involves transmitting dots $*$ and dashes $-$. An agent attempted to send a five-character code five different times, but only one of the five transmissions was correct. However, it was known that each erroneous transmission has a different number of errors than the others, and no transmission had five errors. The first transmission was $* - - * -$, which was**

not correct. The other four transmissions are $*---*$, $--**-$, $*-*-*$, $*****$. Which one is correct?

We know that first transmission was not correct. Then we need to discuss 4 cases. For each case, we will assume one transmission was correct and count number of errors between it and every other transmission.

1st	*	-	-	*	-	2	2	3	3
A	*	-	-	-	*	C	4	1	3
B	-	-	*	*	-	4	C	3	3
C	*	-	*	-	*	1	3	C	2
D	*	*	*	*	*	3	3	2	C

Figure 2: Transmissions

As we can see from the table, transmission A is correct because it is the only case which has different number of errors between it and every other transmission.

11. D **A checker is placed on a 5x5 checkerboard as pictured. The checker may be moved one square at a time but only to the left or down. Also, the checker may not move to any of the three black squares. In how many difference ways can the checkerr be moved to the lower left corner of the board?**

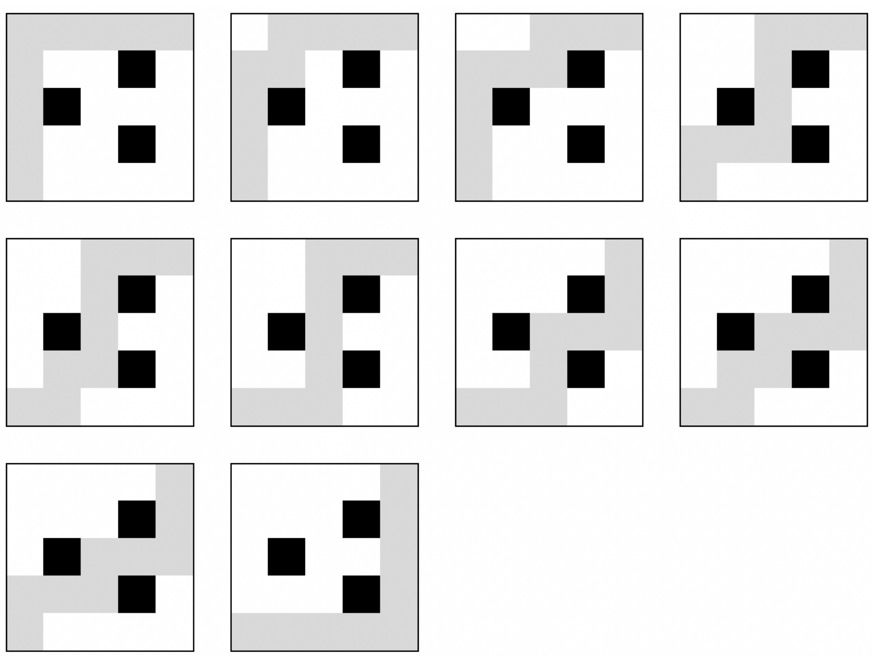

Figure 3: All possible moves for checker.

There are 10 difference ways.

12. C **The function $P(t) = \cos 8t$ can be written as sums and differences of powers of $\cos t$ only. When $P(t)$ is written this way, what is the coefficient of $(\cos t)^3$?**

    Note that $\cos 2t = 2\cos^2 t - 1$. Then

    $$\cos 8t = 2\cos^2 4t - 1 = 2(2\cos^2 2t - 1)^2 - 1 = 2(2(2\cos^2 t - 1)^2 - 1)^2 - 1$$

    We can either simplify it to

    $$128\cos^8 t - 256\cos^6 t + 160\cos^4 t - 32\cos^2 t + 1$$

    or notice that there can't be any odd powers of $\cos t$ after we simplify it.
    Therefore coefficient of $(\cos t)^3$ is 0.

13. B **Find the length of the shortest line segment with one endpoint on the line passing through $(7, 8)$ that is parallel to the vector $\langle 3, -4\rangle$ and the other endpoint on the cicle with equation $(x+3)^2+(y-2)^2 = 3$. Round your answer to nearest tenth.**
    The equation of line passing through $(7, 8)$ that is parallel to the vector $\langle 3, -4\rangle$ is $4x + 3y - 52 = 0$.
    Note that the shortest line segment will be parallel to line $4x + 3y - 52 = 0$ and pass through $(-3, 2)$, center of the circle.
    Distance between line and center of cicle can be computed by

    $$d = \frac{\left|4 * (-3) + 3 \cdot 2 - 52\right|}{\sqrt{4^2 + (-3)^2}} = \frac{58}{5}$$

    Note that radius of circle is $\sqrt{3}$. Therefore length of the shortest line segment with one point on circle and one point on line is $\frac{58}{5} - \sqrt{3} \approx 9.9$.

14. E **Let a and b be positive integers with $a^2 + b^2 = 2019^2$. Find $a + b$.**
    Let

    $$a^2 + b^2 = 2019^2$$

    $$a + b = s$$

    Substitute $a = s - b$ into $a^2 + b^2 = 2019^2$ to get equation

    $$2b^2 - 2sb + (s^2 - 2019^2) = 0$$

    Then the solutions are

    $$b_{1,2} = \frac{-(-2s) \pm \sqrt{(-2s)^2 - 4 \cdot 2(s^2 - 2019^2)}}{2 \cdot 2a}$$

Since b is positive integer, $\sqrt{(-2s)^2 - 4 \cdot 2(s^2 - 2019^2)}$ must be an integer. We can plug in all the options $(2319, 2540, 2711, 2719, 2811)$ into $\sqrt{2 \cdot 2019^2 - s^2}$. The only s that produced an integer is 2811.

15. B **Which describes the graph (in $\mathbb{R}$) of all solutions of this system?**

$$\begin{cases} 2x - 6y - 8z = 15 \\ -8x - 8y + 6z = -65 \\ x - 19y - 17z = 5 \end{cases}$$

$$\begin{bmatrix} 2 & -6 & -8 & 15 \\ -8 & -8 & 6 & -65 \\ 1 & -19 & -17 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & -\frac{25}{16} & \frac{255}{32} \\ 0 & 1 & \frac{13}{16} & \frac{5}{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the solution is $\begin{cases} x = \frac{255}{32} + \frac{25}{16}t \\ y = \frac{5}{32} - \frac{13}{16}t \\ z = t \end{cases}$

Note that this is parametric equation of one line in 3D space.

16. D **The graph of $f(x) = ax^2 + bx + c$ is symmetric about the y-axis and its x and y intercepts form an equilateral triangle. If the maximum value of $f(x)$ is 4, find $a + b + c$.**
Since $f(x)$ is symmetric about y-axis, $b = 0$. Since maximum of $f(x)$ is 4, we know that $f(x) = -ax^2 + 4$ where a is positive.

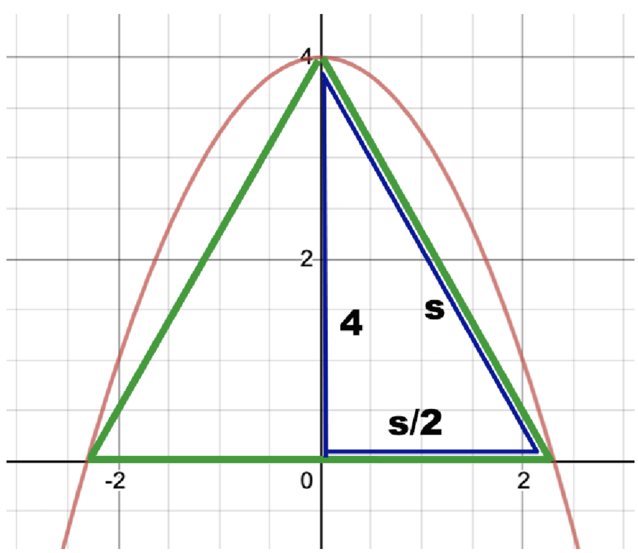

Figure 4: Graph of $f(x)$

Because intersections with axis form equilateral triangle, we know that height of triangle is 4. Then according to Pythagorian Theorem $4^2 + (\frac{s}{2})^2 = s^2$. That is $s = \frac{8}{\sqrt{3}}$.
Therefore intersection with x-axis are $-\frac{4}{\sqrt{3}}$ and $\frac{4}{\sqrt{3}}$. Then from $f(x) = -ax^2 + 4$, we get $0 = -a(\frac{4}{\sqrt{3}})^2 + 4$. That is $a = \frac{3}{4}$.
Our equation is $f(x) = -\frac{3}{4}x^2 + 4$ with coefficients $a = -\frac{3}{4}$, $b = 0$ and $c = 4$. Therefore $a + b + c = -\frac{3}{4} + 0 + 4 = \frac{13}{4}$.

17. C **How many of the following are both a continuous function on $\mathbb{R}$ and also one-to-one?** $g(x) = \ln e^x$, $h(x) = x|x|$, $k(x) = x^2$, $m(x) = \frac{1}{x+1}$, $n(x) = \frac{x}{x^2+1}$, $p(x) = \sin x$, $q(x) = \arctan x$, $r(x) = \frac{x}{|x|+1}$.

- $g(x) = \ln e^x = x$
  continuous and one-to-one
- $h(x) = x|x|$
  continuous and one-to-one
- $k(x) = x^2$
  continuous and not one-to-one
- $m(x) = \frac{1}{x+1}$
  not continuous and one-to-one
- $n(x) = \frac{x}{x^2+1}$
  continuous and not one-to-one
- $p(x) = \sin x$
  continuous and not one-to-one
- $q(x) = \arctan x$
  continuous and one-to-one
- $r(x) = \frac{x}{|x|+1}$
  continuous and one-to-one

Four functions are continuous and one-to-one.

18. E **Let $\{a_n\}$ be an arithmetic sequence with initial value m and common difference d. Let $\{g_n\}$ be a geometric sequence with initial value k and common ratio 2. The sum of the first 100 terms of $\{a_n\}$ and the sum of the first 10 terms of $\{g_n\}$ are equal. If m, d, and k are all positive integers, which of the following numbers $(2, 5, 17, 31, 33)$ must divide m?**

$$m + (m + d) + ... + (m + 99) = k + 2k + ... + 2^9k$$

$$100m + 4950d = 1023k$$

$$(2^2 \cdot 5^2)m + (2 \cdot 3^2 \cdot 5^2 \cdot 11)d = (3 \cdot 11 \cdot 31)k$$

$$(2^2 \cdot 5^2)m + (33 \cdot 2 \cdot 3 \cdot 5^2)d = (33 \cdot 31)k$$

Since terms $(33 \cdot 2 \cdot 3 \cdot 5^2)d$ and $(33 \cdot 31)k$ are both divisible by 33, $(2^2 \cdot 5^2)m$ must be divisible by 33. Therefore m must be divisible by 33.

19. B **Some hikers set out on a hike at noon. At some point, they turn around and follow the same path back to where they began, and arrive there at 8:00 p.m. Their speed is 4 mi/hr on level ground, 3 mi/hr uphill and 6 mi/hr downhill. How many miles did they hike?**
Note that the distance they go uphill there will be equal to the distance they go dowhill back. Therefore, $3t_1 = 6t_4$. That is $t_4 = \frac{t_1}{2}$. Similarly, $t_3 = \frac{t_2}{2}$. The time they go there and time they go back will be equal.

	mi/h	time travelled there	time travelled back
ground	4	t_G	t_G
uphill	3	t_1	t_2
downhill	6	$t_3 = \frac{t_2}{2}$	$t_4 = \frac{t_1}{2}$

Note that trip took 8 hours. Therefore $2t_G + \frac{3}{2}t_1 + \frac{3}{2}t_2 = 8$. That is $2t_G = 8 - \frac{3}{2}t_1 - \frac{3}{2}t_2$. We need to find how many miles thay hiked that is $miles = 4(2t_G) + 3(t_1 + t_2) + 6(\frac{t_1}{2} + \frac{t_2}{2}) = 8t_G + 6t_1 + 6t_2 = 4(8 - \frac{3}{2}t_1 - \frac{3}{2}t_2) + 6t_1 + 6t_2 = 32$. Therefore, they hiked 32 miles.

20. C **In the game of craps, a player (known as the shooter) rolls two fair six-sided dice. The shooter immediately loses if the sum of the dice is 2,3, or 12 and immidiately wins if the sum of the dice is 7 or 11 on the first roll. If the sum is anything else (4,5,6,8,9,or 10), that number becomes the point and the shooter rolls again. The shooter now wins by rolling that same point again and loses by rolling a 7. If any other number is rolled, the shooter rolls again and keeps rolling until the shooter wins by rolling the point or loses by rolling a 7. Find the probability that shooter wins.**

sum on dices	probability
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

During first round:

- probability of winning: $\frac{6}{36} + \frac{2}{36} = \frac{8}{36}$

During if game continues:

- probability of winning when point is 4 or 10
  Probability that sum 4 occurs is $\frac{3}{36}$. Probability that sum 7 occurs is $\frac{6}{36}$. Therefore probability that game will continue is $1 - \frac{3}{36} - \frac{6}{36} = \frac{27}{36}$. The same is true for sum 10.

$$2\cdot\left(\left(\frac{3}{36}\right)^2+\frac{27}{36}\left(\frac{3}{36}\right)^2+\left(\frac{27}{36}\right)^2\left(\frac{3}{36}\right)^2+...\right) = 2\cdot\left(\frac{3}{36}\right)^2\sum_{n=0}^{+\infty}(\frac{27}{36}\Big)^n = 2\cdot\left(\frac{3}{36}\right)^2\cdot\frac{1}{1-\frac{27}{36}} = \frac{1}{18}$$

- probability of winning when point is 5 or 9
  Probability that sum 5 occurs is $\frac{4}{36}$. Probability that sum 7 occurs is $\frac{6}{36}$. Therefore probability that game will continue is $1 - \frac{4}{36} - \frac{6}{36} = \frac{26}{36}$. The same is true for sum 9.

$$2\cdot\left(\left(\frac{4}{36}\right)^2+\frac{26}{36}\left(\frac{4}{36}\right)^2+\left(\frac{26}{36}\right)^2\left(\frac{4}{36}\right)^2+...\right) = 2\cdot\left(\frac{4}{36}\right)^2\cdot\frac{1}{1-\frac{26}{36}} = \frac{4}{45}$$

- probability of winning when point is 6 or 8
  Probability that sum 6 occurs is $\frac{5}{36}$. Probability that sum 7 occurs is $\frac{6}{36}$. Therefore probability that game will continue is $1 - \frac{5}{36} - \frac{6}{36} = \frac{25}{36}$. The same is true for sum 8.

$$2\cdot\left(\left(\frac{5}{36}\right)^2+\frac{25}{36}\left(\frac{5}{36}\right)^2+\left(\frac{25}{36}\right)^2\left(\frac{5}{36}\right)^2+...\right) = 2\cdot\left(\frac{5}{36}\right)^2\cdot\frac{1}{1-\frac{25}{36}} = \frac{25}{198}$$

Therefore probability of winning is

$$\frac{8}{36}+\frac{1}{18}+\frac{4}{45}+\frac{25}{198} = \frac{244}{495}$$