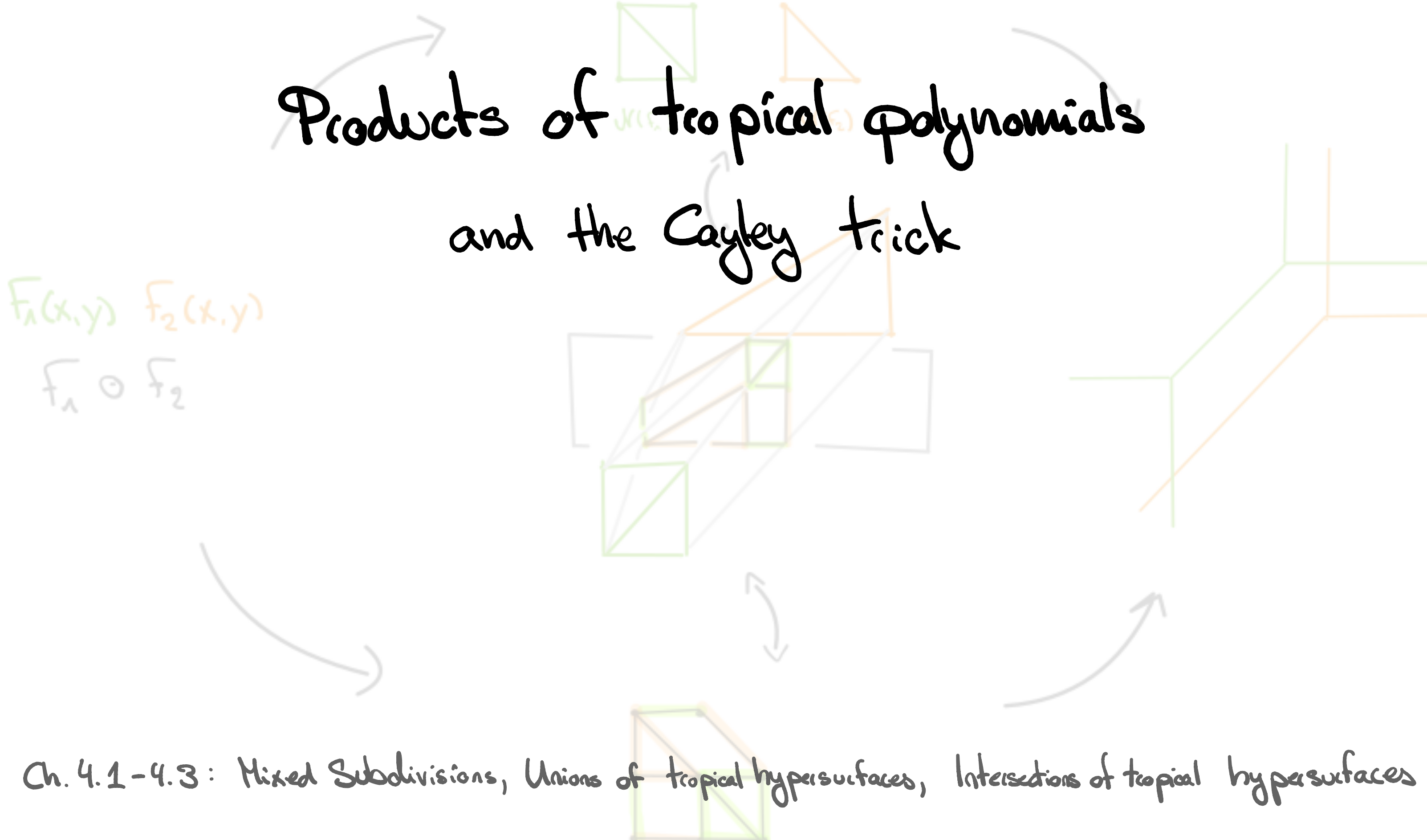
Products of tropical polynomials

and the Cayley trick



Ch. 4.1 – 4.3: Mixed Subdivisions, Unions of tropical hypersurfaces, Intersections of tropical hypersurfaces

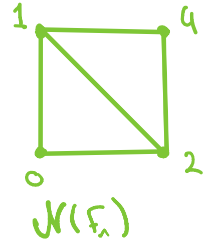
A first example

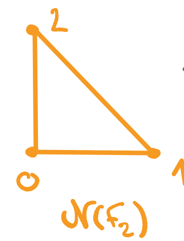
$$F_1(x,y) = 4 \odot x \odot y \oplus 2 \odot x + 1 \odot y \oplus 0$$

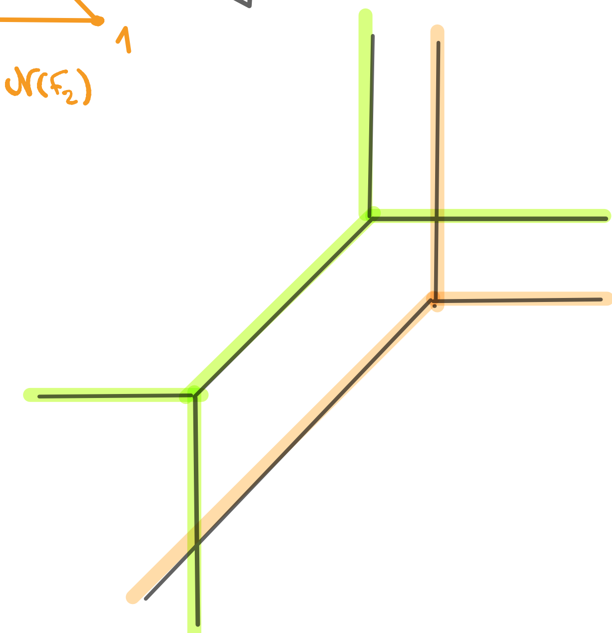
$$F_2(x,y) = 1 \odot x \oplus 2 \odot y \oplus 0$$

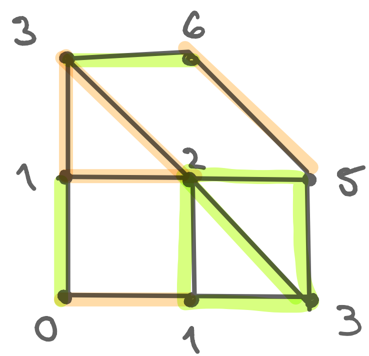
$$F_1(x,y) \odot F_2(x,y)$$

$$\begin{aligned} = \; & 5 \odot x^{\odot 2} \odot y \oplus 6 \odot x \odot y^{\odot 2} \oplus \cancel{4 \odot x \odot y} \\ \oplus \; & 3 \odot x^{\odot 2} \oplus \cancel{4 \odot x \odot y} \oplus \cancel{2 \odot x} \\ \oplus \; & 2 \odot x \odot y \oplus 3 \odot y^{\odot 2} \oplus 1 \odot y \\ \oplus \; & 1 \odot x \oplus \cancel{2 \odot y} \oplus 0 \end{aligned}$$



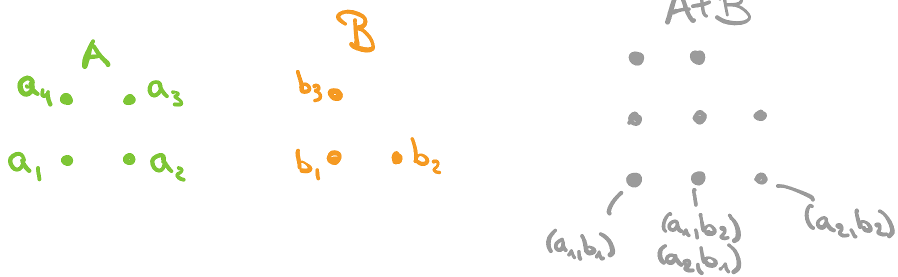




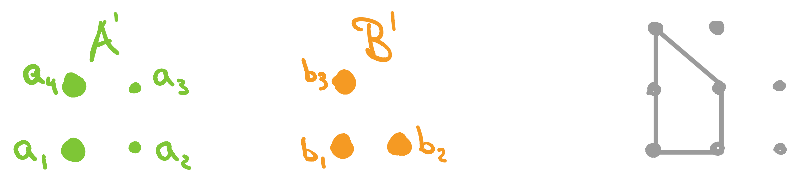


Mixed Subdivisions

Let $A, B \subseteq \mathbb{R}^d$ be finite. The **set of labels** of $x \in A+B$ is $\{(a,b) \mid a \in A, b \in B, x = a+b\}$



Let $A' \subseteq A$, $B' \subseteq B$. The **mixed cell** generated by A', B' is $M(A', B') = \mathrm{conv}\{a+b \mid a \in A', b \in B'\}$



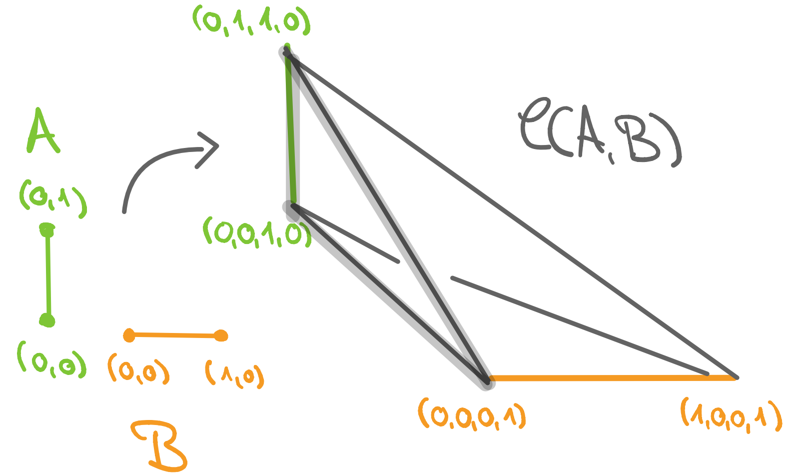
A **mixed subdivision** of A+B is a subdivision in which every cell is a mixed cell.

The Cayley Polytope

The **Cayley embedding** of $A, B \subseteq \mathbb{R}^d$ is

$$\mathcal{C}(A,B) = \{(a,e_1) \mid a \in A\} \cup \{(b,e_2) \mid b \in B\} \subseteq \mathbb{R}^d \times \mathbb{R}^2$$

Let $A' \subseteq A$, $B' \subseteq B$. The **Cayley cell** generated by A' and B' is $\mathrm{conv}(\mathcal{C}(A',B'))$



E.g. $A' = A$, $B' = \{(0,0)\}$

Th. 4.3 (**Cayley Trick**): Polyhedral subdivisions of $\mathcal{C}(A,B)$ are in bijection with mixed subdivisions of $A+B$.

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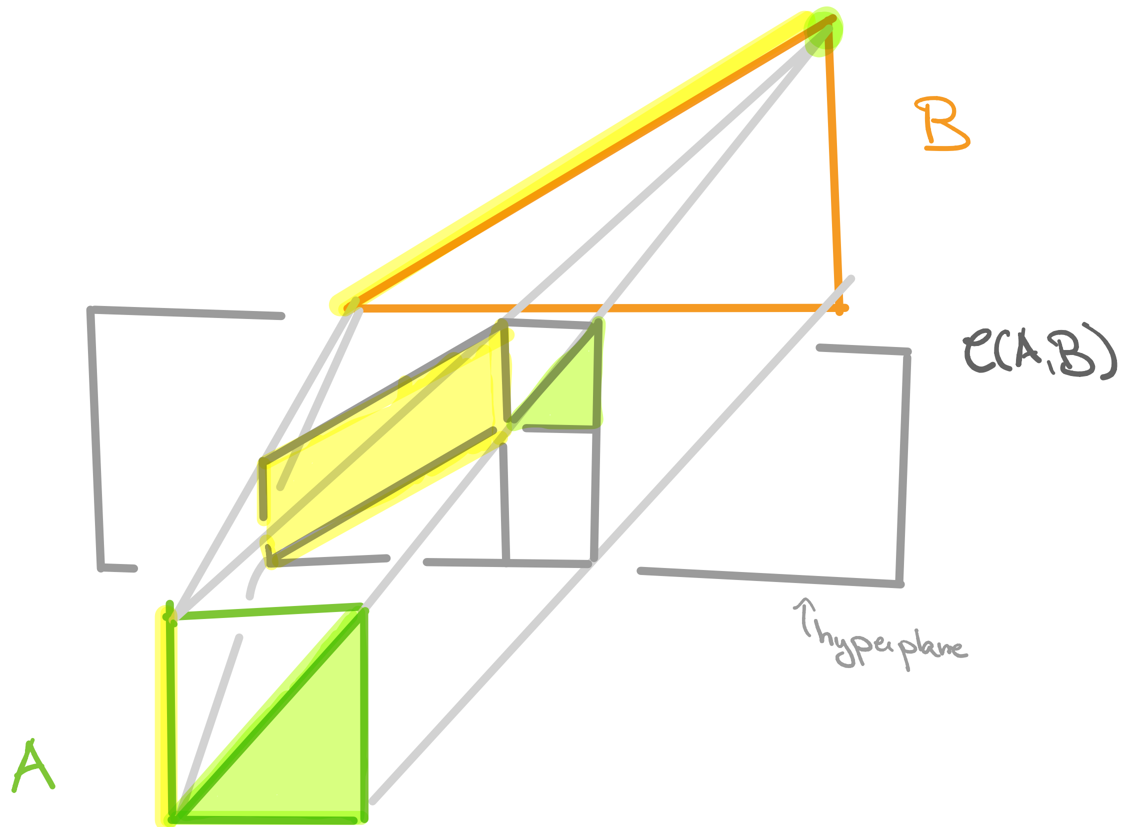


The Cayley Trick

Let Σ_i be a polyhedral subdivision of $\mathcal{C}(A,B)$. Then

$$M(\Sigma_i) = \{ M(A',B') \mid \operatorname{conv}(\mathcal{C}(A',B')) \in \Sigma_i \}$$

mixed cell

cayley cell

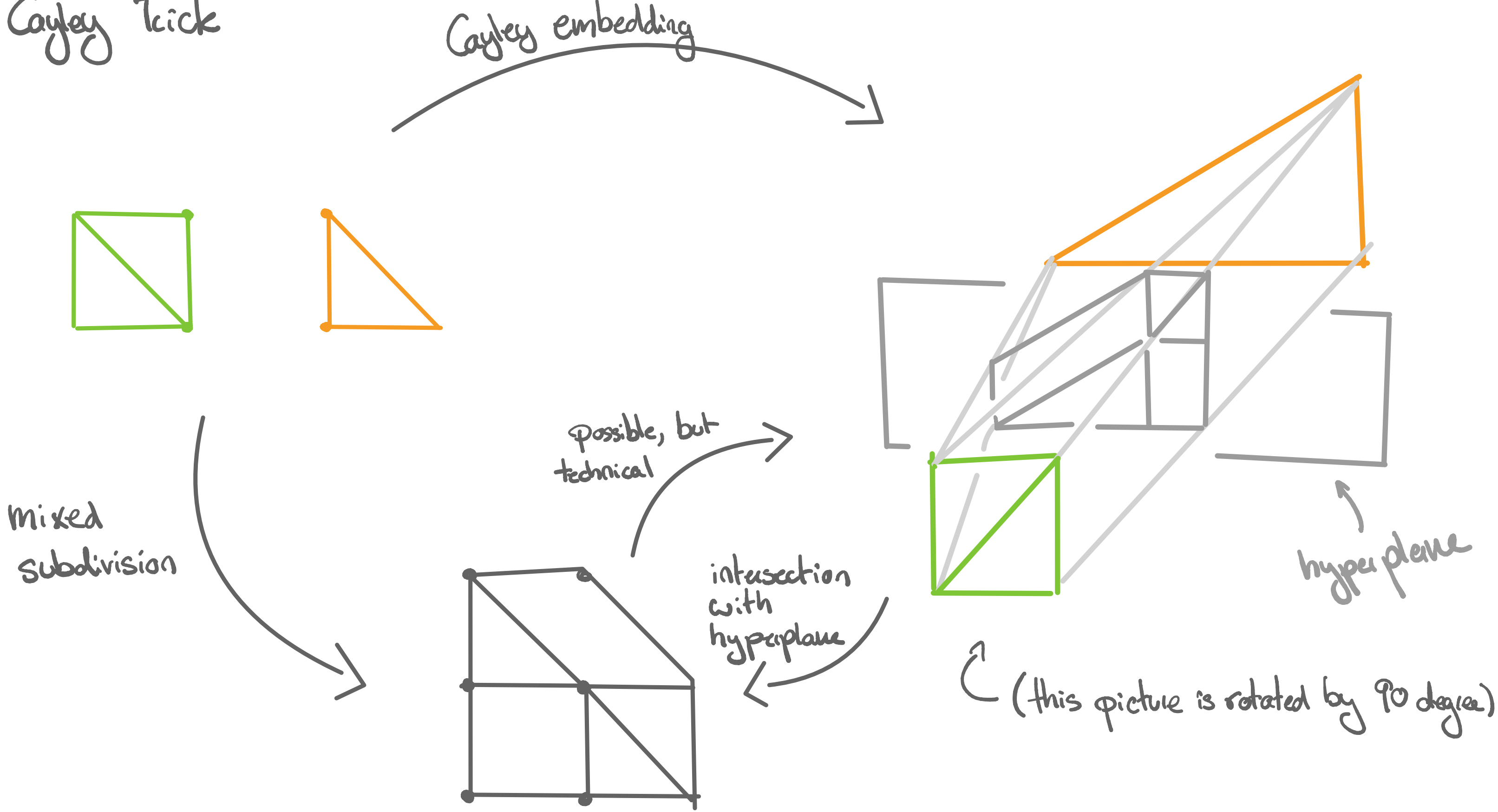
is a subdivision of $\frac{1}{2}(A+B)$, called the **mixed subdivision induced by** Σ_i.

we ignore this factor.

A **fine mixed subdivision** is a mixed subdivision of $A+B$ that corresponds to a triangulation of $\mathcal{C}(A,B)$.

A **coherent subdivision** is a mixed subdivision of $A+B$ that corresponds to a regular subdivision of $\mathcal{C}(A,B)$

The Cayley Trick



Unions of tropical hypersurfaces

Let $\mathbb{K} = K\{\{t\}\}$ field of Puiseux series (eg $\mathbb{K} = \mathbb{C}\{t\}$ or $\mathbb{K} = \mathbb{Q}\{\{t\}\}$)

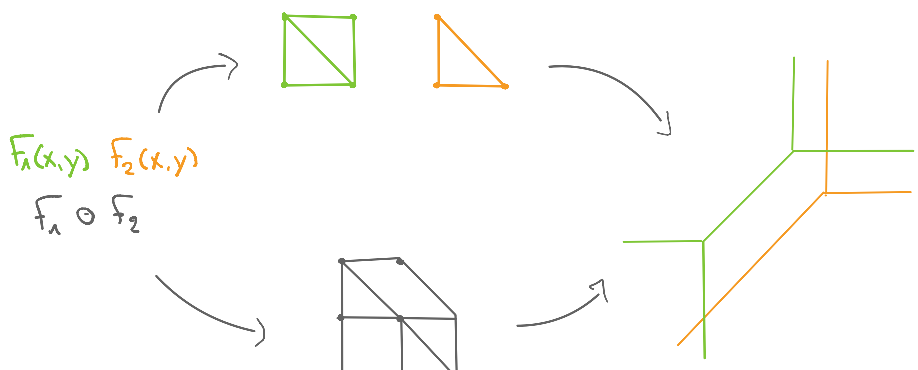
$\Rightarrow \mathbb{K}[x_1^{\pm 1},\dots,x^{\pm d}]$ does not contain zero-divisors

Let $f,g \in \mathbb{K}[x_1^{\pm 1},\dots,x^{\pm d}]$, $F = \mathrm{trop}(f)$, $G = \mathrm{trop}(g)$.

$\mathbb{K}$ has no zero-divisors $\Rightarrow$ $\big((f\cdot g)(x) = 0 \Leftrightarrow f(x) = 0 \lor g(x) = 0\big)$

$\Rightarrow$ $V(f\cdot g) = V(f) \cup V(g)$ variety in $\mathbb{K}$

$\Rightarrow$ $\mathcal{T}(F \odot G) = \mathcal{T}(\mathrm{trop}(f\cdot g)) = \mathcal{T}(F) \cup \mathcal{T}(G)$ tropical hypersurface



The regular subdivision $\mathcal{S}(F\odot G)$ of $\mathcal{N}(F\odot G)$

Let $F = \bigoplus_u a_u \odot x^{\odot u}$, $G = \bigoplus_v b_v \odot x^{\odot v}$.

Newton polytope $\mathcal{N}(F\odot G) = \mathcal{N}(F) + \mathcal{N}(G)$

Consider the lifting function on the point configuration $\mathcal{C}(\mathrm{supp}(F), \mathrm{supp}(G)) \subseteq \mathbb{R}^d \times \mathbb{R}^2$

$$\lambda_{(F,G)} = \begin{cases} (u, e_1) \longmapsto a_u & \text{for } u \in \mathrm{supp}(F) \\ (v, e_2) \longmapsto b_v & \text{for } v \in \mathrm{supp}(G) \end{cases}$$

Cor. 4.10: Let Σ be the regular subdivision of $\mathcal{C}(\mathrm{supp}(F), \mathrm{supp}(G))$ induced by $\lambda_{(F,G)}$.

The corresponding (coherent) mixed subdivision $M(\Sigma)$ of $\mathcal{N}(F)+\mathcal{N}(G)$ coincides with the regular subdivision $\mathcal{S}(F\odot G)$ of $\mathcal{N}(F\odot G)$

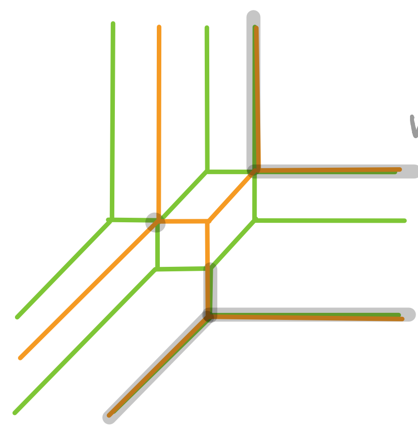
induced by the coefficients of $F\odot G$

Intersections of tropical hypersurfaces

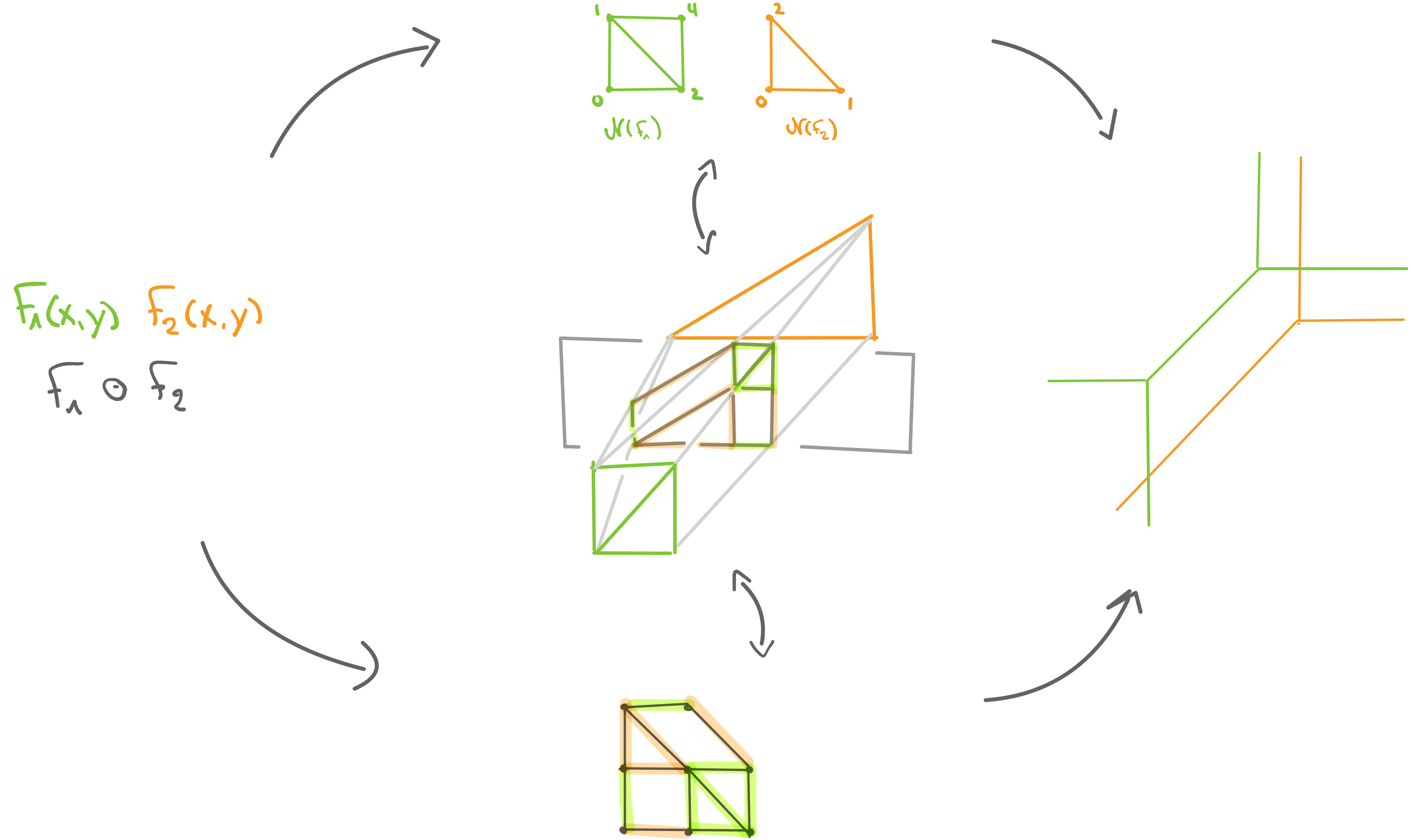
A **tropical prevariety** is the intersection of finitely many tropical hypersurfaces.
A **tropical variety** is the tropicalization of a (algebraic) variety.

Classical algebraic geometry: "A variety is the intersection of hypersurfaces and vice versa"

Tropical geometry: "A tropical prevariety is the intersection of trop. hypersurfaces and vice versa"
"A tropical variety is the intersection of trop. hypersurfaces"



intersections of tropical hypersurfaces can look weird


$F_1(x,y)$ $F_2(x,y)$
$F_1 \odot F_2$
$\mathcal{N}(F_1)$
$\mathcal{N}(F_2)$
1 4 0 2
2 0 1