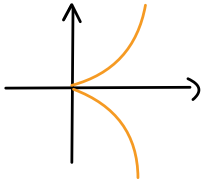
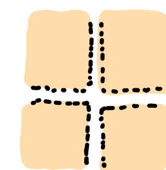
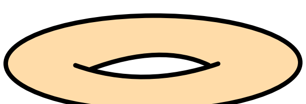
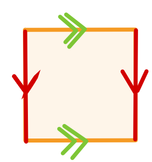


WHAT IS A TORUS?







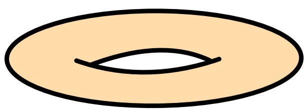
WHAT IS A TORUS?

I The three lives of a torus

II Basic notions in toric geometry

THE THREE LIVES OF A TORUS

Topology



"We show that xy is homeomorphic to a torus"

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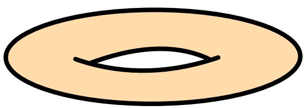

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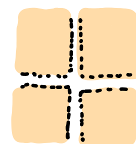
Algebraic Geometry



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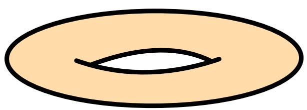

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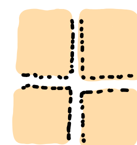

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Lie groups

?

A torus in topology

Definition: The n-dimensional torus T^n is the product of n circles, i.e.

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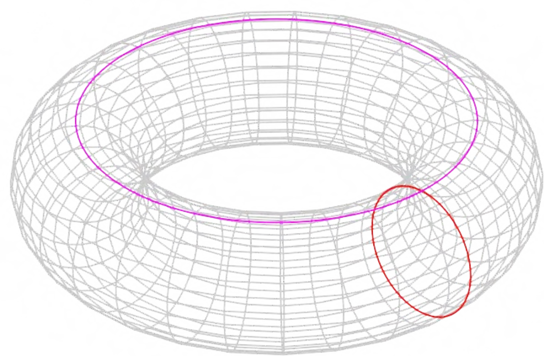
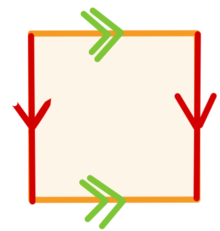
$n=2$:



Parametrization: $$T^2 = \left\{ \begin{pmatrix} \cos\theta \cos\varphi \\ \cos\theta \sin\varphi \\ \sin\theta \end{pmatrix} \in \mathbb{R}^3 : \varphi, \theta \in [0, 2\pi] \right\}$$

A torus in topology

T^2 can be obtained by identifying opposite edges of a square:



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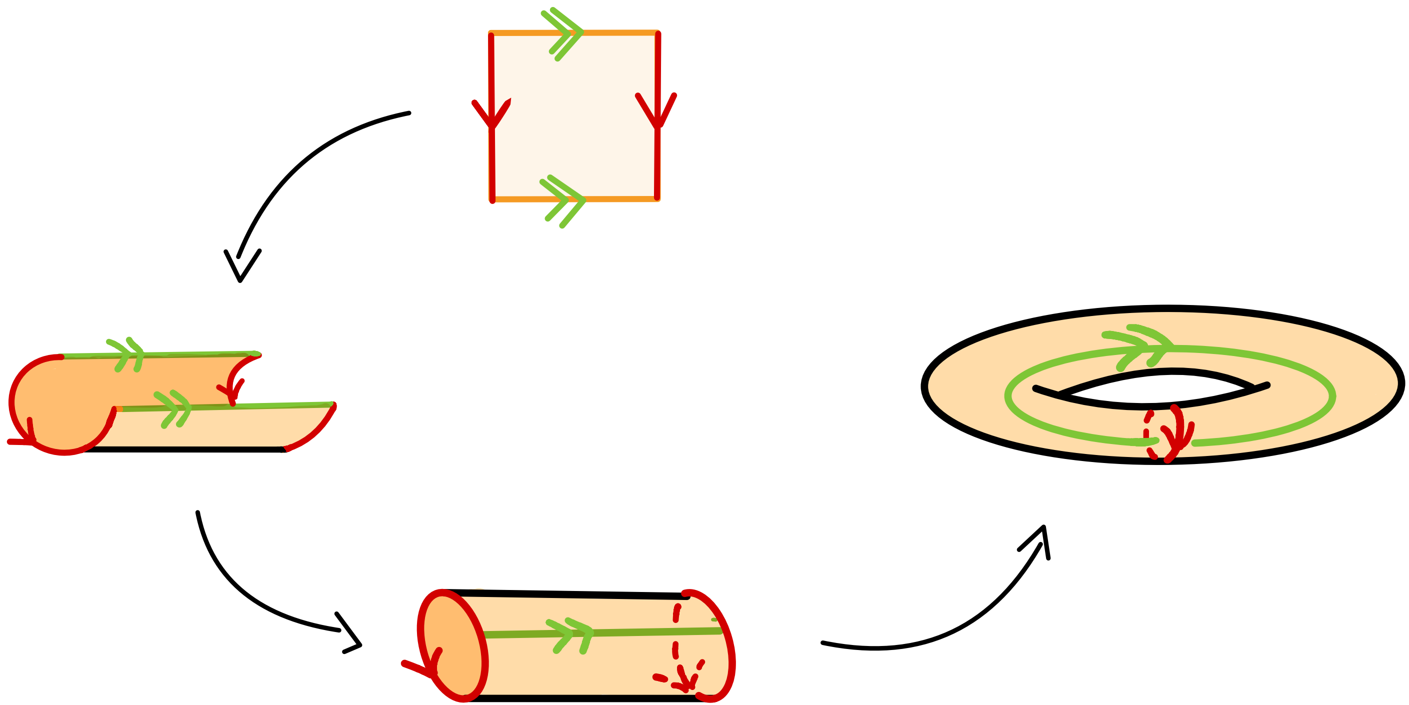

A torus in algebraic geometry

Real life : $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a multiplicative group.

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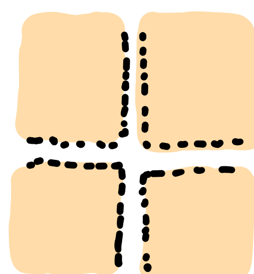
$(\mathbb{C}^*)^n$ is a group under componentwise multiplication.

"The n-dimensional torus" $\rightsquigarrow$ $(\mathbb{C}^*)^n$, together with this group structure (or to an object isomorphic to $(\mathbb{C}^*)^n$).

↳ outside the coordinate hyperplanes



$(\mathbb{C}^*)^2$

A torus in algebraic geometry

Let's get formal!

Let K be an algebraically closed field (e.g. $\mathbb{C}$).

Def.: An affine variety over K is the zero-locus in K^n of some finite family of polynomials in n variables and coefficients in K that generate a prime ideal.

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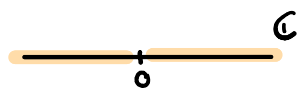
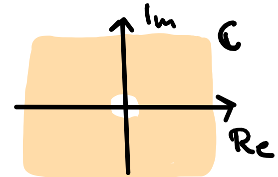
Example: $f: \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = xy - 1$. zero-locus: $\left\{ \binom{x}{y} \in \mathbb{C}^2 \mid xy - 1 = 0 \right\}$

This is isomorphic to $\{ x \in \mathbb{C} : x = \frac{1}{y} \ \exists y \in \mathbb{C} \} = \mathbb{C} \setminus \{0\} = \mathbb{C}^*$





$$(\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$$

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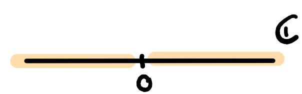

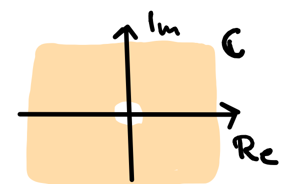

Def: Let K be an algebraically closed field. An algebraic torus T^n is an affine variety isomorphic to $(K^*)^n$, where T^n inherits a group structure from this isomorphism.

The deeper connection...

> **Definition:** A Lie group is a group that is also a finite-dimensional smooth manifold
> s.t. the group operations multiplication and inverses are smooth maps.
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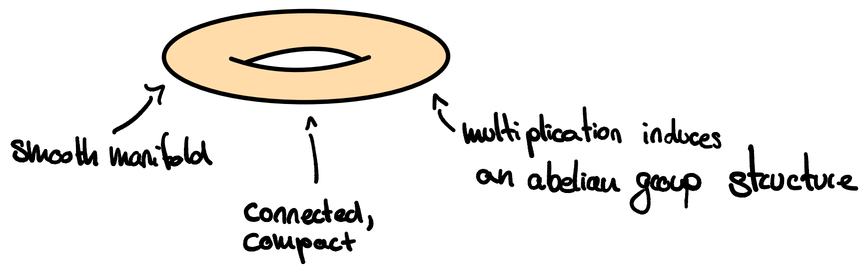

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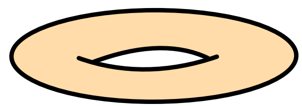
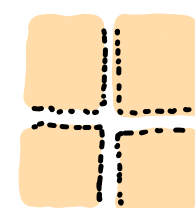


algebraic linear groups

Definition: An algebraic group is a group that is also an algebraic variety, s.t. the group operations multiplication and inverses are given by regular maps on the variety.

An algebraic torus is an abelian, affine algebraic group.



II (SOME) BASIC NOTIONS IN TORIC GEOMETRY

Definition: An affine toric variety V is an irreducible affine variety containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski-open dense subset s.t. the action of T on itself extends to an action on V.

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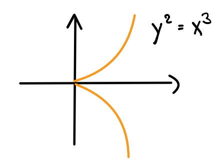
Definition: A variety is irreducible if it is not the union of two smaller varieties.

Definition: Let V be an affine variety. A set S is Zariski-closed in V if S is an affine variety contained in V.
A set U is Zariski-open if $U = V \setminus S$ for some affine variety $S \subseteq V$.

Definition: An affine toric variety V is an irreducible affine variety containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski-open dense subset s.t. the action of T on itself extends to an action on V.

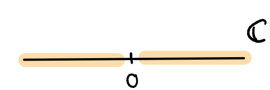
Example: $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \mid x^3 = y^2 \right\} = \left\{ \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \in \mathbb{C}^2 \mid t \in \mathbb{C} \right\}$

- affine variety : zero-locus of $f = x^3 - y^2$.



- irreducible
- torus : $T = \left\{ \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \in \mathbb{C}^2 \mid t \in \mathbb{C}^* \right\} \simeq \mathbb{C}^*$

$$\begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \leftarrow\!\shortmid\; t$$



- Zariski-open: $T = V \setminus \{0\}$, $\{0\}$ is an affine variety
- dense : The smallest (Zariski-closed) affine variety containing T is V.

Definition: An affine toric variety V is an irreducible affine variety containing a torus $T \cong (\mathbb{C}^*)^n$ as a Zariski-open dense subset s.t. the action of T on itself extends to an action on V.

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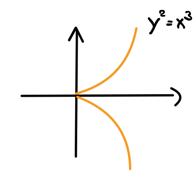


$$T = \left\{ \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \in \mathbb{C}^2 \mid t \in \mathbb{C}^* \right\} \cong \mathbb{C}^*$$

$$\begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \mapsfrom t$$

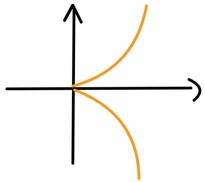
$$\mathbb{C}^* \times \mathbb{C}^* \to T \times T \longrightarrow T \longrightarrow \mathbb{C}^*$$

$$(a, b) \longmapsto a \cdot b$$

$$(a, b) \mapsto \left(\begin{pmatrix} a^2 \\ a^3 \end{pmatrix}, \begin{pmatrix} b^2 \\ b^3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a^2 \cdot b^2 \\ a^3 \cdot b^3 \end{pmatrix} \longmapsto a \cdot b$$

Extend to $V = T \cup \{0\}$: $\quad T \times V \longrightarrow V$

$$(a, 0) \longmapsto 0$$



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Thank you