

# INTERSECTION BODIES OF POLYTOPES

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joint work with Katalin Berlow, Chiara Meroni and Isabelle Shankar

18 November 2021

## RADIAL FUNCTIONS AND STAR BODIES

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Let  $P$  be a polytope. Then the *intersection body*  $IP$  of  $P$  is given by the radial function (restricted to the sphere)

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### Theorem 2

The degree of the irreducible components of the algebraic boundary of *IP* is bounded by

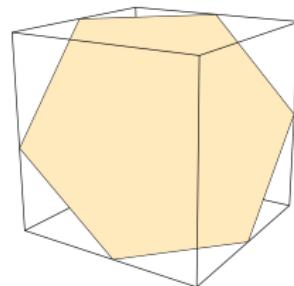
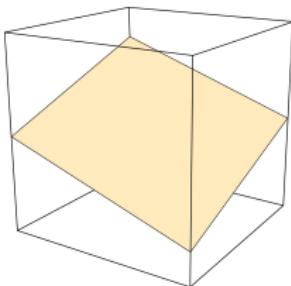
number of edges of  $P - (\dim(P) - 1)$ .

**Part 1:  $IP$  is semialgebraic\*.**

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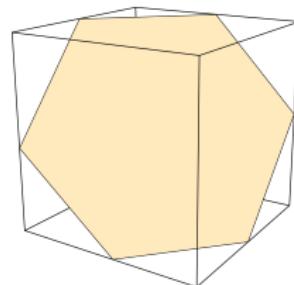
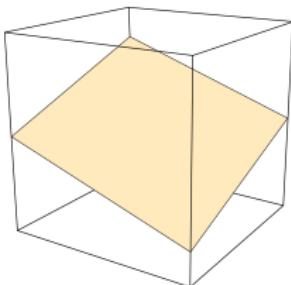
\* i.e. a subset of  $\mathbb{R}^d$  defined by a boolean combination of polynomial inequalities.

## EXAMPLE: THE 3-CUBE



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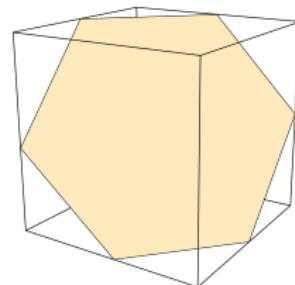
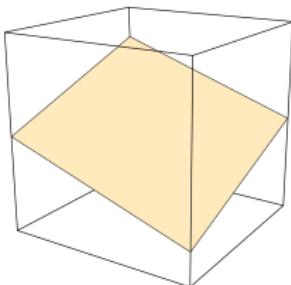
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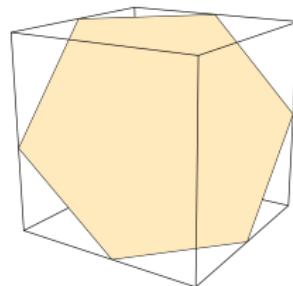
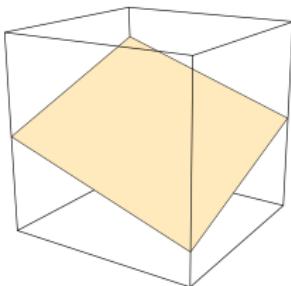


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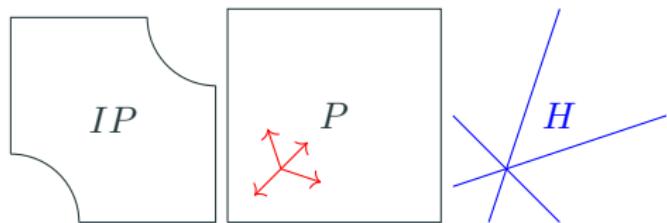
**First question:**

For which  $x \in \mathbb{R}^3$  intersects  $x^\perp$  a **fixed set of edges** of  $P$ ?

# HYPERPLANE ARRANGEMENT $H$

$$H = \{v^\perp \mid v \text{ is a vertex of } P \text{ and } v \neq 0\}.$$

$C$  max chamber of  $H \Rightarrow \forall x \in C : x^\perp$  intersects  $P$  in fixed set of edges

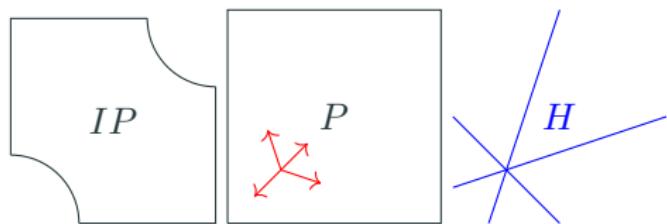


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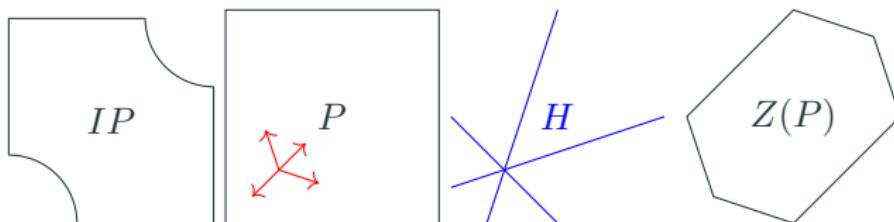
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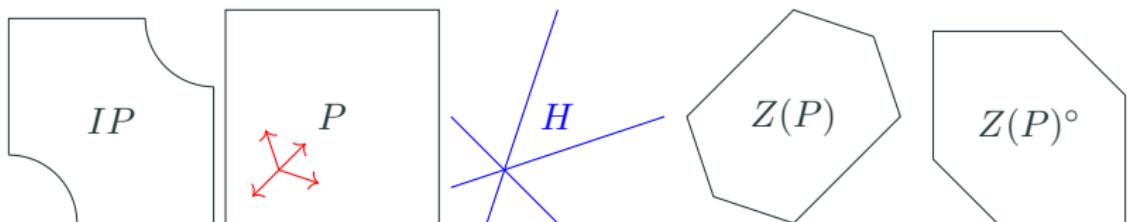
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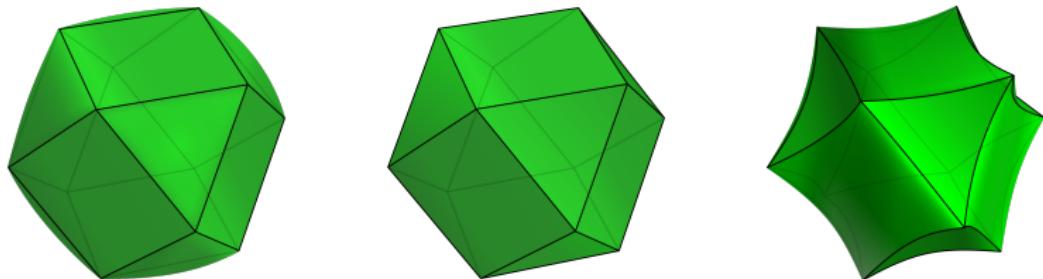
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$\longleftrightarrow$  vertices of  $Z(P)$

$\longleftrightarrow$  facets of  $Z(P)^\circ$



## $Z(P)$ CAN HAVE MANY $P$ s!



**left:**  $IP_1$  for  $P_1 = [-1, 1]^3$

**right:**  $IP_2$  for  $P_2 = \text{conv} \left( \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right)$

**center:**  $Z(P_1)^\circ = Z(P_2)^\circ$

⇒ The zonotope  $Z(P)$  does not determine the polytope  $P$  or the intersection body  $IP$ !

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$$Q = \bigcup_{\text{facets of } Q} \text{conv}(\Delta, 0)$$

is a triangulation of  $Q$ .

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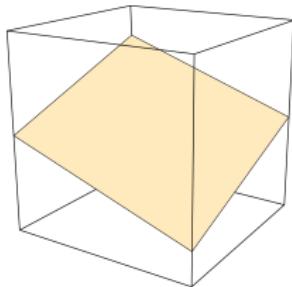
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$$\begin{aligned} IP \cap C &= \{x \in C | \rho(x) \geq 1\} = \{x \in C | \frac{p(x)}{\|x\|^2 q(x)} \geq 1\} \\ &= \{x \in C | \|x\|^2 q(x) - p(x) \leq 0\}. \end{aligned}$$

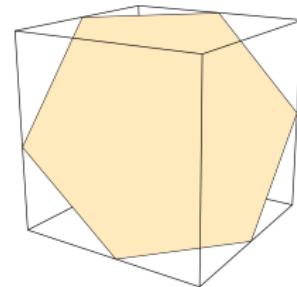
$\Rightarrow IP$  is semialgebraic.

□

EXAMPLE: 3-CUBE |  $\rho(x)|_C = \frac{p(x)}{\|x\|^2 q(x)}$

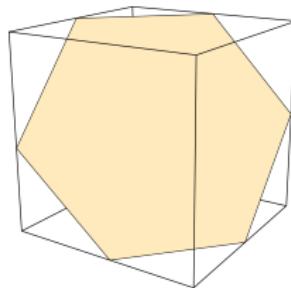
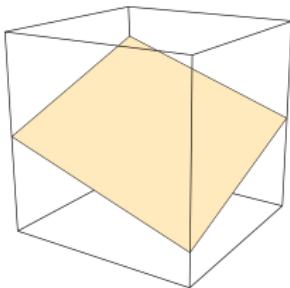


$$C_1 = \text{cone}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right)$$



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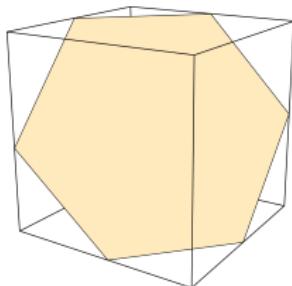
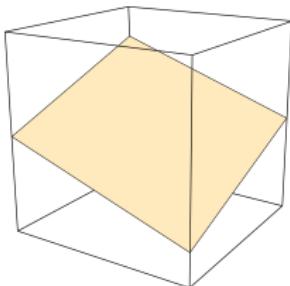


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$$\rho(x, y, z)|_{C_1} = \frac{4(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)3z} = \frac{4}{3z}$$

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$$\begin{aligned} \rho(x, y, z)|_{C_2} &= \frac{-(x^2 - 2xy + y^2 - 2xz - 2yz + z^2)(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)3xyz} \\ &= \frac{-(x^2 - 2xy + y^2 - 2xz - 2yz + z^2)}{3xyz} \end{aligned}$$

## Part 2: The algebraic boundary

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Prop.:  $p$  is divisible by  $\|x\|^2$ .

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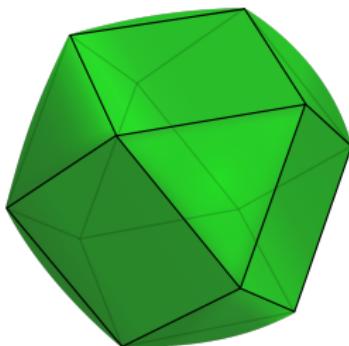
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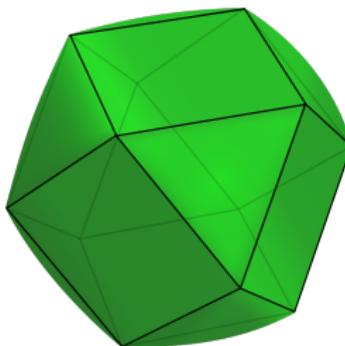
What are the degrees of the irreducible components?

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$$q_1 - \frac{p_1}{\|x\|^2} = 3z - \frac{4(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)} = 3z - 4$$

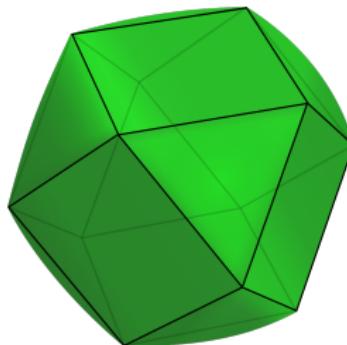
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$\Rightarrow$  If  $P = [-1, 1]^3$  then the possible degrees are 1 and 3.

What happens if we translate  $P$ ?

## DEGREE BOUND

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$$P_2 = \text{conv} \left( \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right)$$

highest degree of irreducible component = 4

$$\text{number of edges of } P_2 - (\dim(P_2) - 1)) = 6 - (3 - 1) = 4$$

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$$\deg \left( q(x) - \frac{p(x)}{\|x\|^2} \right) = \frac{1}{2} (\# \text{ vertices of a hexagon}) = 3$$

## CASE STUDY: $d$ -DIMENSIONAL CENTERED CUBE $[-1, 1]^d$

### Proposition

Let  $P = [-1, 1]^d$ . Then the number of irreducible components of  $IP$  of degree 1 is at least  $2d$ .

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Thank you!