Multivariate Volume, Ehrhart and h^*-Polynomials of Polytropes

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joint work with Leon Zhang (UC Berkeley)

23 July 2020

Goal:

Compute (multivariate versions of) volume, Ehrhart and h^*-polynomials of all polytropes of dimension at most 4.

Overview:

- Background in tropical geometry, tropical polytopes and polytropes
- How to compute multivariate volume polynomials
- How to compute multivariate Ehrhart polynomials
- How to compute multivariate h^*-polynomials
- Understanding the coefficients of the volume polynomials

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Example: $$1 \odot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \min(1+0,5) \\ \min(1+1,2) \\ \min(1+2,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Definition

Let $V = \{v_1, \ldots, v_r\} \subseteq \mathbb{R}^n$. The tropical convex hull of V is

$$\operatorname{tconv}(V) = \{a_1 \odot \mathbf{v}_1 \oplus \cdots \oplus a_r \odot \mathbf{v}_r \mid a_1, \ldots, a_r \in \mathbb{R}\}$$

$$x \in \operatorname{tconv}(V) \implies \lambda \odot x = x + \lambda(1, \ldots, 1)^T \in \operatorname{tconv}(V).$$

Identify $\operatorname{tconv}(V)$ with its image in the tropical projective torus

$$\mathbb{TP}^{n-1} = \mathbb{R}^n / \mathbb{R} \odot (1, \ldots, 1)^T.$$

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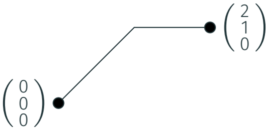

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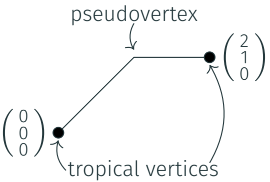

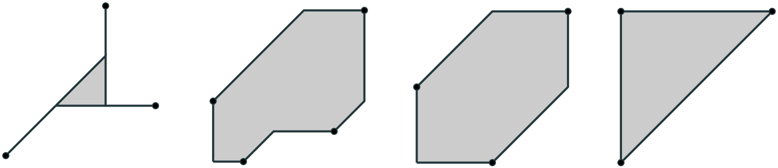

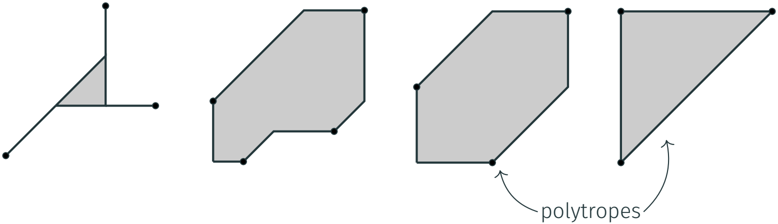

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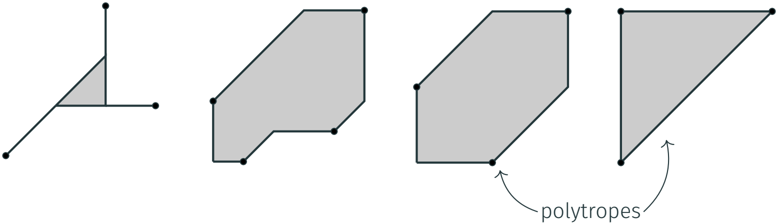
Polytropes are tropical polytropes that are clasically convex.



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Question

Given the tropical vertices of a tropical polytope P, how can we tell if P is a polytrope?

↳ Kleene stars!

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$(n \times n)$-matrix with
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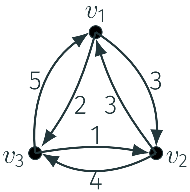

weighted complete digraph K_n

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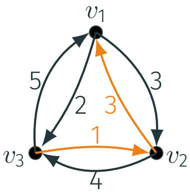

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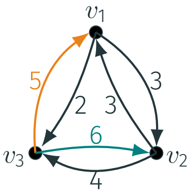

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Kleene Stars and Shortest Paths

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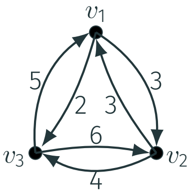

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Definition

A matrix $\mathbf{c} \in \mathbb{R}^{n \times n}$ is a Kleene star if $\mathbf{c} = \mathbf{c}^*$. The polytrope region is the set $\mathcal{P}ol_n = \{\mathbf{c} \in \mathbb{R}^{n \times n} \mid \mathbf{c} \text{ is a Kleene star}\}$.

Proposition (de la Puente '13)

Let $P \subseteq \mathbb{TP}^{n-1}$ be a non-empty set. The following are equivalent:

(1) P is a polytrope.

(2) There is a Kleene star $\mathbf{c} \in \mathcal{P}ol_n$ such that $P = \mathrm{tconv}(\mathbf{c})$.

(3) There is a Kleene star $\mathbf{c} \in \mathcal{P}ol_n$ such that
$P = \{y \in \mathbb{R}^n \mid y_i - y_j \leq c_{ij}, y_n = 0\}$.

Furthermore, the $\mathbf{c}$'s in the last two statements are equal, and are uniquely determined by P.

Maximal Polytropes

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A polytrope is called maximal if it has $\binom{2n-2}{n-1}$ vertices as an ordinary polytope.

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Question

Given two Kleene stars, how can we tell if they define polytropes of the same type?

Kleene Stars and Gröbner Fans

$$\mathcal{P}ol_n = \{\mathbf{c} \in \mathbb{R}^{n\times n} \mid \mathbf{c} \text{ is a Kleene star}\}$$

Gröbner fan $\mathcal{GF}_n$ of the ideal $I = \langle x_{ij}x_{ji} - 1, x_{ij}x_{jk} - x_{ik}\rangle$

$\to$ Subfan $\mathcal{GF}_n|_{\mathcal{P}ol_n}$

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Theorem (Tran, '17)

$$\text{Cones of } \mathcal{GF}_n|_{\mathcal{P}ol_n} \quad \overset{bij.}{\longleftrightarrow} \quad \text{types of polytropes in } \mathbb{TP}^{n-1}$$

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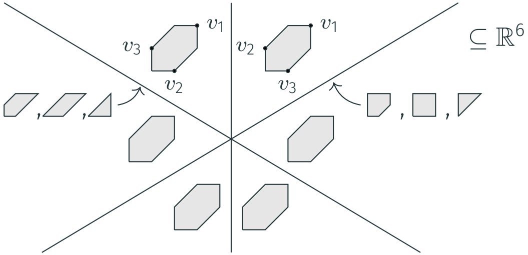

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We want to compute a polynomial

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Theorem

Let X be the smooth toric variety defined by the normal fan Σ of a maximal polytrope $P_{\mathbf{c}}$ and D_{ij} the divisors corresponding to the rays of Σ. Then $[D_{ij}] \in H^2(X, \mathbb{Q})$ and

$$p(a_{12}, a_{13}, \dots, a_{n(n-1)}) = \int_X \left[\sum a_{ij} D_{ij}\right]^{\dim P}$$

is a polynomial such that $p(\mathbf{c}) = \mathrm{Vol}(P_{\mathbf{c}})$.

There is an algorithm (de Loera-Sturmfels '03) that only depends on the choice of the cone.

$$
\begin{aligned}
Vol(\mathbf{a}) = & - a_{12}^2 - a_{13}^2 - a_{21}^2 - a_{23}^2 - a_{31}^2 - a_{32}^2 + 2a_{12}a_{13} \\
& + 2a_{13}a_{23} + 2a_{21}a_{23} + 2a_{21}a_{31} + 2a_{12}a_{32} + 2a_{31}a_{32}
\end{aligned}
$$

$$
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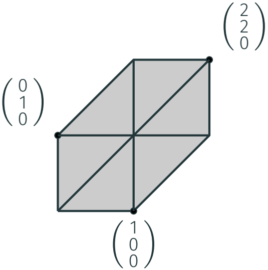

$$
Vol(\mathbf{c}) = 6
$$

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- Use a differential operator - the *Todd Operator*

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Theorem (Khovanskiĭ-Pukhlikov, 1992)

Let $P \subseteq \mathbb{R}^n$ be a unimodular, lattice d-polytope. Then:

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For $\mathbf{h} \in \mathbb{R}^m$, the shifted polytope $P_{\mathbf{h}}$ is defined as

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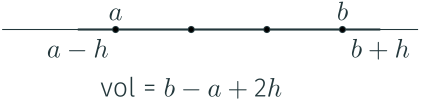
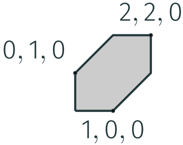


$$\begin{aligned} &\mathrm{Todd}(b-a+2h)|_{h=0} \\ &\quad = [1 + \tfrac{1}{2}\tfrac{d}{dh}](b-a+2h)|_{h=0} \\ &\quad = b-a+2h+1|_{h=0} \\ &\quad = b-a+1 \end{aligned}$$



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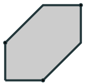
volume in terms of facet heights



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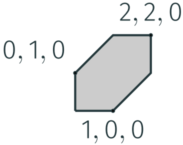

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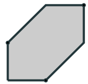
$\rightarrow$ for small $\mathbf{h}$, $\mathrm{vol}(P_{\mathbf{h}}) = \mathrm{vol}(\mathbf{a}+\mathbf{h})$



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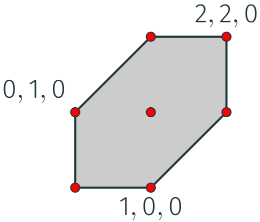
$\rightarrow$volume in terms of facet heights

$\rightarrow$ for small $\mathbf{h}$, $\mathrm{vol}(P_{\mathbf{h}}) = \mathrm{vol}(\mathbf{a}+\mathbf{h})$

$$\begin{aligned}
&\mathrm{Todd}_{\mathbf{h}}\,\mathrm{vol}(\mathbf{a}+\mathbf{h})\\
&= \Big[1 + \frac{1}{2}\frac{\partial}{\partial h_{12}} + \frac{1}{6}\Big(\frac{\partial}{\partial h_{12}}\Big)^2\Big] \cdots \Big[1 + \frac{1}{2}\frac{\partial}{\partial h_{32}} + \frac{1}{6}\Big(\frac{\partial}{\partial h_{32}}\Big)^2\Big]\,\mathrm{vol}(\mathbf{a}+\mathbf{h})\Big|_{\mathbf{h}=0}\\
&= \Big[1 + \frac{1}{2}\frac{\partial}{\partial a_{12}} + \frac{1}{6}\Big(\frac{\partial}{\partial a_{12}}\Big)^2\Big] \cdots \Big[1 + \frac{1}{2}\frac{\partial}{\partial a_{32}} + \frac{1}{6}\Big(\frac{\partial}{\partial a_{32}}\Big)^2\Big]\,\mathrm{vol}(\mathbf{a})\\
&= vol(\mathbf{a}) + \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32}) + 1
\end{aligned}$$

$$
\begin{aligned}
ehr(\mathbf{a}) = &-\frac{1}{2}a_{12}^2 - \frac{1}{2}a_{13}^2 - \frac{1}{2}a_{21}^2 - \frac{1}{2}a_{23}^2 - \frac{1}{2}a_{31}^2 - \frac{1}{2}a_{32}^2 \\
&+ a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32} \\
&+ \frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{23} + a_{31} + a_{32}) + 1
\end{aligned}
$$



$$ehr(1,2,1,2,0,0) = 7$$

$$ehr(t,2t,t,2t,0,0) = 3t^2 + 3t + 1$$

From Ehrhart to h^* polynomials

Ehrhart series of a d-dimensional lattice polytope P:

$$Ehr_P(t) = \sum_{k \geq 0} ehr_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}}$$

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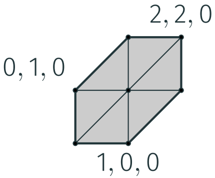
$$\Rightarrow \quad h_P^*(t) = \sum_{j=0}^{d} c_j A_j(t) (1-t)^{d-j}$$

From Ehrhart to h^* : Example

$h^*(t) = \sum_{j=0}^{d} c_j A_j(t)(1-t)^{d-j}$



From Ehrhart to h^* : Example

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$+ (a_{12}a_{13} + a_{13}a_{23} + a_{21}a_{23} + a_{21}a_{31} + a_{12}a_{32} + a_{31}a_{32})t^2$

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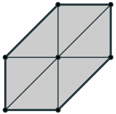
$\rightarrow$ collect terms of like degree



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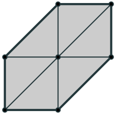

$\rightarrow$ collect terms of like degree

$\rightarrow$ apply formula to find h^*

$$h^*(\mathbf{a}, t) = \left(\sum_{i \neq j \in [3]} -\frac{1}{2}[a_{ij}^2 + a_{ij}] + \sum_{i \neq j \neq k \in [3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] + 1 \right) t^2$$

$$+ \left(\sum_{i \neq j \in [3]} \frac{1}{2}[a_{ij} - a_{ij}^2] + \sum_{i \neq j \neq k \in [3]} [a_{ij}a_{ik} + a_{ji}a_{ki}] - 2 \right) t + 1$$

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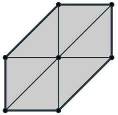

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$h^*(1, 2, 1, 2, 0, 0, t) = t^2 + 4t + 1$

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Multivariate volume, Ehrhart, and h^* polynomials for all polytropes up to dimension 4.

$\longrightarrow$ fast computation time.

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$$
\begin{aligned}
&2a_{12}^3 - 3a_{12}^2a_{13} + a_{13}^3 - 3a_{12}^2a_{14} + 6a_{12}a_{13}a_{14} - 3a_{13}^2a_{14} + a_{21}^3 - 3a_{13}^2a_{23} \\
&+ 6a_{13}a_{14}a_{23} - 3a_{14}^2a_{23} - 3a_{14}a_{23}^2 - 3a_{21}a_{23}^2 + a_{23}^3 - 3a_{21}^2a_{24} + 6a_{14}a_{23}a_{24} \\
&+ 6a_{21}a_{23}a_{24} - 3a_{14}a_{24}^2 - 3a_{23}a_{24}^2 + a_{24}^3 - 3a_{21}^2a_{31} + 6a_{21}a_{24}a_{31} - 3a_{24}^2a_{31} \\
&- 3a_{24}a_{31}^2 + a_{31}^3 - 3a_{12}^2a_{32} + 6a_{12}a_{14}a_{32} - 3a_{14}^2a_{32} - 3a_{31}^2a_{32} - 3a_{14}a_{32}^2 \\
&+ 6a_{14}a_{24}a_{34} + 6a_{24}a_{31}a_{34} + 6a_{14}a_{32}a_{34} + 6a_{31}a_{32}a_{34} - 3a_{14}a_{34}^2 - 3a_{24}a_{34}^2 \\
&- 3a_{31}a_{34}^2 - 3a_{32}a_{34}^2 + 2a_{34}^3 + 6a_{21}a_{31}a_{41} - 3a_{31}^2a_{41} + 6a_{31}a_{32}a_{41} - 3a_{32}^2a_{41} \\
&- 3a_{21}a_{41}^2 - 3a_{32}a_{41}^2 + a_{41}^3 - 3a_{12}^2a_{42} + 6a_{12}a_{13}a_{42} - 3a_{13}^2a_{42} + 6a_{12}a_{32}a_{42} \\
&+ 6a_{32}a_{41}a_{42} - 3a_{13}a_{42}^2 - 3a_{32}a_{42}^2 - 3a_{41}a_{42}^2 + a_{42}^3 - 3a_{21}^2a_{43} + 6a_{13}a_{23}a_{43} \\
&+ 6a_{21}a_{23}a_{43} - 3a_{23}^2a_{43} + 6a_{21}a_{41}a_{43} - 3a_{41}^2a_{43} + 6a_{13}a_{42}a_{43} + 6a_{41}a_{42}a_{43} \\
&- 3a_{13}a_{43}^2 - 3a_{21}a_{43}^2 - 3a_{42}a_{43}^2 + a_{43}^3
\end{aligned}
$$

dimension	2	3	4	5
# comb. types of max. polytropes	1	6	27248	?

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Theorem: Joswig-Schröter ’19

Maximal n-polytropes $\Leftrightarrow$ regular central triangulations of the fundamental polytope FP_{n+1}.

fundamental polytope:

$$FP_n = \operatorname{conv}\{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j \in [n]\}.$$

A regular central triangulation of FP_4 is determined by a triangulation of each of the six square facets.

Understanding the Coeffiecients: dimension 3

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The coefficients of the six 3-dimensional volume polynomials correspond to the six triangulations of the fundamental polytope.

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$e_1 - e_2$ neighbors $e_3 - e_2$
and $e_3 - e_2$ is adjacent to a triangulating edge in the square
$\Rightarrow$ coefficient of $a_{12}^2 a_{32}$ is -3.

$$
\begin{aligned}
&2a_{12}^3 - 3a_{12}^2a_{13} + a_{13}^3 - 3a_{12}^2a_{14} + 6a_{12}a_{13}a_{14} - 3a_{13}^2a_{14} + a_{21}^3 - 3a_{13}^2a_{23} \\
&+ 6a_{13}a_{14}a_{23} - 3a_{14}^2a_{23} - 3a_{14}a_{23}^2 - 3a_{21}a_{23}^2 + a_{23}^3 - 3a_{21}^2a_{24} + 6a_{14}a_{23}a_{24} \\
&+ 6a_{21}a_{23}a_{24} - 3a_{14}a_{24}^2 - 3a_{23}a_{24}^2 + a_{24}^3 - 3a_{21}^2a_{31} + 6a_{21}a_{24}a_{31} - 3a_{24}^2a_{31} \\
&- 3a_{24}a_{31}^2 + a_{31}^3 - 3a_{12}^2a_{32} + 6a_{12}a_{14}a_{32} - 3a_{14}^2a_{32} - 3a_{31}^2a_{32} - 3a_{14}a_{32}^2 \\
&+ 6a_{14}a_{24}a_{34} + 6a_{24}a_{31}a_{34} + 6a_{14}a_{32}a_{34} + 6a_{31}a_{32}a_{34} - 3a_{14}a_{34}^2 - 3a_{24}a_{34}^2 \\
&- 3a_{31}a_{34}^2 - 3a_{32}a_{34}^2 + 2a_{34}^3 + 6a_{21}a_{31}a_{41} - 3a_{31}^2a_{41} + 6a_{31}a_{32}a_{41} - 3a_{32}^2a_{41} \\
&- 3a_{21}a_{41}^2 - 3a_{32}a_{41}^2 + a_{41}^3 - 3a_{12}^2a_{42} + 6a_{12}a_{13}a_{42} - 3a_{13}^2a_{42} + 6a_{12}a_{32}a_{42} \\
&+ 6a_{32}a_{41}a_{42} - 3a_{13}a_{42}^2 - 3a_{32}a_{42}^2 - 3a_{41}a_{42}^2 + a_{42}^3 - 3a_{21}^2a_{43} + 6a_{13}a_{23}a_{43} \\
&+ 6a_{21}a_{23}a_{43} - 3a_{23}^2a_{43} + 6a_{21}a_{41}a_{43} - 3a_{41}^2a_{43} + 6a_{13}a_{42}a_{43} + 6a_{41}a_{42}a_{43} \\
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Partition	Example monomial	Possible coefficients	Coefficient sum
4	a_{12}^4	$-6, -3, -2, -1, 0, 1, 2, 3$	-20
3 + 1	$a_{12}^3 a_{13}$	$-4, 0, 4, 8$	320
2 + 2	$a_{12}^2 a_{13}^2$	$0, 6$	300
2+1+1	$a_{12} a_{13} a_{14}^2$	$-12, 0, 12$	-2160
1+1+1+1	$a_{12} a_{13} a_{14} a_{15}$	$0, 24$	1680

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Thank You!