Data compression by randomized algorithms

SSPM presentation

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Singular Value Decomposition

Definition :

$A \in \mathcal{M}_{m,n}(\mathbb{C})$, its singular value decomposition is :

$$A = U\Sigma V^*$$

- $U \in \mathcal{M}_m(\mathbb{C})$ *(Left singular vectors)*
- $V \in \mathcal{M}_n(\mathbb{C})$ *(Right singular vectors)*
- $\Sigma \in \mathcal{M}_{m,n}(\mathbb{C})$, rectangular diagonal matrix.

Remark :

Result on SVD :

$$\min_{rank(X) \leq j} \|A - X\| = \sigma_{j+1}$$

RSVD

Find a k-dimensional subspace E that captures most of the action of A.

Stage 1 : Range Finder

Given $A \in \mathbb{R}^{m \times n}$, target rank k and ϵ error tolerance.

$$Q \in \mathbb{R}^{m \times k} \qquad \|A - QQ^* A\| \leq \epsilon$$

Construction with result of the SVD

$$\|A - QQ^* A\| \approx \min_{rank(X) \leq j} \|A - X\|$$

Columns of Q are the k dominant left singular vectors of A

Algorithm Randomized range finder

Require: A, k *numerical range*, p *oversampling parameter*

1: Draw a random test matrix $\Omega \in \mathbb{R}^{n\times(k+p)}$
2: Compute $Y = A\Omega$
3: Compute a QR-factorization : $(Q, \_) = QR(Y)$.
4: **return** Q

Product $Y = A\Omega$

Draw random vectors $\{\omega_i \;\; i = 1 \dots k\}$

$$y^{(i)} = A\omega^{(i)} \;\; i = 1 \dots k$$

Stage 2 :

Using $Q \in \mathbb{R}^{m \times k}$ to compute the SVD on a small matrix B.

Algorithm RSVD

Require: A, the orthonormal matrix Q

Construct $B = Q^* A \in \mathbb{R}^{k \times n}$

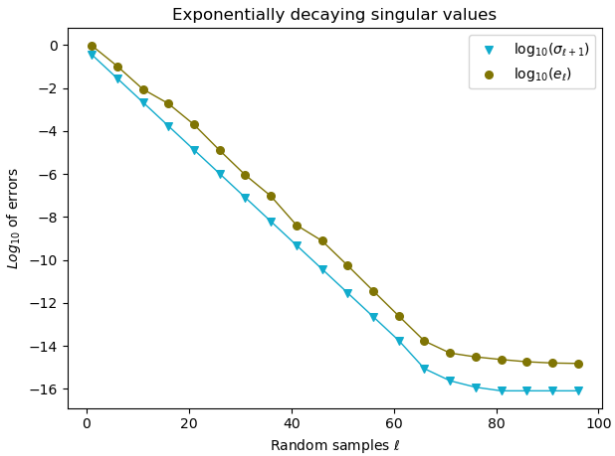
Compute the SVD on B : $B = \tilde{U} \Sigma V^*$

Set $U = Q\tilde{U}$

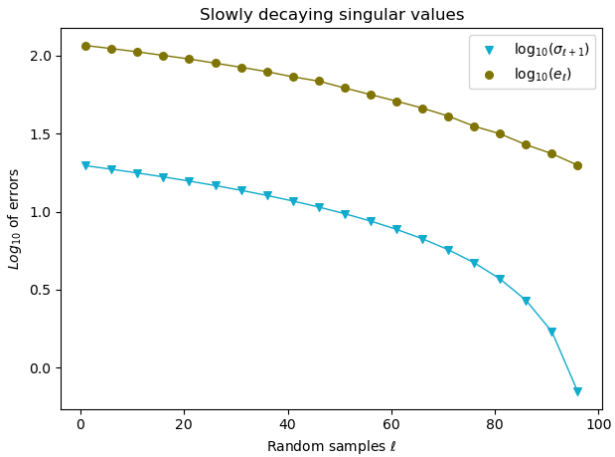
return U, Σ, V^*

Test

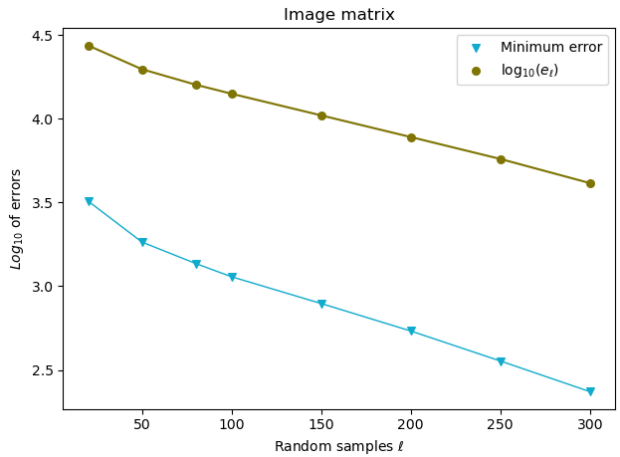
- σ_{k+1} : the theoretical minimum approximation error.
- $e_k = \|(I - Q^{(k)} Q^{(k)*})A\|$



(a) Exponential decay : 0.6^i



(b) Slow decay : $0.2i + 0.3$.



(c) Dense matrix $(768, 1024)$.

Figure – Error of the randomized SVD on different type of matrices.

Adaptive Range Finder

Error Estimation

For $(\omega^{(i)})_{i=1}^{r}$, standard Gaussian vectors, *r=10*

$$\mathbb{P}(||(I - QQ^*)A|| \leq 10\sqrt{\frac{2}{\pi}} \max_{i=1,\dots r} ||(I - QQ^*)Aw^{(i)}||) = 1 - 10^{-r}$$

Given ϵ tolerance error we can construct $Q = Q^{(l)}$ incrementally such as

$$||(I - Q^{(l)}Q^{(l)^*})A|| \leq \epsilon$$

Condition to find l

$$||\tilde{q}|| = ||(I - Q^{(l)}Q^{(l)^*})A\omega|| \leq \frac{\varepsilon\sqrt{\pi}}{10\sqrt{2}}$$

Adaptive Range Finder

Algorithm Adaptive randomized range finder

Require: A, ε a tolerance, r an integer

Draw Gaussian random vectors ω_i of length n, $i = 1, \ldots, r$

for $i = 1, \ldots, r$ **do**

Compute $y_i = A\omega_i$

end for

Take $j = 0$, and an empty basis matrix $Q^{(0)}$

while $\max_{i=1,\ldots,r} \{\|y_{j+i}\|\} \geq \dfrac{\varepsilon\sqrt{\pi}}{10\sqrt{2}}$ **do**

$j = j + 1$

Overwrite $y_j = (I - Q^{(j-1)}(Q^{(j-1)})^*)y_j$

Set $q_j = \dfrac{\tilde{y}_j}{||\tilde{y}_j||}$

Form $Q^{(j)} = [Q^{(j-1)} q_j]$

Draw a standard Gaussian vector ω_{j+r} of length n

$y_{j+r} = (I - Q^{(j)}(Q^{(j)})^*)A\omega_{j+r}$

for $i = j + 1, \ldots, j + r - 1$ **do**

Overwrite $y_i = y_i - q_j\langle q_j, y_i\rangle$

end for

end while

return Q

Adaptive Range Finder

ϵ	l	Approximation error
0.01	20	0.00012
0.1	14	0.0036

Test on a matrix of shape $(600, 600)$.

ϵ	l	Approximation error
0.01	200	$1e^{-13}$
none	199	0.896

Test on a matrix of shape $(200, 200)$.

ϵ	l	Approximation error
0.01	769	$1e^{-12}$
none	767	12.44

Test on a matrix of shape $(768, 1024)$.

Table – required size basis l for ϵ

Approximation error : $\|(I - Q^{(l)} Q^{(l)^*})A\|$

Power Iteration

In the case where the singular spectrum decays slowly :

$$((AA^*)^q)A = (U\Sigma^2U^*)qA = U\Sigma^{2q+1}V^*$$

Hence, this operation forces the spectrum to decay rapidly.

Algorithm Randomized power iteration

Require: p the oversampling parameter, q a small integer

Draw a random $n \times (k+p)$ test matrix Ω

Form the matrix product $Y = (AA^*)^q A\Omega$

Compute a QR-factorization : $(Q, \_) = QR(Y)$

return Q

Subspace iteration

floating-point arithmetic

All information associated with singular values that are small would be extinguished by rounding errors.

Algorithm Subspace iteration

Require: $m \times n$ matrix A, integers p and q

Draw a random $n \times (k+p)$ test matrix Ω

Form the matrix product $Y_0 = A\Omega$ and compute its QR factorization : $Y_0 = Q_0 R_0$

for $j = 1, 2, \ldots, q$ **do**

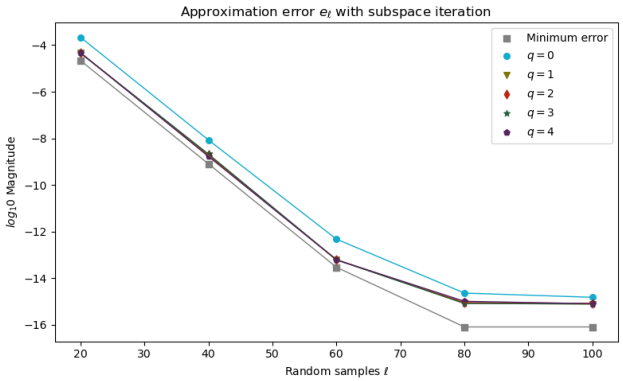
Form $\tilde{Y}_j = A^* Q_{j-1}$ and compute its QR factorization $\tilde{Y}_j = \tilde{Q}_j \tilde{R}_j$

Form $Y_j = A\tilde{Q}_{j-1}$ and compute QR factorisation $Y_j = Q_j R_j$

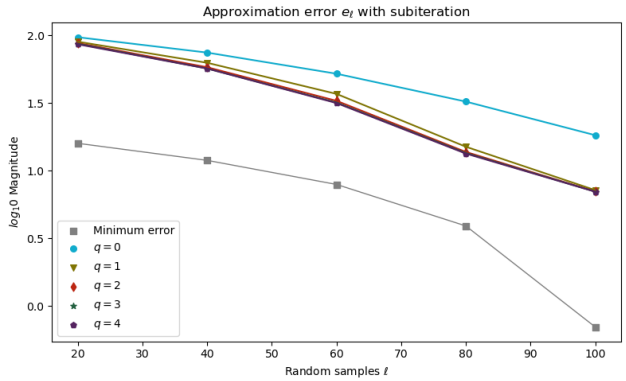
end for

return $Q = Q_q$ (via the QR factorization $Y = QR$)

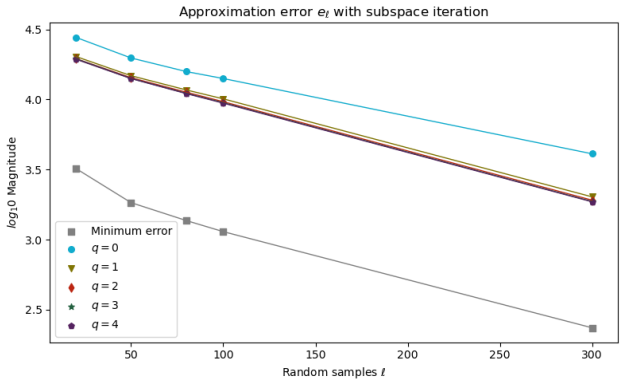
Test



(a) Exponential decay



(b) Slow decay



(c) Dense matrix

Figure – Subspace iteration algorithm with $q \in \{0, \ldots, 4\}$

SVD via Row extraction

Interploative Decomposition on rows

$A \in \mathbb{K}^{m \times n}$ matrix of rank k :

$$A = XA_{(J,:)}$$

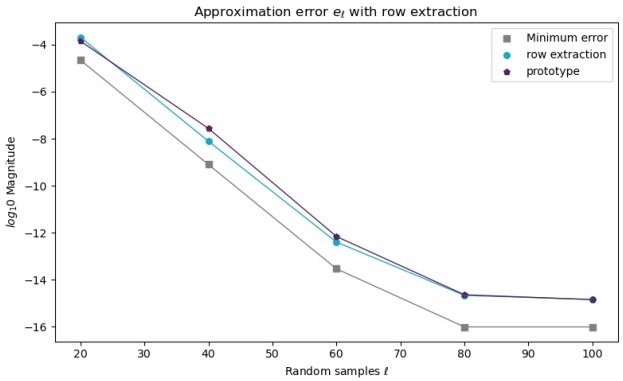
Where

- J, subset of k indices in $[\![1, m]\!]$.
- $A_{(J,:)} \in \mathbb{R}^{k \times n}$ represent J rows of A.
- $X \in \mathbb{R}^{m \times k}$ and $X_{(J,:)} = I_k$

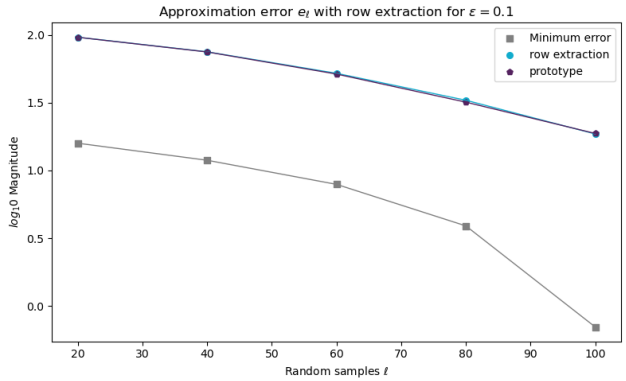
Algorithm SVD via Row Extraction

Require: A and Q such as $\|A - QQ^*A\| \leq \varepsilon$
Compute ID factorization $Q = XQ(J, :)$
Extract $A_{(J,:)}$ and compute the QR factorization $A_{(J,:)} = R^*W^*$
Form the product $Z = XR^*$
Compute the SVD on $Z : Z = U\Sigma\tilde{V}^*$
Form $V = W\tilde{V}$
return U, Σ and V

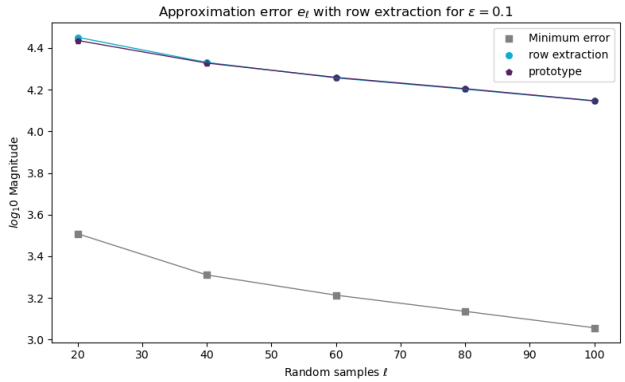
Test



(a) Exponential decay



(b) Slow decay



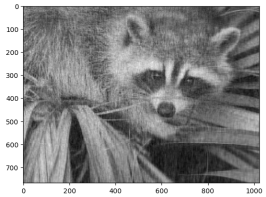
(c) Dense matrix

Figure – Row extraction scheme with $\epsilon = 0.1$

Image Reconstruction

(a) Original matrix

(b) RSVD with $k = 50$



(c) RSVD with $k = 100$

(d) RSVD with $k = 200$

(e) RSVD with $k = 300$

Figure – Reconstruction Image $(768, 1024)$ with the RSVD

CUR decomposition

Definition

$A \in \mathcal{M}_{m\times n}(\mathbb{C})$, then :

$$A = CUR$$

- $C \in \mathcal{M}_{m\times k}(\mathbb{K})$ such as : $C = A(:, J_s)$ (J_s subset of columns of A);
- $R \in \mathcal{M}_{k\times n}(\mathbb{K})$ such as : $R = A(I_s, :)$ (I_s subset of rows of A);
- $U \in \mathcal{M}_{k\times k}(\mathbb{K})$.

Algorithm for CUR decomposition

Algorithm Randomized CUR

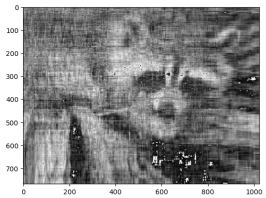
Require: an $m \times n$ matrix A, a target rank k.
Compute the column-ID decomposition of A : $C, Z = \text{column\_ID}(A, k)$.
Compute the QR decomposition of C^T : , $S, P = \text{qr}(C^T)$.
Extract the top k row indices of P : $I = P(1 : k)$.
Extract the top k row from the input matrix : $R = A(I, :)$
Compute : $U = ZR^{\dagger}$
return A $k \times k$ matrix U, a $m \times k$ matrix C and a $k \times n$ matrix R.

Principle :

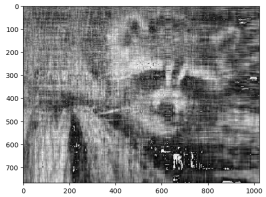
$$A = CZ \Longrightarrow \text{Finding } U \text{ such as } Z = UR$$

Numerical results

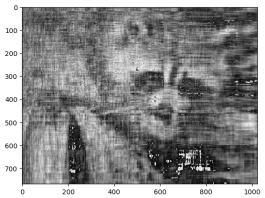
(a) Original matrix



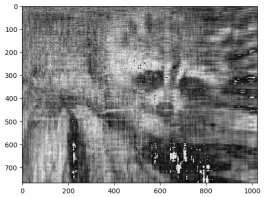
(b) CUR with $k = 50$



(c) CUR with $k = 100$



(d) CUR with $k = 200$



(e) CUR with $k = 300$

Figure – Results for different target ranks after using the CUR decomposition

Numerical results

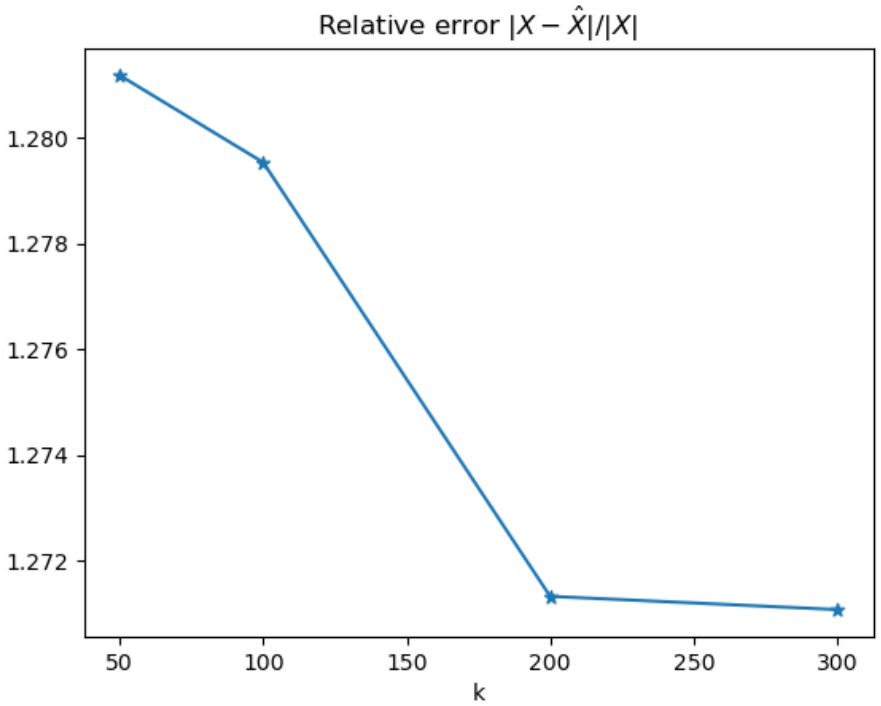


Figure – Relative errors $\frac{\|X - \hat{X}\|_F}{\|X\|_F}$ for each target rank k

Numerical results

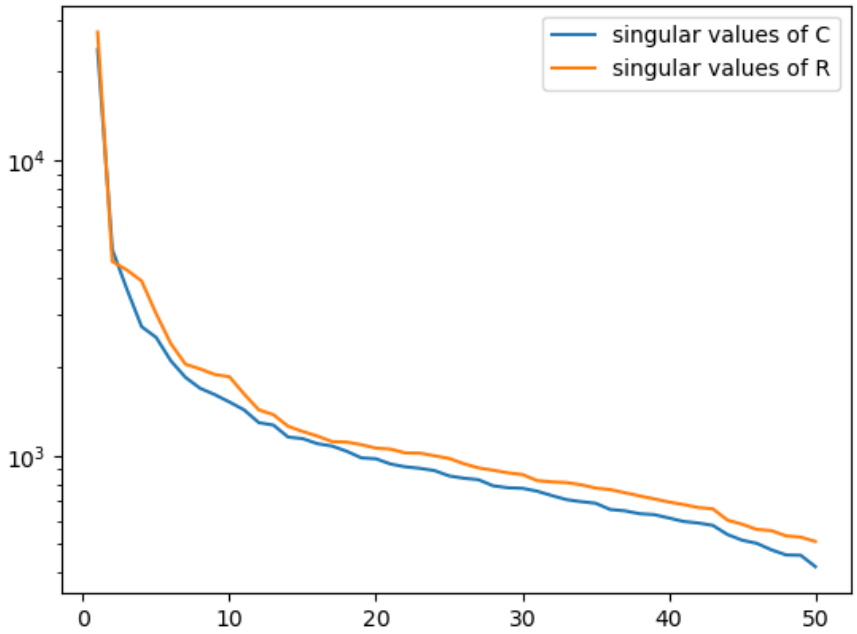


Figure – Singular values of C and R for $k = 50$

Tensors

Definition (order of a tensor)

The order of a tensor is its number of dimensions.

Definition (norm of a tensor)

$X \in \mathbb{R}^{I_1 \times \ldots \times I_N}$. Its norm is defined by :

$$\|X\| = \sqrt{\sum_{i_1=1}^{I_1} \ldots \sum_{i_N=1}^{I_N} x_{i_1 \ldots i_N}^2}$$

Definition (n-rank of a tensor)

The n-rank of $\mathbf{X} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$, denoted $\text{rank}_n(X)$, is the column rank of $X_{(n)}$.
$\Longrightarrow$ If we denote $R_n = \text{rank}_n(X)$ for $n = 1, \ldots, N$, then $\mathbf{X}$ is rank-$(R_1, \ldots, R_N)$.

Matricization of a tensor

Definition (mode-n matricization)

Let $X \in \mathbb{R}^{I_1 \times \ldots \times I_N}$, $n \in \mathbb{N}$, $1 \leq n \leq N$. The element $x_{i_1 i_2 \ldots i_N}$ will become $x_{i_n j}$ where :

$$j = 1 + \sum_{\substack{k=1 \\ k \neq n}}^{N} (i_k - 1) J_k, \text{ with } J_k = \prod_{\substack{m=1 \\ m \neq n}}^{k-1} I_m$$

Example : $X \in \mathbb{R}^{5 \times 3 \times 2}$ given by :

$$X_1 = \begin{bmatrix} 1 & 6 & 11 \\ 2 & 7 & 12 \\ 3 & 8 & 13 \\ 4 & 9 & 14 \\ 5 & 10 & 15 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 16 & 21 & 26 \\ 17 & 22 & 27 \\ 18 & 23 & 28 \\ 19 & 24 & 29 \\ 20 & 25 & 30 \end{bmatrix}$$

$$\Longrightarrow X_{(1)} = \begin{bmatrix} 1 & 6 & 11 & 16 & 21 & 26 \\ 2 & 7 & 12 & 17 & 22 & 27 \\ 3 & 8 & 13 & 18 & 23 & 28 \\ 4 & 9 & 14 & 19 & 24 & 29 \\ 5 & 10 & 15 & 20 & 25 & 30 \end{bmatrix}, \; X_{(2)} = \begin{bmatrix} 1 & 2 & 3 & \ldots & 19 & 20 \\ 6 & 7 & 8 & \ldots & 24 & 25 \\ 11 & 12 & 13 & \ldots & 29 & 30 \end{bmatrix}, \; X_{(3)} = \begin{bmatrix} 1 & 2 & 3 & \cdots & 15 \\ 16 & 17 & 18 & \cdots & 30 \end{bmatrix}$$

Tensor multiplication

Definition (mode-n product)

Let $X \in \mathbb{R}^{I_1 \times \ldots \times I_N}$ and $U \in \mathbb{R}^{J \times I_n}$. The n-mode product of X and U, denoted by $X \times_n U$, is a tensor of size $I_1 \times \ldots \times I_{n-1} \times J \times I_{n+1} \times \ldots \times I_N$, whose elements are :

$$(X \times_n U)_{i_1 \ldots i_{n-1} j i_{n+1} \ldots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2 \ldots I_N} u_{j i_n}$$

Example : $U = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$. The product $Y = X \times_1 U \in \mathbb{R}^{2 \times 3 \times 2}$ is given by :

$$Y_{(1)} = \begin{bmatrix} 95 & 200 & 345 \\ 110 & 260 & 410 \end{bmatrix} \text{ and } Y_{(2)} = \begin{bmatrix} 470 & 595 & 720 \\ 560 & 710 & 860 \end{bmatrix}$$

Tucker decomposition

Definition

Let $\mathcal{G} \in \mathbb{R}^{J_1 \times \ldots \times J_1}$ and $A^{(i)} \in \mathbb{R}^{I_i \times J_i}$, for $i = 1, \ldots, N$. Then the Tucker decomposition of $\mathcal{X} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$ is :

$$\mathcal{X} \approx \underbrace{\mathcal{G}}_{\text{core tensor}} \times_1 A^{(1)} \times_2 A^{(2)} \times_3 \ldots \times_N A^{(N)} = [\![\mathcal{G}; A^{(1)}, \ldots, A^{(N)}]\!]$$

Algorithm HOSVD

Require: $\mathcal{X}$, $(R_1, \ldots, R_N)$
  for $n = 1, \ldots, N$ **do**
    $A^{(n)} \longleftarrow R_n$ leading left singular vectors of the randomised SVD of $\mathcal{X}_{(n)}$
  end for
  $\mathcal{G} \longleftarrow \mathcal{X} \times_1 A^{(1)\,T} \times_2 \ldots \times_N A^{(N)\,T}$
  return $\mathcal{G}, A^{(1)}, \ldots, A^{(N)}$.

Reduce the computational burden of the HOSVD $\Longrightarrow$ ST-HOSVD.

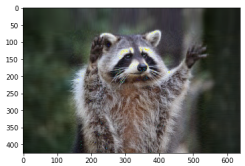
Algorithm ST-HOSVD

Require: $\mathcal{X}$ and target rank $r = (r_1, \ldots, r_n)$
  $\mathcal{G} = \mathcal{X}$
  for $n = 1, \ldots, N$ **do**
    $U_n, \Sigma_n, V_n = TruncatedSVD(\mathcal{G}^{(n)}, r_n)$ *randomized SVD*
    $\mathcal{G}^{(n)} \longleftarrow \Sigma_n V_n^*$
  end for
  return $X_{ST-HOSVD} = [\![\mathcal{G}; U_1, \ldots, U_N]\!]$

Numerical results

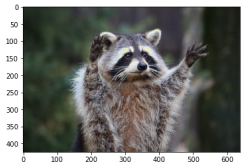
(a) Original tensor of size (425,638,3)

(b) HOSVD with $R = (20, 20, 3)$



(d) HOSVD with $R = (100, 100, 3)$

(c) HOSVD with $R = (50, 50, 3)$



(e) HOSVD with $R = (200, 200, 3)$

Figure – Results for different target ranks after using the HOSVD

Numerical results

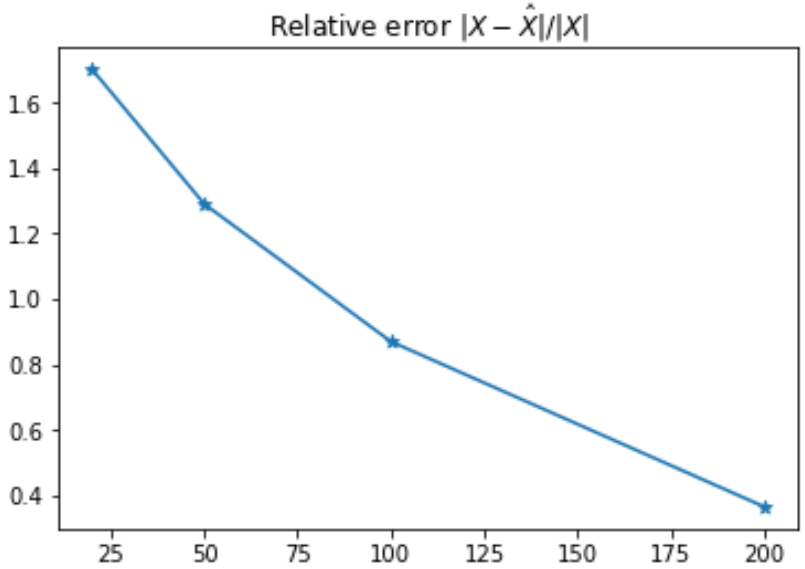


Figure – Relative errors $\frac{||X - \hat{X}||}{||X||}$ for each target rank k

Tools

Dimension Reduction maps (DRMs)

Let $k < N$, the **DRMs** is a matrix $\mathcal{M} \in \mathbb{R}^{k \times N}$

$$\mathbb{E}||\mathcal{M}u||_2^2 = const.||u||_2^2 \quad \text{for all } u \in \mathbb{R}^{\mathbb{k}}$$

Tensor Random Projection (TRP)

Given $S_i \in \mathbb{K}^{d \times m_i}$ for $i = 1, \ldots, k$ independent DRMs, the TRP is defined as

$$\mathbf{S} := S_1 \odot S_2 \odot \ldots \odot S_k \in \mathbb{K}^{d \times n}$$

Where $n = \prod_{i=1}^{k} m_i$.

Tensor compression

Given $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and a target rank $r = (r_1, \ldots, r_N)$ we set the sketch size parameters :

- $k = (k_1, \ldots k_N) \geq r$
- $s = (s_1, \ldots, s_N) \geq k$

Then draw independent, random DRMs :

$$\Omega_1, \ldots \Omega_N \ ; \ \Phi_1 \ldots \Phi_N$$

with $\Omega \in \mathbb{R}^{I_{(-n)} \times k_n}$ and $\Phi \in \mathbb{R}^{I_n \times s_n}$ for $n \in [N]$. We use these DRMs to compute :

$$V_n = X^{(n)} \Omega_n \quad \in \mathbb{R}^{I_n \times k_n}, n \in [N]$$
$$\mathcal{H} = \mathcal{X} \times_1 \Phi_1^T \cdots \times_N \Phi_N^T \quad \in \mathbb{R}^{s_1 \times \cdots \times s_N}$$

Tucker Sketch

Algorithm Tucker Sketch

Require: RDRM a function that generates a random RMD and a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$

function Tucker_Sketch($\mathcal{X}, k, s$)
for , $\in 1, \ldots, N$ **do**
    $\Omega_n = RDRM(I_{(-n)}, k_n)$
    $\Phi_n = RDRM(I_n, s_n)$
    $V_n = X^{(n)} \Omega_n$
end for
$\mathcal{H} = \mathcal{X} \times_1 \Phi_1^T \cdots \times_N \Phi_N^T$
return $(\mathcal{H}, V_1, \ldots, V_N, \{\Phi_n, \Omega_n\}_n)$

Reducing storage for the Ω_n

Generate DRMs $A_n \in \mathbb{R}^{I_n \times k} \forall n \in \{1, \ldots, N\}$ and then set
$\Omega_n = A_1 \odot \ldots A_{n-1} \odot A_{n+1} \odot \cdots \odot A_N \; \forall \; n \in \{1, \ldots, N\}$.

Two Pass

The Two-Pass algorithm uses only the factors sketches V_n, we set then $s = 0$

Algorithm Two-Pass Sketch

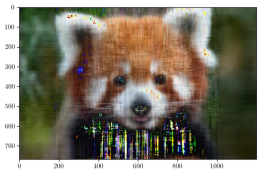
Require: tensor $\mathcal{X}$ and sketch parameters k
$(\mathcal{H}, V_1, \ldots, V_N, \{\Phi_n, \Omega_n\}_n)$ = TuckerSketch($\mathcal{X}$,k,0)
for $n \in \{1, \ldots, N\{$ **do**
$(Q_n, \_) = QR(V_n)$ (QR factorization)
end for
$\mathcal{W} = \mathcal{X} \times_1 Q_1^T \cdots \times_N Q_N^T \in \mathbb{R}^{k_1 \times \cdots \times k_N}$
return $\hat{\mathcal{X}} = [\![\mathcal{W}; Q_1, \ldots, Q_N]\!]$

Remark :

$$[\![\mathcal{W}; Q_1, \ldots, Q_N]\!] = \mathcal{W} \times_1 Q_1 \times_2 \cdots \times_N Q_N$$

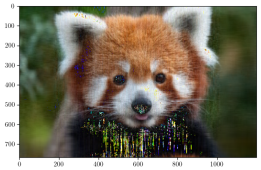
Test

(a) Shape $(1200, 768, 3)$

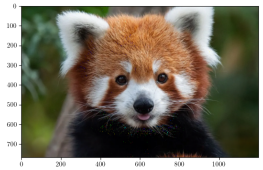


(b) $k = (100, 50, 3)$

(d) $k = (600, 300, 3)$



(c) $k = (300, 100, 3)$



(e) $k = (900, 500, 3)$

Figure – Results for different target ranks after using the Two-Pass

One Pass

For the One-Pass algorithm, we need the core sketch $\mathcal{H}$ defined earlier.

Algorithm One-Pass Sketch

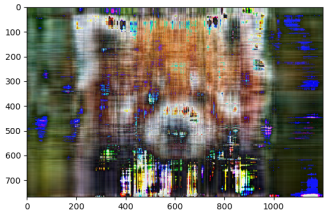
Require: tensor $\mathcal{X}$ and sketches parameters k and s

$(\mathcal{H}, V_1, \ldots, V_N, \{\Phi_n, \Omega_n\}_n) = \text{Tucker\_Sketch}(\mathcal{X}, k, s)$
for $n \in \{1, \ldots, N\{$ **do**
    $(Q_n, \_) = QR(V_n)$
end for
$\mathcal{M} = \mathcal{H} \times_1 (\Phi_1^T Q_1)^{-1} \cdots \times_N (\Phi_N^T Q_N)^{-1}$
return $\hat{\mathcal{X}} = [\![\mathcal{M}, Q_1 \ldots Q_n]\!]$

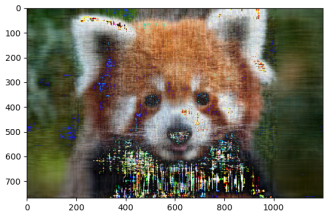
Remark :

We set $s = 2k + 1$ for the test.

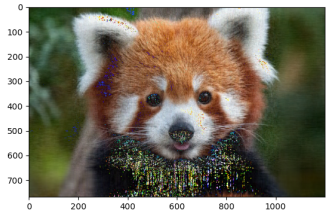
Test



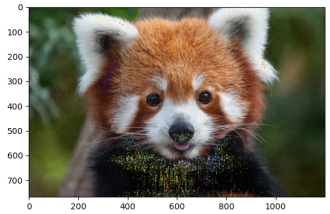
(a) $k = (50, 20, 3)$



(b) $k = (100, 80, 3)$



(c) $k = (300, 200, 3)$



(d) $k = (600, 300, 3)$

Figure – Results for different target ranks after using the One-Pass

Test

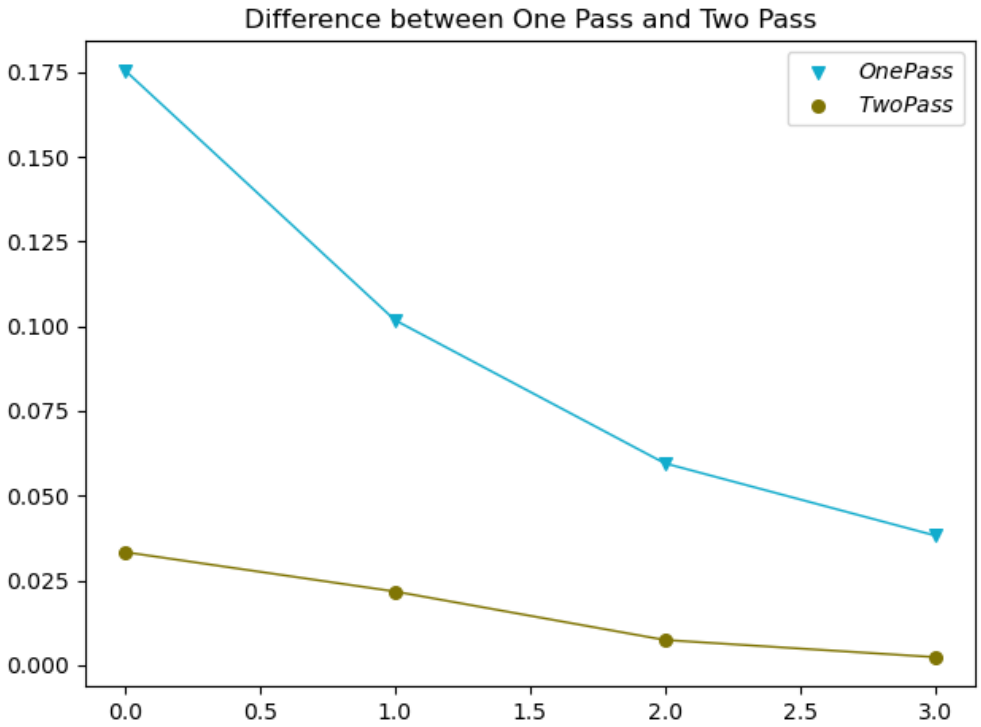


Figure – Approximation error with Two-Pass and One-Pass with the regret metric

Conclusion

Other methods possible :

- Fixed-rank approximations ;
- SRFT for dense matrix to compute the RSVD ;
- Streaming methods : decomposing a tensor into : $A = H_1 + H_2 + \ldots$

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