Honours Differential Equations

Marie Biolková

First Order ODEs

$$y' + p(x)y = g(x)$$

Integrating Factors

$$y = \frac{1}{e^{\int p(x)dx}}\left[\int e^{\int p(x)dx} g(x)dx + C\right]$$

Exact ODEs

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \Longleftrightarrow \quad \psi_x = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \psi_y$$

Find $g(x,y)$ by integrating and comparing $\int M dx$ with $\int N dy$.

Wronskian

$$W[y_1, ..., y_n] = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

The functions $\{y_i\}$ form a fundamental set of solutions if $W \neq 0$ (i.e. if they're linearly independent. Then any solution can be written as their linear combination. If $W(x_0) \neq 0$ then $W(x) \neq 0 \quad \forall x \in [\alpha, \beta]$.

Undetermined Coefficients: Repeated Roots

Let k be a real root with multiplicity s then

$$y = e^{kx}(c_0 + c_1 x + c_2 x^2 + ... + c_{s-1}x^{s-1}).$$

If $k = \lambda + \mu i$ then

$$y = e^{\lambda x}[(c_0 + c_1 x + c_2 x^2 + ... + c_{s-1}x^{s-1}) \cos \mu x \\ +(d_0 + d_1 x + d_2 x^2 + ... + d_{s-1}x^{s-1}) \sin \mu x].$$

For particular solution, if $g(x)$ solves the ODE then multiply the trial function by x^s.

Variation of Parameters

Cramer's rule:

$$y_{\text{par}} = \sum_{j=1}^{n} u_j(x) y_j(x), \quad u_j' = g(x)\frac{W_j[x]}{W[y_1, ..., y_n]}$$

$$y_{\text{par}} = \sum_j \int_{x_0}^{x} g(s)\frac{W_j[s]}{W[y_1(s), ..., y_n(s)]}ds$$

where $W_j[x]$ is the determinant of the matrix where we replace the j-th column by the vector (0,0,...,1).

Laplace Transforms

The Laplace transform of $f(x)$ defined for $x \in [0, \infty)$ is

$$F(s) = \mathcal{L}\{f\}(s) \equiv \int_0^\infty e^{-sx} f(x)dx = \lim_{t\to\infty}\int_0^T e^{-sx} f(x)dx$$

Requires $f \in E$ (f be of exponential type) - need $f(t)$ to be piecewise continuous on any $[0, T]$ where it is defined and $|f(x)| \leq Ae^{Bx} \forall x \in [0, \infty)$ for some constants A, B.
It is a linear operation, i.e.

$$\mathcal{L}\{c_1 f_1(x) + c_2 f_2(x)\}(s) = c_1\mathcal{L}\{f_1(x)\} + c_2\mathcal{L}\{f_2(x)\}.$$

If $f(x)$ is continuous on $[0, \infty)$ and $f, f' \in E$ then

$$\mathcal{L}\{f'(x)\} = s\mathcal{L}\{f(x)\} - f(0)$$

$$\mathcal{L}\{f^{(n)}(x)\} = s^n\mathcal{L}\{f(x)\} - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Let $\mathcal{L}\{f(x)\}(s) = F(s)$:

1. **s-shift**: $\mathcal{L}\{e^{-cx}f(x)\}(s) = F(s+c)$
2. **x-shift**: $\mathcal{L}\{f(x-c)\}(s) = e^{-sc}F(s)$ if $c \geq 0$ and $f(x) = 0$ for $x < 0$.
3. **s-derivative**: $\mathcal{L}\{xf(x)\}(s) = -F'(s)$ or in general $\mathcal{L}\{x^n f(x)\}(s) = (-1)^n F^{(n)}(s)$.
4. **scaling**: $\mathcal{L}\{f(cx)\}(s) = \frac{1}{c}F(\frac{s}{c})$, $F(sc) = \frac{1}{c}\mathcal{L}\{f\left(\frac{x}{c}\right)\}$ if $c > 0$.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

Unit Step Function

Given a function $f(t)$ defined for $t \geq 0$,

$$f(t)u_c(t) = \begin{cases} f(t) & \text{for} \quad t \geq c \\ 0 & \text{for} \quad t < c \end{cases}$$

$$f(t)(u_a(t) - u_b(t)) = \begin{cases} f(t) & \text{for} \quad t \in [a,b) \\ 0 & \text{for} \quad t \notin [a,b) \end{cases}$$

Dirac Distribution

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0, \qquad \int_{-\infty}^{\infty}\delta(t)dt = 1$$

$$\int_{-\infty}^{\infty}\delta(t-t_0)f(t)dt = f(t_0)$$

Convolution

Convolution is commutative, associative and distributive, but $(f * 1) \neq f$ and $(f * f) \neq f^2$.

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\mathcal{L}\{f * g\} = (\mathcal{L}\{f\})(\mathcal{L}\{g\})$$

First-Order Systems of ODEs

$$x_i'(t) = F_i(x_j(t), t) \quad i, j = 1, ..., n$$

From n-th order to system of first-order ODEs

$$y^{(n)} = F(y, y', ..., y^{(n-1)}, t)$$

Change variables to $x_1 = y, x_2 = y', ..., x_n = y^{(n-1)}$ and take derivatives $x_1' = x_2, x_2' = x_3, ..., x_{n-1}' = x_n$ and $x_n' = y^{(n)} = F(x_1, x_2, ..., x_n, t)$.
A first-order ODE system is linear if it has the form

$$x_i' = \frac{dx_i}{dy} = \sum_{j=1}^{n} P_{ij}(t)x_j + g_i(t) \quad i = 1, ..., n$$

Homogeneous Systems of Linear ODEs

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x}$$

The general solution is given by the linear combination of any fundamental set of n solutions

$$\mathbf{x}_{\text{gen}}(t) = \sum_{j=1}^{n} c_j\mathbf{x}_j(t)$$

$$\text{with} \quad W[\mathbf{x}_1, ..., \mathbf{x}_n] = |\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n| = \det\Psi(t) \neq 0.$$

Liouville's Theorem

$$\dot{W} = W\text{tr}P \implies W(t) = e^{\int_{t_0}^{t}\text{tr}P(s)ds}W(t_0)$$

Different Eigenvalues: Real

We look for exponential solutions of $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with $\mathbf{x} = e^{rt}\boldsymbol{\xi} \implies \frac{d\mathbf{x}}{dt} = r\mathbf{x}$.
If the corresponding eigenvalue problem $(A - r\mathbb{I})\boldsymbol{\xi} = 0$ has different eigenvalues r_i, the eigenvectors $\boldsymbol{\xi}^{(i)}$ are linearly independent and the general solution reads

$$\mathbf{x} = \sum_{j=1}^{n} c_j e^{r_j t}\boldsymbol{\xi}^{(j)}.$$

If an eigenvalue r has algebraic multiplicity $s \geq 2$, the method still works if the geometric multiplicity (number of linearly independent eigenvectors) equals s.

Different Eigenvalues: Complex

If $r_1 = \lambda + i\mu$ is and eigenvalue, i.e. $(A - r_1\mathbb{I})\boldsymbol{\xi}_1 = 0$ then the complex conjugate $r_1^* = \lambda - i\mu$ is also an eigenvalue with eigenvector $\boldsymbol{\xi}_1^*$.
To convert into real solutions, write $\boldsymbol{\xi}_1 = \mathbf{a} + i\mathbf{b}$, then the 2 real solutions are

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

and the general solution: $\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + ...$

Fundamental Matrix

A fundamental marix $\Psi(t)$ is an $n \times n$ matrix with fundamental solutions as columns:

$$\Psi(t) = \begin{pmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{pmatrix}$$

- $\det\Psi(t) = W(t) \neq 0$
- $\mathbf{x}(t) = \sum_{j=1}^n c_j\mathbf{x}^{(j)} = \Psi(t)\mathbf{c}, \quad \mathbf{c} = (c_1, ..., c_n)^T.$
- If $\mathbf{x}(t_0) = \Psi(t_0)\mathbf{c} = \mathbf{x}_0 \implies \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}_0$
  $\mathbf{x}(t) = \Psi(t)\mathbf{c} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}_0$ (requires invertible $\Psi(t)$).
- $\Psi' = A\Psi$

Matrix exponential

$$e^{At} = \sum_{n=0}^{\infty}\frac{(At)^n}{n!} = \mathbb{I} + At + \frac{1}{2!}A^2t^2 + ...$$
$$= \lim_{n\to\infty}\left(\mathbb{I} + \frac{1}{n}A\right)^n$$

- $\mathbf{x}(t) = e^{At}\mathbf{x}_0 \iff e^{At} = \Psi(t)\Psi^{-1}(t_0)$
- $e^{At} = \Psi(t)$ for $\Psi(0) = \mathbb{I}$

Let $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ and consider the matrix T with eigenvectors $\boldsymbol{\xi}^{(i)}$ as columns. The matrix AT has columns equal to $A\xi^{(i)} = r_i\xi^{(i)}$, so

$$AT = \begin{pmatrix} r_1\xi_1^{(1)} & \dots & r_n\xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ r_1\xi_n^{(1)} & \dots & r_n\xi_n^{(n)} \end{pmatrix} = T\begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_n \end{pmatrix}$$
$$= T\text{diag}(r_1, ..., r_n) = TD \implies D = T^{-1}AT.$$

Then $\mathbf{x} = T\mathbf{y} \implies \frac{d\mathbf{y}}{dt} = D\mathbf{y}$ and in the new variables, the solution is

$$\mathbf{y}_i = e^{r_i t}\begin{pmatrix} 0 \\ . \\ 1 \\ 0 \\ . \\ 0 \end{pmatrix} \quad 1 \text{ } i\text{-th component.}$$

Thus its fundamental matrix equals $Q(t) = e^{Dt} = \text{diag}(e^{r_1 t}, ..., e^{r_n t})$. The fundamental matrix in the original variables $\mathbf{x}$ is $\Psi(t) = TQ(t)$ and the exponential matrix is $e^{At} = \Psi(t)\Psi^{-1}(0) = TQT^{-1}$. Only works if A is diagonalisable. If we have an eigenvalues with geometric multiplicity $<$ algebraic multiplicity, we cannot diagonalise!

Repeated Eigenvalues

If the algebraic multiplicity $>$ geometric multiplicity, then follow these steps:

1. One solution is $\mathbf{x}_1 = e^{\lambda t}\boldsymbol{\xi}$.
2. Second solution is of the form $\mathbf{x}_2 = te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta}$ with $(A - \lambda\mathbb{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.
3. (Third solution is of the form $\mathbf{x}_3 = \frac{t^2}{2}e^{\lambda t}\boldsymbol{\xi} + te^{\lambda t}\boldsymbol{\eta} + e^{\lambda t}\boldsymbol{\zeta}$ with $(A - \lambda\mathbb{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$.)

Connection to matrix methods Build a matrix T out of $\boldsymbol{\xi}, \boldsymbol{\eta}, (\boldsymbol{\zeta})$, then $T^{-1}AT = J$ where J is an upper triangular matrix (Jordan form). We can then proceed as if it was D above. The exponential has the form

$$e^{J_\lambda t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Nonhomogeneous Systems of ODEs

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$$

The general solution is of the form

$$\mathbf{x}(t) = \sum_i c_i\mathbf{x}^{(1)}(t) + \mathbf{x}_{\text{par}}(t).$$

Diagonalisation

Introduce change of variables $\mathbf{x} = T\mathbf{y}$, so we have

$$T\mathbf{y}' = AT\mathbf{y} + \mathbf{g}(t) \implies \frac{d\mathbf{y}}{dt} = D\mathbf{y} + T^{-1}\mathbf{g} = D\mathbf{y} + \mathbf{h}.$$

This leads to a system of n decoupled equation which we solve by direct integration:

$$y_i' = r_i y_i + h_i \implies y_i(t) = c_i e^{r_i t} + e^{r_i t}\int_{t_0}^t e^{-r_i s}h_i(s)ds.$$

General solution in the original variables equals $\mathbf{x}(t) = T\mathbf{y}(t)$. In case the matrix is not diagonalisable, find its Jordan form and proceed in a similar way (with J instead of D). Note that we will need to integrate from the bottom up.

Method of Undetermined Coefficients

Works if $\mathbf{g}(t)$ is built out of polynomials and exponentials (real or complex). Same rules apply with the exception that if $\mathbf{g}(t) = \mathbf{u}e^{\lambda t}$ where λ is an eigenvalue of A with multiplicity 1, then

$$\mathbf{x}_{\text{par}} = te^{\lambda t}\mathbf{a} + e^{\lambda t}\mathbf{b}.$$

If the multiplicity is n, we must write $\mathbf{x}_{\text{par}} = e^{\lambda t}\sum_{i=0}^n t^i\mathbf{a}_i$.

Variation of Parameters

If $A = P(t)$ is not constant, we look for solutions to the non-homogeneous part of the form

$$\mathbf{x}(t) = \Psi(t)\mathbf{u}(t).$$

Introducing this to the system gives

$$\frac{d\mathbf{x}}{dt} = \Psi'(t)\mathbf{u}(t) + \Psi(t)\frac{d\mathbf{u}}{dt} = P(t)\Psi(t)\mathbf{u}(t) + \mathbf{g}(t).$$

Remembering $\Psi' = P(t)\Psi$,

$$\Psi\frac{d\mathbf{u}}{dt} = \mathbf{g}(t) \implies \frac{d\mathbf{u}}{dt} = \Psi^{-1}\mathbf{g} \implies \mathbf{u}(t) = \int_{t_0}^t \Psi^{-1}(s)\mathbf{g}(s)ds + \mathbf{f}$$

Thus the general solution is

$$\mathbf{x}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}_0 + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{g}(s)ds.$$

Qualitative Theory of ODEs

Consider a nonlinear autonomous system (i.e. F, G have no explicit time dependence)

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y).$$

Critical Points

A point $\mathbf{x}_0 = (x_0, y_0)$ is a critical point if $F(x_0, y_0) = G(x_0, y_0) = 0$. Locally, around any critical point, nonlinear ODEs $\approx$ linear ODEs. Use Taylor expansions (for F and G):

$$F(x, y) = F(x_0, y_0) + \partial_x F(x_0, y_0)(x - x_0)$$
$$+\partial_y F(x_0, y_0)(y - y_0) + \eta_1(x, y)$$

where $\frac{\eta_1(x,y)}{||\mathbf{x}-\mathbf{x_0}||} \to 0$ as $(x, y) \to (x_0, y_0)$. Linear approximation consists of dropping η_1.
Introduce new variables $u_1 \equiv x - x_0$, $u_2 \equiv y - y_0$. These satisfy

$$\frac{d\mathbf{u}(t)}{dt} = \begin{pmatrix} \partial_x F(x_0, y_0) & \partial_y F(x_0, y_0) \\ \partial_x G(x_0, y_0) & \partial_y G(x_0, y_0) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A\mathbf{u}.$$

A is the Jacobian matrix.

	Linear System		Locally Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

A node is proper if it has independent eigenvectors and improper if there is a missing eigenvector. A critical point $\mathbf{x}_0$ is stable if $\forall\epsilon$, $\exists\delta > 0$ s.t. every solution $\mathbf{x} = \phi(t)$ with $||\phi(0) - \mathbf{x}_0|| < \delta$ at $t = 0$ satisfies $||\phi(t) - \mathbf{x}_0|| < \epsilon$, $\forall t > 0$.
A critical point $\mathbf{x}_0$ is asymptotically stable if it is stable and the solution $\mathbf{x} = \phi(t)$ is forced to approach $\mathbf{x}_0$ as $t \to \infty$.
Sometimes a nonlinear ODE system has an exact phase portrait given by

$$\left.\begin{aligned} \frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y) \end{aligned}\right\} \implies \frac{dy}{dx} = \frac{G(x, y)}{F(x, y)} \implies H(x, y) = c.$$

Lyapunov's Theory

Let $E(x, y)$ be defined on a domain D containing (0,0).
$E(x, y)$ is *positive (negative) definite* if $E(0, 0) = 0$ and $E(x, y) > 0 \; \forall(x, y) \in D$ ($E(x, y) < 0 \; \forall(x, y) \in D$).
$E(x, y)$ is *positive (negative) semi-definite* if $E(0, 0) = 0$ and $E(x, y) \geq 0 \; \forall(x, y) \in D$. ($E(x, y) \leq 0 \; \forall(x, y) \in D$.
Theorem Given an autonomous system with critical point (0,0), if $\exists E(x, y)$ continuous with continuous first partial derivatives, positive definite and for which $\frac{dE}{dt}$ is negative definite on some domain D containing (0,0) then (0,0) is asymptotically stable. If $\frac{dE}{dt}$ is negative semi-definite $\implies$ (0,0) is stable (at the non-linear level). $E(x, y)$ is called *Lyapunov function.*

Theorem Given an autonomous system with critical point (0,0), assume $\exists E(x,y)$ continuous with continuous first partial derivatives, such that $E(0,0)=0$ and that in every neighbourhood of (0,0) $\exists$ at least one point (x_1,y_1) where $E(x_1,y_1)$ is positive (negative). If $\exists$ some domain D containing (0,0) where $\frac{dE}{dt}$ is positive definite (negative definite) on $D \implies$ (0,0) is an unstable critical point.

Limit Cycles

Periodic solutions: $f(x+T)=f(x)\ \forall x$ (the smallest possible T is fundamental period). Trajectories form closed curves. A linear combination or product of functions with the same period T also have period T.
Limit cycles are periodic solutions s.t. at least one other non-closed trajectory asymptotes to them as $t\to\infty$ (or $-\infty$ or both).
Let $F(x,y), G(x,y)$ have continuous first partial derivatives in some domain D. The we have the following:
Theorem A closed trajectory must necessarily enclose at least one critical point. If it encloses only one critical point, it cannot be a saddle point. (i.e. no critical points in $D \implies$ no closed trajectories in D; if $\exists$ a unique critical point in D and it is a saddle $\implies$ no closed trajectories in D).
Theorem Let D be simply connected (i.e. without holes). If $\partial_x F+\partial_y G$ has the same sign in $D \implies$ there are no closed trajectories in D.
Poincaré-Bendixon Theorem Let R consist of a bounded subdomain of D and its boundary. Suppose R has no critical points. If a certain trajectory lies entirely in R, then this trajectory either is a periodic (closed) trajectory or spirals towards one. Either way, $\exists$ a closed trajectory.

Fourier Series

Inner Product

$$(u(x),v(x)) \equiv \int_{-L}^{L} u(x)v(x)dx$$

$$(u(x),v(x)) \equiv \int_{\alpha}^{\beta} u^*(x)v(x)dx \qquad \text{(complex functions)}$$

The set $\{1, \sin\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\}$ forms an orthogonal basis. If $S_n(x)=\sin\frac{n\pi x}{L}, S_m(x)=\sin\frac{m\pi x}{L}, C_n(x)=\cos\frac{n\pi x}{L}, C_m(x)=\cos\frac{m\pi x}{L}, C_0=1$, then

$$\left.\begin{aligned}(S_m,S_n) &= 0\\ (S_n,S_n) &= L\end{aligned}\right\} \implies (S_m,S_n)=L\delta_{mn} \quad m,n\neq 0$$

$$\left.\begin{aligned}(C_m,C_n) &= 0\\ (C_n,C_n) &= L\end{aligned}\right\} \implies (C_m,C_n)=L\delta_{mn} \quad m,n\neq 0$$

$$(S_m,C_n)=(S_n,C_n)=(C_0,C_m)=(C_0,S_m)=0, \quad (C_0,C_0)=2L.$$

A periodic function with period $2L$ can be expressed as *Fourier series*

$$\begin{aligned}f(x) &= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\\ &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}\end{aligned}$$

For piecewise continuous functions the series converges to $f(x)\ \forall x$ where $f(x)$ is continuous. At discontinuities, the series converges to $\frac{f(x^+)+f(x^-)}{2}$, not to $f(x)$ - Gibbs phenomenon.

Euler-Fourier Formulas

Projecting the function onto orthogonal basis gives

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \equiv \langle f(x)\rangle = c_0\\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n=0,1,2,...\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\\ c_n &= \frac{a_n-ib_n}{2}, \quad c_{-n}=\frac{a_n+ib_n}{2} \quad (n>0)\\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx, \quad n\in\mathbb{Z}\end{aligned}$$

Even functions ($f(-x)=f(x)$) only have cosine coefficient series. Odd functions ($f(-x)=-f(x)$) only have sine coefficient series. Due to symmetries, even/odd functions only require information about half the interval $[0,L]$.

Parseval's Theorem

$$\begin{aligned}(f,f) &= \int_{-L}^{L}|f(x)|^2dx = 2L\sum_{n=-\infty}^{\infty}|c_n|^2\\ &= L\left[\frac{|a_0|^2}{2}+\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)\right]\end{aligned}$$

Partial Differential Equations

- Assume separation of variables $u(x,t)=X(x)T(t)$.
- Introduce one (or more) separation parameter λ.
- Solve eigenvalue problem(s): quantisation of λ (depends on boundary / initial conditions).
- Write the most general solution as a linear combination of all solutions to the eigenvalue boundary problems.
- Identify any undetermined coefficients using initial conditions.

Heat Equation

$$\partial_t u = \alpha^2\partial_x^2 u, \quad \alpha>0$$

- initial condition: $u(x,0)=f(x)$, $0\leq x\leq L$
- boundary conditions: $u(0,t), u(L,t)$, $t>0$

Homogeneous boundary conditions $u(0,t)=u(L,t)=0$

$$\begin{aligned}u(x,t)=X(x)T(t) \implies X''+\lambda X &= 0\\ T'+\alpha^2\lambda T &= 0\end{aligned}$$

$$X_n(x)=\sin\frac{n\pi x}{L}, \quad n=1,2,3,... \quad \lambda_n=\frac{n^2\pi^2}{L^2}$$

$$T_n=e^{-\frac{n^2\pi^2\alpha^2t}{L^2}}$$

$$u(x,t)=\sum_{n=1}^{\infty}c_nX_n(x)T_n(t)=\sum_{n=1}^{\infty}c_ne^{-\frac{n^2\pi^2\alpha^2t}{L^2}}\sin\frac{n\pi x}{L}$$

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Nonhomogeneous boundary conditions
$u(0,t)=T_1, u(L,t)=T_2$.
Map problem to one with homogeneous boundary conditions. Define time independent function $g(x)=\lim_{t\to\infty}u(x,t)$.

$$g(x)=T_1+(T_2-T_1)\frac{x}{L} \implies u(x,0)=f(x)-g(x)$$

Then $\partial_t g=0$ and it is easy to solve for $g(x)$. The original problem has the form $u(x,y)=g(x)+w(x,t)$ ($w(x)$ satisfies a homogeneous set of boundary conditions with different initial value function).

$$c_n=\frac{2}{L}\int_0^L (f(x)-g(x))\sin\frac{n\pi x}{L}dx$$

$$u(x,t)=T_1+(T_2-T_1)\frac{x}{L}+\sum_{n=1}^{\infty}c_ne^{-\frac{n^2\pi^2\alpha^2t}{L^2}}\sin\frac{n\pi x}{L}$$

Insulated ends $X'(0)=X'(L)=0$
Process is the same but this time the result is a cosine series.

Wave Equation

$$\partial_t^2u=a^2\partial_x^2u \quad a = \text{wave speed}$$

- initial position: $u(x,0)=f(x)$
- initial velocity $u_t(x,0)=g(x)$
- fixed ends: $u(0,t)=u(L,t)=0$

String with initial position No initial velocity, so $u_t(x,0)=0 \implies T'(0)=0$. X_n and c_n are same as homogeneous heat equation.

$$u(x,t)=\sum_{n=1}^{\infty}c_nX_n(x)T_n(t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

String with initial velocity No initial position, so $u(x,0)=0 \implies T(0)=0$. We find that

$$T_n(t)=\sin\frac{n\pi at}{L}$$

$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}$$

$$c_n=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

String with initial position and velocity Let $v(x,t)$ be the solution for the vibrating string with no initial velocity ($g(x)=0$). Let $w(x,t)$ be the solution for the string with no initial displacement ($f(x)=0$). Then $u(x,t)=v(x,t)+w(x,t)$.

Laplace's Equation

$$\nabla^2 u \equiv \partial_x^2 u + \partial_y^2 u = 0$$

Dirichlet boundary conditions: $u(x, y)$ specified at the boundary.
Rectangle Assume separation of variables $u(x, y) = X(x)Y(y)$. Then

$$X'' - \lambda X = 0$$
$$Y'' + \lambda Y = 0$$

Example : $u(x, 0) = u(x, b) = 0, u(0, y) = 0, u(a, y) = f(y), 0 \le x \le a, 0 \le y \le b$

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

$$c_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} dy$$

Disc Change coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0$$

Assume $u(r, \theta) = R(r)\Theta(\theta)$, then

$$r^2 R'' + rR' = \lambda R$$
$$\Theta'' = -\lambda \Theta$$

Example: $u(a, \theta) = f(\theta), x^2 + y^2 = a^2, 0 \le \theta \le 2\pi$ and $u(x, y) = \sqrt{x^2 + y^2} \le a$
Periodicity and boundedness determine:

- $\lambda = 0$ allows a constant solution $u_0(r, \theta) = \frac{c_0}{2}$.
- $\lambda = n^2$ allows solutions of the form $u_n(r, \theta) = r^n(a_n \cos n\theta + b_n \sin n\theta)$

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (e_n \cos n\theta + f_n \sin n\theta)$$

$$u(a, \theta) = f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (e_n \cos n\theta + f_n \sin n\theta)$$

$$a^n e_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$a^n f_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

Sturm-Liouville Boundary Problems

Homogeneous Problems

Consider differential equations of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0$$

Define the differential operator L and rewrite the equation

$$L[y] = -[p(x)y']' + q(x)y$$
$$L[y] = \lambda r(x)y$$
$$a_1 y(0) + a_2 y'(0) = 0 \quad b_1 y(1) + b_2 y'(1) = 0$$

All eigenvalues λ for which there are nontrivial solutions are real. If we have two eigenvalues λ_1 and λ_2 with $\lambda_1 \ne \lambda_2$ and corresponding eigenfunctions ϕ_1, ϕ_2 then

$$\langle \phi_1, \phi_2 \rangle = \int_0^1 r(x)\phi_1(x)\phi_2(x)dx = 0.$$

That is, the pair is orthogonal with respect to the inner product defined by the Sturm-Liouville problem (w.r.t the weight function $r(x)$), denoted by the angled brackets to differentiate from the original inner product. For each eigenvalue, there is a unique linearly independent eigenfunction. They form and infinite ordered sequence $\lambda_1 < \lambda_2 .. < \lambda_n$ and $\lambda_n \to \infty$.
Eigenfunctions satisfying

$$\langle \phi_n, \phi_n \rangle = \int_0^1 r(x)\phi_n^2(x)dx = 1$$

are said to be normalised and form an orthonormal set w.r.t. $r(x)$. A function$f(x)$ can be written as a sum of these eigenfunctions as follows:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Multiplying by $r(x)\phi_m(x)$ and integrating gives

$$\sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_m(x)\phi_n(x)dx = c_m$$

$$c_m = \int_0^1 r(x)\phi_m(x)f(x)dx = \langle f(x), \phi_m \rangle$$

Lagrange's Identity

$$\int_0^1 (L[u]v - uL[v])dx = [-p(x)(u'(x)v(x) - u(x)v'(x))]_0^1 = 0$$

$$(L[u], v) - (u, L[v]) = 0$$

Nonhomogeneous Problems

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x)$$

First look at the homogeneous problem $L[y] = \lambda r(x)y$ with eihenvalues $\lambda_1, \lambda_2 ..$ and eigenfunctions $\phi_1, \phi_2 ...$ Assume the solution $y = \phi(x)$ can be written as

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

$$c_n = \int_0^1 f(x)\phi_n(x)dx$$

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x)$$

If c_n is zero then b_n is arbitrary - infinitely many solutions. If $\lambda_n = \mu$ for some n and $c_n \ne 0$ then there are no solutions.
Example: generalised heat equation

$$r(x)u_t = (p(x)u_x)_x - q(x)u + F(x, t)$$

with two boundary conditions $u_x(0, t) - h_1(0, t) = 0, u_x(1, t) + h_2 u(1, t) = 0$ and initial condition $u(x, 0) = f(x)$. Assume solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

where ϕ_n are eigenfunctions of the problem. Expand $F(x, t)$ in the same basis. It is convenient to consider

$$\frac{F(x, t)}{r(x)} = \sum_n \gamma_n(t)\phi_n(x)$$

$$\text{with} \quad \gamma_n(t) = \int_0^1 r(x) \frac{F(x, t)}{r(x)} \phi_n(x)dx = (F, \phi_n)$$

Substituting we find

$$\dot{b_n} + \lambda_n b_n(t) = \gamma_b(t) \quad n = 1, 2, 3...$$

Using initial conditions

$$u(x, 0) = f(x) = \sum_n \alpha_n \phi_n(x) \implies \alpha_n = \int_0^1 r(x)f(x)\phi_n(x)dx$$

$$b_n(t) = \alpha_n e^{\lambda_n t} + \int_0^t e^{-\lambda_n (t-s)} \gamma_n(s)ds$$

Wave equation in 2D

Rectangle

$$\partial_t^2 u = a^2(\partial_x^2 u + \partial_y^2 u)$$

Separation of variables gives rise to:

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T''}{a^2 T} = -(\lambda + \mu)$$

$$X'' + \lambda X = 0$$
$$Y'' + \mu Y = 0$$

Example : $0 \le x \le L, 0 \le y \le M$ with $u(0, y) = u(L, y) = u(x, 0) = u(x, M) = 0$.

$$X = \sin(m\pi x/L), \quad \lambda_m = m^2\pi^2/L^2 m \quad m = 1, 2, ...$$

$$Y = \sin(n\pi y/M), \quad \mu_n = n^2\pi^2/M^2 m \quad n = 1, 2, ...$$

$$T'' + a^2(\lambda_m + \mu_n)T = 0$$

$$T(t) = T_{mn}(t) = c_{mn} \cos(\omega_{mn} t) + d_{mn} \sin(\omega_{mn} t)$$

where $\omega_{mn} = a\pi\sqrt{m^2/L^2 + n^2/M^2}$.
General solution is $u(x, y, t) = X(x)Y(y)T(t)$.

$$u(x, y, 0) = f(x, y) = \sum_{m,n} c_{mn} \sin(m\pi x/L) sin(n\pi y/M)$$

$$\partial_t u(x, y, 0) = g(x, y) = \sum_{m,n} d_{mn} \sin(m\pi x/L) \sin(n\pi y/M)$$

Disc

$$\partial_t^2 u = a^2 \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \right) u$$

Separation of variables gives rise to:

$$\Theta'' + m^2 \Theta = 0$$
$$T'' + a^2 \mu^2 T = 0$$
$$R'' + \frac{R'}{r} + \left(\mu^2 - \frac{m^2}{r^2} \right) R = 0$$

Example: $0 \le x^2 + y^2 \le 1$ with $u(1, \theta, t) = 0, \partial_t u(x, y, 0) = 0, u(r, \theta, 0) = f(r, \theta)$.

$$T(t) = k_1 \sin(\mu a t) + k_2 \cos(\mu a t)$$
$$\Theta(\theta) = a_1 \cos(m\theta) + a_2 \sin(m\theta)$$
$$R(r) = c_1 J_m(\mu r) + c_2 Y_m(\mu r)$$

Periodicity in θ requires $m = 1, 2, ...$ Boundedness imposes $a_2 = 0$. $u(1, \theta, t)$ imposes $J_m(\mu) = 0$, i.e. $\mu = \mu_{m1}, \mu_{m1}...$ are zeroes of the Bessel function. Initial condition $\partial_t u(r, \theta, 0) = 0$ imposes $k_1 = 0$. So the general solution is

$$u = \sum_m \sum_n (c_{mn} \cos(m\theta) + d_m n \sin(m\theta)) \cos(a\mu_{mn} t) J_m(\mu_{mn} r)$$

$$c_{mn} \propto \int_0^1 \int_0^{2\pi} f(r, \theta) \cos(m\theta) J_{mn}(\mu_{mn} r) r dr d\theta$$

$$d_{mn} \propto \int_0^1 \int_0^{2\pi} f(r, \theta) \sin(m\theta) J_{mn}(\mu_{mn} r) r dr d\theta$$

Third Laplace's Equation in Cylindrical Coordinates

$$\nabla^2 u \equiv \partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 0$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \psi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Using separation of variables $u(\rho, \psi, z) = R(\rho)\Psi(\psi)Z(z)$,

$$\frac{1}{R\rho} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2 \Psi} \frac{d^2 \Psi}{d\psi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$

$$\frac{d^2 Z}{dz^2} = \chi^2 Z$$

$$\frac{d^2 \Psi}{d\psi^2} = -m^2 \Psi$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(1 - \frac{m^2}{\rho^2} \right) R = 0$$

The radial equation is Bessel's equation. So the solution are of the form

$$R_m(\rho) = c_1 J_m(\chi\rho) + c_2 Y_m(\chi\rho)$$

which is a linear combination of Bessel functions of first and second kind.

- $p(\rho) = r(\rho) = \rho$: vanish at origin $\rho = 0$
- $q(\rho) = \frac{m^2}{\rho}$: unbounded as $\rho \to 0$.

Useful Facts

- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\int u dv = uv - \int v du$

Polar coordinates

$$x = r \cos\theta \quad y = r \sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

Cylindrical coordinates

$$x = \rho \cos\psi, \quad y = \rho \sin\psi, \quad z = z$$