FPM Analysis

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The Real Numbers

The Triangle Inequality

- $|a+b| \leq |a| + |b|$
- $||a| - |b|| \leq |a-b|$

Approximation Property If the set $E \subset \mathbb{R}$ has a supremum then for any positive number $\varepsilon > 0$ there exists $a \in E$ such that $\sup E - \varepsilon < a \leq \sup E$.
Remark: If $E \subset \mathbb{N}$ has a supremum then $\sup E \in E$.
Archimedean Principle Given positive real numbers $a, b \in \mathbb{R}$ there is an integer $n \in \mathbb{N}$ such that $b < na$.
The Completeness Axiom If $E \subset \mathbb{R}$ is non empty and bounded above then E has a supremum.

- Set E has a supremum iff the set $-E$ has an infinum and $\inf(-E) = -\sup E$.
- Set E has an infinum iff the set $-E$ has a supremum and $\sup(-E) = -\inf E$.

Monotone Property If $A \subset B$ are two nonempty subsets of $\mathbb{R}$ and B is bounded above then $\sup A \leq \sup B$. If B is bounded below then $\inf A \geq \inf B$.
Bernouilli's Inequality Let $n > 0, x \geq -1$, then

- $(1+x)^n \leq 1 + nx$ if $n \in (0,1]$
- $(1+x)^n \geq 1 + nx$ if $n \in [1, \infty]$.

Sequences

A sequence of real numbers (x_n) is said to *converge* to a real number a if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - a| < \varepsilon$.

- Every convergent sequence is bounded.

The Squeeze Theorem Suppose $(x_n), (y_n), (w_n)$ are real sequences.

- If both $x_n \to a$ and $y_n \to a$ (same a!) as $n \to \infty$ and if
$$x_n \leq w_n \leq y_n \text{ for all } n \geq N_0$$
then $w_n \to a$ as $n \to \infty$.
- If $x_n \to 0$ and (y_n) is bounded then the product $x_n y_n \to 0$ as $n \to \infty$.

Theorem 2.2.3 Let $E \subset \mathbb{R}$. If E has a finite supremum, i.e. E is bounded above, then there is a sequence (x_n) with $x_n \in E$ such that $x_n \to \sup E$ as $n \to \infty$. An analogous statement holds if E has finite infinum (i.e. bounded below).
Comparision Theorem for Sequences Suppose $(x_n), (y_n)$ are real sequences. If both $\lim\limits_{n\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$ exist in $\mathbb{R}^*$ and if $x_n \leq y_n$ for all $n \geq N$ for some $N \in \mathbb{N}$ then $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$.
Monotone Convergence If (x_n) is increasing and bounded above or if it is decreasing and bounded below, then (x_n) is convergent (and converges to the supremum/infimum of the set $\{x_n | n \in \mathbb{N}\}$ respectively.

- $\limsup x_n = \lim\limits_{N\to\infty} \sup\{x_n | n > N\}$
- $\liminf x_n = \lim\limits_{N\to\infty} \inf\{x_n | n > N\}$

Theorem 2.3.7 Let (x_n) be a sequence of real numbers then $\lim\limits_{n\to\infty} x_n$ exists as $\mathbb{R}^*$ iff $\limsup x_n = \liminf x_n$ in which case $\limsup x_n = \liminf x_n = \lim\limits_{n\to\infty} x_n$.

Cauchy Sequences

A sequence (x_n) of numbers $x_n \in \mathbb{R}$ is said to be *Cauchy* if $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq N.$$

A sequence of real numbers x_n is a Cauchy sequence $\iff$ (x_n) converges.

Subsequences

Theorem 2.4.3 Let (x_n) be a sequence of real numbers.

- There exists $t \in \mathbb{R}$ such that $\forall \varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ for which $|x_n - t| < \varepsilon \iff$ there exists a subsequence of (x_n) converging to t.
- The sequence is not bounded above (below) $\iff$ there exists a subsequence converging to ∞ (converging to $-\infty$).

Theorem 2.4.4 Every sequence of real numbers has a monotone subsequence.
Theorem 2.4.5 Every bounded monotone sequence converges.
Bolzano-Weierstrass Every bounded sequence of real numbers has a convergent subsequence.

Useful Limits of Sequences

- $a^{\frac{a}{n}} \to 1$ as $n \to \infty$, provided $a > 0$
- $(1 + \frac{1}{n})^n \to e$ as $n \to \infty$
- $(1 - \frac{1}{n})^n \to \frac{1}{e}$ as $n \to \infty$

Infinite Series

Let $S = \sum\limits_{k=1}^{\infty} a_k$ be an infinite series with terms a_k. For each n define the partial sum by $s_n = \sum\limits_{k=1}^{n} a_k$. S is said to converge $\iff$ the sequence of partial sums (s_n) converges to some $s \in \mathbb{R}$. That is $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ we have

$$|s_n - s| = \left| \sum_{k=1}^{n} a_k - s \right| < \varepsilon.$$

If the sequence of partial sums diverges then S diverges.

Theorem 3.2.1 Suppose $a_k \geq 0$ for large k. Then $\sum\limits_{k=1}^{\infty} a_k$ converges $\iff$ (s_n) is bounded. That is $\exists M > 0$ such that $\left| \sum\limits_{k=1}^{n} a_k \right| \leq M$ for all $n \in \mathbb{N}$.

Cauchy Criterion The infinite series $\sum_{k=1}^{\infty} a_k$ converges $\iff$ $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall m \geq n \geq N$ we have $\left| \sum\limits_{k=n}^{m} a_k \right| < \varepsilon$.

Harmonic Series The series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges.

Divergence Test Let (a_k) be a sequence. If a_k does not converge to 0 then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Geometric Series Let $x \in \mathbb{R}$ and $N \in \{0, 1, 2, ...\}$. Then the series $\sum\limits_{k=N}^{\infty} x^k$ converges $\iff$ $|x| < 1$. In this case $\sum\limits_{k=N}^{\infty} x^k = \frac{x^N}{1-x}$. In particular

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad |x| < 1.$$

Comparison Test Suppose $0 \leq a_k \leq b_k$ for large k.

- If $\sum\limits_{k=1}^{\infty} b_k < \infty$ then $\sum\limits_{k=1}^{\infty} a_k < \infty$.
- If $\sum\limits_{k=1}^{\infty} a_k = \infty$ then $\sum\limits_{k=1}^{\infty} b_k = \infty$.

Limit Comparison Test Suppose $0 \leq a_k, 0 < b_k$ for large k and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists as an extended real number.

- If $L \in (0, \infty)$ then $\sum\limits_{k=1}^{\infty} a_k$ converges $\iff$ $\sum\limits_{k=1}^{\infty} b_k$ converges.
- If $L = 0$ and $\sum\limits_{k=1}^{\infty} b_k$ converges then $\sum\limits_{k=1}^{\infty} a_k$ converges.
- If $L = \infty$ and $\sum\limits_{k=1}^{\infty} b_k$ diverges then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Root Test Suppose that $r = \lim_{k\to\infty} |a_k|^{\frac{1}{k}}$ exists. If

- $r < 1$ then $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely.
- $r > 1$ then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Absolute Convergence

- A series converges *absolutely* if $\sum\limits_{k=1}^{\infty} |a_k| < \infty$.
- A series S converges *conditionally* if S converges but $\sum\limits_{k=1}^{\infty} |a_k|$ diverges.

- A series converges absolutely $\iff$ $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall m \geq n \geq N$, $\sum_{k=1}^{\infty} |a_k| < \varepsilon$.
- If a series converges absolutely then the series converges, but not conversely.

Cauchy's Condensation Test Let $\sum_{k=1}^{\infty} a_k$ be a series of non-negative terms and assume (a_k) is a decreasing sequence. If $\sum_{k=1}^{\infty} 2^n a_{2^n}$ converges then $\sum_{k=1}^{\infty} a_k$ converges.

Telescopic Series Let (b_k) be a convergent sequence. Then $\sum_{k=1}^{\infty}(b_k - b_{k+1}) = b_1 - \lim_{k\to\infty} b_k$.

Ratio Test Let $a_k \in \mathbb{R}$ and assume $r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ exists in $\mathbb{R}^*$:

- $r < 1 \implies \sum_{k=1}^{\infty} a_k$ converges absolutely.
- $r > 1 \implies \sum_{k=1}^{\infty} a_k$ diverges.

Integral Test $f\colon [1,\infty) \to \mathbb{R}$ positive and decreasing on $[1,\infty)$. Let $a_k = f(k)$ then

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} f(k) \text{ converges} \iff \int_1^{\infty} f(x)dx < \infty.$$

p-series Test The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if and only if $p > 1$.

Alternating Sign Series Let (a_k) be non-negative, decreasing series such that $\lim_{k\to\infty} a_k = 0$. Then $\sum_{k=1}^{\infty}(-1)^k a_k$ is convergent.

Continuity

$f\colon \mathrm{dom}(f) \to \mathbb{R}$ is continuous if there exists sequence (x_n) in $\mathrm{dom}(f)$ s.t. $\lim_{n\to\infty} x_n = a$. We have $\lim_{n\to\infty} f(x_n) = f(a)$.

$\varepsilon-\delta$ **definition:** $f\colon \mathrm{dom}(f) \to \mathbb{R}$ continuous at $a \in \mathrm{dom}(f)$ iff

$$\forall \varepsilon > 0\, \exists \delta : |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

- f continuous at a $\iff$ $\lim_{x\to a} f(x) = f(a)$.
- f continuous at $a \in \mathbb{R}$ and g continuous at $f(a)$, then $g \circ f$ continuous at a.

Extreme Value Theorem

$I \subset \mathbb{R}$ *closed and bounded* and f continuous on I, then $\exists x_m, x_M \in I$ such that

- $f(x_m) = \inf\{f(x) | x \in I\}$.
- $f(x_M) = \sup\{f(x) | x \in I\}$.

Lemma 4.2.4: Let I open interval and $f\colon I \to \mathbb{R}$ continuous at $a \in I$ and $f(a) > 0$, then for some $\delta, \varepsilon > 0$ we have

$$f(x) > \varepsilon,\ \forall x \in (a-\delta, a+\delta).$$

Intermediate Value Theorem

I non-degenerate interval and $f : I \to \mathbb{R}$ continuous. Let $a, b \in I$, $a < b$ then:

$$\forall y_0 \in (f(a), f(b))\, \exists x_0 \in (a,b) : f(x_0) = y_0.$$

Bolzano's Theorem f continuous on $[a,b]$ s.t. $f(a)f(b) < 0$, then $\exists c \in (a,b) : f(c) = 0$.

- $f\ [a,b] \to \mathbb{R}$ strictly increasing such that $\mathrm{im}(f)$ is an interval, then f continuous on $[a,b]$.
- $f\ [a,b] \to \mathbb{R}$ continuous strictly increasing, then $f^{-1}\ [f(a), f(b)] \to \mathbb{R}$ continuous strictly increasing.

Limits of Functions

$f\colon \mathrm{dom}(f) \to \mathbb{R}$, $a \in \mathbb{R}^*$, then $\lim_{x\to a} f(x) = L$ for $L \in \mathbb{R}^*$ if for *every* sequence (x_n) in $\mathrm{dom}(f)$ which converges to a we have $\lim_{n\to\infty} f(x_n) = L$.

Comparison Theorem for Functions $a \in \mathbb{R}$ and I open interval s.t. $a \in I$. If f, g are defined everywhere on $I \setminus \{a\}$ and have limits as $x \to a$ then

$$f(x) \leqslant g(x),\ \forall x \in I \setminus \{a\} \implies \lim_{x\to a} f(x) \leqslant \lim_{x\to a} g(x).$$

Differentiability

$f\colon I \to \mathbb{R}$ is differentiable on $a \in \mathbb{R}$ if $a \in I$ and

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

- f differentiable $\implies$ f continuous.
- f continuously differentiable on I if f' exists and continuous on I.

Rolle's Theorem Let $a, b \in \mathbb{R}$, $a < b$. If f continuous on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$, then $\exists c \in (a,b) : f'(c) = 0$.

Mean Value Theorem

Let $a, b \in \mathbb{R}$, $a < b$. If f continuous on $[a,b]$ and differentiable on (a,b) then $\exists c \in (a,b)$ s.t.

$$f(b) - f(a) = f'(c)(b-a).$$

Generalized Mean Value Theorem If f, g continuous on $[a,b]$ and differentiable on (a,b) then $\exists c \in (a,b)$ s.t.

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

L'Hôpital's Rule Let $a \in \mathbb{R}^*$ and I interval that contains a or has endpoint a. Let f, g differentiable on $I \setminus \{a\}$ and

- $\forall x \in I \setminus \{a\} : g(x) \neq 0,\ g'(x) \neq 0$
- $A = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, $A \in \mathbb{R}^*$
- $B = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists with $B \in \mathbb{R}^*$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Monotone functions

Theorem 5.4.3 Let f be injective and continuous on I, then f is strictly monotone on I and f^{-1} is continuous and strictly monotone on $f(I)$.

Inverse Function Theorem Let f be injective and continuous on *open* interval I. If $a \in f(I)$ and f' exists at $f^{-1}(a)$ and is *non-zero*, then f^{-1} differentiable at a and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

Taylor's Theorem

Taylor's Polynomial Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}^*$, $a < b$. If $f\colon (a,b) \to \mathbb{R}$ differentiable n-times at $x_0 \in (a,b)$, then Taylor's polynomial of degree n is

$$P_n^{f,x_0} = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Taylor's Formula Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}^*$, $a < b$. If $f\colon (a,b) \to \mathbb{R}$ and $f^{(n+1)}$ exists on (a,b) then $\forall x, x_0 \in (a,b) \exists c \text{between} x, x_0$ s.t.

$$f(x) = P_n^{f,x_0} + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{(n+1)}.$$

N.B.: c depends on n, x and x_0.

Useful facts

- The series
$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}, \sum_{k=2}^{\infty}(-1)^k \frac{1}{\log k}, \sum_{k=2}^{\infty}(-1)^k \frac{1}{k \log k}$$
are all convergent (Corollary 3.4.2).
- The *radius of convergence* of $\sum_{n=1}^{\infty} c_n (x-a)^n$ can be defined as $R = \frac{1}{\limsup \sqrt[n]{|c_n|}}$.