Projections Review

Orthogonal:

Vectors $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** (perpendicular) if their inner product is 0. We use the following shorthand: $\mathbf{v} \perp \mathbf{w}$.

Example:

$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}'\mathbf{w} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2 \cdot 1 + 1 \cdot (-2) = 0$$

Why?

- If $\mathbf{v}$ and $\mathbf{w}$ are perpendicular, the triangle is a right triangle.
- The Pythagorean theorem says

$$\begin{aligned}
&||\mathbf{v}||^2 + ||\mathbf{w}||^2 = ||\mathbf{v} + \mathbf{w}||^2 \\
\Rightarrow &\left(\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\right)^2 + \left(\sqrt{w_1^2 + w_2^2 + \cdots + w_n^2}\right)^2 = \left(\sqrt{(v_1 + w_1)^2 + (v_2 + w_2)^2 + \cdots + (v_n + w_n)^2}\right)^2 \\
\Rightarrow &v_1^2 + v_2^2 + \cdots + v_n^2 + w_1^2 + w_2^2 + \cdots w_n^2 = v_1^2 + 2v_1 w_1 + w_1^2 + \cdots + v_n^2 + 2v_n w_n + \cdots w_n^2 \\
\Rightarrow &0 = 2v_1 w_1 + \cdots + 2v_n w_n \\
\Rightarrow &0 = 2\mathbf{v}'\mathbf{w}
\end{aligned}$$

$\Rightarrow \mathbf{v}'\mathbf{w} = 0$

Column space:

The **column space** of a matrix $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_p \end{bmatrix}$ is the linear span of its columns: $\mathcal{C}(\mathbf{X}) = \left\{ \mathbf{v} : \mathbf{v} = \sum_{j=1}^{p} a_j \mathbf{x}_j \right\}$

Perpendicular projection operator (matrix)

$\mathbf{H}$ is a **perpendicular projection operator** (matrix) onto $\mathcal{C}(\mathbf{X})$ if and only if

(i) $\mathbf{v} \in \mathcal{C}(\mathbf{X}) \Rightarrow \mathbf{H}\mathbf{v} = \mathbf{v}$ (projection doesn't change things in $\mathcal{C}(\mathbf{X})$)

(ii) $\mathbf{w} \perp \mathcal{C}(\mathbf{X}) \Rightarrow \mathbf{H}\mathbf{w} = \mathbf{0}$ (perpendicular - vectors orthogonal to $\mathcal{C}(\mathbf{X})$ go to 0)

Example:

Suppose $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathcal{C}(\mathbf{X}) = \left\{ \mathbf{v} : \mathbf{v} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ - this means vectors like $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -0.5 \end{bmatrix}$

Let $\mathbf{H} = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}$, which is the perpendicular projection operator onto $\mathcal{C}(\mathbf{X})$.

Verify:

(i) For any $a \in \mathbb{R}$,

$$\mathbf{H} \begin{bmatrix} 2a \\ a \end{bmatrix} = \begin{bmatrix} 0.8 \cdot 2a + 0.4a \\ 0.4 \cdot 2a + 0.2a \end{bmatrix} = \begin{bmatrix} 2a \\ a \end{bmatrix} \text{ (Property (i) holds.)}$$

(ii) For any $a \in \mathbb{R}$,

$$\mathbf{H} \begin{bmatrix} a \\ -2a \end{bmatrix} = \begin{bmatrix} 0.8 \cdot a + 0.4(-2a) \\ 0.4 \cdot a + 0.2(-2a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ (Property (ii) holds.)}$$

Theorem: The (unique) perpendicualr projection operator onto $\mathcal{C}(\mathbf{X})$ is $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

Proof: Long and not helpful for intuition. The setup is to verify conditions (i) and (ii) in the definition.

Example:

If $\mathbf{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}.$$