Exercise 1 Consider the following two dimensional system:

$$dX_t = k_1 X_t dt + \sigma_1 X_t dW_t, \tag{1}$$
$$dY_t = -k_2 Y_t dt + \sigma_2 dZ_t, \tag{2}$$
$$dW_t dZ_t = \rho dt; \tag{3}$$

where dW_t and dZ_t are two correlated Wiener processes.
Simulate the system of equations up to time $T = 1$ with and without the antithetic variates technique. Compute the means, the variances, and the covariance, $\text{Cov}(X_T, Y_T) = \mathbb{E}[(X_T - \mathbb{E}[X_T])(Y_T - \mathbb{E}[Y_T])]$, of the two processes at time T. How do these observables behave when the variance reduction method is used? To make the exercise interesting do not use more than 100 trajectories. What happens if more and more trajectories are used? Use these numerical values for the parameters: $k_1 = 1.2$, $\sigma_1 = 0.9$, $X_0 = 1$, $k_2 = 0.8$, $\sigma_1 = 1.1$, $Y_0 = 2$, $\rho = 0.45$.

Exercise 2 Simulate a geometric brownian motion

$$dX_t = kX_t dt + \sigma X_t dW_t \tag{4}$$

using both the Euler-Mayurama and the Milstein methods. Investigate the strong and weak convergences of the two methods for this equation as a function of the time-step. Use the following parameters: $k = 1.1$, $\sigma = 1$, $X_0 = 1$; use the negative powers of 2 as time-steps. How are the convergences affected by the numerical values of the drift and volatility? Note that to investigate the weak convergence a big computational effort may be necessary in order to obtain self-evident results.

Exercise 3 Assuming that a stock follows a geometric brownian motion under the risk neutral measure with riskless interest rate $r = 0.05$ and volatility $\sigma = 0.6$; price an European call with six months expiration $T = 1/2$ and an Asian put with one year expiration $T = 1$. At $t = 0$ the stock trades at $S_0 = 12$ and the strike price is $K = 10$ for both options. The payoff of an European call is $\max(S_T - K, 0)$ while the payoff of an Asian put is $\max(K - \int_0^T dt S_t, 0)$. Is it worth to use a variance reduction technique in this case?