1 Simplified version of the model from Fourakis, Karabarbounis (2024). The Puzzling Behavior of Spreads during Covid

It is a quantitative model of sovereign default with several types of debt. For full description of the model please see Fourakis, Karabarbounis (2024). In this file we will only describe key elements of the model because we replicate a simplified version of the original model.

1.1 Timing of the model

1. Exogenous states (z_t, d_t, δ_t) are realized: z_t – level of technology, d_t – credit standing of the government, δ_t – promised new official loans
2. Knowing $s_t = (b_t, f_t, z_t, d_t, \delta_t)$, government decides whether to restructure privately held debt $\eta_{t+1} \in \{0, \bar{\eta}\}$.
3. Actual value of new official loans is realized $\hat{\delta}_t$
4. Conditional on not restructuring debt, government chooses new value of privately held debt and transfers to households
5. Households and firms make their choices knowing $x_t = \left(s_t, \eta_{t+1}, \hat{\delta}_t, b_{t+1}, T_t^h\right)$. Foreign lenders price government debt.

1.2 Households

All households are hand-to-mouth. Their problem is essentially static:

$$
\begin{aligned}
U^h\left(w_t, T_t^h\right) = &\max_{c^h, l^h} \frac{\left(c_t^h\right)^{1-\sigma} - 1}{1-\sigma} - \chi^h \frac{\left(l_t^h\right)^{1+1/\varepsilon}}{1+1/\varepsilon} \\
&\text{s.t.} \quad c_t^h = w_t \theta^h l_t^h + T_t^h
\end{aligned}
$$

FOC (necessary and sufficient):

$$
\begin{aligned}
&\chi^h \left(l_t^h\right)^{1/\varepsilon} = w_t \theta^h \left(c_t^h\right)^{-\sigma} \\
&c_t^h = w_t \theta^h l_t^h + T_t^h \\
\Rightarrow &l_t^h = \left(\frac{w_t \theta^h}{\chi^h} \left(c_t^h\right)^{-\sigma}\right)^{\varepsilon} \\
&c_t^h = w_t \theta^h \left(\frac{w_t \theta^h}{\chi^h} \left(c_t^h\right)^{-\sigma}\right)^{\varepsilon} + T_t^h
\end{aligned}
$$

1.3 Firms

Firms are perfectly competitive and operate the following technology:

$$y_t = z_t l_t$$

where l_t are efficiency units of labor. Hence, wage is determined as follows:

$$w_t = z_t$$

1.4 Government

Government has access to two types of debt: official loans and long-term private debt which is bought by competitive risk-neutral foreign investors at price q. When the government is in good credit standing, the maturity rate is λ_p and coupon is κ_p. When the government is in bad credit standing, the maturity rate is λ_d and coupon is κ_d. The maturity rate of official loans is λ_g and coupon is $\kappa_g = r$. Official loans are non-defaultable, so their price is equal to 1.

Private debt is defaultable because government may choose to restructure it. Restructuring is associated with utility costs $C(\eta_{t+1}, s_t)$ and exclusion from issuing debt for at least one period.

$$C(\eta_{t+1}, s_t) = \begin{cases} \mu + \mu_z \log(z_t) & \text{if } \eta_{t+1} > 0 \\ 0 & \text{else} \end{cases}$$

Credit standing for next period is determined as

$$d_{t+1} = \begin{cases} 0 & \text{if } \eta_{t+1} = 0 \text{ and } d_t = 0 \\ 0 & \text{if } \eta_{t+1} = \bar{\eta} \text{ or } d_t = 1, \text{ with probability } \psi \\ 1 & \text{if } \eta_{t+1} = \bar{\eta} \text{ or } d_t = 1, \text{ with probability } 1 - \psi \end{cases}$$

If government begins the period in good credit standing $d_t = 0$ and chooses not to restructure debt $\eta_{t+1} = 0$, then its problem is given by

$$V^n(s_t, \eta_{t+1}, \hat{\delta}_t) = \max_{b_{t+1}, T_t^h} \left\{ U^h(z_t, T_t^h) + \beta_g \mathbb{E}_t \mathcal{V}(s_{t+1} | s_t, \eta_{t+1}, \hat{\delta}_t) \right\}$$
$$T_t^h + (\lambda_p + \kappa_p) b_t + (\lambda_g + \kappa_g) f_t = q_t(s_t, \eta_{t+1}, \hat{\delta}_t, b_{t+1}, T_t^h)(b_{t+1} - (1 - \lambda_p) b_t) + \hat{\delta}_t - i_t$$
$$f_{t+1} = (1 - \lambda_g) f_t + \hat{\delta}_t$$

where i_t is issuance cost introduced for technical reasons (see Chatterjee and Eyigungor (2015) for details):

$$i_t = i\left(\underbrace{\mathbb{E}_t\{\eta_{t+2}(s_{t+1}) > 0\}}_{\text{one period ahead default probability}}\right) q_t(s_t, \eta_{t+1}, \hat{\delta}_t, b_{t+1}, T_t^h)(b_{t+1} - (1 - \lambda_p) b_t)$$
$$i(x) = \begin{cases} \frac{1}{2}\left(1 + \sin\left(\left(\frac{x - \bar{d}}{1 - \bar{d}} - \frac{1}{2}\right)\pi\right)\right) & \text{if } b_{t+1} - (1 - \lambda_p) > 0 \\ 0 & \text{if } b_{t+1} - (1 - \lambda_p) \leqslant 0 \end{cases}$$

If the government begins the period in bad credit standing $d_t = 1$ or chooses to restructure debt in current period $\eta_{t+1} = \bar{\eta}$, then its problem is given by

$$\begin{aligned}
& V^d\left(s_t, \eta_{t+1}, \hat{\delta}_t\right) = U^h\left(z_t, T_t^h\right) - C\left(\eta_{t+1}, s_t\right) + \beta_g \mathbb{E}_t \mathcal{V}\left(s_{t+1} | s_t, \eta_{t+1}, \hat{\delta}_t\right) \\
\text{s.t.}\quad & T_t^h + \left(\lambda_d + \kappa_d\right) b_t + \left(\lambda_g + \kappa_g\right) f_t = \hat{\delta}_t \\
& b_{t+1} = \left(1 - \eta_{t+1}\right)\left(1 - \lambda_d\right) b_t \\
& f_{t+1} = \left(1 - \lambda_g\right) f_t + \hat{\delta}_t
\end{aligned}$$

When government chooses whether to restructure its debt or not, it solves the following problem

$$\begin{aligned}
\mathcal{V}\left(s_t\right) = \max_{\eta_{t+1} \in \{0, \bar{\eta}\}} & \left\{ d_t \bar{V}^d\left(s_t, \eta_{t+1}\right) + \left(1 - d_t\right) \cdot \right. \\
& \left. \cdot \left[\mathbb{I}\{\eta_{t+1} > 0\} \bar{V}^d\left(s_t, \eta_{t+1}\right) + \mathbb{I}\{\eta_{t+1} = 0\} \bar{V}^n\left(s_t, \eta_{t+1}\right)\right] \right\}
\end{aligned}$$

where

$$\begin{aligned}
\bar{V}^d\left(s_t, \eta_{t+1}\right) &= \int V^d\left(s_t, \eta_{t+1}, \hat{\delta}_t\right) dF\left(\hat{\delta}_t | \eta_{t+1}, \delta_t\right) \\
\bar{V}^n\left(s_t, \eta_{t+1}\right) &= \int V^n\left(s_t, \eta_{t+1}, \hat{\delta}_t\right) dF\left(\hat{\delta}_t | \eta_{t+1}, \delta_t\right)
\end{aligned}$$

1.5 Foreign investors

Zero profit condition determines the price:

$$\begin{aligned}
q\left(\underbrace{s_t, \eta_{t+1}, \hat{\delta}_t, b_{t+1}, T_t^h}_{x_t}\right) = & \frac{1}{1+r} \mathbb{E}_t \left\{ \left(1 - d_{t+1}\right) \mathbb{I}\{\eta_{t+2}\left(s_{t+1}\right) = 0\} \left[\lambda_p + \kappa_p + \left(1 - \lambda_p\right) q\left(x_{t+1}\right)\right] \right. \\
& \left. + \left(d_{t+1} + \left(1 - d_{t+1}\right) \mathbb{I}\{\eta_{t+2}\left(s_{t+1}\right) > 0\}\right) \left[\lambda_d + \kappa_d + \left(1 - \eta_{t+2}\left(s_{t+1}\right)\right)\left(1 - \lambda_d\right) q\left(x_{t+1}\right)\right] \right.
\end{aligned}$$

1.6 Process for official loans

Value of new official loans takes one of three values: $\delta = (0, \delta_L, \delta_H)$. Promises of official loans at the beginning of the period evolve according to:

$$\Pi_{\delta_t | \hat{\delta}_{t-1}} = \begin{pmatrix} 1 & 0 & 0 \\ \pi_\delta & 1 - \pi_\delta & 0 \\ 0 & \pi_\delta & 1 - \pi_\delta \end{pmatrix}$$

In other words, if low or high value of official loans realized last period, then the promised amount either stays the same or decreases.

Actual amount of new official loans is determined according to:

$$\Pi_{\hat{\delta}_t|(\delta_t,\eta_{t+1}=\bar{\eta})} = \begin{pmatrix} 1-\pi_{\hat{\delta}} & \pi_{\hat{\delta}}/2 & \pi_{\hat{\delta}}/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Pi_{\hat{\delta}_t|(\delta_t,\eta_{t+1}=0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If low or high amount of official loans was promised, then it is realized. If zero amount of official loans was promised and government does not default, then zero amount is realized. If zero amount of official loans was promised and government defaults, then with some probability positive amount of official loans is realized.

1.7 Definition of equilibrium

Equilibrium consists of policy functions which describe household choices $c^h(x), l^h(x)$, policy functions which describe government choices $\eta'(s), b'\left(s,\hat{\delta}\right), T^h\left(s,\hat{\delta}\right)$ and pricing function $q(x)$ such that

1. Households maximize their values
2. Government maximizes its value
3. Price of debt satisfies zero profit condition of foreign investors

1.8 Computational method

1. Finding optimal consumption and labor involves solving a non-linear equation in consumption. In order to improve computational efficiency, we are going to introduce a grid for the amount of resources which government can get from issuing debt NB_t. When the government is in good credit standing and does not declare default, this amount must be equal to:

   $$NB_t = q_t\left(s_t,\eta_{t+1},\hat{\delta}_t,b_{t+1},T_t^h\right)(b_{t+1}-(1-\lambda_p)b_t)+\hat{\delta}_t-i_t-(\lambda_p+\kappa_p)b_t-(\lambda_g+\kappa_g)f_t$$

   When the government is in bad credit standing or restructured debt in the current period (let's denote it as a state of default, $\text{default}_t = 1$), this value must be equal to

   $$NB_t = \hat{\delta}_t-(\lambda_d+\kappa_d)b_t-(\lambda_g+\kappa_g)f_t$$

   This value determines the amount of transfers T_t^h. Together with realization of z_t it determines equilibrium labor and consumption:

$$
\begin{aligned}
T_t^h &= NB_t \\
l_t^h &= \left(\frac{z_t\theta^h}{\chi^h}\left(c_t^h\right)^{-\sigma}\right)^{\varepsilon} \\
c_t^h &= z_t\theta^h\left(\frac{z_t\theta^h}{\chi^h}\left(c_t^h\right)^{-\sigma}\right)^{\varepsilon} + T_t^h
\end{aligned}
$$

So, before starting iterations we compute optimal labor and consumption on the grid of (NB_t, z_t). Then we are going to use this result in our iterative procedure.

2. When starting a new iteration, we have a guess for expected value function of the government, price function, probability of restructure next period

$$
\begin{aligned}
&\mathbb{E}_t\mathcal{V}\left(s_{t+1}, b_{t+1}|s_t, \eta_{t+1}, \hat{\delta}_t\right) \\
&q\left(s_t, \eta_{t+1}, \hat{\delta}_t, b_{t+1}\right) \\
&\Pr\left(\eta_{t+2}\left(s_{t+1}\right) > 0|s_t, \eta_{t+1}, \hat{\delta}_t\right)
\end{aligned}
$$

Initial guess is zero expected value for all combinations of state variables, price equal to $\frac{1}{1+r}$ for all combinations of state variables and zero restructuring probability for all state variables.

We are going to store value and policy functions in the following matrix:

$$
\begin{array}{cccccccc}
 & (z_1, 0) & (z_1, \delta_L) & (z_1, \delta_H) & (z_2, 0) & (z_2, \delta_L) & (z_2, \delta_H) & \dots \\
(f_1, b_1) & & & & & & & \\
(f_1, b_2) & & & & & & & \\
\dots & & & & & & & \\
(f_1, b_n) & & & & & & & \\
(f_2, b_1) & & & & & & & \\
(f_2, b_2) & & & & & & & \\
\dots & & & & & & & \\
(f_2, b_n) & & & & & & & \\
\dots & & & & & & &
\end{array}
$$

It would become evident that such a matrix is sufficient to store all the information we need. In particular, we will store all value functions and policy functions as the following objects:

$$V^{d,nr}\left(b_t, f_t, z_t, \hat{\delta}_t\right) = V^d\left(b_t, f_t, z_t, \hat{\delta}_t, d_t = 1, \eta_{t+1} = 0\right)$$
$$V^{d,r}\left(b_t, f_t, z_t, \hat{\delta}_t\right) = V^d\left(b_t, f_t, z_t, \hat{\delta}_t, d_t, \eta_{t+1} = \bar{\eta}\right)$$
$$V^n\left(b_t, f_t, z_t, \hat{\delta}_t\right) = V^n\left(b_t, f_t, z_t, \hat{\delta}_t, d_t = 0, \eta_{t+1} = 0\right)$$
$$\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t = 1\right) = \mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}, d_{t+1}}_{\text{take expectation over these}} | z_t, \hat{\delta}_t, \text{default}_t = 1\right)$$
$$\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t = 0\right) = \mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 0 | z_t, \hat{\delta}_t, \text{default}_t = 0\right)$$
$$\mathcal{V}^{bc}\left(b_t, f_t, z_t, \delta_t\right) = \mathcal{V}\left(b_t, f_t, z_t, \delta_t, d_t = 1\right)$$
$$\mathcal{V}^{gc}\left(b_t, f_t, z_t, \delta_t\right) = \mathcal{V}\left(b_t, f_t, z_t, \delta_t, d_t = 0\right)$$
$$q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t = 0\right)$$
$$q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t = 1\right)$$
$$\Pr\left(\eta_{t+2} > 0 | b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t = 1\right)$$
$$\Pr\left(\eta_{t+2} > 0 | b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t = 0\right)$$

3. Overall the procedure works as follows:

   (a) Enter iteration n with a guess for prices: $q^n\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t\right)$. For $n = 0$ guess that price equals $\frac{1}{1+r}$ for all combinations of state variables.

   (b) Enter iteration m with a guess for expected value function $\mathbb{E}_t^{n,m}\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$ and probabilities of debt restructuring $\Pr^{n,m}\left(\eta_{t+2} > 0 | b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$. For $n = 0, m = 0$ use zero expected value and zero restructuring probability for all combinations of state variables. For arbitrary n and $m = 0$ use $\mathbb{E}_t^{n-1,\bar{m}}\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$ and $\Pr^{n-1,\bar{m}}\left(\eta_{t+2} > 0 | b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$ as guesses, i.e. expected value function and probabilities which we converged to using the previous guess for prices.

   (c) Update all value functions, policy functions and restructuring probabilities in accordance with points 4-7 described below. Overall, we proceed as follows:

      i. Solve the problem of the government after new official loans has been realized. Compute $V^{d,nr}\left(b_t, f_t, z_t, \hat{\delta}_t\right)$, $V^{d,r}\left(b_t, f_t, z_t, \hat{\delta}_t\right)$, $V^n\left(b_t, f_t, z_t, \hat{\delta}_t\right)$ and policy function for new private debt conditional on good credit standing and not restructuring.

      ii. Solve the problem of the government after promise of new official loans has been realized. Compute $\mathcal{V}\left(b_t, f_t, z_t, \delta_t, d_t\right)$ and policy function which determines whether the government chooses to restructure its debt or not.

      iii. Compute expected value functions $\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$ and probabilities of restructuring $\Pr\left(\eta_{t+2} > 0 | b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$.

   (d) Check convergence of expected value functions $\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$. If there is no convergence, then update the guess for the new iteration as follows:

$$\mathbb{E}_t^{n,m+1}\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right) =$$
$$= \alpha_F \mathbb{E}_t^{n,m}\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right) + (1-\alpha_F)\,\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$$

(e) Update prices $q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t\right)$ in accordance with policy functions obtained from iterative procedure on expected value function (see point 8 below for details). Check convergence of prices. If there is no convergence, then update the guess for the new iteration as follows:

$$q^{n+1}\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t\right) =$$
$$= \tilde{\alpha}_F q^n\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t\right) + (1-\tilde{\alpha}_F)\, q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t\right)$$

4. When solving the problem of the government after new official loans has been realized for each $s_t, \eta_{t+1}, \hat{\delta}_t$, we proceed as follows.
If the government begins the period in bad credit standing $d_t = 1$ and chose not to restructure its debt in the current period $\eta_{t+1} = 0$, then it does not make any choices. Value function is given by

$$V^{d,nr}\left(b_t, f_t, z_t, \hat{\delta}_t\right) = U^h\left(z_t, NB_t\right) + \beta_g \mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}, d_{t+1}}_{\text{take expectation over these}} |z_t, \hat{\delta}_t, \text{default}_t = 1\right)$$

$$\text{s.t.}\quad NB_t = \hat{\delta}_t - (\lambda_d + \kappa_d)\, b_t - (\lambda_g + \kappa_g)\, f_t$$
$$b_{t+1} = (1-\lambda_d)\, b_t$$
$$f_{t+1} = (1-\lambda_g)\, f_t + \hat{\delta}_t$$

If the government chose to restructure debt $\eta_{t+1} = \bar{\eta}$ in the current period, then it does not make any choices

$$V^{d,r}\left(b_t, f_t, z_t, \hat{\delta}_t\right) = U^h\left(z_t, NB_t\right) - \mu - \mu_z \log\left(z_t\right) +$$
$$+ \beta_g \mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}, d_{t+1}}_{\text{take expectation over these}} |z_t, \hat{\delta}_t, \text{default}_t = 1\right)$$

$$\text{s.t.}\quad NB_t = \hat{\delta}_t - (\lambda_d + \kappa_d)\, b_t - (\lambda_g + \kappa_g)\, f_t$$
$$b_{t+1} = (1-\bar{\eta})(1-\lambda_d)\, b_t$$
$$f_{t+1} = (1-\lambda_g)\, f_t + \hat{\delta}_t$$

If the government enters the period in good credit standing and does not restructure its debt, then it can choose new value of debt

$$
\begin{aligned}
& V^n\left(b_t, f_t, z_t, \hat{\delta}_t\right) = \max_{b_{t+1}} \left\{U^h\left(z_t, NB_t\right) + \right. \\
& \left. +\beta_g \mathbb{E}_t \mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 0 | z_t, \hat{\delta}_t, \text{default}_t = 0\right)\right\} \\
\text{s.t.} \quad & NB_t = q_t\left(b_{t+1} - \left(1-\lambda_p\right) b_t\right) + \hat{\delta}_t - i_t\left(\mathbb{E}_t\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, d_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}\right) > 0\right\}, q_t, b_{t+1}, b_t\right) \\
& -\left(\lambda_p + \kappa_p\right) b_t - \left(\lambda_g + \kappa_g\right) f_t \\
& q_t = q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t = 0\right) \\
& f_{t+1} = \left(1-\lambda_g\right) f_t + \hat{\delta}_t
\end{aligned}
$$

5. When solving the problem of the government after promise of new official loans has been realized for each s_t, we proceed as follows.
   If the government begins period in bad credit standing $d_t = 1$, then the problem is

$$
\begin{aligned}
& \mathcal{V}\left(b_t, f_t, z_t, \delta_t, d_t = 1\right) = \mathcal{V}^{bc}\left(b_t, f_t, z_t, \delta_t\right) = \\
& = \max_{\eta_{t+1} \in \{0, \bar{\eta}\}} \left\{\mathbb{I}\{\eta_{t+1} > 0\} \bar{V}^{d,r}\left(b_t, f_t, z_t, \delta_t\right) + \mathbb{I}\{\eta_{t+1} = 0\} V^{d,nr}\left(b_t, f_t, z_t, \underbrace{\delta_t}_{=\hat{\delta}_t}\right)\right\} \\
& \bar{V}^{d,r}\left(b_t, f_t, z_t, \delta_t\right) = \sum_{\hat{\delta}_t} \Pr\left(\hat{\delta}_t | \eta_{t+1} = \bar{\eta}, \delta_t\right) V^{d,r}\left(b_t, f_t, z_t, \hat{\delta}_t\right)
\end{aligned}
$$

   Solution to this problem is the policy function $\eta'_{bc}\left(b_t, f_t, z_t, \delta_t\right) = \eta'\left(b_t, f_t, z_t, \delta_t, d_t = 1\right)$
   If the government begins period in good credit standing $d_t = 0$, then the problem is

$$
\begin{aligned}
& \mathcal{V}\left(b_t, f_t, z_t, \delta_t, d_t = 0\right) = \mathcal{V}^{gc}\left(b_t, f_t, z_t, \delta_t\right) = \\
& = \max_{\eta_{t+1} \in \{0, \bar{\eta}\}} \left\{\mathbb{I}\{\eta_{t+1} > 0\} \bar{V}^{d,r}\left(b_t, f_t, z_t, \delta_t\right) + \mathbb{I}\{\eta_{t+1} = 0\} V^{n}\left(b_t, f_t, z_t, \underbrace{\delta_t}_{=\hat{\delta}_t}\right)\right\} \\
& \bar{V}^{d,r}\left(b_t, f_t, z_t, \delta_t\right) = \sum_{\hat{\delta}_t} \Pr\left(\hat{\delta}_t | \eta_{t+1} = \bar{\eta}, \delta_t\right) V^{d,r}\left(b_t, f_t, z_t, \hat{\delta}_t\right)
\end{aligned}
$$

   Solution to this problem is the policy function $\eta'_{gc}\left(b_t, f_t, z_t, \delta_t\right) = \eta'\left(b_t, f_t, z_t, \delta_t, d_t = 0\right)$

6. To compute expected value functions, we use the following formulas:

$$\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}, d_{t+1}}_{\text{take expectation over these}} |z_t, \hat{\delta}_t, \text{default}_t = 1\right) =$$
$$= \psi \sum_{\delta_{t+1}} \sum_{z_{t+1}} \Pr\left(z_{t+1}|z_t\right) \Pr\left(\delta_{t+1}|\hat{\delta}_t\right) \mathcal{V}\left(b_{t+1}, f_{t+1}, z_{t+1}, \delta_{t+1}, d_{t+1} = 0\right) +$$
$$+ (1-\psi) \sum_{\delta_{t+1}} \sum_{z_{t+1}} \Pr\left(z_{t+1}|z_t\right) \Pr\left(\delta_{t+1}|\hat{\delta}_t\right) \mathcal{V}\left(b_{t+1}, f_{t+1}, z_{t+1}, \delta_{t+1}, d_{t+1} = 1\right)$$

$$\mathbb{E}_t\mathcal{V}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}, d_{t+1}}_{\text{take expectation over these}} |z_t, \hat{\delta}_t, \text{default}_t = 0\right) =$$
$$= \sum_{\delta_{t+1}} \sum_{z_{t+1}} \Pr\left(z_{t+1}|z_t\right) \Pr\left(\delta_{t+1}|\hat{\delta}_t\right) \mathcal{V}\left(b_{t+1}, f_{t+1}, z_{t+1}, \delta_{t+1}, d_{t+1} = 0\right)$$

7. To compute probabilities of restructuring, we use the following formulas:

$$\Pr\left(\eta_{t+2}\left(s_{t+1}\right) > 0|s_t, \eta_{t+1}, \hat{\delta}_t\right) = \Pr\left(\eta_{t+2} > 0|b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right)$$
$$= \mathbb{E}_t\left\{\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, d_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}\right) > 0\right\} |b_{t+1}, f_{t+1}, z_t, \hat{\delta}_t, \text{default}_t\right\}$$

default_t takes value of 1 if $d_t = 1$ or $\eta_{t+1} = \bar{\eta}$ and 0 otherwise. If $\text{default}_t = 1$, then probability of restructure next period is

$$\Pr\left(\eta_{t+2}\left(s_{t+1}\right) > 0|s_t, \eta_{t+1}, \hat{\delta}_t\right) =$$
$$= \psi \sum_{\delta_{t+1}} \sum_{z_{t+1}} \Pr\left(z_{t+1}|z_t\right) \Pr\left(\delta_{t+1}|\hat{\delta}_t\right) \mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, z_{t+1}, \delta_{t+1}, d_{t+1} = 0\right) > 0\right\} +$$
$$+ (1-\psi) \sum_{\delta_{t+1}} \sum_{z_{t+1}} \Pr\left(z_{t+1}|z_t\right) \Pr\left(\delta_{t+1}|\hat{\delta}_t\right) \mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, z_{t+1}, \delta_{t+1}, d_{t+1} = 1\right) > 0\right\}$$

If $\text{default}_t = 0$, then probability of restructure next period is

$$\Pr\left(\eta_{t+2}\left(s_{t+1}\right) > 0|s_t, \eta_{t+1}, \hat{\delta}_t\right) =$$
$$= \sum_{\delta_{t+1}} \sum_{z_{t+1}} \Pr\left(z_{t+1}|z_t\right) \Pr\left(\delta_{t+1}|\hat{\delta}_t\right) \mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, z_{t+1}, \delta_{t+1}, d_{t+1} = 0\right) > 0\right\} +$$

8. To compute prices $q\left(\underbrace{b_t, f_t, z_t, d_t, \delta_t, \eta_{t+1}, \hat{\delta}_t, b_{t+1}, T_t^h}_{x_t}\right) = q\left(b_{t+1}, f_t, z_t, d_t, \eta_{t+1}, \hat{\delta}_t\right)$, we use the following formulas (notice that price is defined for f_t, not f_{t+1})

$$
\begin{aligned}
& q\left(\underbrace{b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t}_{\tilde{x}_t}\right) = \\
& = \frac{1}{1+r}\mathbb{E}_t\left\{(1-d_{t+1})\,\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1}\right)=0\right\}\cdot\right. \\
& \cdot\left[\lambda_p + \kappa_p + (1-\lambda_p)\, q\left(\tilde{x}_{t+1}\right)\right] + \\
& + \left(d_{t+1} + (1-d_{t+1})\,\mathbb{I}\{\eta_{t+2}\left(s_{t+1}\right) > 0\}\right)\cdot \\
& \cdot\left[\lambda_d + \kappa_d + \left(1-\eta_{t+2}\left(s_{t+1}\right)\right)(1-\lambda_d)\, q\left(\tilde{x}_{t+1}\right)\right] = \\
& = \frac{1}{1+r}\mathbb{E}_t\left\{(1-d_{t+1})\,\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1}\right)=0\right\}\cdot\right. \\
& \cdot\left[\lambda_p + \kappa_p + (1-\lambda_p)\, q\left(\underbrace{b_{t+2}, z_{t+1}, \hat{\delta}_{t+1}}_{\text{take expectation over these}}, f_{t+1}, \text{default}_{t+1}=0\right)\right] \\
& + \left(d_{t+1} + (1-d_{t+1})\,\mathbb{I}\{\eta_{t+2}\left(s_{t+1}\right) > 0\}\right)\cdot \\
& \left.\cdot\left[\lambda_d + \kappa_d + \left(1-\eta_{t+2}\left(s_{t+1}\right)\right)(1-\lambda_d)\, q\left(b_{t+2}, f_{t+1}, z_{t+1}, \hat{\delta}_{t+1}, \text{default}_{t+1}=1\right)\right]\right\}
\end{aligned}
$$

If $\text{default}_t = 0$, then the price is

$$
\begin{aligned}
&q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t = 0\right) = \\
&= \frac{1}{1+r}\mathbb{E}_t\left\{(1-d_{t+1})\,\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1}\right) = 0\right\}\cdot\right. \\
&\cdot\left[\lambda_p + \kappa_p + (1-\lambda_p)\,q\left(\tilde{x}_{t+1}\right)\right] + \\
&+\left(d_{t+1} + (1-d_{t+1})\,\mathbb{I}\{\eta_{t+2}\left(s_{t+1}\right) > 0\}\right)\cdot \\
&\left.\cdot\left[\lambda_d + \kappa_d + (1-\eta_{t+2}\left(s_{t+1}\right))\left(1-\lambda_d\right) q\left(\tilde{x}_{t+1}\right)\right]\right\} = \\
&= \frac{1}{1+r}\mathbb{E}_t\left\{\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 0\right) = 0\right\}\cdot\right. \\
&\cdot\left[\lambda_p + \kappa_p + (1-\lambda_p)\,q\left(\underbrace{b_{t+2}, z_{t+1}, \hat{\delta}_{t+1}}_{\text{take expectation over these}}, f_{t+1}, \text{default}_{t+1} = 0\right)\right] + \\
&+\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 0\right) > 0\right\}\cdot \\
&\left.\cdot\left[\lambda_d + \kappa_d + (1-\bar{\eta})\left(1-\lambda_d\right) q\left(\underbrace{b_{t+2}, z_{t+1}, \hat{\delta}_{t+1}}_{\text{take expectation over these}}, f_{t+1}, \text{default}_{t+1} = 1\right)\right]\right\}
\end{aligned}
$$

If $\text{default}_t = 1$, then the price is

$$q\left(b_{t+1}, f_t, z_t, \hat{\delta}_t, \text{default}_t = 1\right) =$$

$$= \frac{1}{1+r}\mathbb{E}_t\left\{(1-d_{t+1})\,\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1}\right) = 0\right\}\cdot\right.$$

$$\cdot\left[\lambda_p + \kappa_p + (1-\lambda_p)\,q\,(\tilde{x}_{t+1})\right] +$$

$$+\left(d_{t+1} + (1-d_{t+1})\,\mathbb{I}\{\eta_{t+2}(s_{t+1}) > 0\}\right)\cdot$$

$$\left.\cdot\left[\lambda_d + \kappa_d + (1-\eta_{t+2}(s_{t+1}))\,(1-\lambda_d)\,q\,(\tilde{x}_{t+1})\right]\right\} =$$

$$= \psi\frac{1}{1+r}\mathbb{E}_t\left\{\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 0\right) = 0\right\}\cdot\right.$$

$$\left[\lambda_p + \kappa_p + (1-\lambda_p)\,q\left(\underbrace{b_{t+2}, z_{t+1}, \hat{\delta}_{t+1}}_{\text{take expectation over these}}, f_{t+1}, \text{default}_{t+1} = 0\right)\right] +$$

$$+\,\mathbb{I}\left\{\eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 0\right) > 0\right\}\cdot$$

$$\left.\cdot\left[\lambda_d + \kappa_d + (1-\bar{\eta})\,(1-\lambda_d)\,q\left(\underbrace{b_{t+2}, z_{t+1}, \hat{\delta}_{t+1}}_{\text{take expectation over these}}, f_{t+1}, \text{default}_{t+1} = 1\right)\right]\right\} +$$

$$+\,(1-\psi)\,\frac{1}{1+r}\mathbb{E}_t\left\{\lambda_d + \kappa_d + \left(1 - \eta_{t+2}\left(b_{t+1}, f_{t+1}, \underbrace{z_{t+1}, \delta_{t+1}}_{\text{take expectation over these}}, d_{t+1} = 1\right)\right)\cdot\right.$$

$$\left.\cdot\,(1-\lambda_d)\,q\left(\underbrace{b_{t+2}, z_{t+1}, \hat{\delta}_{t+1}}_{\text{take expectation over these}}, f_{t+1}, \text{default}_{t+1} = 1\right)\right\}$$

1.9 Results

For this exercise we choose the same model parameters as the ones used by Fourakis, Karabarbounis (2024). We use three-point grid for technology z, five-point grid for accumulated level of official loans f and fifteen-point grid for bonds b.

First, let's examine price schedules. Figure 1 shows how realization of z impacts pricing bond schedule for different values of private debt b' which can be chosen by the government.

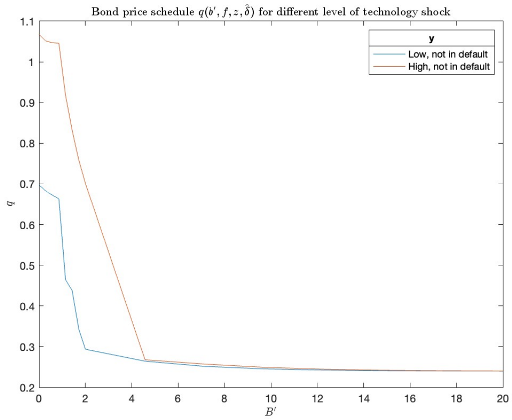
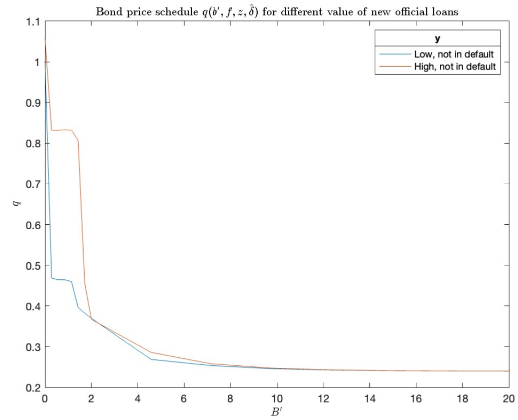


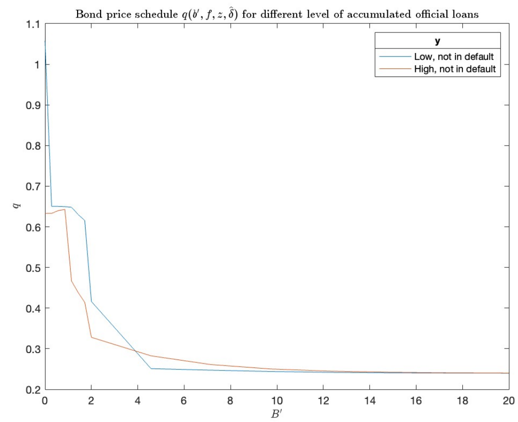
Figure 1: Price schedule for private bonds conditional on $f = 10$, $\hat{\delta} = \delta_L$: comparison of high and low realization of technology shock

Higher level of debt reduces the price as it increases probability of restructuring next period. In addition, price is higher in case of higher value of technology shock as persistency of technological process ensures that probability of high state is more likely after high state, and probability of restructuring is lower in case of high state realization.

Figure 2 shows how realized new official loans $\hat{\delta}$ and accumulated value of official loans f impact the price schedule for privately held bonds.



(a) Comparison for different values of realized official loans $\hat{\delta} = 0$ and $\hat{\delta} = \delta_H$(z at its mean of 7.4, $f = 10$)

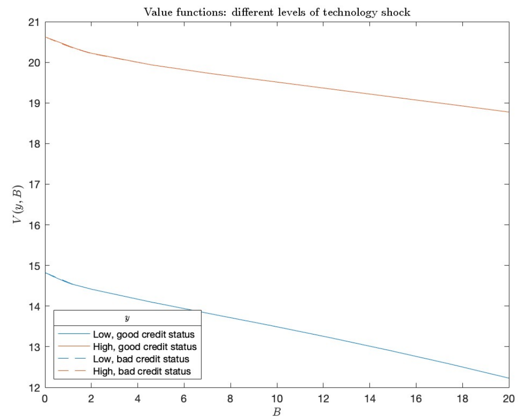


(b) Comparison for high and low value of accumulated official loans (z at its mean of 7.4, $\hat{\delta} = \delta_L$)

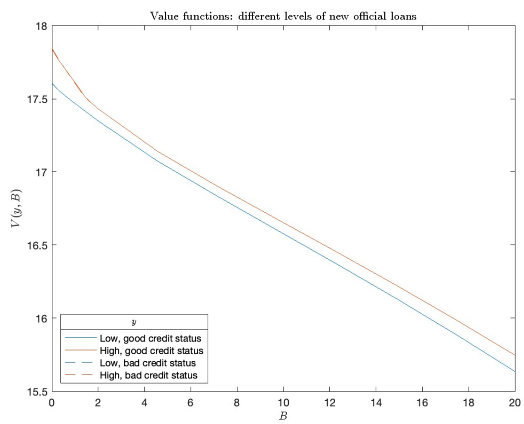
Figure 2: Price schedule for private bonds

As before, higher level of debt decreases the price. Higher realized new loans decrease the probability of restructuring tomorrow and as a consequence shift price schedule upwards. Moreover, for positive debt holdings realization of zero official loans reduces prices dramatically as then promised new official loans tomorrow are zero as well. Restructuring the debt leads to the chance of actually realized new official loans being positive and creates strong incentive to restructure private debt next period. We can see the consequences of this moral hazard issue reflected in bond pricing schedule. Higher accumulated official loans also decrease prices of private debt as it becomes difficult to service overall debt and probability of restructuring increases.

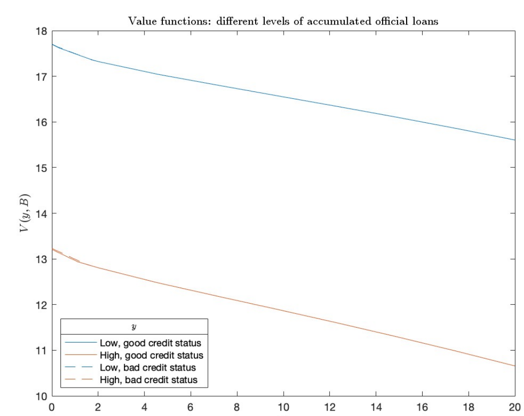
Figure 3 shows value functions $\mathcal{V}(.)$ in case of good and bad credit standing. Lower realization of technology shocks, lower new official loans and higher accumulated official loans reduce the value: both for good and bad credit standing. For higher level of accumulated debt there is no difference between good and bad credit standing as probability of restructuring is 1 (and it is priced by the market). For lower values of debt there are some differences. For example, the value of good credit is larger than the value of bad credit standing conditional on high state of technology. In this case probability of restructuring is relatively low, and access to private debt market is important. The opposite is true for low realization of z.



(a) Comparison for high and low realization of technology shock



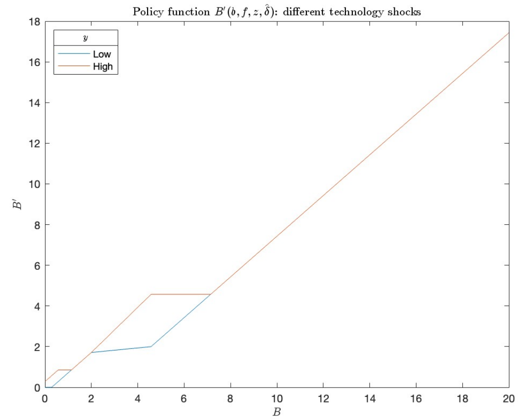
(b) Comparison for different values of realized official loans $\hat{\delta} = 0$ and $\hat{\delta} = \delta_H$



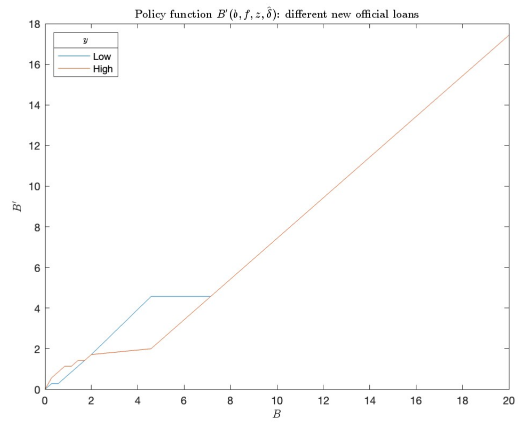
(c) Comparison for high and low value of accumulated official loans

Figure 3: Value functions at the beginning of the period when government decides whether to restructure debt

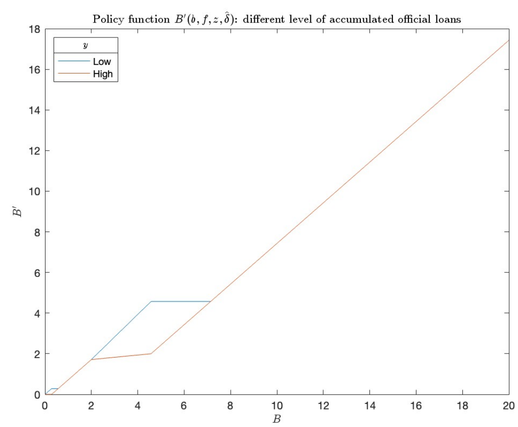
Figure 4 plots policy function for private bonds. Higher level of current debt leads to higher level of debt tomorrow. High state of technology incentivized the government to issue more debt as prices are higher. For different values of realized official loans we can see two forces. If current level of government debt is low enough, then government chooses higher level of debt when it receives higher new official loans. This is the same price effect we have seen for z. However, for intermediate levels of current debt government chooses lower level of debt when it receives higher new official loans because choosing higher levels of debt is associated with significant drop in price of debt. The sensitivity of debt price to the level of debt is lower for $\hat{\delta} = 0$, so government chooses higher level of debt when $\hat{\delta} = 0$ in comparison with $\hat{\delta} = \delta_H$.



(a) Comparison for high and low realization of technology shock



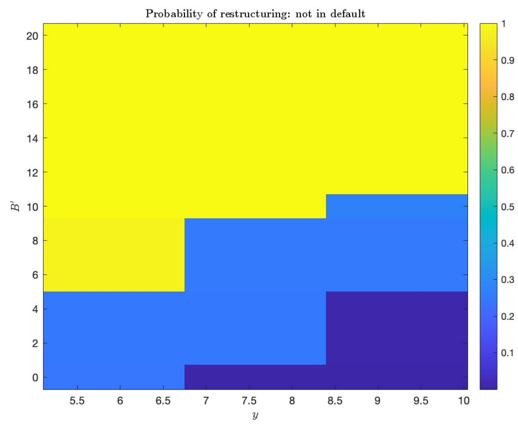
(b) Comparison for different values of realized official loans $\hat{\delta} = 0$ and $\hat{\delta} = \delta_H$



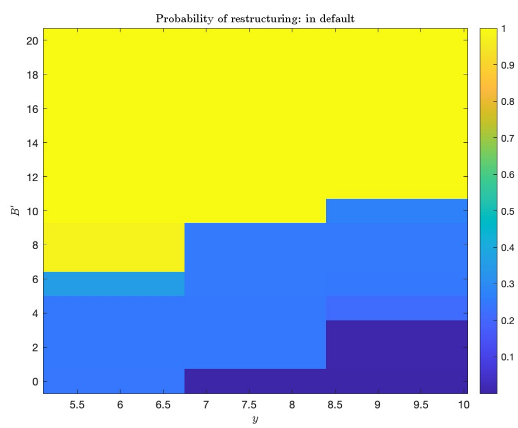
(c) Comparison for high and low value of accumulated official loans

Figure 4: Policy functions for private bonds (conditional on good credit standing and ability to issue private debt)

Figures 5-7 show probabilities of restructuring. As discussed before, higher level of accumulated private debt and lower realization of technology shock increase probability of restructuring. Similarly, higher accumulated official loans increase probability of restructuring. For positive value of debt, zero realized value of new official loans implies that government restructures its debt for sure next period. The reason is the moral hazard problem discussed before. For low level of official loans there is a region when probability of restructuring is less that 1. If the level of accumulated private debt is low enough, there is a chance that government chooses to not to restructure debt in order to keep its access to market for privately held bonds. For high level of official loans probability of restructuring in this region is even lower as subsidized loans make it easier to service debt.

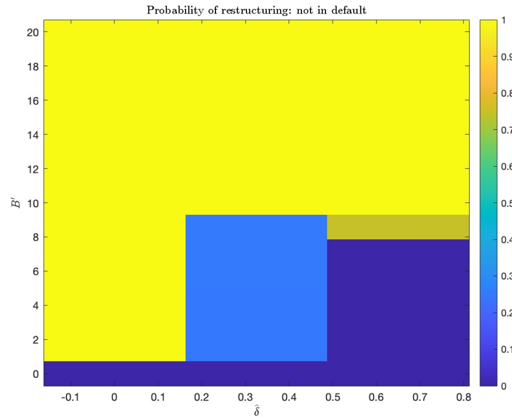


(a) Comparison for high and low realization of technology shock: $\text{default}_t = 0$

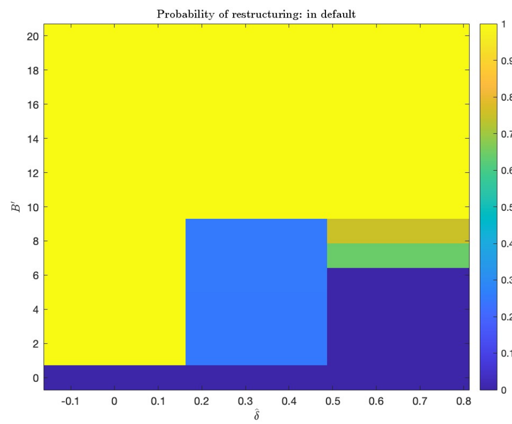


(b) Comparison for high and low realization of technology shock: $\text{default}_t = 1$

Figure 5: Probabilities of restructuring privately held debt next period

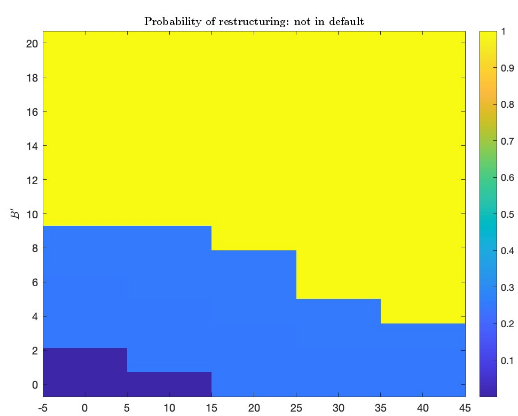


(a) Comparison for different values of realized official loans, $\hat{\delta} = 0$ and $\hat{\delta} = \delta_H$: $\text{default}_t = 0$

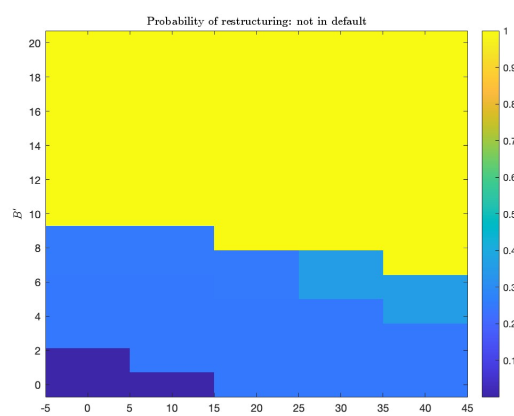


(b) Comparison for different values of realized official loans, $\hat{\delta} = 0$ and $\hat{\delta} = \delta_H$: $\text{default}_t = 1$

Figure 6: Probabilities of restructuring privately held debt next period



(a) Comparison for high and low value of accumulated official loans:$\text{default}_t = 0$



(b) Comparison for high and low value of accumulated official loans:$\text{default}_t = 1$

Figure 7: Probabilities of restructuring privately held debt next period