Image Denoising using Ising Model

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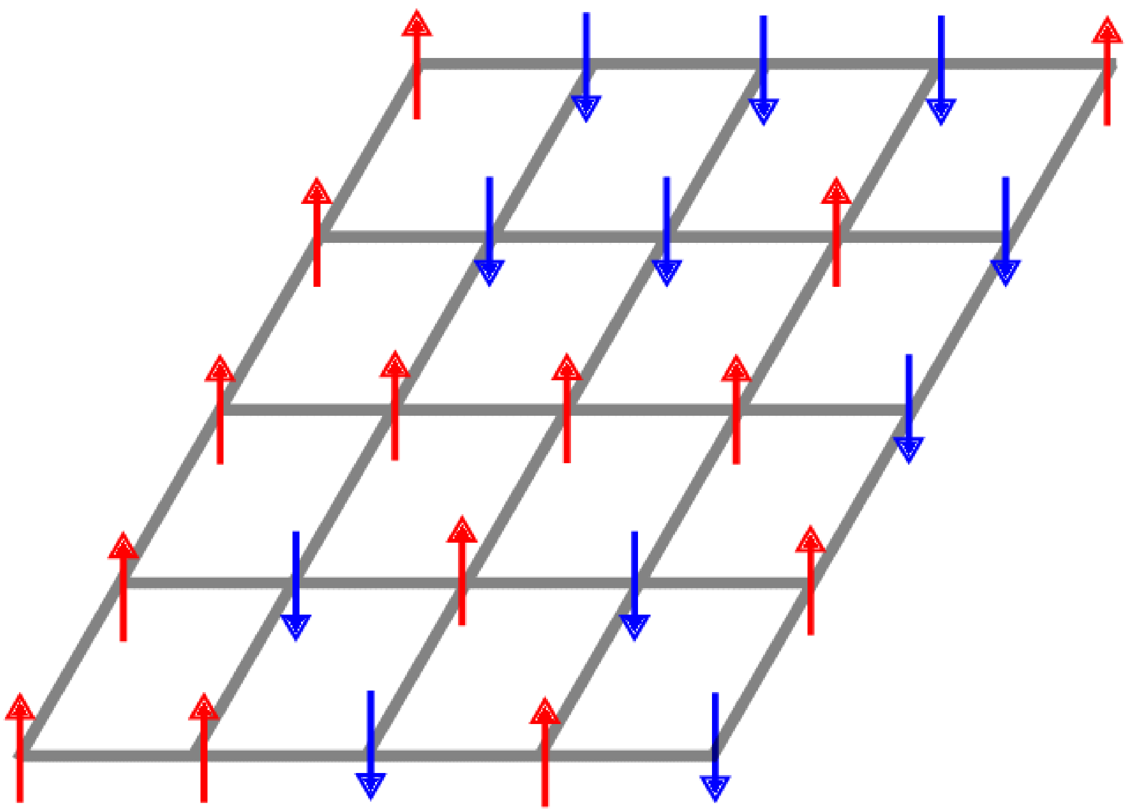
Ising model

- It’s a mathematical model describing ferromagnetism
- It consists of discrete random variables, taking values in $\{-1, +1\}$ arranged in a lattice

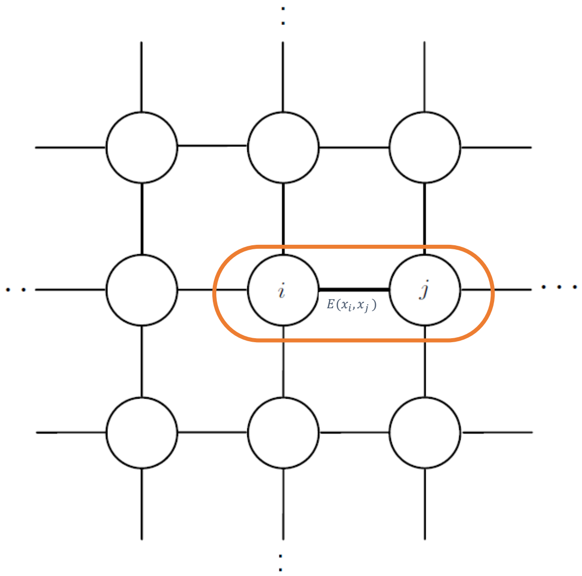


Edge potential

Each edge connecting adjacent sites carries a certain energy, given by

$$E(x) = -J \sum_{i \sim j} x_i x_j$$

- J is a constant denoting the *strenght* of the correlation between pairs of spins
- $i \sim j$ referes to all pairs of spins such that i and j are neighbours



Prior distribution

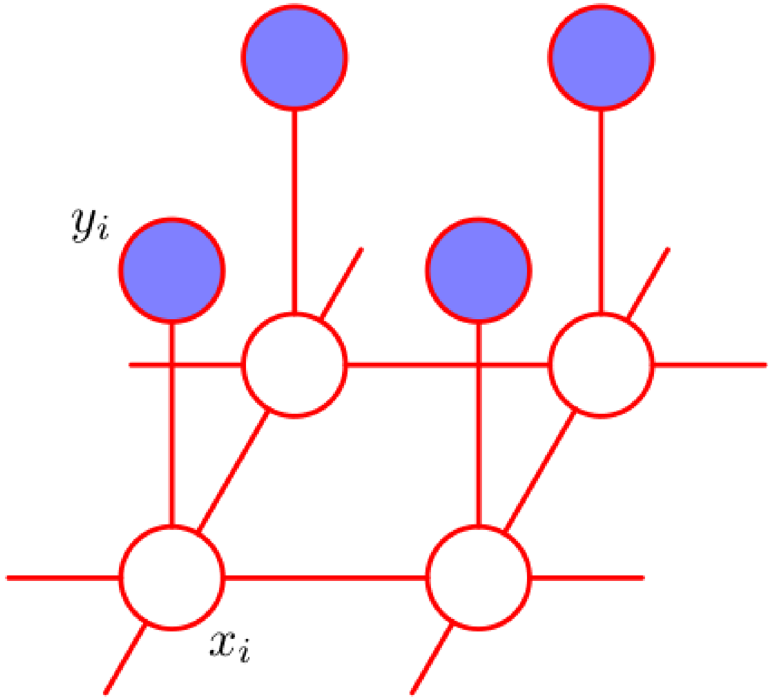
Using Boltzman distribution, we can define a probability measure over the configuration of spins, given by

$$P(X) = \frac{1}{Z}\exp\{-E(X)\} = \frac{1}{Z}\exp\left\{J\sum_{i\sim j} x_i x_j\right\}$$

Ising model for Image Denoising

- x_i are the pixels in the *unknown* noisy free image

- y_i are the pixels in the *observed* noisy image



Likelihood

We assume that the pixels in the noise free image are corrupted with an independent zero-Gaussian noise, which means

$$y_i = x_i + \varepsilon_i \qquad\qquad \varepsilon_i \sim N(0, \sigma^2) \;\; iid$$

Then the likelihood $\mathrm{P(Y|X)}$ is given by

$$P(Y|X) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - x_i)^2}{2\sigma^2}\right\}$$

Posterior Distribution

Assuming the prior distribution is given by the Ising model, we can apply Bayes formula and obtain the following posterior distribution

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

$$= \frac{1}{ZP(Y)}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - x_i)^2}{2\sigma^2} + J\sum_{i\sim j} x_i x_j\right\}$$

Approximate Inference: Sampling

Since exact inference is intractable, we will use two well-known sampling techniques to obtain an approximation of the posterior distribution

- **Metropolis-Hastings sampling**
- **Gibb’s sampling**

Metropolis-Hastings Sampling

To reach the noisy-free image X

1. Choose a random pixel y_i from the noisy image
2. Compute an acceptance probability of flipping this pixel
3. Flip the pixel with the obtained probability
4. Repeat the process until convergence is reached

Acceptance probability

Assume a bit flip for pixel x_i is proposed at iteration t ($x_i' \leftarrow -x_i^{(t)}$). Then the probability of accepting this flip is

$$\alpha = \min\left\{1, \frac{P(X'|Y)}{P(X|Y)}\right\}$$

where $\frac{P(X'|Y)}{P(X|Y)} = \exp\left\{-\frac{2x_i^{(t)}y_i}{\sigma^2} - 2J\sum_{i\sim j} x_i^{(t)}x_j^{(t)}\right\}$

Gibb's sampling

To obtain a sample from $\mathrm{P}(\mathrm{X}|\mathrm{Y})$, we will sample from $\mathrm{P}\big(x_i|\mathrm{N}(x_i)\big)$ at each iterations.

To sample from $\mathrm{P}\big(x_i|\mathrm{N}(x_i)\big)$

1. Pick a number u from $U{\sim}Unif([0,1])$
2. If $u \leq \mathrm{P}\big(x_i = 1|\mathrm{N}(x_i)\big)$, set $\tilde{x}_i = 1$
   else, set $\widetilde{x}_i = -1$

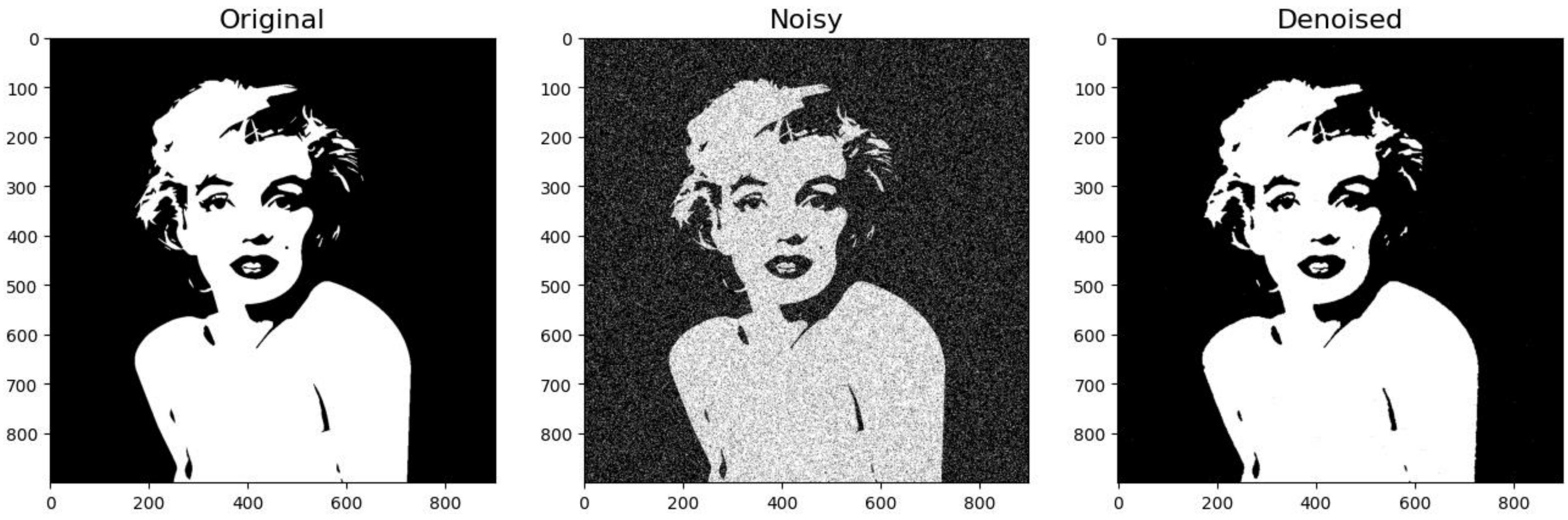
Experimental evaluation

We used a 900×900 binary image and artificially corrupted it by adding white noise.

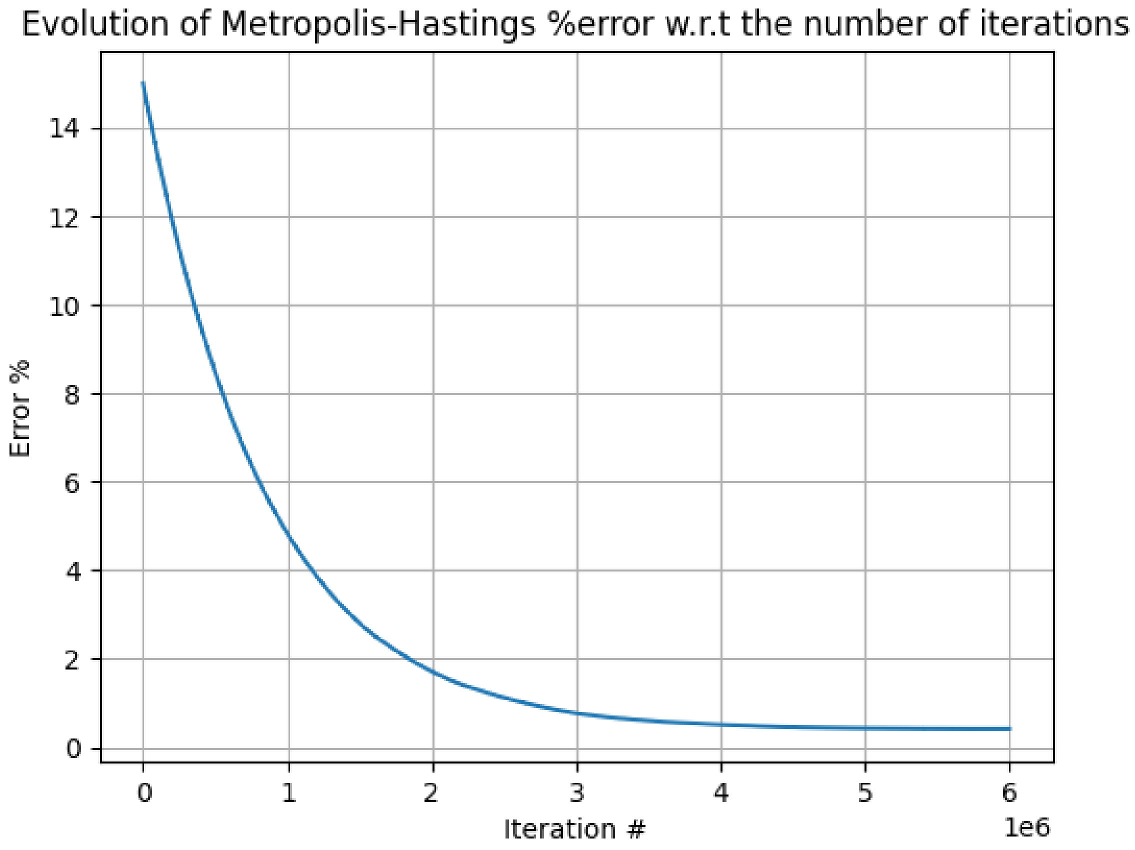
To evaluate the performances of both procedures, we used a simple loss function defined as

$$err = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\,(x_i = y_i)$$

Results using Metropolis-Hastings

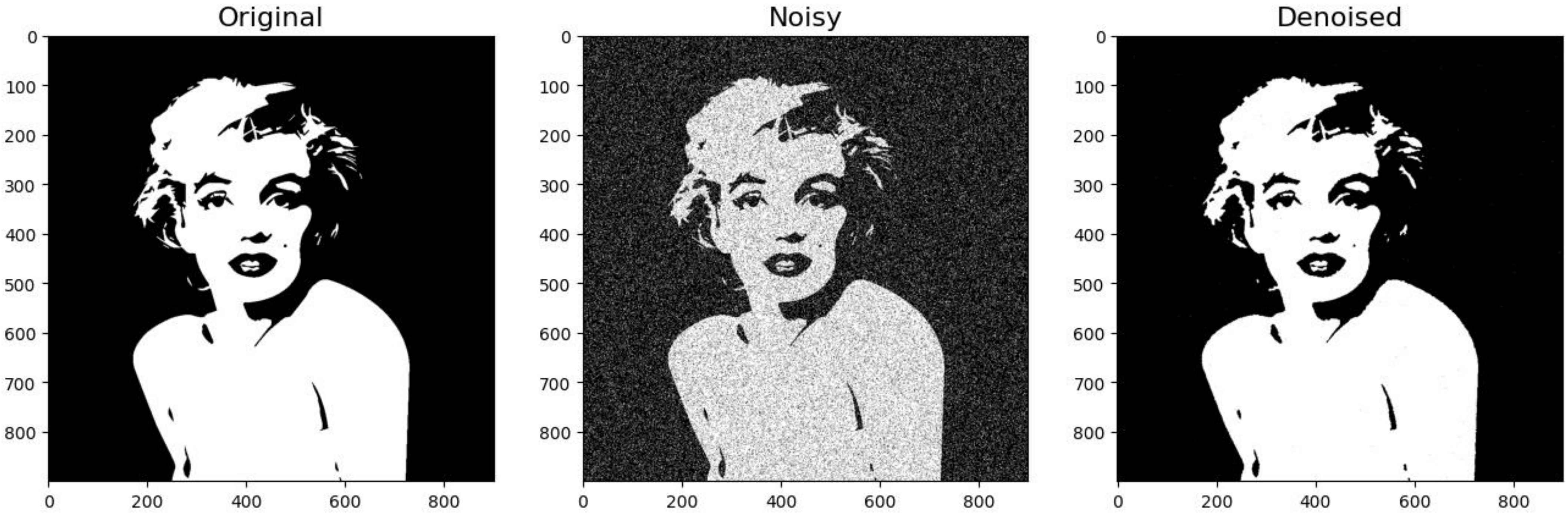


Convergence of Metropolis-Hastings

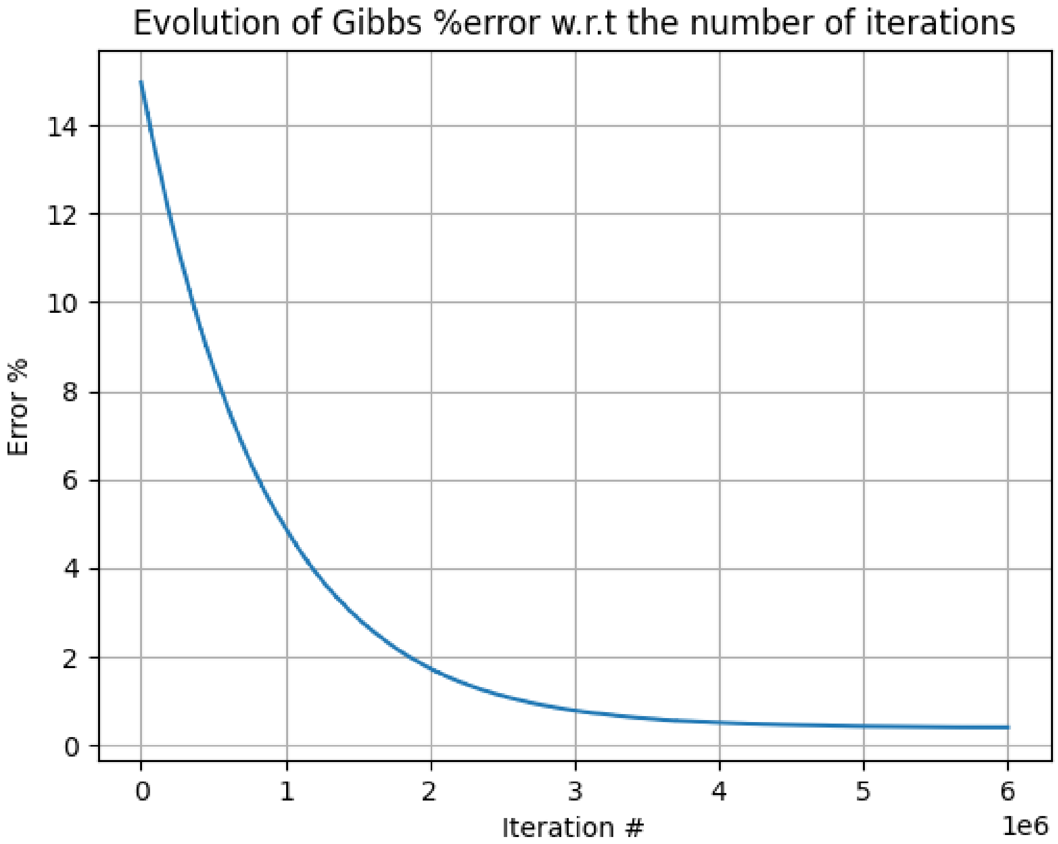


After 6×10^6 iterations, the final error is equal to $0.418\ \%$

Results using Gibbs

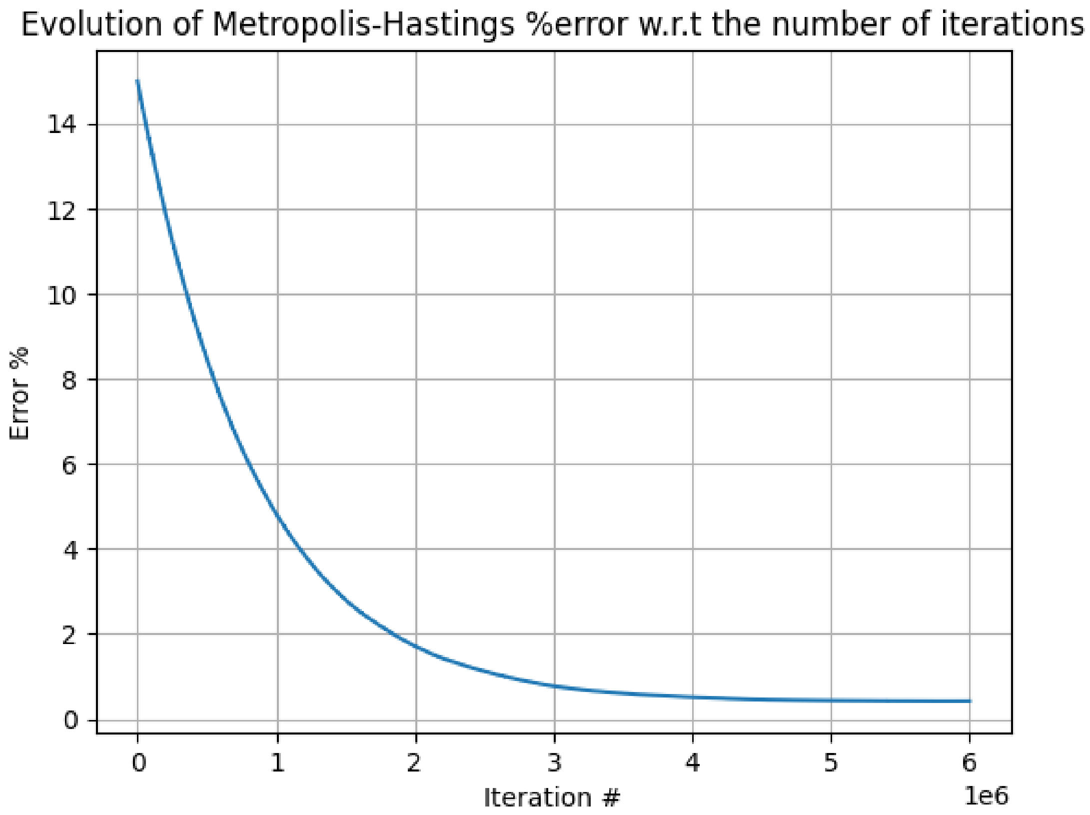


Convergence of Gibbs

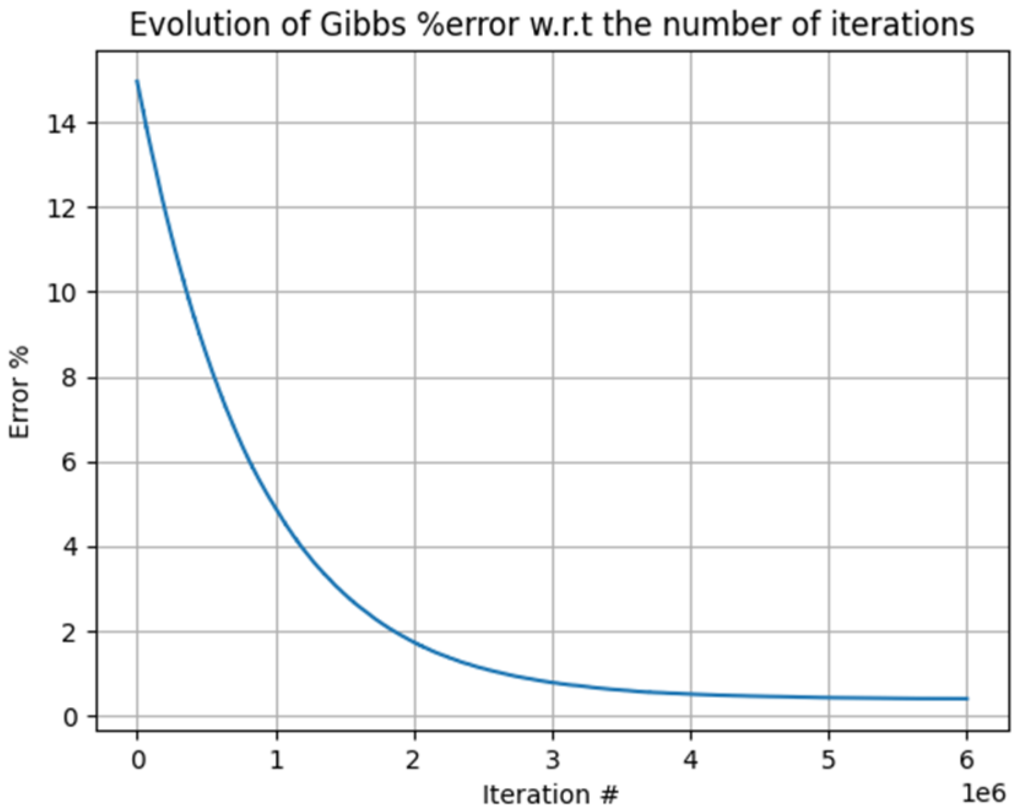


After 6×10^6 iterations, the final error is equal to $0.394\ \%$

Convergence

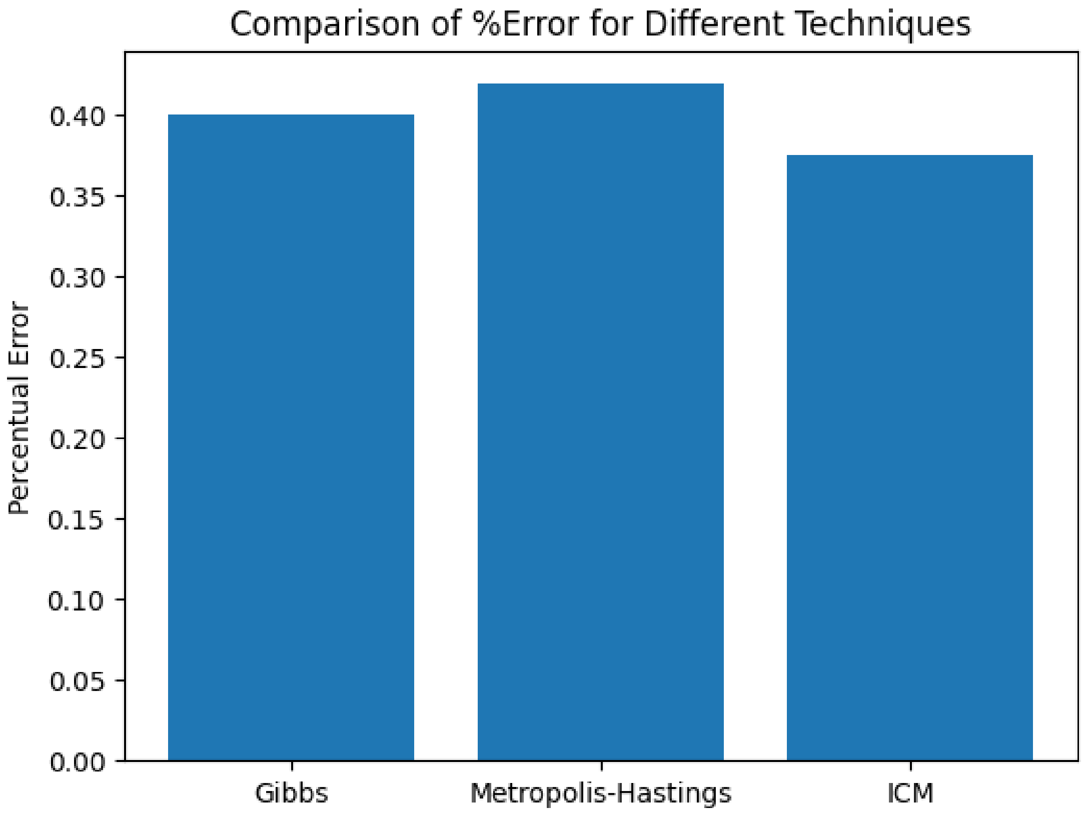


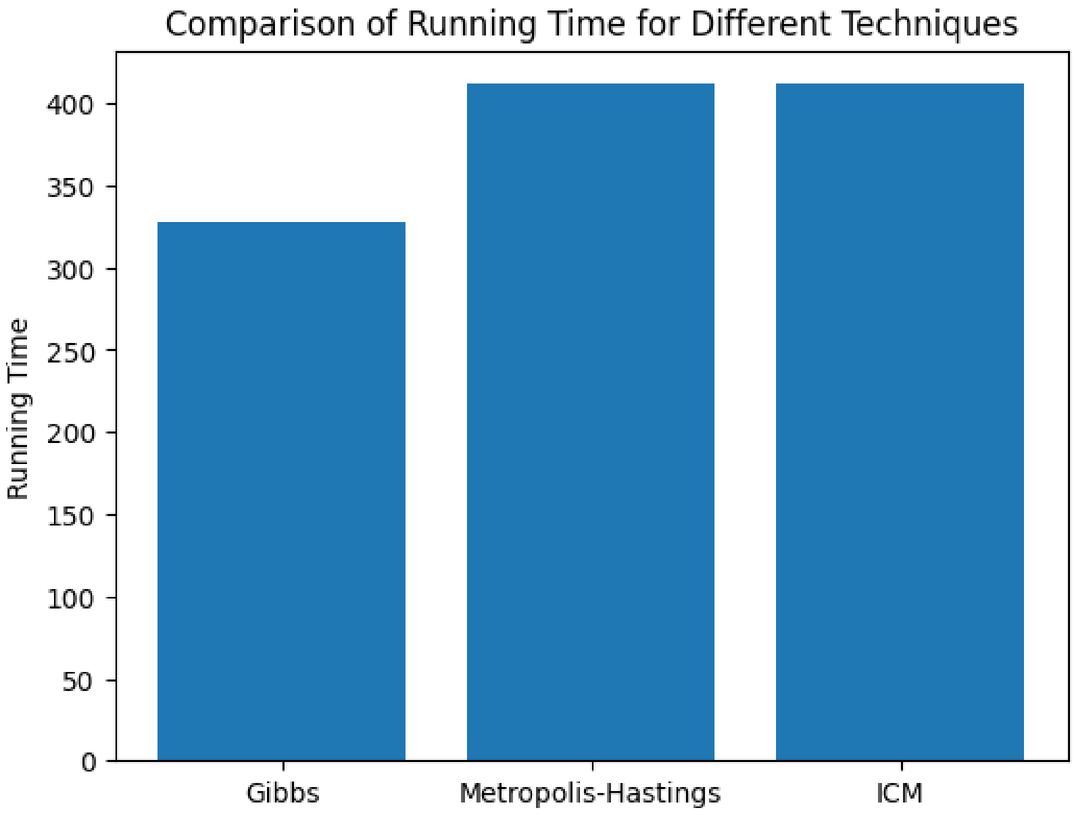
After 6×10^6 iterations, the final error is $0.418\ \%$



After 6×10^6 iterations, the final error is equal to $0.394\ \%$

Comparison


Comparison of %Error for Different Techniques
Percentual Error
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
Gibbs
Metropolis-Hastings
ICM


Comparison of Running Time for Different Techniques
Running Time
0
50
100
150
200
250
300
350
400
Gibbs
Metropolis-Hastings
ICM