BAYESIAN ANALYSIS OF ORDER UNCERTAINTY IN ARIMA MODELS

BY RICARDO S. EHLERS AND STEPHEN P. BROOKS

Federal University of Paraná, Brazil and University of Cambridge, UK

Abstract. In this paper we extend the work of Brooks and Ehlers (2002) and Brooks et al. (2003) by constructing adaptive proposal schemes for reversible jump MCMC in the context of autoregressive moving average models. In particular, the full conditional distribution is not available for the added parameters and approximations to it are provided by suggesting an adaptive updating scheme which automatically selects proposal parameter values to improve the efficiency of between-model moves. The performance of the proposed algorithms is assessed by simulation studies and the methodology is illustrated by applying it to a real data set.

Keywords. Bayesian model selection, posterior model probability, Markov chain Monte Carlo, reversible jump MCMC, autoregressive moving average.

1 Introduction

Autoregressive moving average (ARMA) processes provide a very useful and parsimonious class of models for describing time series data. For a time series of equally spaced observations y_t, $(t = 1, 2, \ldots, n)$, the general Gaussian ARMA(k, q) model takes the form

$$y_t = \sum_{j=1}^{k} a_j y_{t-j} + \sum_{j=1}^{q} b_j \epsilon_{t-j} + \epsilon_t \tag{1}$$

where the error terms ϵ_t are *i.i.d.* $N(0, \sigma_\epsilon^2)$. ARMA-type models are also relevant for modelling volatility, for example a GARCH(k, q) model can be interpreted as an ARMA(m, k) model for ϵ_t^2 where $m = \max(k, q)$ (see Franses and van Dijk 2000). The process is stationary and invertible if the roots of both the AR and MA characteristic polynomials lie outside the unit circle. These conditions impose a set of restrictions on the ARMA coefficients which are difficult to incorporate into a prior distribution.

Though other forms are available (see for example Box and Jenkins 1976) we shall adopt the following associated likelihood approximation for the ARMA(k, q) model

$$\tilde{p}(\boldsymbol{y}|k, q, \mathbf{a}^{(k)}, \mathbf{b}^{(q)}, \sigma_\epsilon^2) = (2\pi\sigma_\epsilon^2)^{-(n-k_{\max})/2} \exp\left(-\frac{1}{2\sigma_\epsilon^2} \sum_{t=k_{\max}+1}^{n} \epsilon_t^2\right) \tag{2}$$

where $k_{\max}$ is the maximum value allowed for k. We then use the same sample size for different models by computng this likelihood function conditional on the first $k_{\max}$ observations. The inclusion of MA terms in the model introduces complicated non-linearities in

the likelihood function as each ϵ_{t-j} depends on the whole set of coefficients in a complicated non-linear way. This complexity is inherited by the posterior distribution of the model parameters and approximation methods are necessary to gain posterior inference.

Alternatively, when modelling ARMA processes, it is often necessary to impose stationarity and invertibility restrictions. This can be done in several ways (see for example Chib and Greenberg 1994, Barbieri and O'Hagan 1996 and Huerta and West 1999). here, we focus upon the method which reparameterises the model in terms of the reciprocal roots of the characteristic equations, as in Huerta and West (1999). In this case, the ARMA(k, q) model may be rewritten as

$$\prod_{i=1}^{k}(1-\lambda_i L)y_t = \prod_{j=1}^{q}(1-\delta_j L)\epsilon_t \quad t = 1, \ldots, n. \tag{3}$$

where L denotes the lag operator so that $L^i y_t = y_{t-i}$. The λ_i and δ_j (either real or occurring in complex conjugate pairs) are then referred to as the reciprocal roots and the process is stationary and invertible if $|\lambda_i| < 1$, $i = 1, \ldots, k$ and $|\delta_j| < 1$, $j = 1, \ldots, q$. We assume, as is commonly the case, that the roots are distinct and non-zero.

If one or more of the autoregressive roots are unity the resulting models are of great value in representing homogeneous non-stationary time series. If d of these roots are unity and the remainder lie outside the unit circle then the resulting process is called an ARIMA(k, d, q) process. Frequently, small values for k, q and d will be appropriate.

In this paper we will show how reversible jump MCMC methods can be used to discriminate between models and will examine their performance both on simulated and real data. It is worth noting that the methods of Troughton and Godsill (1997) cannot be used here because of the MA component. Finally, we discuss how the output of our algorithm may then be used to carry out either Bayesian model averaging or model selection.

The paper is organised as follows. In Section 2 we discuss the implementation of Bayesian methods for fitting ARMA models via MCMC, including model order assessment. As we shall see, estimation is complicated computationally since the inclusion of MA terms introduces complex non-linearities in the likelihood function. A detailed description of model parameterisation and updating mechanisms for within and between model moves is provided. In Section 3 we do not impose stationarity/invertibility constraints and update the ARMA coefficients directly. Section 4 addresses the problem of order assessment in ARMA models when stationarity and invertibility restrictions are to be imposed. We propose a parameterisation in terms of reciprocal roots of the characteristic equations where it is straightforward to assign priors that satisfy these restrictions and to deal with unit roots.

2 The Bayesian Approach

2.1 Parameter Estimation and Model Probabilities

In many applications, in addition to the estimation of model parameters, there is substantial prior uncertainty concerning the choice of models that are the most appropriate for any given dataset. The classical approach is to use information criteria such as the AIC (Akaike, 1974) to discriminate between competing models. In the Bayesian framework, model uncertainty can be handled in a parametric fashion by indexing all models under consideration, treating this index as another parameter and using posterior model probabilities and/or Bayes factors (Kass and Raftery 1995). In realistically complex cases the number of competing models may be large and the corresponding analysis complex, so numerical techniques are used to efficiently explore model space.

We assume that a data vector $\boldsymbol{y}$ is observed and can be described by any of M candidate models. Associated with each model is a likelihood function $p(\boldsymbol{y}|\boldsymbol{\theta}^{(k)}, k)$ depending upon an unknown parameter vector $\boldsymbol{\theta}^{(k)}$, where $k \in \{1, \ldots, M\}$ is a model indicator determining the parameter dimension, which may vary from model to model. We shall focus upon the Bayesian framework here and assign prior distributions $p(\boldsymbol{\theta}^{(k)}|k)$ to each parameter vector and a prior distribution $p(k)$ to the model number. We are interested in computing the joint posterior distribution of all unknown quantities, i.e. the model indicator and parameters, denoted by

$$\pi(k, \boldsymbol{\theta}^{(k)}) \propto p(\boldsymbol{y}|\boldsymbol{\theta}^{(k)}, k)p(\boldsymbol{\theta}^{(k)}|k)p(k).$$

Based on a sample from this distribution, marginal posterior model probabilities can be approximated by the sample proportion of models visited. Also, samples from the posterior distribution within each model are automatically available for inference by simply conditioning on samples where the chain is in state k.

So, we need to simulate Markov chains whose state vector may change dimension as the simulation proceeds. An important innovation in the MCMC literature was the introduction of trans-dimensional algorithms to explore model space. Although alternatives exist (e.g. Stephens, 2000), we shall focus upon the use of the reversible jump MCMC algorithm proposed by Green (1995) for trans-dimensional transitions. We discuss the difficulty in implementing these algorithms efficiently and suggest an adaptive updating scheme which automatically selects proposal parameter values to improve the efficiency of between-model moves.

Here the parameter vector $\boldsymbol{\theta}$ contains σ_ϵ^2, the ARMA coefficients (or the reciprocal roots) plus the associated hyperparameters and both k and q as unknown parameters. We use reversible jump MCMC for moving between the competing ARMA models with (k, q) playing the role of model indicator and determining the model dimension. Note that σ_ϵ^2 will be present in every model. Upper bounds on model order (i.e. $k_{\max}$ and $q_{\max}$) are fixed *a priori* and we assume that the model parameters are *a priori* independent and that each model is equally likely. So, for the purposes of this paper we use uniform priors for the model order.

2.2 MCMC Basics

A general method of constructing a Markov chain with the desired stationary distribution is to draw new candidate values $\boldsymbol{\theta}'$ from an arbitrary distribution and apply some sort of correction mechanism so that they behave as a random sample from the target distribution. Metropolis et al. (1953) first introduced the Metropolis algorithm which was later generalised by Hastings (1970). In practical terms, the Metropolis-Hastings algorithm involves generating a new value $\boldsymbol{\theta}'$ from the density $q(\boldsymbol{\theta}, \boldsymbol{\theta}')$ and then evaluating the acceptance probability

$$\alpha(\boldsymbol{\theta}, \boldsymbol{\theta}') = \min\left(1, \frac{\pi(\boldsymbol{\theta}')q(\boldsymbol{\theta}', \boldsymbol{\theta})}{\pi(\boldsymbol{\theta})q(\boldsymbol{\theta}, \boldsymbol{\theta}')}\right).$$

We then generate $u \sim U(0, 1)$ and accept the move if $u \leq \alpha$, otherwise we reject the move and the chain stays at the current value $\boldsymbol{\theta}$. In this paper we shall use random walk Metropolis updates, adopting a proposal distribution centered on the current parameter value. When the so called full conditional posterior distribution of $\boldsymbol{\theta}$ is of standard form then the simplest approach is to sample new values from this distribution, in which case the acceptance probability equals 1. For further details on fixed dimensional MCMC see for example Brooks (1998).

2.3 Reversible Jump MCMC

The reversible jump algorithm (Green 1995) is a general strategy for generating samples from the joint posterior distribution $\pi(k, \boldsymbol{\theta}^{(k)})$. This is based upon the standard Metropolis-Hastings approach of proposing a move and defining a probability of accepting that move. In any typical application, both traditional MCMC for within-model moves and reversible jump updates for between-model moves will be employed in order to explore both parameter and model space.

There are simple versions of the reversible jump algorithm that can be applied in model discrimination problems. Suppose that the current state of the Markov chain is $(k, \boldsymbol{\theta}^{(k)})$, where $\boldsymbol{\theta}^{(k)}$ has dimension n_k, and we have defined one or more different move types allowing transitions between spaces of different dimensions. A move type r is performed with probability $p_k(r)$ by generating $\boldsymbol{u}$ from a specified proposal density $q(\cdot)$ and setting $(\boldsymbol{\theta}^{(k')}, \boldsymbol{u}') = g(\boldsymbol{\theta}^{(k)}, \boldsymbol{u})$. Here, g is a specified invertible function and $n_k + |\boldsymbol{u}| = n_{k'} + |\boldsymbol{u}'|$ where $|\boldsymbol{u}|$ denotes the dimension of $\boldsymbol{u}$. Then, we accept $(k', \boldsymbol{\theta}^{(k')})$ as the new state of the chain with probability $\min(1, A)$ where

$$A = \frac{\pi(k', \boldsymbol{\theta}^{(k')}) p_{k'}(r') q(\boldsymbol{u}')}{\pi(k, \boldsymbol{\theta}^{(k)}) p_k(r) q(\boldsymbol{u})} \left| \frac{\partial g(\boldsymbol{\theta}^{(k)}, \boldsymbol{u})}{\partial(\boldsymbol{\theta}^{(k)}, \boldsymbol{u})} \right| \tag{4}$$

is called the acceptance ratio.

A class of moves most commonly used for transitions between nested models consists of adding or deleting parameters from the current model to the next. In this special case, if we assume that $n_{k'} > n_k$ and $|\boldsymbol{u}| = n_{k'} - n_k$, then the transition from the larger model to the smaller one is entirely deterministic and the acceptance ratio (4) reduces to

$$A = \frac{\pi(k', \boldsymbol{\theta}^{(k')} | \boldsymbol{y}) p_{k'}(r')}{\pi(k, \boldsymbol{\theta}^{(k)} | \boldsymbol{y}) p_k(r) q(\boldsymbol{u})} \left| \frac{\partial g(\boldsymbol{\theta}^{(k)}, \boldsymbol{u})}{\partial(\boldsymbol{\theta}^{(k)}, \boldsymbol{u})} \right| \tag{5}$$

The applications in this paper will focus on a particular implementation in which the increase in dimensionality of the parameter space is accomplished through an identity function. In this case, the Jacobian term in (4) and (5) is equal to one. If we replace the posterior densities by the appropriate product of prior density and likelihood function then the acceptance ratio can be written as

$$A = \text{likelihood ratio} \times \text{prior ratio} \times \text{proposal ratio}. \tag{6}$$

The problem of order uncertainty in pure AR and ARMA models has been addressed previously using reversible jump MCMC (e.g. Troughton and Godsill 1997 and Barbieri and O'Hagan 1996) and stochastic search variable selection (e.g. Barnett et al. 1996) methods. More recently, (Philippe 2001) applied an algorithm developed in Stephens (2000) based on the construction of a continuous time Markov birth-death process to deal with order uncertainty in AR models. Also, Vermaak et al. (2004) develop reversible jump MCMC strategies in AR models for both the non-stationary and stationary cases. We extend their methods considerably here, in a non-trivial way, by including MA terms and using adaptive methods for the proposal parameters in the reversible jumps.

In this paper, we use reversible jump MCMC for moving between the different possible models. Though not necessary, we shall assume that the values of σ_ϵ^2 remain unchanged under model moves. Whatever the parameterisation, the updating scheme is implemented in two steps by firstly updating the AR coefficients via random walk Metropolis, and then proposing to add new coefficients or deleting existing ones. In the second step, this same

scheme is applied to the MA component. At each iteration a random choice between the birth or death move is made with probability 1/2.

Of course moves to higher autoregressive orders must be rejected with probability 1 when $k = k_{\max}$, and likewise in the MA component when $q = q_{\max}$. We also propose what we call "arima moves" by proposing a new value for the number of differences d and updating k accordingly. Further details on this move type are given in Section 4.3.1

2.4 Adaptive Proposals

The performance of the resulting Markov chain, in particular the ability to jump between models, will depend critically upon the choice of the proposal distribution. While for within-model moves it is fairly easy to choose proposals that lead both to high acceptance rates and rapid mixing, this is considerably more difficult for trans-dimensional algorithms as there is no Euclidean structure between models to guide proposal choice. In practice, the proposals are typically tuned on the basis of short pilot runs.

There have been several recent suggestions as to how to construct efficient proposals in trans-dimensional MCMC (see Green 2003, for a review on these methods). In particular, Brooks et al. (2003) develop methods to try and find parameters for the proposal distribution based upon a Taylor series expansion of the acceptance ratio for certain canonical jumps. Their method is an attempt to translate the natural ideas for proposal construction from a Euclidean space to the union of model spaces. Brooks and Ehlers (2002) discuss how to choose optimal values for the proposal parameters in AR models and we extend their development here to choose optimal values for the proposal parameters in ARMA models for both parameterisations in (1) and (3).

3 Non-Stationary/Invertible Models

3.1 Priors

In this section we assign unconstrained prior distributions to the coefficients. For fixed values of $k \in \{0, \ldots, k_{\max}\}$ and $q \in \{0, \ldots, q_{\max}\}$, each ARMA coefficient in (1) is assumed normally distributed with mean zero and variances σ_a^2 and σ_b^2 respectively, while σ_ϵ^2 is assumed inverse-Gamma distributed. Also, all model parameters are assumed to be *a priori* independent and prior inverse-Gamma distributions are specified for the hyperparameters σ_a^2 and σ_b^2. The inverse-Gamma family of prior distributions is conditionally conjugate, i.e. the full posterior conditional distribution is also inverse-Gamma. This conditional conjugacy means that the variances are easily updated. A common choice in the literature is the vague (but proper) inverse-Gamma(ϵ, ϵ) prior with small values for ϵ.

3.2 Within-Model Moves

Within each model, i.e. with k and q fixed, parameters are updated using traditional MCMC methods. In this case, it is easy to show that the full conditional distributions of the variance components remain inverse Gamma and these parameters are then updated by a Gibbs move. We note that the full conditional distribution of σ_ϵ^2 is conditional on the first $k_{\max}$ observations so that it is based on the same number of observations at each iteration.

However, full conditional distributions of standard form for the individual ARMA coefficients cannot be derived analytically. Though the complete conditional distribution of $\mathbf{a}^{(k)}$ may be obtained analytically and is of standard form this would involve performing a

computationally demanding matrix inversion at every iteration. Keeping the updating procedure simple and straightforward is a desirable computational feature, so we update the whole vector of AR coefficients sequentially via the Metropolis-Hastings algorithm by combining univariate moves. We use random walk Metropolis updates with a normal proposal density centred on the current parameter value. Similarly, the full conditional distributions for the individual MA coefficients are not of standard form and, again, we use random walk Metropolis updates.

3.3 Between Model Moves

Suppose that we are currently in model ARMA(k, q) and we must generate a new value k' to which we will propose a jump. It is often desirable to place higher probability on models closest to the current one so as to avoid spending too much time proposing moves to models with very low posterior mass (Troughton and Godsill 1997). This can be accomplished by using a discretised Laplacian distribution so that the distribution for k' is given by

$$p(k') \propto \exp(-\beta|k - k'|), \quad k' \in [1, \ldots, k_{\max}],$$

where $\beta \geq 0$ denotes a scale parameter. Of course, taking $\beta = 0$, we obtain the uniform proposal. However, if we take $\beta = 0.5$, say, then a jump to $k \pm 1$ is three times more likely than a jump to $k \pm 3$, for example. Without loss of generality, let us suppose that $k' > k$, and that we generate $\boldsymbol{u} = (u_1, \ldots, u_{k'-k})$ from some proposal density q.

One type of move to be explored here is what Brooks and Ehlers (2002) call a non-deterministic down move. In this case we propose a jump from k to k' by generating the whole vector of AR coefficients in the k'-dimensional space, so that moves to lower dimensional spaces are no longer deterministic. We note that in this case the current values of $\mathbf{a}^{(k)}$ will not be used to determine the new values. In terms of dimension matching, this is equivalent to setting the change of variables as $\mathbf{a}' = \boldsymbol{u}$ and $\boldsymbol{u}' = \mathbf{a}$, which has unity Jacobian. This move would then be accepted with probability $\min(1, A)$ where, from (4),

$$A = \frac{p(\boldsymbol{y}|k', q, \boldsymbol{u}, \mathbf{b}, \sigma_\epsilon^2)}{p(\boldsymbol{y}|k, q, \mathbf{a}, \mathbf{b}, \sigma_\epsilon^2)} \frac{p(\boldsymbol{u}|k', \sigma_a^2)}{p(\mathbf{a}|k, \sigma_a^2)} \frac{r_{k',k} q(\mathbf{a})}{r_{k,k'} q(\boldsymbol{u})} \tag{7}$$

and this remains unchanged whether or not $k < k'$. Similar comments follow for the MA component with a whole set of new coefficients being proposed.

For this type of model move, taking the proposal density for $\boldsymbol{u}$ to be a multivariate normal density then expressions for the mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{C}$ are given in the Appendix A. Note that we need to calculate the proposal parameters for both the proposed move and the corresponding reverse (non-deterministic) move in order to evaluate this acceptance ratio.

Having described all the steps necessary for the MCMC simulations, we now present both a simulation study and the analysis of a real data set where we give details concerning prior specification and examine the performance of the updating schemes described in this section.

3.4 Simulation Study

To assess the performance of our RJMCMC algorithm for order selection problems in ARMA models we simulated 20 data sets from AR(3), MA(3) and ARMA(3,3) stationary processes each with 1000 data points. For each data set we ran our full updating scheme with 1,000,000 iterations (discarding the first 500,000 as burn-in) using the first 20, 50, 100, 500 and 1000 observations and recorded the estimated posterior probability of the true model. We also

recorded the proportion of time that the algorithm correctly selects the true model for each sample size. These are shown in Table 1 where, for each process, the first row shows the average posterior probability of the true model while the second row shows the proportion of correct choices. It is clear that, as the sample size increases the performance of the algorithm improves. Acceptable performances seem to be achieved for datasets comprising at least 200 observations.

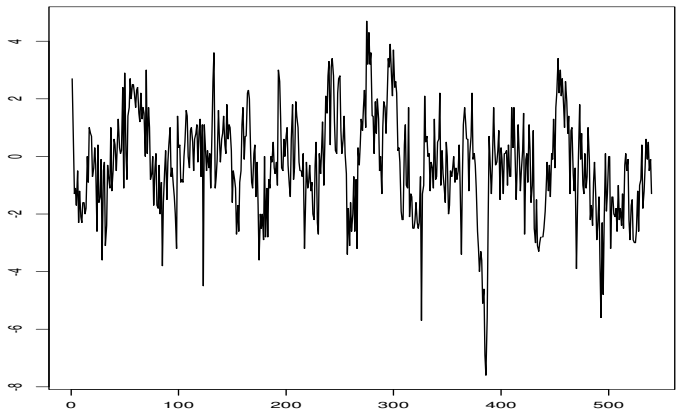
Table 1: Changes in model probabilities and proportion of correct model for simulated AR(3), MA(3) and ARMA(3,3) processes as the sample size increases.

	Sample size					
model	20	50	100	200	500	1000
AR(3)	0.0561	0.1837	0.2842	0.4401	0.6034	0.7336
	0.1429	0.8095	0.9048	0.7619	0.9048	0.9524
MA(3)	0.0364	0.0438	0.0736	0.1028	0.2189	0.3367
	0.0476	0.0476	0.2857	0.3333	0.8095	0.8571
ARMA(3,3)	0.0202	0.0285	0.0524	0.1232	0.2663	0.3967
	0.0000	0.0000	0.0476	0.3333	0.8571	0.9524

3.5 A Real Data Example

In this section we illustrate our updating scheme for ARMA models on one data set which has been analysed in the time series literature: the Southern oscillation index (SOI). The series appears in Trenberth and Hoar (1996) and Huerta and West (1999) and consists of 540 monthly observations registered from 1950 to 1995 and is related to sea surface temperature. The original SOI series is plotted in Figure 1.

Figure 1: Southern oscillation index. 540 measurements taken between 1950-1995 of the difference of the departure from the long-term monthly mean sea level pressures at Tahiti and Darwin.



In order to compare the different methods for specifying proposal parameters we fit ARMA models to this data set with both AR and MA orders varying from 0 to 5, thus considering a set of competing parsimonious models. The variance of the proposal distribution for model moves was set, after pilot-tuning, as $\sigma^2 = 0.01$, and we then used an approximate

likelihood based on fixed error terms for proposing model moves based on the second order method of Brooks et al. (2003). The within-model moves (with k and q fixed) were performed by updating the AR and MA coefficients via the random-walk Metropolis algorithm as described in Section 3.2 using proposal variances $\sigma_n^2 = 0.1$. We assigned inverse Gamma priors with parameters 0.01 for the error and prior variances. Then, 1,000,000 iterations of the algorithm were run discarding the first 500,000 as burn-in.

The posterior distribution of model order for the full updating scheme appears in Table 2 where we can see that when we include MA terms in the model the ARMA(1,1) is identified as the most likely one with decreasing posterior support for models AR(3), AR(4), ARMA(2,2), ARMA(2,1), AR(3,1) and ARMA(4,1) and low probabilities for other models. Of course for this case an exhaustive search over the 35 possible ARMA models can be performed and the classical information criteria (AIC and BIC) also select the ARMA(1,1) model.

Table 2: Posterior model order probabilities for the SOI data based on 500,000 iterations after a 500,000 burn-in. Top model highlighted in bold.

	MA order					
AR order	0	1	2	3	4	5
1	0.0000	**0.4428**	0.0259	0.0042	0.0031	0.0056
2	0.0178	0.0360	0.0717	0.0141	0.0102	0.0136
3	0.0971	0.0254	0.0151	0.0097	0.0057	0.0066
4	0.0817	0.0336	0.0124	0.0063	0.0034	0.0033
5	0.0239	0.0153	0.0063	0.0040	0.0024	0.0028

4 Imposing Stationarity and Invertibility

When reparameterising the model in terms of reciprocal roots as in (3) updating one (or a conjugate pair) of the reciprocal roots changes the whole vector of coefficients (either AR or MA). Note that it is not necessary to impose an identifying ordering on the roots since the vector of coefficients is the same irrespective of ordering. The set of all variables in the problem is now $(\boldsymbol{y}, k, q, \boldsymbol{\lambda}, \boldsymbol{\delta}, \sigma_\epsilon^2)$ and the likelihood function has the same form in terms of ARMA coefficients and is evaluated by first mapping from $\boldsymbol{\lambda}$ to $\mathbf{a}$ and from $\boldsymbol{\delta}$ to $\mathbf{b}$ since $p(\boldsymbol{y}|k, q, \boldsymbol{\lambda}, \boldsymbol{\delta}, \sigma_\epsilon^2) = p(\boldsymbol{y}|k, q, \mathbf{a}(\boldsymbol{\lambda}), \mathbf{b}(\boldsymbol{\delta}), \sigma_\epsilon^2)$. In the next sections we present details of priors and proposals in terms of the AR reciprocal roots and the development for the MA component is analogous.

4.1 Parameter Priors

Conditional on model order we assume independent priors for the real and any pairs of complex conjugate reciprocal roots. A real reciprocal root r has a continuous prior density over the support (-1,1) while a pair of complex conjugates $(\lambda_j, \lambda_{j^*})$ can be written as

$$\begin{aligned} \lambda_j &= r\cos\theta + ir\sin\theta \\ \lambda_{j^*} &= r\cos\theta - ir\sin\theta \end{aligned}$$

and here we specify the prior distribution of the complex pair in terms of the two defining parameters, θ and r, over a support in the stationary region. So, assuming that θ lies in the

interval $(0, \pi)$ then $r \in (-1, 1)$ as in the case for real roots. Here we shall also assume that θ and r are *a priori* independent so that the prior density for the complex pair is given by

$$p(\lambda_j, \lambda_{j^*}) = p(\theta)p(r) \left| \frac{\partial(\lambda_j, \lambda_{j^*})}{\partial(\theta, r)} \right|^{-1}.$$

While we can consider a variety of priors for r we will rarely, if ever, have any prior information concerning θ and we shall therefore assume a $U(0, \pi)$ prior distribution throughout.

To place prior information on r, we can reparameterise the reciprocal roots by taking a real quantity x without restriction and try and find a suitable function that maps the real line onto the interval (-1,1). Here we use the function

$$r = \frac{2e^x}{1 + e^x} - 1 \tag{8}$$

with inverse $x = \log((1+r)/(1-r))$. We can now place suitable priors on x, noting that very large values of $|x|$ correspond to values of $|r|$ very close to 1. A $N(0, \sigma_a^2)$ provides a reasonable family of prior distributions, centred on zero (corresponding to $r = 0$), with σ_a^2 determining the shape of the prior distribution of r. This becomes more concentrated around zero as σ_a^2 decreases and becomes U-shaped and more concentrated near -1 and 1 as σ_a^2 increases. Thus, this prior can be used to describe a broad range of prior beliefs. We refer to this as a logistic-based prior.

For this prior specification the full conditional distribution of the hyperparameter σ_a^2 can be obtained analytically and has a standard form. Under the assumed conditionally independent prior and assigning an inverse Gamma prior to σ_a^2 its full conditional distribution is given by

$$IG\left(\alpha + \frac{n_r + n_c}{2}, \beta + \frac{1}{2}\left[\sum_{i:\lambda_i \in \mathbb{R}} x_i^2 + \sum_{j:\lambda_j \in \mathbb{C}} x_j^2\right]\right)$$

where n_r and n_c are the number of real roots and the number of complex conjugate pairs respectively, and α and β are the prior parameters.

Model Priors

For particular values of k and q the root structure is not unique, except when $k = 1$ or $q = 1$. For example, if $k = 4$ we can have 4 real roots, 2 real and 2 complex roots or 4 complex roots in the AR component, so that there are 3 possible configurations of real and complex roots corresponding to $k = 4$. Therefore, in order to assign a uniform prior on the AR order, the prior probability for a certain value of k should be split uniformly over the possible configurations of real and complex roots corresponding to that order. Likewise for the MA component.

If k is even, then the roots can be divided into $d = k/2$ pairs and each of them can be either real or complex. Since the order is irrelevant and the number of pairs of one type (real or complex) can vary from 0 to d it follows that the number of possible configurations is given by $d+1$. If k is odd, there are $d = [k/2]$ pairs of roots (where $[x]$ denotes the integer part of x) plus one real root and so the number of possible configurations is again $d+1$ since the number of pairs of one type varies from 0 to d. Therefore, given the value of model order k, it follows that

$$\mathbb{P}(r \text{ real and } c \text{ complex roots}) \propto \frac{1}{[k/2] + 1}.$$

This prior specification differs from Huerta and West (1999) where a uniform distribution is assigned to the possible configurations of real and complex roots thus leading to a non-uniform prior distribution on model order.

4.2 Within-Model Moves

For k and q fixed, it is easy to see that the full conditional distribution of σ_ϵ^2 has the same inverse Gamma form as in Section 3.2 with the vector of ARMA coefficients computed from the reciprocal roots. This parameter is then updated by a Gibbs move.

In order to update the ARMA coefficients we randomly choose one of the reciprocal roots and use Metropolis-Hastings updates with the proposal density centred on the current value as follows. If the chosen λ_j is real we propose a new value by sampling λ_j' from $U[\max(\lambda_j - \delta, -1), \min(\lambda_j + \delta, 1)]$. Of course when $\lambda_j - \delta < -1$ and $\lambda_j + \delta > 1$ the proposal distribution is simply the $U(-1, 1)$ and the proposal ratio in the acceptance probability is equal to 1.

A similar approach is adopted when the chosen λ_j is complex. We propose a new value for the pair $(\lambda_j, \lambda_{j^*})$ by sampling θ^* from $U[\max(0, \theta - \delta), \min(\pi, \theta + \delta)$ and r^* from $U[\max(r - \delta, -1), \min(r + \delta), 1]$, and setting the proposed new values as $r^* \cos\theta^* \pm ir^* \sin\theta^*$.

The above schemes ensure that the new values of the reciprocal roots (either real or complex), are proposed in a neighbourhood of the current ones and are restricted to stationarity.

4.3 Between-Model Moves

We employ model moves that increase or decrease the model dimension by one or two by proposing the addition (or deletion) of one real or a pair of complex conjugate roots. The reciprocal roots common to the current and proposed model remain unchanged. Model moves are performed in two steps by first deciding on the birth or death of roots and then deciding on a single real or a pair of conjugate complex roots to be added or deleted. So, four model move types are allowed: real birth, complex birth, real death and complex death. Here each move type is proposed with the same probability 1/4 so that they cancel out in the proposal ratio.

Suppose that we propose a move from ARMA(k, q) to ARMA$(k + 1, q)$ by adding one real reciprocal root r sampled from a continuous distribution over the support (-1,1). The models are nested in terms of reciprocal roots and these are assumed *a priori* independent.

Suppose now that we propose adding a pair of complex reciprocal roots $(u, \bar{u})$ where

$$\begin{aligned} u &= r\cos\theta + ir\sin\theta \\ \bar{u} &= r\cos\theta - ir\sin\theta. \end{aligned}$$

Here, we specify the proposal distribution of the complex pair in terms of the two defining parameters θ and r over the support $(0, \pi) \times (-1, 1)$. So, we shall propose new values not for $(u, \bar{u})$ directly but for (θ, r) so that the Jacobian term will cancel in the acceptance ratio with that arising from the prior.

Note that, under this parameterisation and updating scheme, the models can be treated as nested so that the Jacobian of the transformation from $(\lambda_1, \ldots, \lambda_k)$ to either $(\lambda_1, \ldots, \lambda_k, r)$ or $(\lambda_1, \ldots, \lambda_k, u, \bar{u})$ equals 1 and does not appear in the proposal ratios. Also, the likelihood ratio is computed by first mapping the set of reciprocal roots to $(a_1', \ldots, a_{k+1}')$ or $(a_1', \ldots, a_{k+2}')$.

Conversely, a real (or complex) death move is proposed by randomly selecting one of the real (or complex) roots and deleting it (or the pair of complex conjugates).

4.3.1 Updating the Number of Unit Roots

For ARIMA(k, d, q) models, a change in d implies a change in k only, so this move type changes both d and k simultaneously. For example, we can propose a move from ARIMA$(k, 0, q)$ to ARIMA$(k-1, 1, q)$ or ARIMA$(k-2, 2, q)$, so we allow unit roots (if they exist) to be either complex or real.

Proposing d to $d+1$ for example is equivalent to saying that one of the roots equals 1 and the other $k-1$ roots (equivalently $k-1$ AR coefficents) are to be estimated. However, after we accept or reject $d+1$, new values of k and q are proposed and accepted or rejected. So it is possible to go from (k, d, q) to $(k, d+1, q)$ in two steps. In practice, values other than $d = 0$, 1 or 2 would not make much sense, so this is the range of possible values that we adopt here.

The criteria for proposing arima moves are as follows: we randomly choose one root which is greater (in absolute value) than a prespecified lower bound L and propose 1 or 2 differences depending on the root being real or complex (this implies deleting 1 or 2 roots). Otherwise, the number of differences is decreased by 1 or 2, which implies adding 1 or 2 roots by sampling from U(-1,-L) or U(L,1) with probability 1/2.

4.3.2 Proposals

Here we consider three families of proposal densities. The first proposal samples a value for the new root r by generating a realisation from a $N(\mu, \sigma^2)$ distribution truncated to the interval $(-1, 1)$. Our second proposal samples $u \sim \beta(\alpha_1, \alpha_2)$ and sets $r = 2u - 1$. Our third proposal samples a new value for

$$\rho = \log\left(\frac{1+r}{1-r}\right),$$

from a $N(\mu, \sigma^2)$ distribution and maps to r using function (8). We refer to these as truncated normal, beta-based and logistic-based proposals respectively. Each of these proposals are characterised by two proposal parameters that can be determined by pilot tuning or using the methods described in Brooks and Ehlers 2002.

Of course, for either real or complex cases if we take both prior and proposal in the same family (logistic-based) the Jacobian term will cancel with that arising from the prior. We can also take the prior as a proposal for θ in which case we sample a new value from a $U(0, \pi)$ and $q(\theta)$ cancels with $p(\theta)$ in the acceptance ratio.

4.3.3 Adaptive Proposals for Reciprocal Roots

Brooks and Ehlers (2002) discuss how to choose optimal values for the proposal parameters in AR models parametrised in terms of reciprocal roots. Since our scheme updates the AR and MA components separately we can extend those developments to choose proposal parameters in ARMA models too. Further details are given in Appendix B.

Suppose that we propose a move from ARMA(k, q) to ARMA$(k+1, q)$ by adding one real reciprocal root r. Then using the representation (3) and denoting the error terms in the higher dimensional model by ϵ'_t we obtain

$$(1 - rL)\prod_{i=1}^{k}(1 - \lambda_i L)y_t = \prod_{j=1}^{q}(1 - \delta_j L)\epsilon'_t$$

so that $\epsilon'_t = (1 - rL)\epsilon_t = \epsilon_t - r\epsilon_{t-1}$ where the ϵ_t denote the error terms in the original model. Thus, the likelihood function under the larger model is simply,

$$L(\boldsymbol{y} \mid k, q, \boldsymbol{\lambda}, \boldsymbol{\delta}, \sigma_\epsilon^2) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum(\epsilon_t - r\epsilon_{t-1})^2\right] \tag{9}$$

where the proportionality constant includes terms that do not depend on r. This is exactly the same expression that appears in Brooks and Ehlers (2002) with the error terms redefined here for ARMA models. So, we can use the expressions given there for the various combinations of prior and proposal distributions.

Suppose now that we propose a move from ARMA(k, q) to ARMA$(k + 2, q)$ by adding a pair of complex reciprocal roots $(u, \bar{u})$. Then, it is easy to show that the error terms in the higher dimensional model can be written as $\epsilon'_t = \epsilon_t - 2r\cos\theta\epsilon_{t-1} + r^2\epsilon_{t-2}$ and the likelihood function for θ and r is again given by the same expression as in Brooks and Ehlers (2002). Their second order method suggests taking the prior of θ as a proposal and they give expressions for the proposal parameters of r for various combinations of prior and proposal distributions.

4.3.4 Updating the MA Component

Consider now jumps that alter the dimension of the MA component and we begin by considering a move from ARMA(k, q) to ARMA$(k, q + 1)$ by adding one real reciprocal root, r. Then the representation for the higher dimensional model is

$$\prod_{i=1}^{k}(1 - \lambda_i L)y_t = (1 - rL)\prod_{j=1}^{q}(1 - \delta_j L)\epsilon'_t$$

so that $\epsilon_t = (1 - rL)\epsilon'_t = \epsilon'_t - r\epsilon'_{t-1}$ where the ϵ_t denote the error terms in the original model. Thus, the derivatives needed to apply the second order method are not available since each error term ϵ'_t depends on r in a complicated non-linear way. Here, we approximate these derivatives by treating ϵ'_{t-1} as if it were fixed in the larger model. In this case, the likelihood function under the larger model is given by,

$$L(\boldsymbol{y} \mid k, q, \boldsymbol{\lambda}, \boldsymbol{\delta}, \sigma_\epsilon^2) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum(\epsilon_t + r\epsilon_{t-1})^2\right].$$

To apply the second order method of Brooks et al. 2003 we take first and second derivatives of the logarithm of the likelihood with respect to r. Comparing with likelihood (9) it is easy to see that when we take the derivatives the term $\sum \epsilon_t\epsilon_{t-1}$ is multiplied by -1. Likewise, when we propose a move from q to $q + 2$ by adding a pair of complex reciprocal roots a term $\sum \epsilon_t\epsilon_{t-2}$ will appear multiplied by -1. So, the expressions given in Brooks and Ehlers (2002) can still be used here with slight modifications.

4.4 Simulation Study

To assess the performance of our RJMCMC algorithm for order selection problems in ARIMA models we turn to the simulated AR(3), MA(3) and ARMA(3,3) data sets of Section 3.4. For each data set and each of the three proposal families we ran our algorithm for 1,000,000 iterations (discarding the first 500,000 as burn-in) using the first 20, 50, 100, 500 and 1000 observations and recorded the estimated posterior probability of the true model. Here the maximum ARMA model orders are $k_{\max} = q_{\max} = 5$ and $d = 0, 1, 2$, so the number of possible

ARIMA models is quite large and an exhaustive enumeration would be cumbersome. We also recorded the proportion of time that the algorithm correctly selects the true model for each sample size. In Table 3 we show the results for the simulated AR(3) processes where, for each proposal distribution the first row refers to the average posterior probability of the true model while the second row shows the proportion of correct choices of the true model. We repeat this experiment with various simulated non-stationary series and the results for simulated ARIMA(2,1,0) processes are shown in Table 4.

Clearly, the performance of the algorithm improves as the sample size increases and a quite similar pattern is observed for the three proposals considered. Acceptable performances seem to be achieved for data sets with at least 200 observations.

Table 3: Changes in model probabilities and proportion of correct model for a simulated AR(3) process as the sample size increases considering the three families of proposals.

	Sample size					
proposal	20	50	100	200	500	1000
Truncated normal	0.0092	0.0398	0.1074	0.2677	0.4565	0.5480
	0.0000	0.1500	0.3500	0.6000	0.9000	0.9500
Beta-based	0.0096	0.0441	0.1059	0.2702	0.4812	0.5404
	0.0000	0.1500	0.3500	0.6500	0.9500	0.9000
Logistic-based	0.0092	0.0414	0.1058	0.2628	0.4822	0.5436
	0.0000	0.1500	0.3000	0.6000	0.9500	0.9000

Table 4: Changes in model probabilities and proportion of correct model for a simulated ARIMA(2,1,0) process as the sample size increases considering the three families of proposals.

	Sample size					
proposal	20	50	100	200	500	1000
Truncated normal	0.0401	0.0985	0.1423	0.2125	0.3235	0.4362
	0.1500	0.4000	0.6500	0.7500	0.9000	0.9000
Beta-based	0.0396	0.0917	0.1402	0.2140	0.3173	0.4173
	0.2500	0.4000	0.6000	0.7000	0.8500	0.9500
Logistic-based	0.0458	0.0924	0.1380	0.2200	0.3352	0.4210
	0.3000	0.3500	0.6000	0.7000	0.9000	0.9500

4.5 The SOI Data Revisited

In this section, we illustrate the application of our algorithm based upon the parameterisation in terms of reciprocal roots with a real data set. The Southern Oscillation Index (SOI) data, described in Section 3.5, is analysed here. The analysis is based on an ARIMA model with maximum order $k_{\max} = q_{\max} = 5$ and $d_{\max} = 2$ posterior inference is based on 500,000 samples after discarding the initial 500,000 as burn-in.

The posterior distribution of model order appears in Table 5, for the logistic-based proposal family of distributions and $d = 0, 1$. We can see that the ARIMA(1,0,1) model is identified as the most likely one with much lower posterior support for other models (as in the previous parameterisation). We note also that, although ARIMA$(p, 2, q)$ models were in the set of candidate models they were not visited often enough after the burn-in period and are thus excluded from the table.

Table 5: Posterior model order probabilities for the SOI data based on 500,000 iterations after a 500,000 burn-in. Top model highlighted in bold.

			MA order					
Proposal	d	AR order	0	1	2	3	4	5
Logistic-based	0	1	0.0000	**0.2675**	0.0575	0.0598	0.0393	0.0355
		2	0.0017	0.0207	0.0323	0.0234	0.0173	0.0168
		3	0.0142	0.0208	0.0216	0.0236	0.0206	0.0183
		4	0.0103	0.0094	0.0117	0.0109	0.0106	0.0101
		5	0.0102	0.0078	0.0101	0.0089	0.0086	0.0082
	1	0	0.0000	0.0157	0.0107	0.0189	0.0193	0.0240
		1	0.0000	0.0031	0.0042	0.0076	0.0089	0.0124
		2	0.0001	0.0019	0.0028	0.0061	0.0078	0.0105
		3	0.0001	0.0012	0.0022	0.0043	0.0050	0.0068
		4	0.0001	0.0011	0.0023	0.0045	0.0046	0.0057

5 Discussion

In this paper we illustrate the Bayesian approach to simultaneous parameter estimation and model order selection for the class of ARIMA time series models. In particular, we address the problem of order selection in a MCMC framework the via reversible jump algorithm.

We present an alternative parameterisation in terms of reciprocal roots of the characteristic equations. This allowed us to enforce stationarity and invertibility constraints in the model parameters in a very straightforward way. Since the stationary/invertible region is convex if each set of reciprocal roots generated via MCMC satisfies those constraints, then so do their means. So, the parameter estimates are guaranteed to impose stationarity/invertibility.

Even for fixed dimensions, the presence of moving average terms in the model introduces complex non-linearities in the likelihood function and classical estimation of model parameters would require numerical optmization methods to be used. In the Bayesian approach, the posterior density of model parameters is of the same complexity as the likelihood function and cannot be directly computed, thus approximation methods are necessary to derive posterior inferences on these parameters. Another difficulty in ARMA models concerns the problem of roots cancellation, i.e. there may be common factors cancelling out if there are similar AR and MA roots. This is a generic phenomenon with ARMA models and the likelihood function is very badly behaved if we overparameterise.

It has been shown via simulated examples that the approach can reliably select the best model for reasonable sample sizes, and it has performed well with a real data set. The approach developed in this paper can be extended to other classes of models (e.g. threshold autoregression, smooth transition autoregression, stochastic volatility models) and this is object of current and future research.

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A Adaptive Proposals for ARMA Models

Here we seek to generalise the adaptive construction of proposal distributions in Brooks and Ehlers (2002) to ARMA models. We concentrate on their second order method which involves setting to zero the first and second derivatives of the acceptance ratio. In this case, each error term ϵ_t depends on the whole set of ARMA coefficients in a complicated non-linear way. So, in order to apply the adaptive proposal methods in this context we need to make some simplifying assumptions. We shall assume that, when proposing to add new coefficients or delete existing ones, the previous values of the error term are kept fixed.

A.1 Updating the AR component

Consider jumps from ARMA(k, q) to ARMA(k', q) by generating new values for the whole vector of AR coefficients directly in the k'-dimensional space while keeping the MA component fixed.

Defining the $(n - k_{\max}) \times q$ matrix of errors

$$\mathbf{E} = \begin{bmatrix} \epsilon_{k_{\max}} & \dots & \epsilon_{k_{\max}-q+1} \\ \vdots & & \vdots \\ \epsilon_{n-1} & \dots & \epsilon_{n-q} \end{bmatrix}$$

the Gaussian autoregressive moving average model of order (k', q) can be written as

$$\boldsymbol{y} = \mathbf{Y}\boldsymbol{u} + \mathbf{E}\mathbf{b} + \boldsymbol{\epsilon}$$

where $\boldsymbol{y} = (y_{k_{\max}+1}, \dots, y_n)'$, $\boldsymbol{\epsilon} = (\epsilon_{k_{\max}+1}, \dots, \epsilon_n)'$ and

$$\mathbf{Y} = \begin{bmatrix} y_{k_{\max}} & \dots & y_{k_{\max}-k'+1} \\ \vdots & & \vdots \\ y_{n-1} & \dots & y_{n-k'} \end{bmatrix}.$$

Then, for $\boldsymbol{u}$ sampled from a multivariate Normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{C}$ and using the *a priori* independence assumption, the acceptance ratio is given by

$$A_{k,k'} \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}(\boldsymbol{\epsilon}^* - \mathbf{Y}\boldsymbol{u})'(\boldsymbol{\epsilon}^* - \mathbf{Y}\boldsymbol{u})\right] \exp\left[-\frac{1}{2\sigma_a^2}\boldsymbol{u}'\boldsymbol{u}\right] \exp\left[\frac{1}{2}(\boldsymbol{u} - \boldsymbol{\mu})'\boldsymbol{C}^{-1}(\boldsymbol{u} - \boldsymbol{\mu})\right] \tag{10}$$

where $\boldsymbol{\epsilon}^* = \boldsymbol{y} - \mathbf{E}\mathbf{b}$ and the terms in the proportionality constant do not depend on $\boldsymbol{u}$.

The second order method can be applied by setting to zero the first and second order derivatives of $\log A_{k,k'}$ with respect to $\boldsymbol{u}$ and ignoring the zeroth order term. However, these derivatives are not available since each error term depends on the whole set of coefficients in a complicated non-linear way. So, $\partial\boldsymbol{\epsilon}^*/\partial\boldsymbol{u}$ is too complex and we approximate by treating $\mathbf{E}$ as if it were fixed in order to get $\boldsymbol{\mu}$ and $\boldsymbol{C}$. We then obtain that

$$\begin{aligned} \nabla \log A_{k,k'} &= \sigma_\epsilon^{-2}\mathbf{Y}'(\boldsymbol{\epsilon}^* - \mathbf{Y}\boldsymbol{u}) - \sigma_a^{-2}\boldsymbol{u} + \boldsymbol{C}^{-1}(\boldsymbol{u} - \boldsymbol{\mu}) \\ \nabla^2 \log A_{k,k'} &= -\sigma_\epsilon^{-2}\mathbf{Y}'\mathbf{Y} - \sigma_a^{-2}\mathbf{I}_{k'} + \boldsymbol{C}^{-1}. \end{aligned}$$

Of course, since $\log A_{k,k'}$ is a quadratic function of $\boldsymbol{u}$, the second derivative does not depend on the value of $\boldsymbol{u}$ and setting it to zero we obtain

$$\boldsymbol{C}^{-1} = \sigma_\epsilon^{-2}\mathbf{Y}'\mathbf{Y} + \sigma_a^{-2}\mathbf{I}_{k'}.$$

Similarly, setting the first derivative to zero and using the above expression for $\boldsymbol{C}^{-1}$ it follows that

$$\begin{aligned}\boldsymbol{C}^{-1}(\boldsymbol{u}-\boldsymbol{\mu}) &= \sigma_a^{-2}\mathbf{I}_{k'}\boldsymbol{u} - \sigma_\epsilon^{-2}\mathbf{Y}'(\boldsymbol{\epsilon}^* - \mathbf{Y}\boldsymbol{u}) \\ &= (\sigma_\epsilon^{-2}\mathbf{Y}'\mathbf{Y} + \sigma_a^{-2}\mathbf{I}_{k'})\boldsymbol{u} - \sigma_\epsilon^{-2}\mathbf{Y}'\boldsymbol{\epsilon}^* = \boldsymbol{C}^{-1}\boldsymbol{u} - \sigma_\epsilon^{-2}\mathbf{Y}'\boldsymbol{\epsilon}^*\end{aligned}$$

so the proposal mean is given by $\boldsymbol{\mu} = \sigma_\epsilon^{-2}\boldsymbol{C}\mathbf{Y}'(\boldsymbol{y} - \mathbf{E}\mathbf{b})$. Note also that, only the proposal mean depends on the current values of the MA coefficients $\mathbf{b}$.

A.2 Updating the MA Component

In an MA component full updating scheme we propose a move from q to q' by generating new values for the whole vector of MA coefficients conditional on the current AR coefficients.

The Gaussian autoregressive moving average model of order (k, q') can be written in matrix form as

$$\boldsymbol{y} = \mathbf{Y}\mathbf{a} + \mathbf{E}\boldsymbol{u} + \boldsymbol{\epsilon}.$$

where the matrix $\mathbf{Y}$ is now $(n - k_{\max}) \times k$ and the $(n - k_{\max}) \times q'$ matrix $\mathbf{E}$ is defined as

$$\mathbf{E} = \begin{bmatrix} \epsilon_{k_{\max}} & \dots & \epsilon_{k_{\max}-q'+1} \\ \vdots & & \vdots \\ \epsilon_{n-1} & \dots & \epsilon_{n-q'} \end{bmatrix}.$$

The acceptance ratio is given by

$$A_{q,q'} \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}(\boldsymbol{\epsilon}^* - \mathbf{E}\boldsymbol{u})'(\boldsymbol{\epsilon}^* - \mathbf{E}\boldsymbol{u})\right] \exp\left[-\frac{1}{2\sigma_b^2}\boldsymbol{u}'\boldsymbol{u}\right] \exp\left[\frac{1}{2}(\boldsymbol{u}-\boldsymbol{\mu})'\boldsymbol{C}^{-1}(\boldsymbol{u}-\boldsymbol{\mu})\right]$$

where $\boldsymbol{\epsilon}^* = \boldsymbol{y} - \mathbf{Y}\mathbf{a}$ and the terms in the proportionality constant do not depend on $\boldsymbol{u}$.

Using the approximation based on $\mathbf{E}$ fixed, the first and second order derivatives of $\log A_{q,q'}$ with respect to $\boldsymbol{u}$ are given by

$$\begin{aligned}\nabla \log A_{q,q'} &= \sigma_\epsilon^{-2}\mathbf{E}'(\boldsymbol{\epsilon}^* - \mathbf{E}\boldsymbol{u}) - \sigma_b^{-2}\boldsymbol{u} + \boldsymbol{C}^{-1}(\boldsymbol{u}-\boldsymbol{\mu}) \\ \nabla^2 \log A_{q,q'} &= -\sigma_\epsilon^{-2}\mathbf{E}'\mathbf{E} - \sigma_b^{-2}\mathbf{I}_{q'} + \boldsymbol{C}^{-1}.\end{aligned}$$

Setting the second derivative, which does not depend on the value of $\boldsymbol{u}$, to zero we obtain

$$\boldsymbol{C}^{-1} = \sigma_\epsilon^{-2}\mathbf{E}'\mathbf{E} + \sigma_b^{-2}\mathbf{I}_{q'}.$$

Similarly, setting the first derivative to zero and using the above expression for $\boldsymbol{C}^{-1}$ it follows that

$$\boldsymbol{\mu} = \sigma_\epsilon^{-2}\boldsymbol{C}\mathbf{E}'(\boldsymbol{y} - \mathbf{Y}\mathbf{a}).$$

B Adaptive Proposals for Reciprocal Roots

Here we give expressions for the proposal parameters when adding or deleting one or two reciprocal roots. Expressions are given by assuming the logistic-based prior described in Section 4.1.

B.1 Updating the AR Component

When proposing a move from k to $k+1$ by adding one real reciprocal root r to the autoregressive component, the likelihood function under the larger model is given by (9). Using a logistic-based proposal for r, we sample a new value $z = \log((1+r)/(1-r))$ from a $N(\mu, \sigma^2)$ distribution. In this case, the acceptance ratio takes the form

$$A_{k,k+1} \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum(\epsilon_t - r\epsilon_{t-1})^2\right]\exp(-z^2/2\sigma_z^2)\exp((z-\mu)^2/2\sigma^2) \tag{11}$$

where terms in the proportionality constant do not depend on r, μ and σ^2.

In order to construct adaptive proposals we use Brooks and Ehlers (2002) second order method by setting to zero the first and second order derivatives of $\log A_{k,k+1}$ with respect to r and evaluating at $r = 0$ (since the likelihoods under the current and new model are identical at this point). It is not difficult to see that the proposal parameters are given by

$$\mu = \frac{2S_2}{S_1 + 4\sigma_\epsilon^2/\sigma_z^2} \quad \text{and} \quad \sigma^2 = \frac{4\sigma_\epsilon^2}{S_1 + 4\sigma_\epsilon^2/\sigma_z^2}$$

where $S_1 = \sum \epsilon_{t-1}^2$ and $S_2 = \sum \epsilon_t\epsilon_{t-1}$.

If the proposed move is from k to $k+2$ by adding a pair of complex reciprocal roots $(u, \bar{u})$, the error terms in the higher dimensional model can be written as

$$\epsilon_t' = (1-uL)(1-\bar{u}L)\prod_{i=1}^{k}(1-\lambda_i L)y_t = (1-uL)(1-\bar{u}L)\epsilon_t = (1-2r\cos\theta L + r^2L^2)\epsilon_t$$

and, given the sequence $\{\epsilon_t\}$, the likelihood function for (θ, r) is given by

$$p(\boldsymbol{\epsilon}|\theta, r, \sigma_\epsilon^2, \boldsymbol{\lambda}^{(k)}) \propto \exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum(\epsilon_t - 2r\cos\theta\epsilon_{t-1} + r^2\epsilon_{t-2})^2\right]. \tag{12}$$

The second order method in Brooks and Ehlers (2002) suggests taking the prior of θ as a proposal in which case we define a proposal ratio $A_{k,k+2}$ replacing the likelihood function in (11) by the one in (12). Setting to zero the first and second derivatives of $A_{k,k+2}$ evaluated at $(\theta, r) = (0, 0)$ it is not difficult to obtain the proposal parameters as

$$\mu = \frac{4S_2}{2(2S_1 + S_3) + 4\sigma_\epsilon^2/\sigma_z^2} \quad \text{and} \quad \sigma^2 = \frac{4\sigma_\epsilon^2}{2(2S_1 + S_3) + 4\sigma_\epsilon^2/\sigma_z^2}$$

where $S_3 = \sum \epsilon_t\epsilon_{t-2}$.

B.2 Updating the MA Component

Here we propose a jump from q to $q+1$ or $q+2$ by adding a new real reciprocal root or a pair of complex ones to the MA component. In this case, treating the past error terms in the higher dimensional model as fixed we can solve the associated derivative equations as the acceptance ratios $A_{q,q+1}$ and $A_{q,q+2}$ take the form (11) with the likelihood terms replaced by

$$\exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum(\epsilon_t + r\epsilon_{t-1})^2\right] \quad \text{and} \quad \exp\left[-\frac{1}{2\sigma_\epsilon^2}\sum(\epsilon_t + 2r\cos\theta\epsilon_{t-1} - r^2\epsilon_{t-2})^2\right]$$

respectively. Be setting to zero the first and second derivatives of $A_{q,q+1}$ and $A_{q,q+2}$ evaluated at $r = 0$ and $(\theta, r) = (0, 0)$ we obtain the proposal parameters as

$$\mu = \frac{-2S_2}{S_1 + 4\sigma_\epsilon^2/\sigma_z^2} \quad \text{and} \quad \sigma^2 = \frac{4\sigma_\epsilon^2}{S_1 + 4\sigma_\epsilon^2/\sigma_z^2}$$

for adding a real root, and

$$\mu = \frac{-4S_2}{2(2S_1 - S_3) + 4\sigma_\epsilon^2/\sigma_z^2} \quad \text{and} \quad \sigma^2 = \frac{4\sigma_\epsilon^2}{2(2S_1 - S_3) + 4\sigma_\epsilon^2/\sigma_z^2}$$

for adding a complex pair.

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