1. v)

Za $u \sim U[0,1]$, $y \sim g(y)$ i $x \sim f(x)$ važi $\frac{f}{g} < M$ (M je gornje ogranicenje).

Slučajan broj se prihvata $Acceptance\ Rejection$ metodom ako važi $u \leq \frac{f(y)}{Mg(y)}$.

Verovatnoća prihvatanja je onda:

$$P\left\{u \leq \frac{f(y)}{Mg(y)}\right\} = \int_{-\infty}^{+\infty} \int_{0}^{\frac{f(y)}{Mg(y)}} g(y) du dy = \int_{-\infty}^{+\infty} \frac{f(y)}{Mg(y)} \cdot g(y)\, dy = \frac{1}{M} \int_{-\infty}^{+\infty} f(y) dy = \frac{1}{M}$$

$\frac{f}{g}$ je normalizujuća konstanta i tada je $\frac{f}{g} = k \cdot \frac{\hat{f}}{\hat{g}}$, gde je $\frac{\hat{f}}{\hat{g}} \leq \widehat{M}$

$\widehat{M}$ se razlikuje od M zbog nedostatka normalizujuće konstante

Pošto $y \sim g(y)$:

$$P\left\{u \leq \frac{\hat{f}(y)}{\widehat{M}\hat{g}(y)}\right\} = \int_{-\infty}^{+\infty} \int_{0}^{\frac{\hat{f}(y)}{\widehat{M}\hat{g}(y)}} g(y) du dy = \int_{-\infty}^{+\infty} \frac{\hat{f}(y)}{\widehat{M}\hat{g}(y)} \cdot g(y)\, dy = \int_{-\infty}^{+\infty} \frac{f(y)}{k\widehat{M}g(y)} g(y) dy = \frac{1}{k\widehat{M}}$$

Nedostajuća normalizujuća konstanta je :

$$k = \frac{1}{\widehat{M} \cdot P\left\{u \leq \frac{\hat{f}(y)}{\widehat{M}\hat{g}(y)}\right\}}$$

2. a)

Apriorna gustina raspodele parametra θ je $\pi(\theta)$.

Aposteriorna gustina je

$$\pi(\theta|\boldsymbol{X}) = \pi(\theta) \prod_{i=1}^{n} f(x_i|\theta)$$

Funkcija verodostojnosti je

$$L(\theta, \boldsymbol{X}) = \prod_{i=1}^{n} f(x_i|\theta)$$

Gornja granica za M u $Acceptance - Rejection$ algoritmu gustine $\pi(\theta|\boldsymbol{X})$ i pomoćne gustine $\pi(\theta)$ mora da zadovoljava uslov $\frac{\pi(\theta|\boldsymbol{X})}{\pi(\theta)} \leq M$.

Iz Bajesovog pravila i ignorisnjem konstanti(one ne uticu na rezultat u $Acceptance - Rejection$ algoritmu) dobijamo $\frac{\pi(\theta|\boldsymbol{X})}{\pi(\theta)} = \prod_{i=1}^{n} f(x_i|\theta)$.

Tada se najmanja (i optimalna) vrednost za M dobija kada maksimizujemo desnu stranu u odnosu na θ što je u stvari maksimum funkcije verodostojnost, tj dobijamo

$$\frac{\pi(\theta|\boldsymbol{X})}{\pi(\theta)} \leq \max_{\theta} \prod_{i} f(x_i|\theta)$$

4. Dat je uzorak $\boldsymbol{X} = (X_1, \dots, X_n)$ iz $N(m, \sigma^2)$ raspodele ,gde je m poznato. Apriorna raspodela za σ^2 je $IG(a, b)$.Traži se aposteriorna raspodela za σ^2.

Obelezimo $\theta = \sigma^2$.

$X_i|\theta: N(m, \theta)\ , i = 1, \dots, n$
$\theta: IG(a, b)$

$$f(X|\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(x-m)^2}{2\theta}}$$

$$L(\theta, \boldsymbol{X}) = \prod_{i=1}^{n} f(X_i|\theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n e^{-\frac{1}{2\theta}\sum_{i=1}^{n}(x_k-m)^2}$$

$$q(\theta) = \frac{\theta^{-a-1} e^{-\frac{b}{\theta}} b^a}{\Gamma(a)}$$

$$q(\theta|\boldsymbol{X}) = \frac{L(\theta, \boldsymbol{X}) \cdot q(\theta)}{\int L(\theta, \boldsymbol{X}) \cdot q(\theta) d\theta}$$

$$q(\theta|\boldsymbol{X}) = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \theta^{-\frac{n}{2}} \cdot e^{-\frac{1}{2\theta}\sum_{i=1}^{n}(x_k-m)^2} \cdot \frac{\theta^{-a-1} e^{-\frac{b}{\theta}} b^a}{\Gamma(a)}}{\int \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \theta^{-\frac{n}{2}} \cdot e^{-\frac{1}{2\theta}\sum_{i=1}^{n}(x_k-m)^2} \cdot \frac{\theta^{-a-1} e^{-\frac{b}{\theta}} b^a}{\Gamma(a)} d\theta}$$

$$q(\theta|\boldsymbol{X}) \ = K \cdot \theta^{-\frac{n}{2}-a-1} \cdot e^{-\frac{1}{2\theta}\left(\sum_{i=1}^{n}(x_k-m)^2 + 2b\right)}$$

$$-\alpha - 1 = -\frac{n}{2} - a - 1 \quad \to \quad \alpha = \frac{n}{2} + a$$

$$-\frac{\beta}{\theta} = -\frac{1}{2\theta}\left(\sum_{i=1}^{n}(x_k - m)^2 + 2b\right) \quad \to \quad \beta = \frac{1}{2}\sum_{i=1}^{n}(x_k - m)^2 + b$$

$\theta|\boldsymbol{X} \ : \quad IG(\alpha, \beta)$

Konačno, aposteriorna raspodela za σ^2 je:

$$\sigma^2|\boldsymbol{X} \ : \quad IG(\frac{n}{2} + a, \frac{1}{2}\sum_{i=1}^{n}(x_k - m)^2 + b)$$