Direct Linear Transformation from Comparator Coordinates into Object Space Coordinates in Close-Range Photogrammetry*

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When does a scientific paper become a "classic"? In the case of the below article, there is a following that has born the test of time. Each year at the beginning of the school semester, we receive many requests from professors for reprints of this article to use in class. The original article was published in 1971 as part of committee work within ASP (the original name for ASPRS), but it has not appeared in PE&RS *or other journals, as far as we know. Unfortunately, one of the authors has passed away and we have not been able to reach the other, but if you were a student or colleague of either we would appreciate hearing from you. Dr Karara was an important member of ASPRS and there is a lengthy In Memoriam in the July 2001 issue of* PE&RS*, but we have located little further information about Y.I. Abdel-Aziz. In light of the continuing demand for their paper, it makes its* PE&RS *debut in this issue. We hope you enjoy this "classic."*

–Dr. Michael Hauck, *ASPRS Executive Director*

Abstract
A method for photogrammetric data reduction without the necessity for neither fiducial marks nor initial approximations for inner and outer orientation parameters of the camera has been developed. This approach is particularly suitable for reduction of data from non-metric photography, but has also distinct advantages in its application to metric photography. Preliminary fictitious data tests indicate that the approach is promising. Experiments with real data are underway.

1. INTRODUCTION
In analytical photogrammetry, measurements of image points are normally done on comparators. The transformation of comprarator coordinates into object space coordinates is usually performed in two steps:

a) Transformation from comparator coordinates into image coordinates, and
b) Transformation from image coordinates into object space (ground coordinates)

For the transformation from comparator coordinates into image coordinates, it is necessary to calibrate and measure fiducial marks. For the transformation from image coordinates into object space coordinates, an iterative solution is generally used, for which one needs initial approximations for the unknown parameters (elements of outer orientation and in some cases also elements of inner orientation of the camera).

In working with hand-held non-metric cameras, neither of the above two requirements are satisfied. In view of the ever increasing use of non-metric cameras in close-range photogrammetry, particularly in cases of medium to low accuracy requirements, it was deemed desirable to develop a method suitable for data reduction from non-metric photography.

The proposed method involves a direct linear transformation from comparator coordinates into object space coordinates. In a sense, it is a simultaneous solution for the two aforementioned transformations. Since the image coordinate system is not involved in the approach, fiducial marks are not needed. Furthermore, the method is a direct solution and does not involve initial approximations for the unknown parameters of inner and outer orientation of the camera.

The proposed method is thus particularly suitable for reduction of data in non-metric photogrammetry. When applied to metric photography, the proposed approach yields at least the same accuracy as the conventional methods, but is easier to program (no linearization necessary) and uses less computer memory and executing time.

2. Mathematical Basis of the Proposed Method
As mentioned above, the proposed method involves a simultaneous solution of two transformations which are usually done separately in conventional analytical photogrammetry.

The transformation of comparator coordinates into image coordinates is generally done in the following forms:

$$\begin{aligned}\bar{x} &= a_1 + a_2 x + a_3 y \\ \bar{y} &= a_4 + a_5 x + a_6 y,\end{aligned} \tag{1}$$

where:
$\bar{x}$, $\bar{y}$ are image coordinates
x, y are comparator coordinates

Such a transformation takes into account errors in perpendicularity between the x and y comparator coordinate axes, and possible differential linear distortions along the x and y comparator coordinate axes (due to lens distortion, film deformation and comparator unadjustment).

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The transformation from image coordinates into object space coordinates is usually done using the following equation:

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ -c \end{bmatrix} = \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \bullet \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}, \ldots \tag{2}$$

Where:
$\bar{x}$, $\bar{y}$ are image coordinates of points,
X,Y,Z are object-space coordinates of points,
X_O, Y_O, Z_O are the object-space coordinates of exposure stations,
C is the camera constant,
λ is a scale factor, and
a_{ij} are the coefficients of transformation

Equation (2) may be expressed as:

$$\bar{x} + c\,\frac{a_{11}(X-X_O)+a_{12}(Y-Y_O)+a_{13}(Z-Z_O)}{a_{31}(X-X_O)+a_{32}(Y-Y_O)+a_{33}(Z-Z_O)} = 0,$$

and (3)

$$\bar{y} + c\,\frac{a_{21}(X-X_O)+a_{22}(Y-Y_O)+a_{23}(Z-Z_O)}{a_{31}(X-X_O)+a_{32}(Y-Y_O)+a_{33}(Z-Z_O)} = 0.$$

Substituting equation (1) into equation (3) one gets:

$$a_1 + a_2 x + a_3 y + c\,\frac{a_{11}(X-X_O)+a_{12}(Y-Y_O)+a_{13}(Z-Z_O)}{a_{31}(X-X_O)+a_{32}(Y-Y_O)+a_{33}(Z-Z_O)} = 0,$$

and (4)

$$a_4 + a_5 x + a_6 y + c\,\frac{a_{21}(X-X_O)+a_{22}(Y-Y_O)+a_{23}(Z-Z_O)}{a_{31}(X-X_O)+a_{32}(Y-Y_O)+a_{33}(Z-Z_O)} = 0.$$

Eliminating y from equations (4), one gets:

$$(a_1a_6 - a_4a_3) + (a_2a_6 - a_5a_3)x + c\,\frac{(a_{11}a_6 - a_3a_{21})(X-X_O) + (a_{12}a_6 - a_{22}a_3)(Y-Y_O) + (a_{13}a_6 - a_{23}a_3)(Z-Z_O)}{a_{31}(X-X_O)+a_{32}(Y-Y_O)+a_{33}(Y-Y_O)} = 0. \tag{5a}$$

Eliminating x from equations (4) one gets:

$$(a_1a_5 - a_4a_2) + (a_3a_5 - a_6a_6)y + c\,\frac{(a_{11}a_5 - a_{21}a_2)(X-X_O) + (a_{12}a_5 - a_{22}a_2)(Y-Y_O) + (a_{13}a_5 - a_{23}a_2)(Z-Z_O)}{a_{31}(X-X_O)+a_{32}(Y-Y_O)+a_{33}(Y-Y_O)} = 0. \tag{5b}$$

Equations (5a) and (5b) may be expressed as:

$$d_1 + d_2\,x\,\frac{b_1 X + b_2 Y + b_3 Z + b_4}{b_9 X + b_{10} Y + b_{11} Z + b_{12}} = 0,$$

and (6)

$$d_3 + d_4\,x\,\frac{b_5 X + b_6 Y + b_7 Z + b_8}{b_9 X + b_{10} Y + b_{11} Z + b_{12}} = 0.$$

Eliminating d_1 and d_3 from equations (6) one gets:

$$d_2\,x + \frac{b_{1'} X + b_{2'} Y + b_{3'} Z + b_{4'}}{b_{9'} X + b_{10'} Y + b_{11'} Z + b_{12'}} = 0,$$

and (7)

$$d_4\,y + \frac{b_{5'} X + b_{6'} Y + b_{9'} Z + b_{8'}}{b_{9'} X + b_{10'} Y + b_{11'} Z + b_{12'}} = 0.$$

Eliminating d_2 and d_4 from equations (7), one gets:

$$x + \frac{b_{1*} X + b_{2*} Y + b_{3*} Z + b_{4*}}{b_{9'} X + b_{10'} Y + b_{11'} Z + b_{12'}} = 0,$$

and (8)

$$y + \frac{b_{5*} X + b_{6*} Y + b_{7*} Z + b_{8*}}{b_{9'} X + b_{10'} Y + b_{11'} Z + b_{12'}} = 0.$$

Eliminating $b_{12'}$ from equations (8), one gets:

$$x + \frac{\ell_1 X + \ell_2 Y + \ell_3 Z + \ell_4}{\ell_9 X + \ell_{10} Y + \ell_{11} Z + 1} = 0,$$

and (9)

$$y + \frac{\ell_5 X + \ell_6 Y + \ell_7 Z + \ell_8}{\ell_9 X + \ell_{10} Y + \ell_{11} Z + 1} = 0.$$

Equations (9) are the basis of the proposed method.

3. Mathematical Model in the Conventional (Collinearity) Approach

As mentioned above, the transformation from comparator coordinates (x, y) into image coordinates ($\bar{x}$, $\bar{y}$) is usually done using equations (1):

$$\bar{x} = a_1 + a_2 x + a_3 y$$
$$\bar{y} = a_4 + a_5 x + a_6 y \tag{1}$$

Since the selection of the image coordinate axes is arbitrary, let us select the definition shown in Fig. 1, where the $\bar{y}$ image coordinate axis is parallel to the y comparator coordinate axis and passes through the image principal point (0). The $\bar{x}$ image coordinate axis is perpendicular to the $\bar{y}$ axis and intersects it at the image principal point.

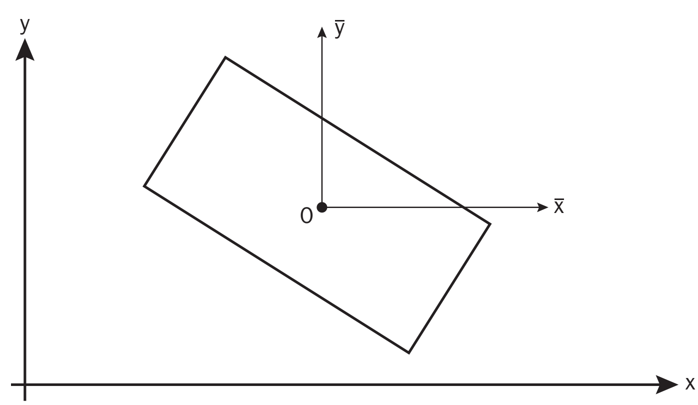

Figure 1. Coordinated Axes. (x & y: comparator coordinate aces; $\bar{x}$ & $\bar{y}$: image coordinate axes).

In this case, a5 in equations (1) becomes zero and the relationship between comparator coordinates and image coordinates can be expressed as:

$$\begin{aligned}\bar{x} &= a_1 + a_2 x + a_3 y \\ \bar{y} &= a_4 + a_6 y\end{aligned} \tag{10}$$

Combining equations (10) and (3) one gets

$$\begin{aligned}&a_1 + a_2 x + a_3 y - c \cdot \frac{a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)} = 0 \\ &a_4 + a_6 y - c \cdot \frac{a_{21}(X - X_0) + a_{22}(Y - Y_0) + a_{23}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)} = 0\end{aligned} \tag{11}$$

Equation (11) has 12 unknowns, but they are not linearly independent. These unknowns can be reduced to 11 linearly independent unknowns by eliminating a_2 and a_6 and introducing two unknowns C_x and C_y to replace C;

($C_x = \frac{c}{a_2}; C_y = \frac{c}{a_6}$, C_x and C_y reflect possible differential linear distortions along x and y comparator axes). Equation 11 can thus be rewritten as:

$$\begin{aligned}&\bar{a}_1 + \bar{a}_2 x - C_x \frac{a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)} = 0 \\ &\bar{a}_3 + y - C_y \frac{a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)} = 0\end{aligned} \tag{12}$$

Equations (12) represent the basic equations in the conventional (collinearity) approach. As explained above, these equations take into consideration the non-prependicularity between comparator axes, and differential linear distortions along x and y axes.

4. Observation Equations

Expanding equations (12) by Taylor's series and neglecting second and higher order items, one gets:

$$V_x + a_{1y}V_y + b_{1x}\Delta\omega + b_{2x}\Delta\phi + b_{3x}\Delta\kappa + b_{4x}\Delta X_0 + b_{5x}\Delta Y_0 + b_{6x}\Delta Z_0 + b_{7x}\Delta C_x + b_{8x}\Delta C_y + b_{9x}\Delta\bar{a}_1 + b_{10x}\Delta\bar{a}_2 + b_{11x}\Delta\bar{a}_3 + F_x{}^{\circ} = 0$$

and (13)

$$V_y + b_{1y}\Delta\omega + b_{2y}\Delta\phi + b_{3y}\Delta\kappa + b_{4y}\Delta Y_0 + b_{5y}\Delta Y_0 + b_{6y}\Delta Z_0 + b_{7y}\Delta C_y + b_{8y}\Delta C_y + b_{9y}\Delta\bar{a}_1 + b_{10y}\Delta\bar{a}_2 + b_{11y}\Delta\bar{a}_3 + F_y{}^{\circ} = 0$$

Where

V_x, V_y are errors in x and y
$a_{1y} = a_2$, is partial derivative of F_x w.r.t. y
b_{1x}, b_{1y} are the partial derivatives of F_x and F_y (see footnote below) w.r.t. ω
b_{2x}, b_{2y} are the partial derivatives of F_x and F_y w.r.t. ϕ
b_{3x}, b_{3y} are the partial derivatives of F_x and F_y w.r.t. κ
b_{4x}, b_{4y} are the partial derivatives of F_x and F_y w.r.t. X_0
b_{5x}, b_{5y} are the partial derivatives of F_x and F_y w.r.t. Y_0
b_{6x}, b_{6y} are the partial derivatives of F_x and F_y w.r.t. Z_0
b_{7x}, b_{7y} are the partial derivatives of F_x and F_y w.r.t. C_x
b_{8x}, b_{8y} are the partial derivatives of F_x and F_y w.r.t. C_y
b_{9x}, b_{9y} are the partial derivatives of F_x and F_y w.r.t. $\bar{a}_1$
b_{10x}, b_{10y} are the partial derivatives of F_x and F_y w.r.t. $\bar{a}_2$
b_{11x}, b_{11y} are the partial derivatives of F_x and F_y w.r.t. $\bar{a}_3$

F_{x^o} and F_{y^o} are functions of approximate values of the unknown parameters.

Equations (13) represent the observation equations in the conventional collinearity approach. The observation equations in the proposed direct approach may be obtained by expanding equations (9) and including all the zero terms (e.g. $0\ell_5$ and $0\ell_2$) for ease of reference:

$$w_1 v_x + X\ell_1 + Y\ell_2 + Z\ell_3 + \ell_4 + 0\ell_5 + 0\ell_6 + 0\ell_7 + 0\ell_8 + xX\ell_9 + xY\ell_{10} + xZ\ell_{11} + x = 0,$$

and (14)

$$w_2 v_y + 0\ell_1 + 0\ell_2 + 0\ell_3 + 0\ell_4 + X\ell_5 + Y\ell_6 + Z\ell_7 + \ell_8 + yX\ell_9 + yY\ell_{10} + yZ\ell_{11} + y = 0.$$

In equations (14) the factor w1 and w2 may be considered as weight factors, and their value can be easily determined in the solution.

A comparison between equations (13) and equations (14) indicate the simplicity of the proposed solution.

5. Analysis of Errors

Both the conventional and proposed approaches are influenced by the following errors:

a. Uncertainties in comparator measurements and errors in object space coordinates of control points.
b. Errors in mathematical modeling of film and lens distortions (random errors as well as unrepresented – or residual – systematic errors).

In addition, the conventional iterative approach is subject to computational errors due to:

a. Iteration criteria
b. Neglecting of second and higher terms in the linearization of the observation equations (13).

Obviously, the proposed direct solution is not subject to these computational errors.

6. Fictitious Data Tests

A number of fictitious data tests were conducted to assess the capabilities of the proposed solution (equations 9) and compare them to the capabilities of the conventional approach (equations 12.) As a datum for comparison of the two approaches, data from the collinearity approach with 9 unknowns (only parameters of inner and outer orientation are included, errors due to comparator adjustment, lens distortion, and film deformation are not considered) were used. The test covered the following aspects: handling of differential linear distortions along the x and y comparator coordinate axes, correction for non-prependicularity of the comparator axes, accuracy of determination of the unknowns (standard error of unit weight), and computer executing time.

Tables I through V summarize the results of these preliminary tests.

$$F_x = \bar{a}_1 + x + \bar{a}_2 y - C_x \cdot \frac{a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)} = 0$$

$$F_y = \bar{a}_3 + y - C_y \cdot \frac{a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0)}{a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)} = 0$$

TABLE I. COMPUTER EXECUTING TIME. THE NUMBER OF ITERATIONS INDICATED IN THE FIRST COLUMN REFLECT THE QUALITY OF APPROXIMATIONS USED IN THE VARIOUS EXPERIMENTS WITH THE COLLINEARITY SOLUTION.

Number of Iterations in Collinearity Approach	Number of Points Involved	Executing Time (secs)	
		Direct	Collinearity
2	43	6.16	12.56
4		6.16	22.14
5		6.16	27.21
No Convergence		6.16	No Solution
2	12	3.77	5.80
4		3.77	9.67
6		3.77	13.56
No Convergence		3.77	No Solution

TABLE II. STANDARD ERROR OF UNIT WEIGHT σ_0. IMAGE COORDINATES PERTURBED BY A RANDOM ERROR OF NORMAL DISTRIBUTION WITH THE STANDARD ERRORS INDICATED IN THE COLUMN "INPUT" AND MEAN ZERO.

Number of Points	σ_0(µm)		
		Output	
	Input	Direct Method	Conventional
43	3.000	2.781	2.995
	5.000	4.635	4.992
	10.000	9.271	9.985
	20.000	18.545	19.969
12	3.000	2.435	2.692
	10.000	8.117	8.972

TABLE III. ACCURACY OF OBJECT-SPACE COORDINATES OBTAINED BY THE DIRECT AND THE COLLINEARITY APPROACHES FOR DIFFERENT NUMBERS OF CONTROL POINTS. IMAGE COORDINATES PERTURBED BY A RANDOM ERROR OF NORMAL DISTRIBUTION WITH STANDARD ERROR: 3.00 µM AND MEAN ZERO. Z-COORDINATE AXIS ALONG CAMERA AXIS. COLLINEARITY ACCORDING TO EQUATION 12. (*MEAN SQUARE ERROR = D $\sqrt{\frac{\text{Sum of squares of residual errors}}{\text{number of points} - 1}}$).

Method	Number of Control Points	Mean Square Error* (µm)			Estimated Standard Error of Unit Weight (µm)
		X	Y	Z	
Direct	5	NO SOLUTION POSSIBLE			
Collinearity	5	NO SOLUTION POSSIBLE			
Direct	6	205	179	408	1.770
Collinearity	6	200	174	407	2.005
Direct	10	165	135	334	2.293
Collinearity	10	172	137	353	2.261
Direct	20	157	94	342	2.792
Collinearity	20	161	95	356	2.744
Direct	30	138	100	301	2.888
Collinearity	30	141	101	310	2.892
Direct	43	135	94	297	2.782
Collinearity	43	135	92	303	2.995

TABLE IV. THE EFFECT OF NONPERPENDICULARITY OF THE X AND Y COMPARATOR AXES. IMAGE COORDINATES PERTURBED BY A RANDOM ERROR OF NORMAL DISTRIBUTION WITH STANDARD ERROR: 3.000 UM, AND MEAN ZERO.
(*COLLINEARITY I:9 UNKNOWNS, COLLINEARITY II, 11 UNKNOWNS).

Angle between x and y	Estimated Standard Error of Unit Weight (µm)		
	Collinearity I*	Collinearity II**	Direct Method
90°	2.970	2.995	2.781
91°	19.616	2.995	2.781
95°	97.616	2.995	2.781
99°	194.096	2.995	2.781

TABLE V. THE EFFECT OF SYSTEMATIC DIFFERENTIAL LINEAR DISTORTION ALONG THE X AND Y COMPARATOR AXES. IMAGE COORDINATES PERTURBED BY A RANDOM ERROR OF NORMAL DISTRIBUTION WITH STANDARD ERROR: 3.000 UM AND MEAN ZERO.
(*COLLINEARITY I: 9 UNKNOWNS, ** COLLINEARITY II: 11 UNKNOWNS.)

Scale Factor		Estimated Standard Error of Unit Weight (µm)		
x	y	Collinearity I*	Collinearity II**	Direct Method
1.000	1.000	2.970	2.995	2.781
1.000	1.0001	3.234	2.995	2.781
1.0001	1.0001	2.970	2.995	2.781
1.000	1.0002	3.896	2.995	2.781
1.0002	1.0002	2.970	2.995	2.781

Extensive testing of the proposed method using real data is currently underway.

7. Summary of Comparisons Between the Proposed and the Conventional Approaches

The proposed approach is particularly suitable for non-metric photography, where no fiducial marks are available, and can also be applied with distinct advantages for data reduction in metric photography. Following are some comments comparing the proposed approach to the conventional collinearity approach.

a. The proposed method yields at least the same accuracy as the conventional solution.
b. The proposed approach is a direct solution involving no iterations and needs no initial approximations for the unknowns. Thus a solution is obtained even in cases where the conventional collinearity approach fails due to the lack of reasonable approximations for the unknown parameters (inner and/or outer orientation elements). A case in point here would be metric photography for which the outer orientation is not known. It follows further that the proposed solution is not subject to computational errors due to iteration criteria nor to errors due to neglecting of second and higher order terms in linearizing the observation equations.
c. The proposed solution is relatively easy to program since it does not involve partial derivatives of the coefficients of the observation equation.
d. The computer executing time and the computer memory used are less in the proposed method than in the conventional collinearity solution.
e. The number of unknowns in the proposed direct method is the same as in the conventional approach, i.e. 11 (eleven). Thus a minimum of 6 well distributed control points are needed for a solution.
f. The proposed method is at a disadvantage in case of low accuracy requirements where one can neglect the comparator calibration errors and lens and film distortions. In this case the collinearity approach will have 9 unknowns compared to the 11 unknowns of the proposed method.

8. Concluding Remarks

Preliminary fictitious data tests indicate that the proposed approach is promising. Even though the method was originally developed for data reduction in close-range non-metric photography, it can be used with distinct advantages in conjunction with close-range metric photography. Experiments with real data are underway, and it is planned to report on these tests in the near future.

9. References

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This article has been reproduced exactly as originally published, with minor text corrections and formatting errors fixed.

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