In [1]:
```
# Data used: the 2019 Index of Economic Freedom (ief) public data from http
s://www.heritage.org/index/ is used
# Here, new metrics with new criteria (features) is introduced for arriving at
a score ranking the world countries
# Examining this new metrics is the motivation for this project based on the f
ollowing reasoning:
    # In our days, new metrics are introduced to rank or explain different thi
ngs and phenomena
        # It is not always clear that the new metrics and its results have any
logic behind or make any sense
    # So, our goal is:
        # to examine the new metrics feature behavior
        # to determine if these features or part of them play a significant ro
le in the overall score
        # to determine whether the overall score can be accurately predicted u
sing these features
# We use Linear Regression model for our analysis
# If the answers from our investigation are positive, then we can take into co
nsideration the new metrics and results
# If the answers are negative, then we should be discard this new metrics and
 the results presented
```

In [2]:
```
# Import libraries

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
%matplotlib inline
sns.set(style = "whitegrid", font_scale = 1.5)
```

In [3]:
```
# Read ief data

data = pd.read_excel('index2019_data.xls')

data.info()
```

```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 186 entries, 0 to 185
Data columns (total 34 columns):
 #   Column                        Non-Null Count  Dtype  
---  ------                        --------------  -----  
 0   CountryID                     186 non-null    int64  
 1   Country Name                  186 non-null    object 
 2   WEBNAME                       186 non-null    object 
 3   Region                        186 non-null    object 
 4   World Rank                    180 non-null    float64
 5   Region Rank                   180 non-null    float64
 6   2019 Score                    180 non-null    float64
 7   Property Rights               185 non-null    float64
 8   Judical Effectiveness         185 non-null    float64
 9   Government Integrity          185 non-null    float64
 10  Tax Burden                    180 non-null    float64
 11  Gov't Spending                183 non-null    float64
 12  Fiscal Health                 183 non-null    float64
 13  Business Freedom              185 non-null    float64
 14  Labor Freedom                 184 non-null    float64
 15  Monetary Freedom              184 non-null    float64
 16  Trade Freedom                 182 non-null    float64
 17  Investment Freedom            184 non-null    float64
 18  Financial Freedom             181 non-null    float64
 19  Tariff Rate (%)               182 non-null    float64
 20  Income Tax Rate (%)           183 non-null    float64
 21  Corporate Tax Rate (%)        183 non-null    float64
 22  Tax Burden % of GDP           179 non-null    float64
 23  Gov't Expenditure % of GDP    182 non-null    float64
 24  Country                       186 non-null    object 
 25  Population (Millions)         186 non-null    object 
 26  GDP (Billions, PPP)           185 non-null    object 
 27  GDP Growth Rate (%)           184 non-null    float64
 28  5 Year GDP Growth Rate (%)    183 non-null    float64
 29  GDP per Capita (PPP)          184 non-null    object 
 30  Unemployment (%)              181 non-null    object 
 31  Inflation (%)                 182 non-null    float64
 32  FDI Inflow (Millions)         181 non-null    float64
 33  Public Debt (% of GDP)        182 non-null    float64
dtypes: float64(25), int64(1), object(8)
memory usage: 49.5+ KB
```

In [4]:
```
data.head(10)
```

Out[4]:

| | CountryID | Country Name | WEBNAME | Region | World Rank | Region Rank | 2019 Score | Property Rights | Judical Effectiveness |
|---|---|---|---|---|---|---|---|---|---|
| **0** | 1 | Afghanistan | Afghanistan | Asia-Pacific | 152.0 | 39.0 | 51.5 | 19.6 | 29.6 |
| **1** | 2 | Albania | Albania | Europe | 52.0 | 27.0 | 66.5 | 54.8 | 30.6 |
| **2** | 3 | Algeria | Algeria | Middle East and North Africa | 171.0 | 14.0 | 46.2 | 31.6 | 36.2 |
| **3** | 4 | Angola | Angola | Sub-Saharan Africa | 156.0 | 33.0 | 50.6 | 35.9 | 26.6 |
| **4** | 5 | Argentina | Argentina | Americas | 148.0 | 26.0 | 52.2 | 47.8 | 44.5 |
| **5** | 6 | Armenia | Armenia | Europe | 47.0 | 24.0 | 67.7 | 57.2 | 46.3 |
| **6** | 7 | Australia | Australia | Asia-Pacific | 5.0 | 4.0 | 80.9 | 79.1 | 86.5 |
| **7** | 8 | Austria | Austria | Europe | 31.0 | 16.0 | 72.0 | 84.2 | 71.3 |
| **8** | 9 | Azerbaijan | Azerbaijan | Asia-Pacific | 60.0 | 13.0 | 65.4 | 59.1 | 53.1 |
| **9** | 10 | Bahamas | Bahamas | Americas | 76.0 | 15.0 | 62.9 | 42.2 | 46.9 |

10 rows × 34 columns

In [5]:
```
data.columns
```

Out[5]:
```
Index(['CountryID', 'Country Name', 'WEBNAME', 'Region', 'World Rank',
       'Region Rank', '2019 Score', 'Property Rights', 'Judical Effectivenes
s',
       'Government Integrity', 'Tax Burden', 'Gov't Spending', 'Fiscal Healt
h',
       'Business Freedom', 'Labor Freedom', 'Monetary Freedom',
       'Trade Freedom', 'Investment Freedom ', 'Financial Freedom',
       'Tariff Rate (%)', 'Income Tax Rate (%)', 'Corporate Tax Rate (%)',
       'Tax Burden % of GDP', 'Gov't Expenditure % of GDP ', 'Country',
       'Population (Millions)', 'GDP (Billions, PPP)', 'GDP Growth Rate (%)',
       '5 Year GDP Growth Rate (%)', 'GDP per Capita (PPP)',
       'Unemployment (%)', 'Inflation (%)', 'FDI Inflow (Millions)',
       'Public Debt (% of GDP)'],
      dtype='object')
```

In [6]:
```
# One can see that the data comprises of:
    # a) well-known criteria such as 'Population (Millions)', 'GDP (Billions,
 PPP)', 'GDP Growth Rate (%)', \
        # '5 Year GDP Growth Rate (%)', 'GDP per Capita (PPP)','Unemployment
 (%)', 'Inflation (%)', 'FDI Inflow (Millions)', \
        # 'Public Debt (% of GDP)'
    # b) new criteria which determine the ief score
# Here, we focus on these new criteria and new ranking score
```

In [7]:
```
# Select the new features contributing to the overall ief score

data_ief = data[['Property Rights', 'Judical Effectiveness', 'Government Integ
rity', 'Tax Burden', "Gov't Spending",
              'Fiscal Health', 'Business Freedom', 'Labor Freedom', 'Monetary
 Freedom', 'Trade Freedom', 'Investment Freedom ',
                 'Financial Freedom', '2019 Score']]
data_ief.head(5)
```

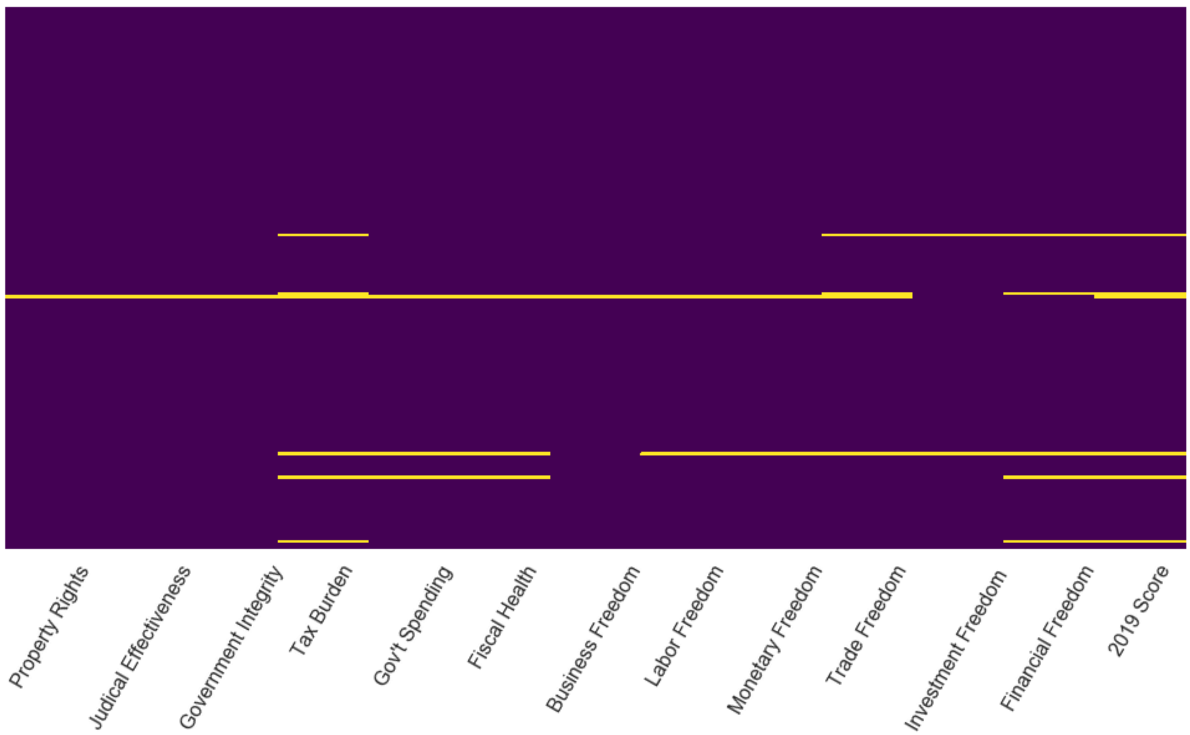
Out[7]:

| | Property Rights | Judical Effectiveness | Government Integrity | Tax Burden | Gov't Spending | Fiscal Health | Business Freedom | Labor Freedom | Monetar Freedoı |
|---|---|---|---|---|---|---|---|---|---|
| **0** | 19.6 | 29.6 | 25.2 | 91.7 | 80.3 | 99.3 | 49.2 | 60.4 | 76. |
| **1** | 54.8 | 30.6 | 40.4 | 86.3 | 73.9 | 80.6 | 69.3 | 52.7 | 81. |
| **2** | 31.6 | 36.2 | 28.9 | 76.4 | 48.7 | 18.7 | 61.6 | 49.9 | 74. |
| **3** | 35.9 | 26.6 | 20.5 | 83.9 | 80.7 | 58.2 | 55.7 | 58.8 | 55. |
| **4** | 47.8 | 44.5 | 33.5 | 69.3 | 49.5 | 33.0 | 56.4 | 46.9 | 60. |

In [8]:
```
# The target is '2019 Score' and the rest of the columns are the predictors (f
eatures)
```

In [9]:
```
# 1) EDA
```

In [10]:
```
# Visualize missing data points
# Please, note that this visualization method works only with relatively small
number of rows

plt.figure(figsize = (17, 8))
sns.heatmap(data_ief.isnull(), yticklabels = False,cbar = False, cmap ='viridi
s')
plt.tick_params(labelsize = 16, rotation = 60)

plt.show()
```




In [11]:
```
# The yellow bars in the plot represent missing data points
```

In [12]:
```
# Another way to find if there is missing data in different columns is by call
ing .info() on data -->
    # if the number of non-null entries is smaller than the total number of en
tries, then that feature has missing values

data_ief.info()
```

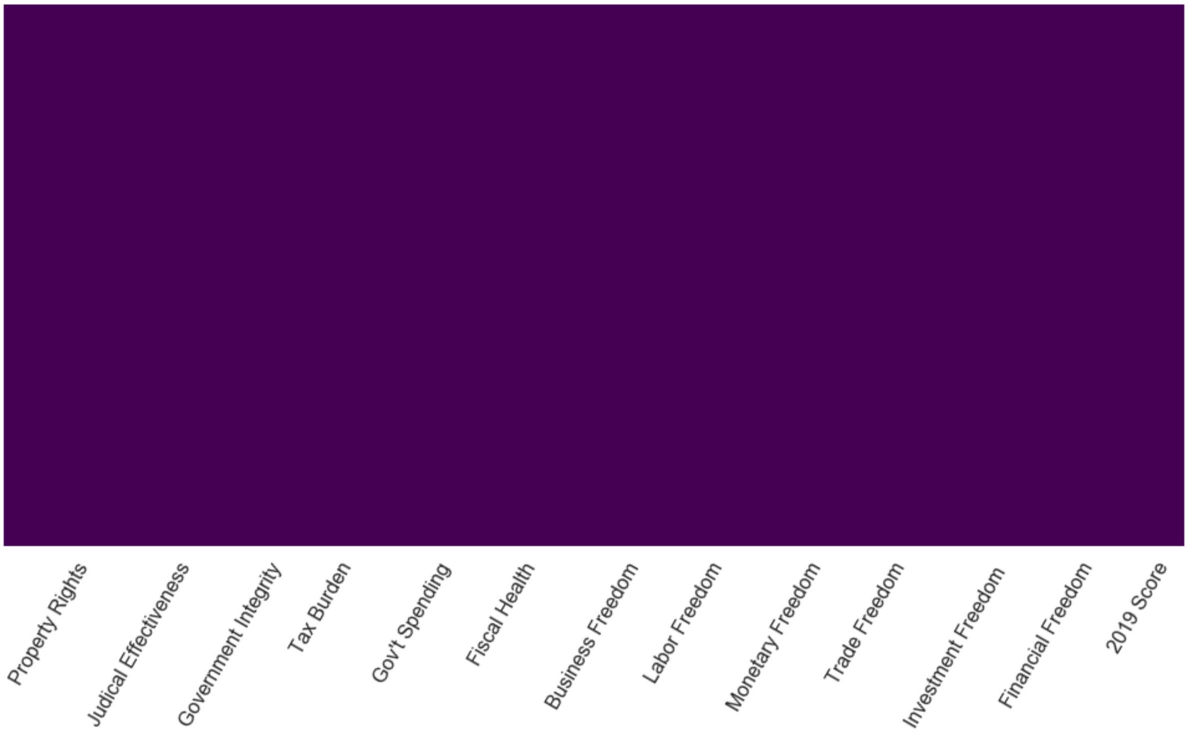
```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 186 entries, 0 to 185
Data columns (total 13 columns):
 #   Column                 Non-Null Count  Dtype  
---  ------                 --------------  -----  
 0   Property Rights        185 non-null    float64
 1   Judical Effectiveness  185 non-null    float64
 2   Government Integrity   185 non-null    float64
 3   Tax Burden             180 non-null    float64
 4   Gov't Spending         183 non-null    float64
 5   Fiscal Health          183 non-null    float64
 6   Business Freedom       185 non-null    float64
 7   Labor Freedom          184 non-null    float64
 8   Monetary Freedom       184 non-null    float64
 9   Trade Freedom          182 non-null    float64
 10  Investment Freedom     184 non-null    float64
 11  Financial Freedom      181 non-null    float64
 12  2019 Score             180 non-null    float64
dtypes: float64(13)
memory usage: 19.0 KB
```

In [13]:
```
# Maximum number of missing data is in the target column '2019 Score' - six mi
ssing points out of 186
# All other missing points are in the rows with missing target data points - s
ee missing data map above
# Since we cannot work without a target value dropping the few missing data po
ints is justified
```

In [14]:
```
# Drop nulls

data_c = data_ief.dropna().reset_index(drop = True)

# Always use .reset_index(drop=True) after dropna() or any time a raw is dropp
ed to avoid index mix up between different cols!!!

data_c.info()
```

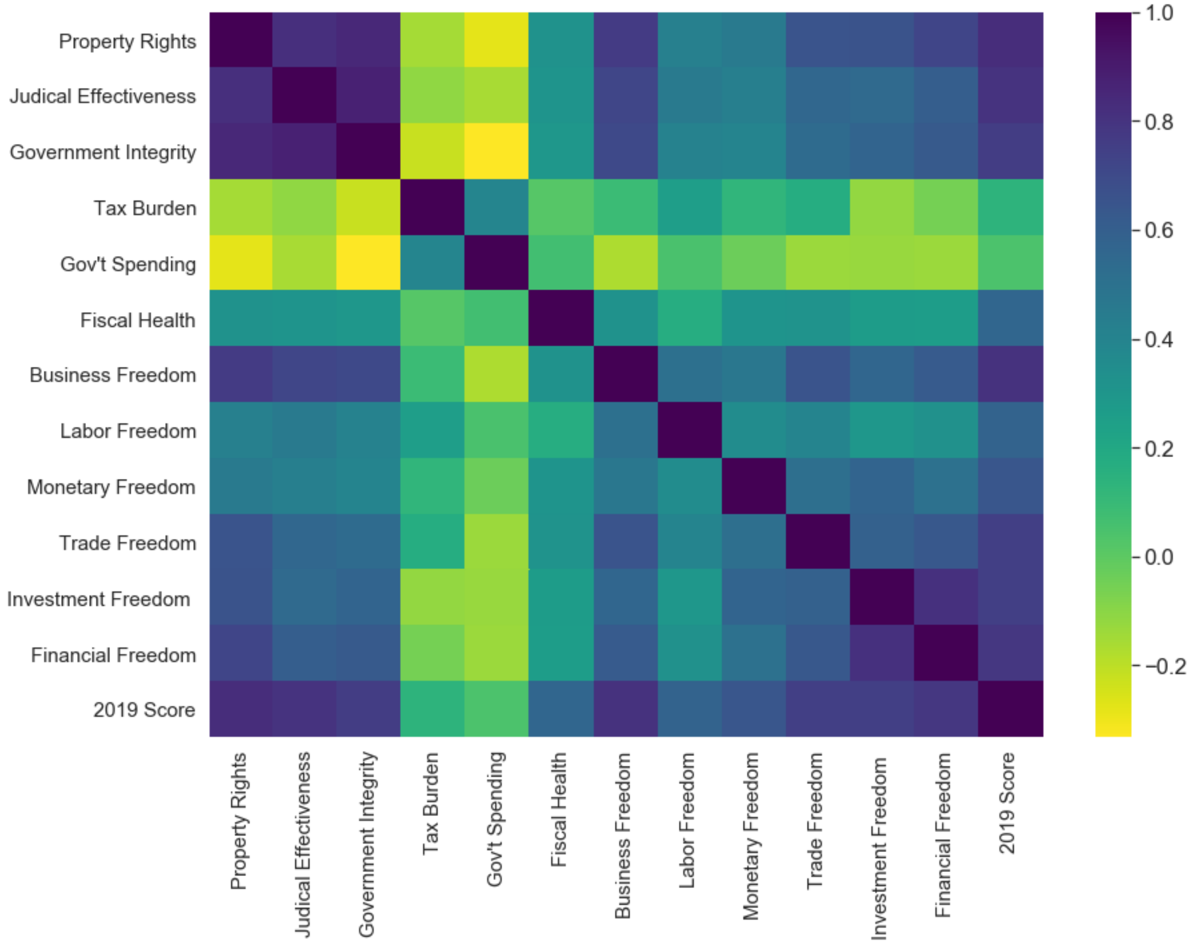
```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 180 entries, 0 to 179
Data columns (total 13 columns):
 #   Column                  Non-Null Count  Dtype  
---  ------                  --------------  -----  
 0   Property Rights         180 non-null    float64
 1   Judical Effectiveness   180 non-null    float64
 2   Government Integrity    180 non-null    float64
 3   Tax Burden              180 non-null    float64
 4   Gov't Spending          180 non-null    float64
 5   Fiscal Health           180 non-null    float64
 6   Business Freedom        180 non-null    float64
 7   Labor Freedom           180 non-null    float64
 8   Monetary Freedom        180 non-null    float64
 9   Trade Freedom           180 non-null    float64
 10  Investment Freedom      180 non-null    float64
 11  Financial Freedom       180 non-null    float64
 12  2019 Score              180 non-null    float64
dtypes: float64(13)
memory usage: 18.4 KB
```

In [15]:
```
# All missing data points are gone, but still do a visual check --> solid colo
r heatmap indicates no nulls

plt.figure(figsize = (17, 8))
sns.heatmap(data_c.isnull(),yticklabels=False,cbar=False,cmap ='viridis')
plt.tick_params(labelsize = 16, rotation = 60)

plt.show()
```




In [16]:
```
# Excellent - no missing data!
```

In [17]:
```
# Plot correlation matrix to examine for highly-correlated features

plt.figure(figsize = (14, 10))
sns.heatmap(data_c.corr(), cmap = 'viridis_r')
plt.tick_params(labelsize = 15)
plt.show()
```




In [18]:
```
# The correlation matrix shows:
    # high degree of correlation between 'Property Rights', 'Judicial Effectiv
eness' and 'Government Integrity' -->
        # keep only 'Property Rights'
    # high degree of correlation between 'Investment Freedom' and 'Financial F
reedom' -->
        # keep only 'Investment Freedom'
# Keeping only one of several highly-correlated features eliminates the possib
ility of having a degenerate matrix -->
    # degenerate matrix creates a problem for the linear regression algorithm
 when solving for the coefficients of each feature
```

In [19]:
```
# Print columns to have them handy for the code that follows
data_c.columns
```

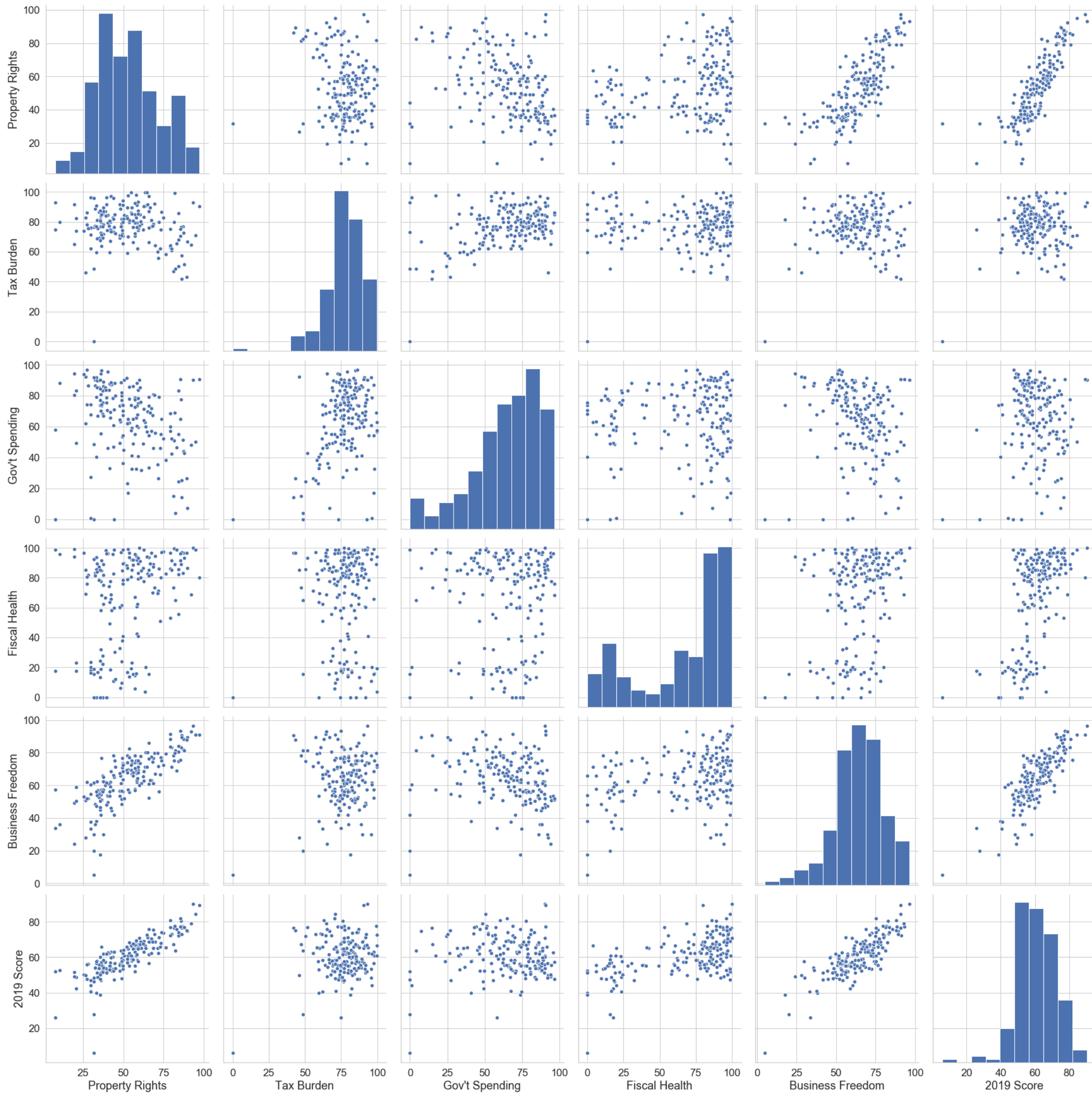
Out[19]:
```
Index(['Property Rights', 'Judical Effectiveness', 'Government Integrity',
       'Tax Burden', 'Gov't Spending', 'Fiscal Health', 'Business Freedom',
       'Labor Freedom', 'Monetary Freedom', 'Trade Freedom',
       'Investment Freedom ', 'Financial Freedom', '2019 Score'],
      dtype='object')
```

In [20]:
```
# Selecting only the uncorrelated features

data_c = data_c[['Property Rights', 'Tax Burden', "Gov't Spending", 'Fiscal He
alth', 'Business Freedom',
       'Labor Freedom', 'Monetary Freedom', 'Trade Freedom', 'Investment Freed
om ', '2019 Score']]
data_c.head(5)
```

Out[20]:

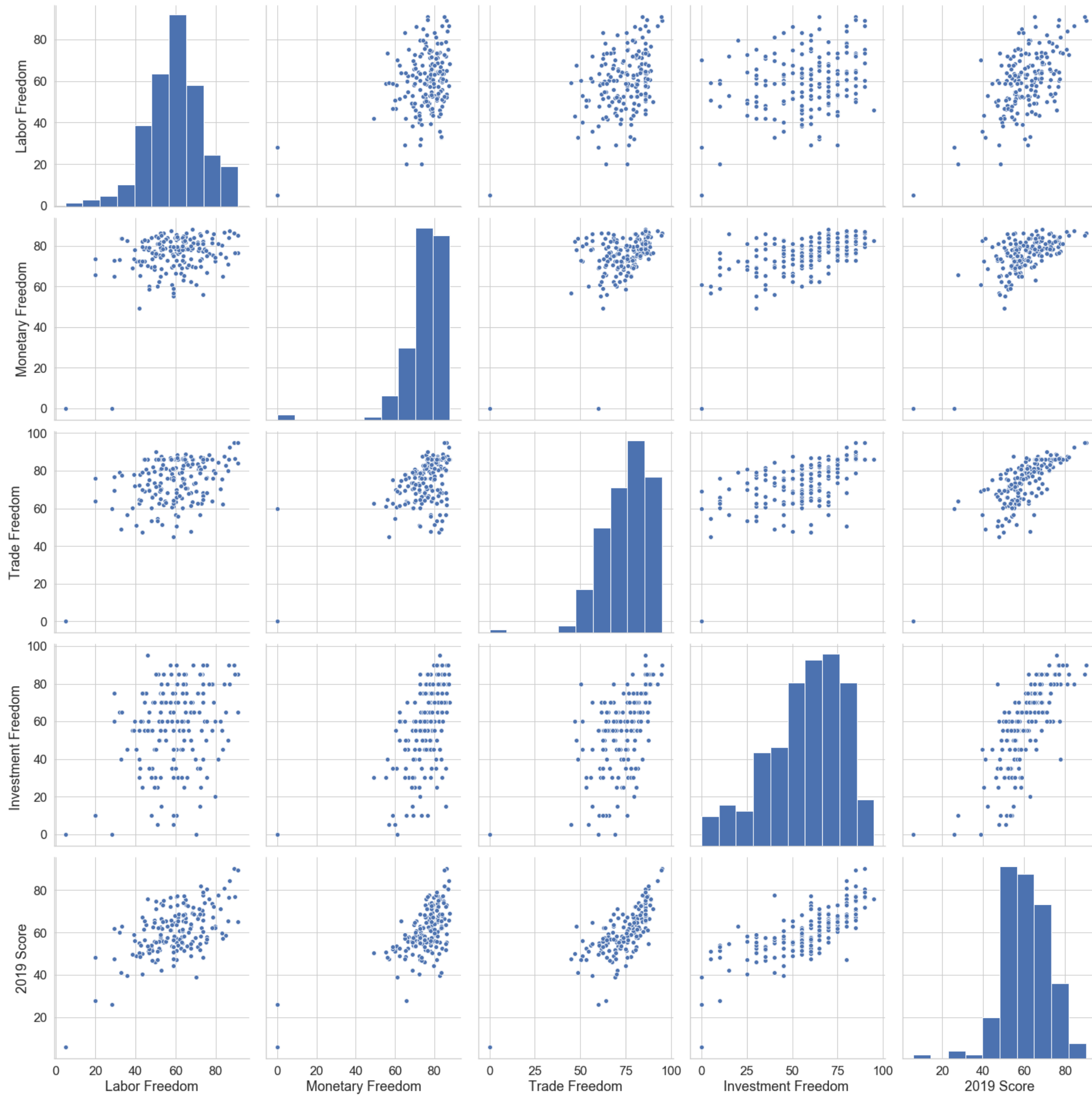
| | Property Rights | Tax Burden | Gov't Spending | Fiscal Health | Business Freedom | Labor Freedom | Monetary Freedom | Trade Freedom | Investment Freedom | 2 Sc |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 19.6 | 91.7 | 80.3 | 99.3 | 49.2 | 60.4 | 76.7 | 66.0 | 10.0 | ! |
| **1** | 54.8 | 86.3 | 73.9 | 80.6 | 69.3 | 52.7 | 81.5 | 87.8 | 70.0 | ( |
| **2** | 31.6 | 76.4 | 48.7 | 18.7 | 61.6 | 49.9 | 74.9 | 67.4 | 30.0 | ∠ |
| **3** | 35.9 | 83.9 | 80.7 | 58.2 | 55.7 | 58.8 | 55.4 | 61.2 | 30.0 | ! |
| **4** | 47.8 | 69.3 | 49.5 | 33.0 | 56.4 | 46.9 | 60.2 | 70.0 | 55.0 | ! |

In [21]:
```
# Create pairplot with first half of data only for better readability; include
target '2019 Score'

sns.pairplot(data_c.iloc[:, [0, 1, 2, 3, 4, 9]], height = 4, aspect = 1)
plt.tight_layout
plt.show()
```




In [22]:
```
# The plots do not show any abnormal feature behavior; no outliers are observe
d
# However, there are points with value of 0 in 'Tax Burden', 'Gov't Spending'
 and 'Fiscal Health'
# In addition, the scatter plots show:
    # clear relationship between 'Property Rights' and 'Business Freedom'
    # the same two features are also correlated to the target '2019 Score'; 'F
iscal Health' shows some degree of correlation
    # no evident relationship between the other features
```

In [23]:
```
# Create pairplot with second half of data

sns.pairplot(data_c.iloc[:, 5:], height = 4, aspect = 1)
plt.tight_layout
plt.show()
```




In [24]:
```
# Similarly here, some data points in 'Monetary Freedom', 'Trade Freedom', and 
'Financial Freedom' are 0
# It is very unlikely that these are real 0 scores given how far off the 0 poi
nts are from the rest of the data points
# The conclusion is that these 0s represent missing data and we need to replac
e them with something more appropriate
    # since there are no outliers, as a first order approximation we will use
 the corresponding mean values
# Regarding correlation, all plotted features appear correlated with the targe
t '2019 Score'
```

In [25]:
```
# Print columns to have them handy for the code that follows
data_c.columns
```

Out[25]:
```
Index(['Property Rights', 'Tax Burden', 'Gov't Spending', 'Fiscal Health',
       'Business Freedom', 'Labor Freedom', 'Monetary Freedom',
       'Trade Freedom', 'Investment Freedom ', '2019 Score'],
      dtype='object')
```

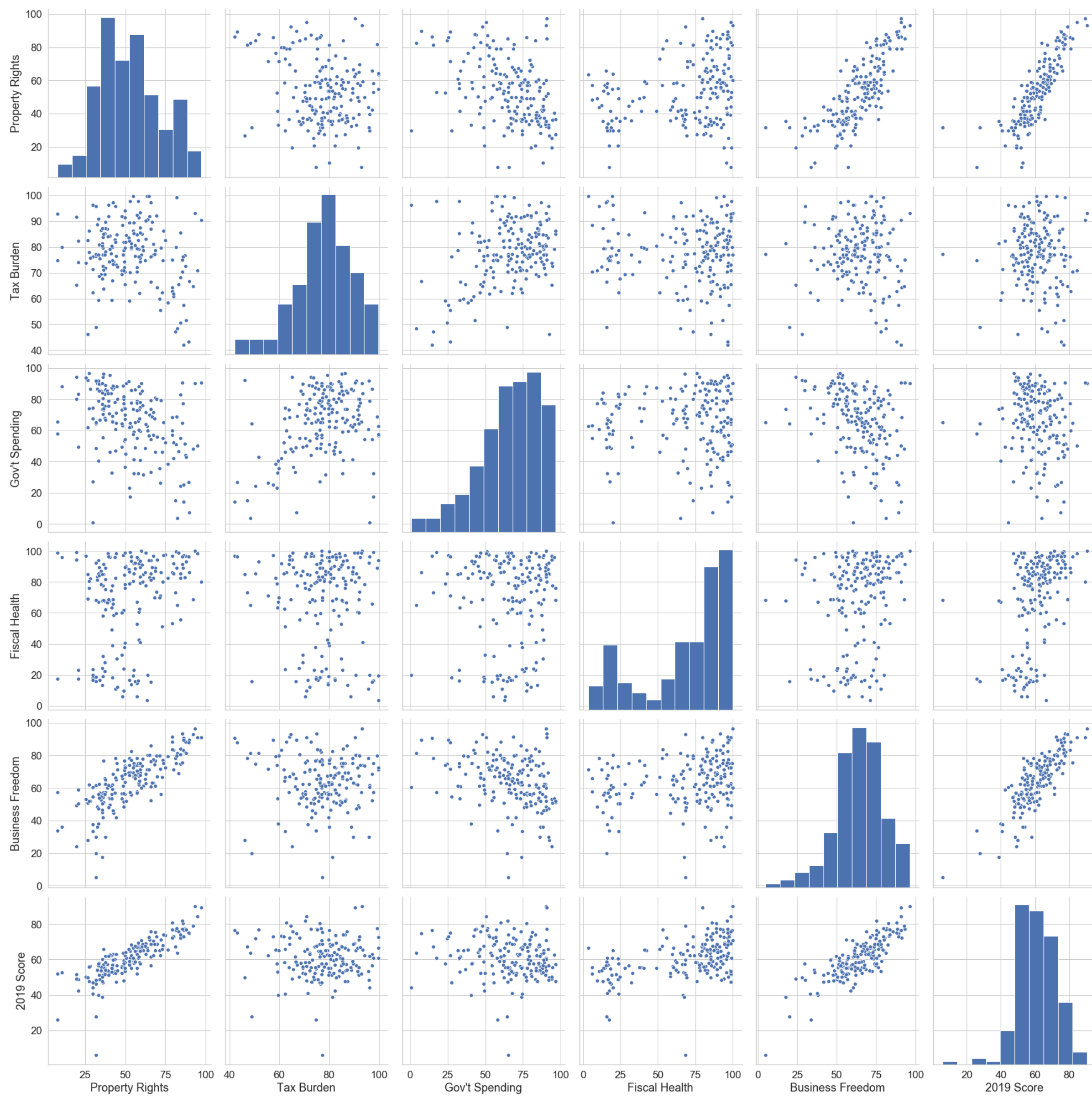
In [26]:
```
# Define function for replacing the zero values

def replaceZeroValues(data):

    for col in ['Property Rights', 'Tax Burden', "Gov't Spending", 'Fiscal Hea
lth', 'Business Freedom',
                'Labor Freedom', 'Monetary Freedom', 'Trade Freedom', 'Investm
ent Freedom ', '2019 Score']:

        for i in range(len(data)):

            if data[col].iloc[i] == 0:

                print(col) # allows us to see which columns have data points =
0

                print(i) # and the row index for these data points

                data[col].iloc[i] = round((data[col].mean()), 1)

        else:

            data[col].iloc[i] = data[col].iloc[i]
```

In [27]:
```
# Replace data points = 0

replaceZeroValues(data_c)
```

```
Tax Burden
87
Gov't Spending
42
Gov't Spending
86
Gov't Spending
87
Gov't Spending
110
Fiscal Health
38
Fiscal Health
50
Fiscal Health
53
Fiscal Health
61
Fiscal Health
87
Fiscal Health
94
Monetary Freedom
87
Monetary Freedom
176
Trade Freedom
87
Investment Freedom
53
Investment Freedom
87
Investment Freedom
176
```

In [28]:
```
# Quick check by creating the same pairplots again
```

In [29]:
```
# First data half + target

sns.pairplot(data_c.iloc[:, [0, 1, 2, 3, 4, 9]], height = 4, aspect = 1)
plt.tight_layout
plt.show()
```

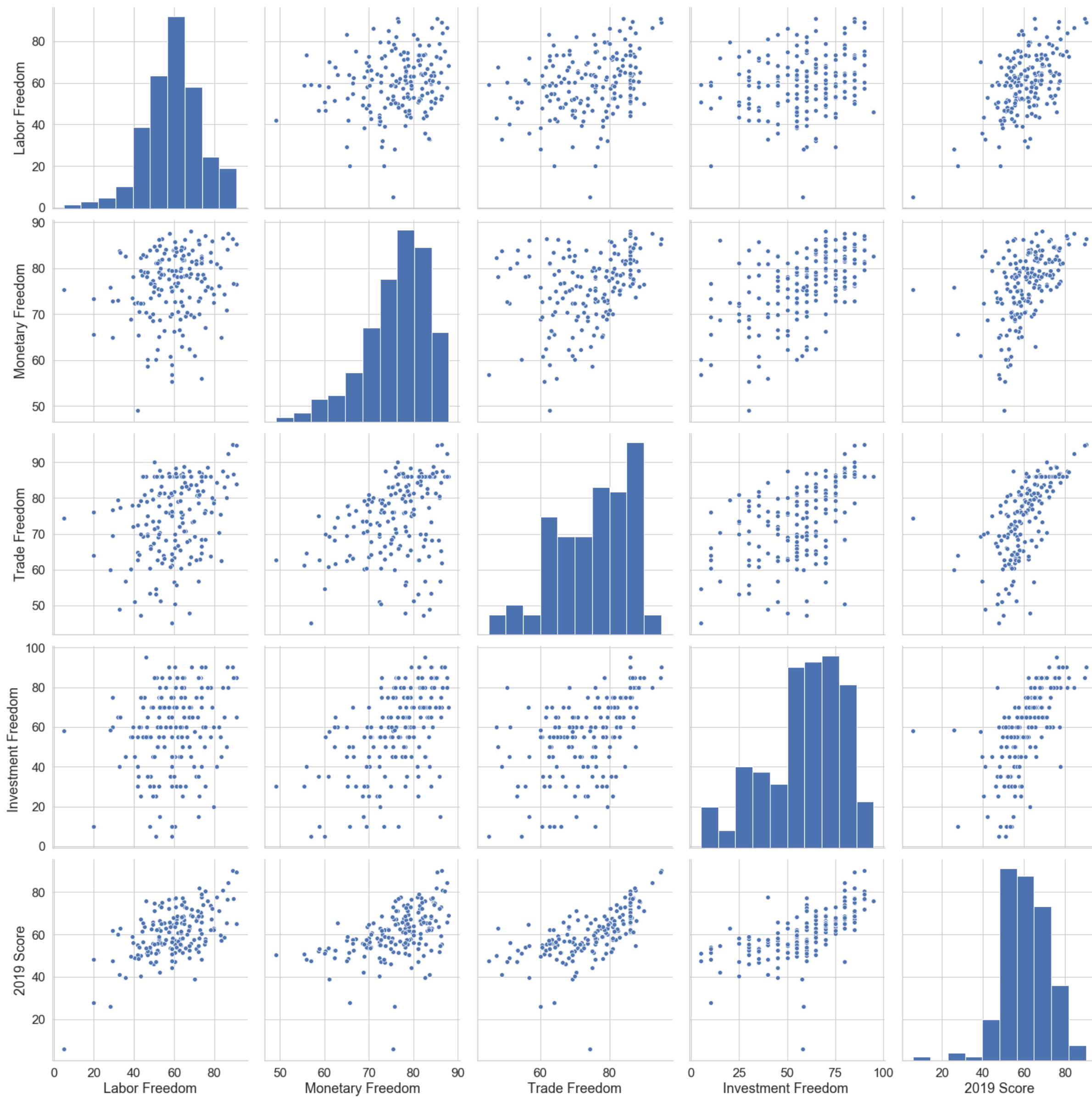



In [30]:
```
# 'Gov't Spending' has a point which is very close to 0 - let's quickly check
 its value

min(data_c["Gov't Spending"])
```

Out[30]: 0.9

In [31]:
```
# This value is close to 0
# However, since we assumed that the missing data entries were writen as 0s, w
e assume that this is a true value
```

In [32]:
```
# Second data half

sns.pairplot(data_c.iloc[:, 5:], height = 4, aspect = 1)
plt.tight_layout
plt.show()
```




In [33]:
```
# Check the data point with a very small value in '2019 Score'
min(data_c.iloc[:, -1])
```

Out[33]: 5.9

In [34]:
```
# Here too, we assume that this is a true data point
```

In [35]:
```
# 2) Create a model and use it with the data - in this project we will use Lin
ear Regression
```

In [36]:
```
# Separate data into features, X, and target, y

X = data_c.iloc[:, :-1].values # features - all data columns, but last
y = data_c.iloc[:, -1].values # target - last data column
```

In [37]:
```
# Split data in train/test subsets

from sklearn.model_selection import train_test_split

X_train, X_test, y_train, y_test = train_test_split(X, y, test_size = 0.2, ran
dom_state = 0)
```

In [38]:
```
# Create Linear Regession model

from sklearn.linear_model import LinearRegression
regressor = LinearRegression()

# Train model with training set
regressor.fit(X_train, y_train)
```

Out[38]:
```
LinearRegression(copy_X=True, fit_intercept=True, n_jobs=None, normalize=Fals
e)
```

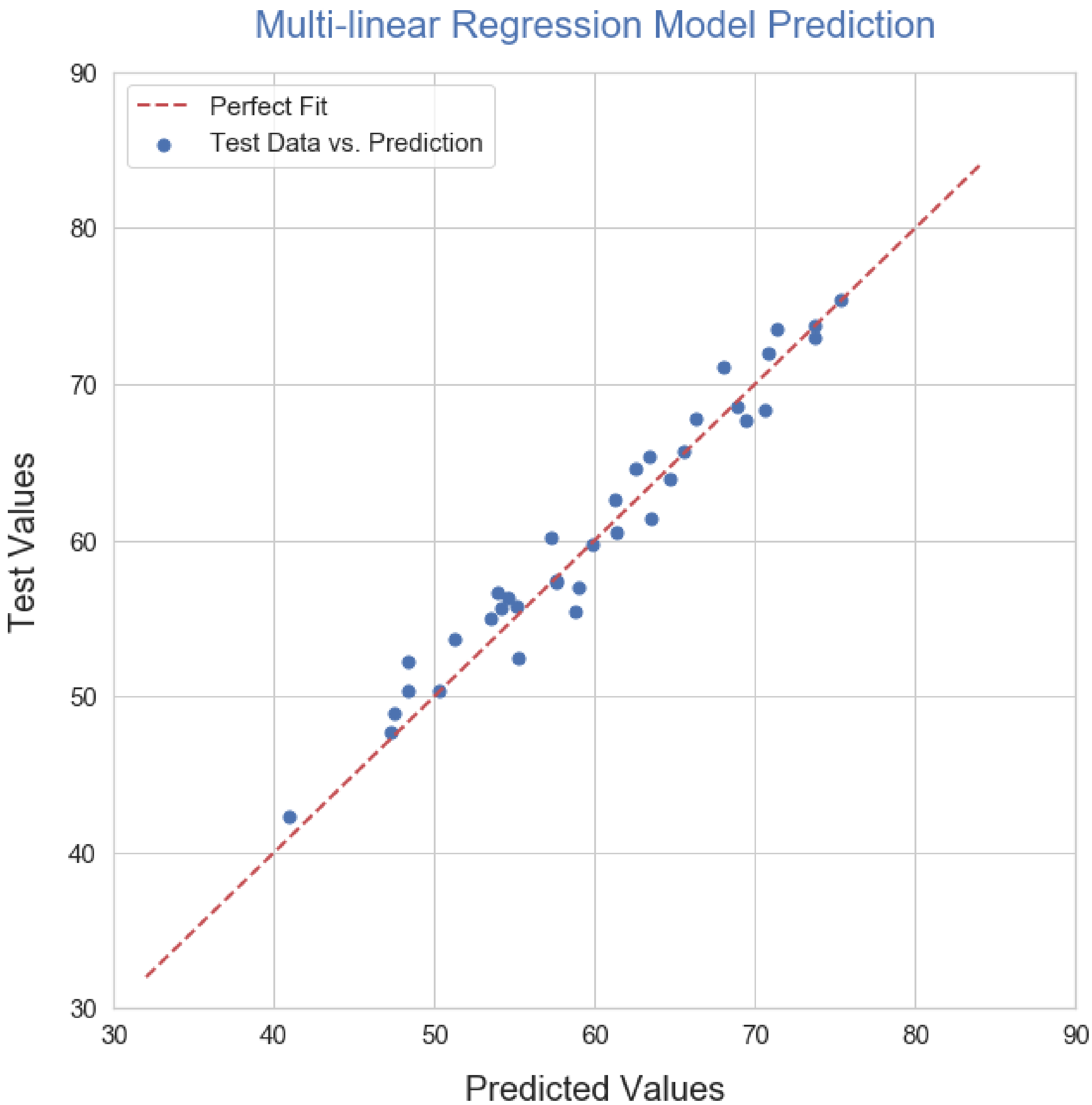
In [39]:
```
# Make predictions using test set
y_pred_1 = regressor.predict(X_test) # indexing '_1' is used to be able to com
pare to results from another model
```

In [40]:
```
# Plot coefficients

coeff_data = pd.DataFrame(regressor.coef_, data_c.iloc[:, :-1].columns, column
s=['Coefficient'])
coeff_data
```

Out[40]:

| | Coefficient |
|---|---|
| **Property Rights** | 0.183020 |
| **Tax Burden** | 0.031414 |
| **Gov't Spending** | 0.111794 |
| **Fiscal Health** | 0.077224 |
| **Business Freedom** | 0.260371 |
| **Labor Freedom** | 0.139410 |
| **Monetary Freedom** | 0.126166 |
| **Trade Freedom** | 0.058219 |
| **Investment Freedom** | 0.104326 |

In [41]:
```
# Compare predictions to the test target, y_test

# Create data points for a straight line representing a perfect fit to the tes
t data points
y_line = np.arange(int(y_test.min()) - 10, int(y_test.max()) + 10)

# Set axes limits - adjust if necessary
x_min = 30
x_max = 90
d_x = 10

y_min = 30
y_max = 90
d_y = 10

plt.figure(figsize = (10, 10))
ax = plt.axes()

ax.set_xlim(x_min, x_max)
ax.set_xticks(np.arange(x_min, x_max + d_x, d_x))

ax.set_ylim(y_min, y_max)
ax.set_yticks(np.arange(y_min, y_max + d_y, d_y))

plt.scatter(y_pred_1, y_test, s = 50, c = 'b', label = 'Test Data vs. Predicti
on')
plt.plot(y_line, y_line, 'r--', lw = 2, label = 'Perfect Fit')
plt.xlabel('Predicted Values', fontsize = 20, labelpad = 15)
plt.ylabel('Test Values', fontsize = 20, labelpad = 15)
plt.title('Multi-linear Regression Model Prediction', fontsize = 22, c = 'b',
pad = 20)
plt.legend(fontsize = 15)
plt.tick_params(labelsize = 15)
plt.show()
```




In [42]:
```
# Predictions are very close to the true target values, y_test
# The straight red dash line represents the ideal case when the prediction poi
nts are equal to the target values
# However, in reality we can never expect to have perfect fit
# In fact, we should be looking for something abnormal/wrong with the data, if
we get a perfect fit from the model!
```

In [43]:
```
# 3) Model optimization

# Question: can we do (a little) better?
# Apply Backward Elimination using features p-value
```

In [44]:
```
# Because we will use statsmodels we need to add a column of ones to X to simu
late the constant term in the regression

X = np.append(arr = np.ones((len(X), 1)).astype(int), values = X, axis = 1)
X
```

Out[44]:
```
array([[ 1. , 19.6, 91.7, ..., 76.7, 66. , 10. ],
       [ 1. , 54.8, 86.3, ..., 81.5, 87.8, 70. ],
       [ 1. , 31.6, 76.4, ..., 74.9, 67.4, 30. ],
       ...,
       [ 1. , 49.8, 79.7, ..., 68.9, 79.2, 30. ],
       [ 1. , 45. , 72.3, ..., 70.3, 72.6, 55. ],
       [ 1. , 29.7, 62.3, ..., 72.4, 70. , 25. ]])
```

In [45]:
```
# First column now is ones and the rest of the columns are the features from X
```

In [46]:
```
# Set an array for the Backward Elimination

X_opt = X[:, [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]] # we start with all features
```

In [47]:
```
# Create model from statsmodels.api

import statsmodels.api as sm

regressor_OLS = sm.OLS(endog = y, exog = X_opt).fit() # using Ordinary Least S
quares (OLS)
regressor_OLS.summary()
```

Out[47]:

OLS Regression Results

| | | | |
|---|---|---|---|
| **Dep. Variable:** | y | **R-squared:** | 0.920 |
| **Model:** | OLS | **Adj. R-squared:** | 0.916 |
| **Method:** | Least Squares | **F-statistic:** | 217.2 |
| **Date:** | Sat, 30 May 2020 | **Prob (F-statistic):** | 2.35e-88 |
| **Time:** | 10:30:15 | **Log-Likelihood:** | -463.36 |
| **No. Observations:** | 180 | **AIC:** | 946.7 |
| **Df Residuals:** | 170 | **BIC:** | 978.7 |
| **Df Model:** | 9 | | |
| **Covariance Type:** | nonrobust | | |

| | coef | std err | t | P>\|t\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| **const** | -6.8642 | 3.639 | -1.886 | 0.061 | -14.048 | 0.320 |
| **x1** | 0.1880 | 0.025 | 7.428 | 0.000 | 0.138 | 0.238 |
| **x2** | 0.0318 | 0.024 | 1.338 | 0.183 | -0.015 | 0.079 |
| **x3** | 0.1110 | 0.014 | 7.870 | 0.000 | 0.083 | 0.139 |
| **x4** | 0.0809 | 0.009 | 8.835 | 0.000 | 0.063 | 0.099 |
| **x5** | 0.2540 | 0.027 | 9.571 | 0.000 | 0.202 | 0.306 |
| **x6** | 0.1331 | 0.020 | 6.551 | 0.000 | 0.093 | 0.173 |
| **x7** | 0.0999 | 0.040 | 2.515 | 0.013 | 0.021 | 0.178 |
| **x8** | 0.0552 | 0.034 | 1.641 | 0.103 | -0.011 | 0.121 |
| **x9** | 0.1079 | 0.017 | 6.326 | 0.000 | 0.074 | 0.142 |

| | | | |
|---|---|---|---|
| **Omnibus:** | 195.499 | **Durbin-Watson:** | 1.785 |
| **Prob(Omnibus):** | 0.000 | **Jarque-Bera (JB):** | 8586.184 |
| **Skew:** | -3.988 | **Prob(JB):** | 0.00 |
| **Kurtosis:** | 35.882 | **Cond. No.** | 3.03e+03 |

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 3.03e+03. This might indicate that there are
strong multicollinearity or other numerical problems.

In [48]:
```
# Examine p-values from table; set significance threshold to 0.05 - everything above is non-significant
# The variable with largest p-value here is x2 with column index = 2; so, for the next step we will remove it from X_opt
```

In [49]:
```
X_opt = X[:, [0, 1, 3, 4, 5, 6, 7, 8, 9]] # remove colum with index = 2 and re
peat

regressor_OLS = sm.OLS(endog = y, exog = X_opt).fit()
regressor_OLS.summary()
```

Out[49]:

OLS Regression Results

| | | | |
|---|---|---|---|
| **Dep. Variable:** | y | **R-squared:** | 0.919 |
| **Model:** | OLS | **Adj. R-squared:** | 0.915 |
| **Method:** | Least Squares | **F-statistic:** | 243.0 |
| **Date:** | Sat, 30 May 2020 | **Prob (F-statistic):** | 3.54e-89 |
| **Time:** | 10:30:15 | **Log-Likelihood:** | -464.31 |
| **No. Observations:** | 180 | **AIC:** | 946.6 |
| **Df Residuals:** | 171 | **BIC:** | 975.4 |
| **Df Model:** | 8 | | |
| **Covariance Type:** | nonrobust | | |

| | **coef** | **std err** | **t** | **P>\|t\|** | **[0.025** | **0.975]** |
|---|---|---|---|---|---|---|
| **const** | -5.3714 | 3.472 | -1.547 | 0.124 | -12.225 | 1.482 |
| **x1** | 0.1807 | 0.025 | 7.294 | 0.000 | 0.132 | 0.230 |
| **x2** | 0.1167 | 0.013 | 8.661 | 0.000 | 0.090 | 0.143 |
| **x3** | 0.0800 | 0.009 | 8.742 | 0.000 | 0.062 | 0.098 |
| **x4** | 0.2589 | 0.026 | 9.831 | 0.000 | 0.207 | 0.311 |
| **x5** | 0.1380 | 0.020 | 6.893 | 0.000 | 0.098 | 0.178 |
| **x6** | 0.0985 | 0.040 | 2.474 | 0.014 | 0.020 | 0.177 |
| **x7** | 0.0652 | 0.033 | 1.984 | 0.049 | 0.000 | 0.130 |
| **x8** | 0.1044 | 0.017 | 6.180 | 0.000 | 0.071 | 0.138 |

| | | | |
|---|---|---|---|
| **Omnibus:** | 185.943 | **Durbin-Watson:** | 1.804 |
| **Prob(Omnibus):** | 0.000 | **Jarque-Bera (JB):** | 7309.994 |
| **Skew:** | -3.710 | **Prob(JB):** | 0.00 |
| **Kurtosis:** | 33.325 | **Cond. No.** | 2.67e+03 |

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 2.67e+03. This might indicate that there are
strong multicollinearity or other numerical problems.

In [50]:
```
# Results show that the constant term (column of ones) has largest p --> remov
e it, as well
```

In [51]:
```
X_opt = X[:, [1, 3, 4, 5, 6, 7, 8, 9]] # remove colum with index = 0 and repea
t

regressor_OLS = sm.OLS(endog = y, exog = X_opt).fit()
regressor_OLS.summary()
```

Out[51]:

OLS Regression Results

| | | | |
|---|---|---|---|
| **Dep. Variable:** | y | **R-squared (uncentered):** | 0.997 |
| **Model:** | OLS | **Adj. R-squared (uncentered):** | 0.997 |
| **Method:** | Least Squares | **F-statistic:** | 7927. |
| **Date:** | Sat, 30 May 2020 | **Prob (F-statistic):** | 1.65e-216 |
| **Time:** | 10:30:15 | **Log-Likelihood:** | -465.56 |
| **No. Observations:** | 180 | **AIC:** | 947.1 |
| **Df Residuals:** | 172 | **BIC:** | 972.7 |
| **Df Model:** | 8 | | |
| **Covariance Type:** | nonrobust | | |

| | coef | std err | t | P>\|t\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| **x1** | 0.1905 | 0.024 | 7.922 | 0.000 | 0.143 | 0.238 |
| **x2** | 0.1085 | 0.012 | 8.721 | 0.000 | 0.084 | 0.133 |
| **x3** | 0.0807 | 0.009 | 8.791 | 0.000 | 0.063 | 0.099 |
| **x4** | 0.2543 | 0.026 | 9.679 | 0.000 | 0.202 | 0.306 |
| **x5** | 0.1345 | 0.020 | 6.733 | 0.000 | 0.095 | 0.174 |
| **x6** | 0.0535 | 0.027 | 1.961 | 0.052 | -0.000 | 0.107 |
| **x7** | 0.0411 | 0.029 | 1.415 | 0.159 | -0.016 | 0.098 |
| **x8** | 0.1107 | 0.016 | 6.721 | 0.000 | 0.078 | 0.143 |

| | | | |
|---|---|---|---|
| **Omnibus:** | 188.606 | **Durbin-Watson:** | 1.782 |
| **Prob(Omnibus):** | 0.000 | **Jarque-Bera (JB):** | 7599.142 |
| **Skew:** | -3.790 | **Prob(JB):** | 0.00 |
| **Kurtosis:** | 33.916 | **Cond. No.** | 27.6 |

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

In [52]:
```
# The feature x7 with column index = 8 in X has largest p --> remove it and re
peat

X_opt = X[:, [1, 3, 4, 5, 6, 7, 9]]

regressor_OLS = sm.OLS(endog = y, exog = X_opt).fit()
regressor_OLS.summary()
```

Out[52]:

OLS Regression Results

| | | | |
|---|---|---|---|
| **Dep. Variable:** | y | **R-squared (uncentered):** | 0.997 |
| **Model:** | OLS | **Adj. R-squared (uncentered):** | 0.997 |
| **Method:** | Least Squares | **F-statistic:** | 9007. |
| **Date:** | Sat, 30 May 2020 | **Prob (F-statistic):** | 4.52e-218 |
| **Time:** | 10:30:15 | **Log-Likelihood:** | -466.60 |
| **No. Observations:** | 180 | **AIC:** | 947.2 |
| **Df Residuals:** | 173 | **BIC:** | 969.5 |
| **Df Model:** | 7 | | |
| **Covariance Type:** | nonrobust | | |

| | coef | std err | t | P>\|t\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| **x1** | 0.1973 | 0.024 | 8.346 | 0.000 | 0.151 | 0.244 |
| **x2** | 0.1109 | 0.012 | 8.973 | 0.000 | 0.087 | 0.135 |
| **x3** | 0.0817 | 0.009 | 8.896 | 0.000 | 0.064 | 0.100 |
| **x4** | 0.2623 | 0.026 | 10.190 | 0.000 | 0.211 | 0.313 |
| **x5** | 0.1361 | 0.020 | 6.805 | 0.000 | 0.097 | 0.176 |
| **x6** | 0.0757 | 0.022 | 3.379 | 0.001 | 0.031 | 0.120 |
| **x7** | 0.1137 | 0.016 | 6.946 | 0.000 | 0.081 | 0.146 |

| | | | |
|---|---|---|---|
| **Omnibus:** | 175.293 | **Durbin-Watson:** | 1.733 |
| **Prob(Omnibus):** | 0.000 | **Jarque-Bera (JB):** | 5875.642 |
| **Skew:** | -3.432 | **Prob(JB):** | 0.00 |
| **Kurtosis:** | 30.135 | **Cond. No.** | 22.7 |

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

In [53]:
```
# All remaining features meet the significance threshold!
# More importantly, even though we have eliminated a feature Adjusted R2 remai
n as high as in the previous step
# Thus, a multiple linear regression model using the last selected features wi
ll be most accurate
```

In [54]:
```
# Due to the addition of ones, features with indexes from X correspond to feat
ures from data_c as follows
# X[:, [1, 3, 4, 5, 6, 7, 9]] --> data_c[:, [0, 2, 3, 4, 5, 6, 8]]

data_c.iloc[:, [0, 2, 3, 4, 5, 6, 8]].head(5)
```

Out[54]:

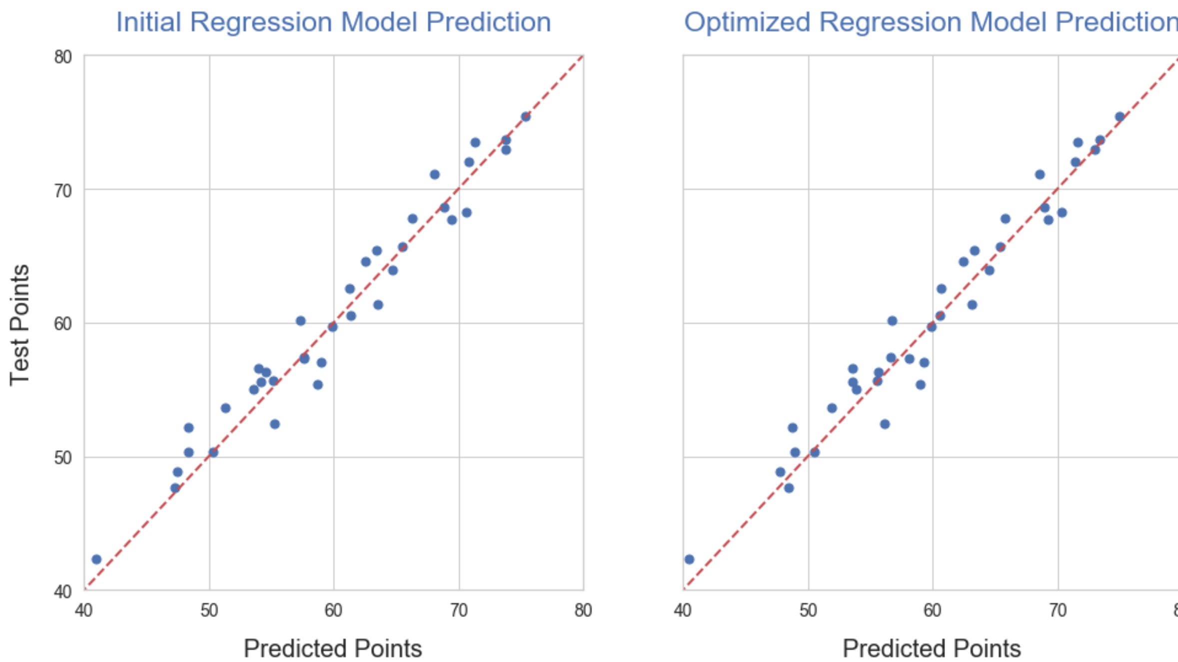
| | Property Rights | Gov't Spending | Fiscal Health | Business Freedom | Labor Freedom | Monetary Freedom | Investment Freedom |
|---|---|---|---|---|---|---|---|
| **0** | 19.6 | 80.3 | 99.3 | 49.2 | 60.4 | 76.7 | 10.0 |
| **1** | 54.8 | 73.9 | 80.6 | 69.3 | 52.7 | 81.5 | 70.0 |
| **2** | 31.6 | 48.7 | 18.7 | 61.6 | 49.9 | 74.9 | 30.0 |
| **3** | 35.9 | 80.7 | 58.2 | 55.7 | 58.8 | 55.4 | 30.0 |
| **4** | 47.8 | 49.5 | 33.0 | 56.4 | 46.9 | 60.2 | 55.0 |

In [55]:
```
# The above seven features, out of 12 total initial features, play significant
role in determining the overall score!
# Note:
    # it is surprising that 'Gov't Spending' made the cut since it did not app
ear to be strongly correlated with the target
```

In [56]:
```
# Use these features with the linear model and see if model predictions will i
mprove

X_r = data_c.iloc[:, [0, 2, 3, 4, 5, 6, 8]].values # new reduced number of fea
tures

X_train, X_test, y_train, y_test = train_test_split(X_r, y, test_size = 0.2, r
andom_state = 0) # replace X with the new X_r
```

In [57]:
```
# Train and predict

regressor.fit(X_train, y_train)
y_pred_2 = regressor.predict(X_test) # change the index to "_2" to be able to
 compare with initial predictions "_1"
```

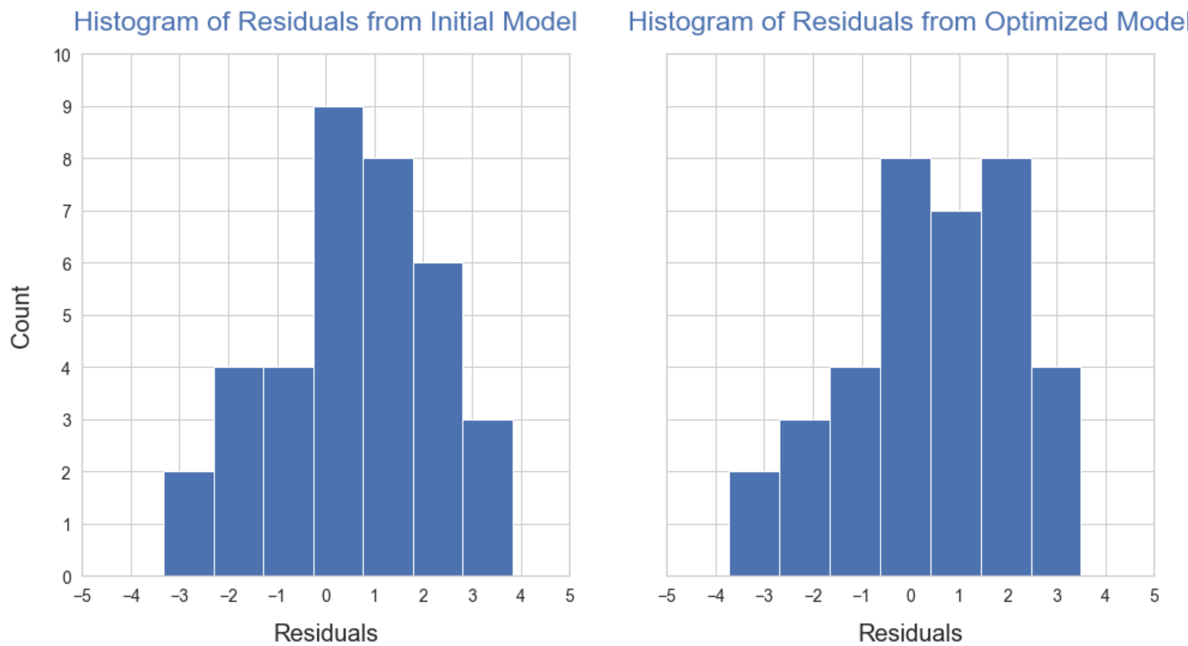
In [58]:
```
# Compare predictions from Initial and Optimized model to test points, y_test

# Create data points for a straight line representing a perfect fit to the y_t
est data points
y_line = np.arange(int(y_test.min()) - 10, int(y_test.max()) + 10)

# set axes limits - adjust if necessary
x_min = 40
x_max = 80
d_x = 10

y_min = 40
y_max = 80
d_y = 10

fig, axes = plt.subplots(1, 2, sharey=True, figsize=(16,8))

# Initial Model
axes[0].scatter(y_pred_1, y_test, s = 50, c = 'b')
axes[0].plot(y_line, y_line, 'r--', lw = 2)
axes[0].set_title('Initial Regression Model Prediction', fontsize = 23, c =
'b', pad = 20)
axes[0].set_xlabel('Predicted Points', fontsize = 20, labelpad = 15)
axes[0].set_ylabel('Test Points', fontsize = 20, labelpad = 15)
axes[0].set_xlim(x_min, x_max)
axes[0].set_xticks(np.arange(x_min, x_max + d_x, d_x))
axes[0].set_ylim(y_min, y_max)
axes[0].set_yticks(np.arange(y_min, y_max + d_y, d_y))
axes[0].tick_params(labelsize = 14)


# Optimized Model
axes[1].scatter(y_pred_2, y_test, s = 50, c = 'b')
axes[1].plot(y_line, y_line, 'r--', lw = 2)
axes[1].set_title('Optimized Regression Model Prediction', fontsize = 23, c =
'b', pad = 20)
axes[1].set_xlabel('Predicted Points', fontsize = 20, labelpad = 15)
axes[1].set_xlim(x_min, x_max)
axes[1].set_xticks(np.arange(x_min, x_max + d_x, d_x))
axes[1].set_ylim(y_min, y_max)
axes[1].set_yticks(np.arange(y_min, y_max + d_y, d_y))
axes[1].tick_params(labelsize = 14)

plt.show()
```




In [59]:
```
# It is difficult to visually discern significant differences between the two
 scatter plots
# That's why we will plot the distributions of the residuals for each model
    # word of caution: the number of observations here is small - not a good c
ase for comparison with normal distribution
```

In [60]:
```
# Get the residuals

# from Initial regression model
res_1 = y_test - y_pred_1

# from Optimized regression model
res_2 = y_test - y_pred_2
```

In [61]:
```
# Plot the histograms of the residuals --> use small number of bins because of
the limited number of data points

# Set axes limits - adjust if necessary
x_min = -5
x_max = +5
d_x = 1

y_min = 0
y_max = 10
d_y = 1

fig, axes = plt.subplots(1, 2, sharey=True, figsize=(16,8))

# Initial Model
axes[0].hist(res_1, bins = 7, color = 'b')
axes[0].set_title('Histogram of Residuals from Initial Model', fontsize = 23,
c = 'b', pad = 20)
axes[0].set_xlabel('Residuals', fontsize = 20, labelpad = 15)
axes[0].set_ylabel('Count', fontsize = 20, labelpad = 15)
axes[0].set_xlim(x_min, x_max)
axes[0].set_xticks(np.arange(x_min, x_max + d_x, d_x))
axes[0].set_ylim(y_min, y_max)
axes[0].set_yticks(np.arange(y_min, y_max + d_y, d_y))
axes[0].tick_params(labelsize = 14)


# Optimized Model
axes[1].hist(res_2, bins = 7, color = 'b')
axes[1].set_title('Histogram of Residuals from Optimized Model', fontsize = 23
, c = 'b', pad = 20)
axes[1].set_xlabel('Residuals', fontsize = 20, labelpad = 15)
axes[1].set_xlim(x_min, x_max)
axes[1].set_xticks(np.arange(x_min, x_max + d_x, d_x))
axes[1].set_ylim(y_min, y_max)
axes[1].set_yticks(np.arange(y_min, y_max + d_y, d_y))
axes[1].tick_params(labelsize = 14)

plt.show()
```




In [62]:
```
# Because of the small number of of observations the histograms are not very s
mooth
# It appears that both models slightly underestimate the target since there ar
e more positive than negative residuals
```

In [63]:
```
# Print the means and the standard deviations (std) of the residuals

print("Mean of Residuals_1:", round(res_1.mean(),2))
print("std of Residuals_1:", round(res_1.std(),2))
print("\n")
print("Mean of Residuals_2:", round(res_2.mean(),2))
print("std of Residuals_2:", round(res_2.std(),2))
```

```
Mean of Residuals_1: 0.51
std of Residuals_1: 1.73


Mean of Residuals_2: 0.47
std of Residuals_2: 1.76
```

In [64]:
```
# The mean of the residuals of the Optimized model is slightly smaller than th
at of the Initial model
# Taking into account the improved Adjusted R2 score as well, the optimized mo
del should be selected as the final model
# Note: for the magnitude of the target values, the mean and the std of the re
siduals are extremely small --> very accurate model
```

In [65]:
```
# This concludes our investigation of the new metrics introduced
# We find that:
    # 1) the new features do not show abnormal behavior
    # 2) most of these features play significant role in determining the overa
l ranking score
    # 3) from these feature using Multiple Linear Regression model one can pre
dict with high accuracy the ranking score
# Thus, we can conclude that the new metrics introduced and the ranking based
 on it are sound
```