Proposition A.1. *There exists a sequence of random numbers* $\{\gamma_{n,T}\}_T$*, that converges to* 0 *as* $T \to \infty$*, such that*

$$\mathbb{P}\Big(\big|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\big| > \gamma_{n,T}\Big) = o(1). \tag{1}$$

Proof of Proposition A.1. Straightforward calculations yield that

$$\begin{aligned}\big|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\big| &\leq \max_{1\leq i<j\leq n} \max_{(u,h)\in\mathcal{G}_T} \left(\left|\frac{\widehat{\widehat{\phi}}_{ij,T}(u,h)}{(\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2)^{1/2}} - \frac{\widehat{\widehat{\phi}}_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}}\right| + \left|\frac{\widehat{\widehat{\phi}}_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} - \frac{\widehat{\phi}_{ij,T}(u,h)}{\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{1/2}}\right|\right) \leq \\ &\leq \max_{1\leq i<j\leq n}\left(\big|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\big| \max_{(u,h)\in\mathcal{G}_T}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right|\right) + \\ &\quad + \max_{1\leq i<j\leq n}\left(\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \max_{(u,h)\in\mathcal{G}_T}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h) - \widehat{\phi}_{ij,T}(u,h)\right|\right).\end{aligned}$$

Then, consider the difference of the kernel averages:

$$\begin{aligned}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h) - \widehat{\phi}_{ij,T}(u,h)\right| &= \Big|\sum_{t=1}^T w_{t,T}(u,h)\big\{(\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top(\mathbf{X}_{it} - \bar{\mathbf{X}}_i) - (\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j)^\top(\mathbf{X}_{jt} - \bar{\mathbf{X}}_j)\big\}\Big| \leq \\ &\leq \Big|\sum_{t=1}^T w_{t,T}(u,h)(\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top(\mathbf{X}_{it} - \bar{\mathbf{X}}_i)\Big| + \Big|\sum_{t=1}^T w_{t,T}(u,h)(\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j)^\top(\mathbf{X}_{jt} - \bar{\mathbf{X}}_j)\Big| = \\ &= |\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top\Big|\sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{it} - \bar{\mathbf{X}}_i)\Big| + |\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j|^\top\Big|\sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{jt} - \bar{\mathbf{X}}_j)\Big| = \\ &= |\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top\Big|\sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{it} - \bar{\mathbf{X}}_i)\Big| + |\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j|^\top\Big|\sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{jt} - \bar{\mathbf{X}}_j)\Big| \leq \\ &\leq |\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top\Big|\sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{it}\Big| + \big|(\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top\bar{\mathbf{X}}_i\big|\Big|\sum_{t=1}^T w_{t,T}(u,h)\Big| + \\ &\quad + |\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j|^\top\Big|\sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{jt}\Big| + \big|(\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j)^\top\bar{\mathbf{X}}_j\big|\Big|\sum_{t=1}^T w_{t,T}(u,h)\Big|\end{aligned}$$

Hence,

$$\begin{aligned}\big|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\big| &\leq \max_{1\leq i<j\leq n}\big|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\big| \max_{1\leq i<j\leq n}\max_{(u,h)\in\mathcal{G}_T}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right| + \\ &\quad + 2\max_{1\leq i<j\leq n}\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\max_{1\leq i\leq n}\left(|\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top \max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{it}\Big|\right) + \\ &\quad + 2\max_{1\leq i<j\leq n}\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\max_{1\leq i\leq n}\big|(\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top\bar{\mathbf{X}}_i\big| \max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=1}^T w_{t,T}(u,h)\Big|\end{aligned} \tag{2}$$

We start by evaluating the second summand in (2).
First, by our assumptions $\widehat{\sigma}_i^2 = \sigma_i^2 + o_P(\rho_T)$. Moreover, for all $i \in \{1, \ldots, n\}$ we know $\sigma_i^2 \neq 0$. Hence,

$$\max_{1\leq i<j\leq n}\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} = O_P(1). \tag{3}$$

Then, by Theorem **??**, we know that

$$|\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i| = O_P(1/\sqrt{T}). \tag{4}$$

Now consider the term $\big| \sum_{t=1}^{T} w_{t,T}(u,h)\mathbf{X}_{it} \big|$. Without loss of generality, we can regard the covariates $\mathbf{X}_{it}$ to be scalars X_{it}, not vectors. The proof in case of vectors proceeds analogously.

By construction the weights $w_{t,T}(u,h)$ are not equal to 0 if and only if $T(u-h) \le t \le T(u+h)$. We can use this fact to rewrite

$$\Big| \sum_{t=1}^{T} w_{t,T}(u,h) X_{it} \Big| = \Big| \sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} w_{t,T}(u,h) X_{it} \Big|.$$

Note that

$$\begin{aligned}
\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} w_{t,T}^2(u,h) &= \sum_{t=1}^{T} w_{t,T}^2(u,h) = \\
&= \sum_{t=1}^{T} \frac{K^2\Big(\frac{\frac{t}{T}-u}{h}\Big)\Big[S_{T,2}(u,h) - \Big(\frac{\frac{t}{T}-u}{h}\Big) S_{T,1}(u,h)\Big]^2}{\Big\{ \sum_{s=1}^{T} K^2\Big(\frac{\frac{s}{T}-u}{h}\Big)\Big[S_{T,2}(u,h) - \Big(\frac{\frac{s}{T}-u}{h}\Big) S_{T,1}(u,h)\Big]^2 \Big\}} = \\
&= 1.
\end{aligned}$$

Denoting by $D_{T,u,h}$ the number of integers between $\lfloor T(u-h) \rfloor$ and $\lceil T(u+h) \rceil$ incl. (with obvious bounds $2Th \le D_{T,u,h} \le 2Th + 2$), we can normalize the weights as follows:

$$\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} \big(\sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)\big)^2 = D_{T,u,h}.$$

According to Theorem **??** (Theorem 2(ii) in **?**), if we denote the weights from the theorem as $a_t = \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)$, we can bound the following probability:

$$\begin{aligned}
&\mathbb{P}\left(\Big| \sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h) X_{it} \Big| \ge x \right) \le \\
&\le C_1 \frac{\big(\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} |\sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)|^{q'} \big) ||X_{i\cdot}||_{q',\alpha}^{q'}}{x^{q'}} + C_2 \exp\left(- \frac{C_3 x^2}{D_{T,u,h} ||X_{i\cdot}||_{2,\alpha}^2} \right) = \\
&= C_1 \frac{(\sqrt{D_{T,u,h}})^{q'} \big(\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} |w_{t,T}(u,h)|^{q'} \big) ||X_{i\cdot}||_{q',\alpha}^{q'}}{x^{q'}} + C_2 \exp\left(- \frac{C_3 x^2}{D_{T,u,h} ||X_{i\cdot}||_{2,\alpha}^2} \right)
\end{aligned}$$

Now take any $\delta > 0$:

$$\mathbb{P}\left(\frac{\max_{(u,h)\in\mathcal{G}_T}\left|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\right|}{\sqrt{T}} \geq \delta\right) =$$

$$= \mathbb{P}\left(\max_{(u,h)\in\mathcal{G}_T}\left|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\right| \geq \delta\sqrt{T}\right) \leq$$

$$\overset{\text{Boole's inequality}}{\leq} \sum_{(u,h)\in\mathcal{G}_T} \mathbb{P}\left(\left|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\right| \geq \delta\sqrt{T}\right) =$$

$$\overset{\text{``normalisation''}}{=} \sum_{(u,h)\in\mathcal{G}_T} \mathbb{P}\left(\left|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} \sqrt{D_{T,u,h}}\cdot w_{t,T}(u,h)X_{it}\right| \geq \delta\sqrt{D_{T,u,h}T}\right) \leq$$

$$\overset{\text{Wu's Theorem}}{\leq} \sum_{(u,h)\in\mathcal{G}_T}\left[C_1\frac{\big(\sqrt{D_{T,u,h}}\big)^{q'}\big(\sum|w_{t,T}(u,h)|^{q'}\big)||X_{i\cdot}||_{q',\alpha}^{q'}}{\big(\delta\sqrt{D_{T,u,h}T}\big)^{q'}} + C_2\exp\left(-\frac{C_3\big(\delta\sqrt{D_{T,u,h}T}\big)^2}{D_{T,u,h}||X_{i\cdot}||_{2,\alpha}^2}\right)\right] =$$

$$\overset{\text{simplification}}{=} \sum_{(u,h)\in\mathcal{G}_T}\left[C_1\frac{\big(\sum|w_{t,T}(u,h)|^{q'}\big)||X_{i\cdot}||_{q',\alpha}^{q'}}{\big(\delta\sqrt{T}\big)^{q'}} + C_2\exp\left(-\frac{C_3\delta^2 T}{||X_{i\cdot}||_{2,\alpha}^2}\right)\right] \leq$$

$$\leq C_1\frac{T^{\theta}||X_{i\cdot}||_{q',\alpha}^{q'}}{T^{q'/2}\cdot\delta^{q'}}\max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil}|w_{t,T}(u,h)|^{q'}\right) + C_2T^{\theta}\exp\left(-\frac{C_3\delta^2T}{||X_{i\cdot}||_{2,\alpha}^2}\right) =$$

$$= C\frac{T^{\theta-q'/2}}{\delta^{q'}} + CT^{\theta}\exp\left(-CT\delta^2\right).$$

where the symbol C denotes a universal real constant that does not depend neither on T nor on δ and that takes a different value on each occurrence. Here in the last equality we used the following facts:

1. $||X_{i\cdot}||_{q',\alpha}^{q'} = \sup_{t\geq 0}(t+1)^{\alpha}\sum_{s=t}^{\infty}\delta_{q'}(H_i,s) < \infty$ holds true since $\sum_{s=t}^{\infty}\delta_{q'}(H_i,s) = O(t^{-\alpha})$ by Assumption **??**;

2. $\max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil}|w_{t,T}(u,h)|^{q'}\right) < \infty$ because for every $x\in[0,1]$ we have $0 \leq |x|^{q'/2} \leq x \leq 1$. Thus, since $\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}^2(u,h) = 1$, we have $0 \leq w_{t,T}^2(u,h) \leq 1$ for all t and

$$0 \leq |w_{t,T}(u,h)|^{q'} = |w_{t,T}^2(u,h)|^{q'/2} \leq w_{t,T}^2(u,h) \leq 1.$$

   This leads us to a bound:

$$\max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil}|w_{t,T}(u,h)|^{q'}\right) \leq \max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil}|w_{t,T}(u,h)|^{2}\right) = 1 < \infty.$$

3. $||X_{i\cdot}||_{2,\alpha}^2 < \infty$ (follows from 1).

By Assumption **??**, $\theta - q'/2 < 0$ and the term on the RHS of the above inequality is converging to 0 as $T \to \infty$ for any fixed $\delta > 0$. Hence,

$$\max_{(u,h)\in\mathcal{G}_T} \Big| \sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h) X_{it} \Big| = o_P(\sqrt{T}),$$

and similarly

$$\max_{(u,h)\in\mathcal{G}_T} \Big| \sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h) \mathbf{X}_{it} \Big| = o_P(\sqrt{T}). \tag{5}$$

Combining (3), (4) and (5), we get the following:

$$\begin{aligned} &2 \max_{1\le i<j\le n} \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \max_{1\le i\le n} \Big(|\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top \max_{(u,h)\in\mathcal{G}_T} \Big| \sum_{t=1}^{T} w_{t,T}(u,h)\mathbf{X}_{it} \Big| \Big) = \\ &= O_P(1) \cdot O_P(1/\sqrt{T}) \cdot o_P(\sqrt{T}) = o_P(1). \end{aligned} \tag{6}$$

Now, consider the third summand in (2).
Similarly as before,

$$\max_{1\le i<j\le n} \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} = O_P(1) \tag{7}$$

and

$$|\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i| = O_P(1/\sqrt{T}). \tag{8}$$

Then, by Proposition **??** $\bar{\mathbf{X}}_i = o_P(1)$.
Finally, consider the local linear kernel weights $w_{t,T}(u,h)$ defined in (**??**). Again, by construction the weights $w_{t,T}(u,h)$ are not equal to 0 if and only if $T(u-h) \le t \le T(u+h)$. We can use this fact to bound $\left|\sum_{t=1}^{T} w_{t,T}(u,h)\right|$ for all $(u,h) \in \mathcal{G}_T$ using the Cauchy-Schwarz inequality:

$$\begin{aligned} \Big| \sum_{t=1}^{T} w_{t,T}(u,h) \Big| &= \Bigg| \sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h) \Bigg| = \Bigg| \sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h) \cdot 1 \Bigg| \le \\ &\le \sqrt{\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}^2(u,h)} \sqrt{\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} 1^2} = \\ &= \sqrt{1} \cdot \sqrt{D_{T,u,h}} \le \sqrt{2Th+2} \le \sqrt{2Th_{\max}+2} \le \sqrt{T+2}. \end{aligned}$$

Hence,

$$\max_{(u,h)\in\mathcal{G}_T} \Big| \sum_{t=1}^{T} w_{t,T}(u,h) \Big| = O(\sqrt{T}). \tag{9}$$

Combining (7), (8), Proposition **??** and (9), we get the following:

$$
\begin{aligned}
&2 \max_{1\le i<j\le n} \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \max_{1\le i\le n} \left|(\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \bar{\mathbf{X}}_i\right| \max_{(u,h)\in\mathcal{G}_T} \left|\sum_{t=1}^T w_{t,T}(u,h)\right| = \\
&= O_P(1)\cdot O_P(1/\sqrt{T})\cdot o_P(1)\cdot O(\sqrt{T}) = o_P(1).
\end{aligned} \tag{10}
$$

Lastly, we look at the first summand in (2). Since $\widehat{\widehat{\sigma}}_i^2 = \sigma_i^2 + o_P(\rho_T)$ and $\widehat{\sigma}_i^2 = \sigma_i^2 + o_P(\rho_T)$ by our assumptions, we have that

$$
\max_{1\le i<j\le n} \left|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\right| = o_P(\rho_T). \tag{11}
$$

Then since $\widehat{\widehat{\phi}}_{ij,T}(u,h)$ has the same distribution as $\widetilde{\phi}_{ij,T}(u,h)$ for each $1\le i<j\le n$ and each $(u,h)\in\mathcal{G}_T$, we can look at $\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h)\right|$ instead:

$$
\begin{aligned}
&\mathbb{P}\left(\left|\max_{(u,h)\in\mathcal{G}_T}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right|\right| \ge c_T\right) = \mathbb{P}\left(\left|\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h)\right|\right| \ge c_T\right) = \\
&\quad = \mathbb{P}\left(\left|\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h)\right| - \max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right| + \max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right|\right| \ge c_T\right) \le \\
&\quad \le \mathbb{P}\left(\left|\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h)\right| - \max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right|\right| \ge c_T/2\right) + \mathbb{P}\left(\left|\max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right|\right| \ge c_T/2\right) \le \\
&\quad \le \mathbb{P}\left(\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h) - \phi'_{ij,T}(u,h)\right| \ge c_T/2\right) + \mathbb{P}\left(\max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right| \ge c_T/2\right).
\end{aligned} \tag{12}
$$

Here we will need one result that we will prove further: by (**??**) we have

$$
\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h) - \phi'_{ij,T}(u,h)\right| = o_P\left(\frac{T^{1/q}}{\sqrt{Th_{\min}}}\right).
$$

Furthermore, $\phi'_{ij,T}(u,h)$ is distributed as $N(0,\sigma_i^2+\sigma_j^2)$ for all $(u,h)\in\mathcal{G}_T$ and all $1\le i<j\le n$ and $|\mathcal{G}_T| = O(T^\theta)$ for some large but fixed constant θ by Assumption **??**. By the standard results from the probability theory, we know that

$$
\max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right| = O_P(\sqrt{\log T}).
$$

Hence, if we take $c_T = o(\sqrt{\log T})$ in (12), we will get the following:

$$
\begin{aligned}
&\mathbb{P}\left(\left|\max_{(u,h)\in\mathcal{G}_T}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right|\right| \ge c_T\right) \le \\
&\quad \le \mathbb{P}\left(\max_{(u,h)\in\mathcal{G}_T}\left|\widetilde{\phi}_{ij,T}(u,h) - \phi'_{ij,T}(u,h)\right| \ge c_T/2\right) + \mathbb{P}\left(\max_{(u,h)\in\mathcal{G}_T}\left|\phi'_{ij,T}(u,h)\right| \ge c_T/2\right) = \\
&\quad = o(1) + o(1) = o(1),
\end{aligned}
$$

which means that

$$
\left|\max_{(u,h)\in\mathcal{G}_T}\left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right|\right| = o_P(\sqrt{\log T}) \tag{13}
$$

Combining (11) and (13), we get the following:

$$\begin{aligned}
&\max_{1\le i<j\le n} \left|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\right| \max_{1\le i<j\le n} \max_{(u,h)\in\mathcal{G}_T} \left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right| = \\
&\quad = o_P(\rho_T) \cdot o_P(\sqrt{\log T}) = \\
&\quad = o_P(1).
\end{aligned} \tag{14}$$

Plugging (6), (10) and (14) in (2), we get that $\left|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\right| = o_P(1)$ and the statement of the theorem follows. $\square$