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Robustifying multivariate trend tests to nonstationary volatility

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ABSTRACT

This article studies inference of multivariate trend model when the volatility process is nonstationary. Within a quite general framework we analyze four classes of tests based on least squares estimation, one of which is robust to both weak serial correlation and nonstationary volatility. The existing multivariate trend tests, which either use non-robust standard errors or rely on non-standard distribution theory, are generally non-pivotal involving the unknown time-varying volatility function in the limit. Two-step residual-based i.i.d. bootstrap and wild bootstrap procedures are proposed for the robust tests and are shown to be asymptotically valid. Simulations demonstrate the effects of nonstationary volatility on the trend tests and the good behavior of the robust tests in finite samples.

1. Introduction

Deterministic trend models have been extensively used to capture the nonstationary trending behavior in economic and financial time series; see Canjels and Watson (1997), Vogelsang (1998), Roy et al. (2004), Bunzel and Vogelsang (2005) and Harvey et al. (2007), inter alia, for relevant work on the univariate trend models. The extension to the multivariate models was recently considered by Vogelsang and Franses (2005a, hereinafter VF). In the empirical application they examined convergence across European GDP by testing whether the growth series in several countries share a common linear deterministic trend (c.f. Carlino and Mills, 1993; Hobijn and Franses, 2000). In the finance literature, VF's common trend tests were used by Carrieri et al. (2004) to explain the dynamics in the gains from sectoral versus cross-country diversification of equity markets, and by Eun and Lee (2006) to study risk-return convergence of several developed stock markets. Other applications of VF's tests to environmental data and geodetic data can be found in Fomby and Vogelsang (2003), Vogelsang and Franses (2005b) and Bacigál (2005).

VF's testing procedures hinge heavily on the assumption of asymptotic stationarity on the innovations that drive the time series. Although a high degree of temporal dependence of the underlying time series is allowed by their tests, the controlled amount of heteroskedasticity is quite limited. In particular, they rule out nonstationarity that may appear in the volatility process. Examples include permanent structural changes in the volatility process (Loretan and Phillips, 1994; Kim and Nelson, 1999; McConnell and Perez Quiros, 2000; Busetti and Taylor, 2003; Sensier and van Dijk, 2004; Cavaliere and Taylor, 2007) and the cases when variances decline monotonically, as observed in the Great Moderation for output growth and inflation rates in most industrialized countries (Blanchard and Simon, 2001). It is thus desirable to develop a robust and accurate testing procedure for the multivariate trends that can be used in the presence of such nonstationary volatility.

The current paper addresses this issue by modeling the time series under investigation as a linear deterministic trend with innovations that follow a semi-parametric vector autoregressive (VAR) process with nonstationary volatility. The volatility specification is quite general allowing for a broad range of patterns of nonstationary behaviors in volatility such as jumps or trending variances mentioned above. We derive the limit distributions of the standard trend tests in VF under nonstationary volatility and show that they are generally non-pivotal, involving the unknown time-varying volatility function in the limit. This can be explained by either of the two facts. First, under nonstationary volatility the non-robust standard errors incorrectly estimate the asymptotic variances of the trend coefficients. This is analogous to the failure of the traditional t-test in the presence of heteroskedasticity. Second, the maintained assumption in VF that a standard invariance principle holds for model innovations is violated under nonstationary volatility. A class of robust tests is then suggested. Although the robust tests proposed are asymptotically pivotal under a quite general type of nonstationary volatility, they may suffer from large size

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distortions in small samples. We then propose for practical use two types of two-step residual-based bootstrap procedures, i.i.d. bootstrap and wild bootstrap, that can be applied to the robust tests.

Robust inference under nonstationary volatility has attracted substantial attention recently in the econometrics literature; see Phillips and Xu (2006), Cavaliere and Taylor (2007), Cavaliere et al. (2010), Beare (2008) and Xu and Phillips (2008). They mainly focus on the coefficients in the stable or unstable autoregressive models. The current work contributes to the literature in two aspects. First, it considers the effects of nonstationary volatility on the tests of multivariate trend coefficients. Second, to our best knowledge, it is the first paper to formally show the non-robustness of the heteroskedasticity and autocorrelation (HAC) "robust" tests with nonstandard fixed-bandwidth (fixed-b) limit distribution theory (Kiefer et al., 2000) to nonstationary variances.

The outline of the paper is as follows. In Section 2, we introduce the model and assumptions. The limit distribution of the OLS trend coefficient estimate is derived. Four classes of trend tests are analyzed and compared in Section 3. Bootstrap inference is discussed in Section 4. Monte Carlo experiments conducted in Section 5 illustrate the finite-sample behaviors of various tests. Section 6 concludes and discusses some possible extensions.

2. Model, assumptions and OLS estimation

Consider m time series $y_t = (y_{1t}, \ldots, y_{mt})'$ which are modeled as

$$y_t = \mu + \beta t + u_t, \tag{1}$$

where μ and β are $m \times 1$ unknown parameters.[1] The innovation process u_t follows a stable VAR(p) process with nonstationary volatility:

$$u_t - A_1 u_{t-1} - \cdots - A_p u_{t-p} := A(L)u_t = v_t, \quad \text{where } v_t = \sigma_t \varepsilon_t. \tag{2}$$

In (2), $A(L) = I_m - \sum_{j=1}^{p} A_j L^j$ where L is the lag operator. The coefficient matrices $A_1, \ldots, A_p$ are such that all roots of $\det(A(\lambda)) = 0$ are outside the unit circle.[2] The sequences σ_t and ε_t satisfy the following conditions.

Assumption 1. ε_t is an $m \times 1$ vector i.i.d. sequence with $\mathrm{E}\varepsilon_t = 0$, $\mathrm{E}\varepsilon_t\varepsilon_t' = I_m$, $\sup_t \mathrm{E}\|\varepsilon_t\|^4 < \infty$, where $\|\cdot\|$ means the vector or matrix norm.

Assumption 2. σ_t is an $m \times m$ non-random sequence of matrices and satisfies $\sigma_t := \sigma(t/T)$ for all $t = 1, \ldots, T$, where $\sigma(r)$ is an unknown matrix of bounded càdlàg functions defined on $(0, 1]$. Moreover, $\sigma(r)\sigma(r)'$ is positive definite for all $r \in (0, 1]$.

The independence condition in Assumption 1 is used for convenience and can be relaxed at the cost of lengthier proofs. The key Assumption 2, which is due to Cavaliere et al. (2010), models the volatility process σ_t as a nonparametric function of time, thereby allows for a rich family of nonstationary dynamics in the innovation variance–covariance matrix; see Cavaliere et al. (2010) for more discussions. The scalar version of Assumption 2 was employed by Cavaliere and Taylor (2007), Beare (2008) and Xu and Phillips (2008), inter alia, to analyze the effects of nonstationary volatility on the inference of univariate time series. The following lemma is useful in developing the limit theory later.

Lemma 1. *Under Assumptions 1 and 2, the partial sum processes of v_t and u_t satisfy $T^{-1/2}\sum_{t=1}^{[Tr]} v_t \Rightarrow \int_0^r \sigma(s)dW_m(s) = \overline{\sigma}W_m^{\sigma}(r)$ and $T^{-1/2}\sum_{t=1}^{[Tr]} u_t \Rightarrow A(1)^{-1}\int_0^r \sigma(s)dW_m(s) := \Omega^{1/2}W_m^{\sigma}(r)$, as $T \to \infty$, where $W_m(s)$ is an m-dimensional standard Brownian motion, $\Omega = A(1)^{-1}\overline{\sigma}^2 A(1)^{-1\prime}$ and $W_m^{\sigma}(r) = \overline{\sigma}^{-1}\int_0^r \sigma(s)dW_m(s)$ with $\overline{\sigma}^2 = \int_0^1 \sigma(s)\sigma'(s)ds$.*

The first weak convergence result in Lemma 1 appeared in Cavaliere et al. (2010). The diffusion process $W_m^{\sigma}(\cdot)$ has the instantaneous covariance matrix $\overline{\sigma}^{-1}\sigma(\cdot)\sigma'(\cdot)\overline{\sigma}^{-1}$, and is equivalent in distribution to a vector time-changed Brownian motion (Davidson, 1994) if $\sigma(\cdot)$ can be rewritten as $\Upsilon_1\Upsilon_2(\cdot)$ where Υ_1 is a constant matrix while $\Upsilon_2(\cdot)$ is a diagonal matrix function. It reduces to a vector standard Brownian motion when $\sigma(\cdot)$ is a constant function.

2.1. The OLS estimator of β

Suppose we observe the time series y_t at T consecutive time points $t = 1, \ldots, T$. The model (1) is estimated by using OLS equation by equation. By the Frisch–Waugh–Lovell Theorem, the OLS estimator of β, $\widehat{\beta}$, can be written as $\widehat{\beta} = (\sum_{t=1}^{T}\widetilde{t}^2)^{-1}\sum_{t=1}^{T}\widetilde{t}y_t$, where $\widetilde{t} = t - \bar{t}$ with $\bar{t} = T^{-1}\sum_{t=1}^{T} t$. Let $V_m^{\sigma}(r) = \int_0^r (s - 1/2)dW_m^{\sigma}(s)$. Under Assumptions 1 and 2, by Lemma 1 and the fact that $T^{-3}\sum_{t=1}^{T}\widetilde{t}^2 \to 1/12$, we have

$$T^{3/2}(\widehat{\beta} - \beta) = \left(T^{-3}\sum_{t=1}^{T}\widetilde{t}^2\right)^{-1} T^{-3/2}\sum_{t=1}^{T}\widetilde{t}u_t \Rightarrow 12\Omega^{1/2}V_m^{\sigma}(1), \tag{3}$$

which equals $12A(1)^{-1}\int_0^1 (s - 1/2)\sigma(s)dW_m(s)$ or m-dimensional Gaussian distribution $\mathcal{N}_m(0, Q)$, where $Q = 12^2 A(1)^{-1}\int_0^1 (s - 1/2)^2\sigma(s)\sigma(s)'ds \cdot A(1)^{-1\prime}$. The integral in the middle involves the interaction between the time trend and the volatility function. When the volatility function $\sigma(\cdot)$ is constant, the asymptotic variance matrix Q reduces to 12Ω, where Ω is the long run variance matrix (LRV) of u_t.

Note that the convergence rate of $\widehat{\beta}$ is unaffected by the bounded nonstationary volatility as specified in Assumption 2. Under more general assumptions on σ_t, for instance, allowing for explosive behavior in volatility, the convergence rate of $\widehat{\beta}$ would be generally slowed down (Chung and Park, 2007; Xu, 2008).

In what follows, we assume that the hypothesis of interest on the trend coefficients β is $H_0 : R\beta = r$, where R and r are $q \times m$ and $q \times 1$ matrices of known constants. The VAR coefficients in the dynamics of the error process can be consistently estimated. Denote the OLS detrended residuals of y_t as $\widehat{u}_t = y_t - \widehat{\mu} - \widehat{\beta}t$. Then $A_1, \ldots, A_p$ can be estimated by $\widehat{A}_1, \ldots, \widehat{A}_p$, which are obtained by regressing $\widehat{u}_t$ on $\widehat{u}_{t-1}, \ldots, \widehat{u}_{t-p}$ for $t = p + 1, \ldots, T$. Denote the residuals as $\widehat{v}_t = \widehat{u}_t - \sum_{j=1}^{p}\widehat{A}_j\widehat{u}_{t-j}$. The following lemma shows that $\widehat{A}_1, \ldots, \widehat{A}_p$ is $\sqrt{T}$-consistent.

Lemma 2. *Under Assumptions 1 and 2, $\sqrt{T}([\widehat{A}_1, \ldots, \widehat{A}_p] - [A_1, \ldots, A_p]) = O_p(1)$.*

3. Tests of multivariate trend coefficients

In this section we shall consider a collection of tests of the hypothesis H_0 under nonstationary volatility. They distinguish

[1] The methodologies developed in this paper could be extended to cover other trend models instead of a linear trend studied here. For instance, adding an indicator function of time trend in (1) and testing its coefficient cross-sectionally would lead to a common broken trend test at a given break point.

[2] The assumption that the VAR coefficients should satisfy the stable condition requires some caution for the practical use of the robust methods developed below. For example, in the study of economic convergence across European countries, Vogelsang and Franses (2005a,b) considered the log ratios of real per capita GDP series given a benchmark country (instead of log levels) so that the stable condition is satisfied.

from each other in the two aspects: the ways in which the estimators of Q, the asymptotic variance of $\widehat{\beta}$, are constructed and the asymptotics that are relied on for the decision rules. The tests of classes 1, 2 and 4 were considered earlier by VF under the assumption of asymptotic stationarity. We investigate their behavior when stationarity assumption fails and propose a class of robust tests (Class 3 tests).

For an observed time series a_t, define the long run variance matrix estimator as $\widehat{LRV}(a_t, M) := \widehat{\Gamma}_0 + \sum_{j=1}^{T} k(j/M)(\widehat{\Gamma}_j + \widehat{\Gamma}_j')$, where $\widehat{\Gamma}_j = T^{-1}\sum_{t=j+1}^{T} a_t a_{t-j}'$, $k(x) = (1-|x|)\cdot I(|x| \le 1)$ is the Bartlett kernel and $M \le T$ is the truncation number. The test statistics considered in this section are of the following form:

$$F = T^3(R\widehat{\beta} - r)'(R\widehat{Q}R')^{-1}(R\widehat{\beta} - r), \tag{4}$$

where $\widehat{Q}$ is the estimator of Q.

3.1. The Class 1 tests

The first two classes of tests exploit the fact that $Q = 12\Omega$, which is true under the asymptotic stationarity of u_t. We will show that neither of them is valid under nonstationary volatility. Class 1 test uses the variance estimator $\widehat{Q}_1 = (T^{-3}\sum_{t=1}^{T}\widetilde{t}^2)^{-1}\widehat{LRV}(\widehat{u}_t, M)$ with the truncation number M increasing at a slower rate than T. The test statistic is defined as F_1 with $\widehat{Q}_1$ plugged in (4). F_1 deals with the serial correlation nonparametrically without utilizing the structure assumed in the VAR specification. Let $\widehat{A}(L) = I_m - \widehat{A}_1 L - \cdots - \widehat{A}_p L^p$. The prewhitened version of F_1 (under correct specification, therefore utilizing the VAR structure) is denoted as F_1^{pw} with $\widehat{Q}_1^{pw} = (T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2)^{-1}\widehat{A}(1)^{-1}\widehat{LRV}(\widehat{v}_t, M = 0)\cdot \widehat{A}(1)^{-1\prime}$ plugged in (4), where $\widehat{LRV}(\widehat{v}_t, M=0) = T^{-1}\sum_{t=p+1}^{T}\widehat{v}_t\widehat{v}_t'$.

The following results describe the asymptotic behaviors of F_1 and F_1^{pw} under nonstationary volatility[3]:

Theorem 1. (i). *Under Assumptions 1, 2 and H_0, when $M/T \to 0$ as $T \to \infty$, $\widehat{Q}_1 \xrightarrow{p} 12\Omega$ and $F_1 \Rightarrow 12[R\Omega^{1/2}V_m^{\sigma}(1)]'[R\Omega R']^{-1}[R\Omega^{1/2}V_m^{\sigma}(1)] \overset{d}{=} Z'Z$, where*

$$Z \sim \mathcal{N}_q\left(0,\ 12(R\Omega R')^{-1/2}RA(1)^{-1} \times \int_0^1 (s-1/2)^2\sigma(s)\sigma(s)'ds\cdot A(1)^{-1\prime}R'(R\Omega R')^{-1/2}\right).$$

(ii). *Under the same assumptions,* (i) *is also true for $\widehat{Q}_1^{pw}$ and F_1^{pw} as $T \to \infty$.*

Remark 1. The presence of nonstationary volatility makes the test statistics F_1 and F_1^{pw} not pivotal any more simply because the asymptotic variance of $\widehat{\beta}$ is not consistently estimated. They have a $\chi^2(q)$ distribution (as in VF) when the volatility function $\sigma(\cdot)$ is constant.

Remark 2. To see the potential effects of nonstationary volatility on the Class 1 tests, consider the univariate trend model, i.e. $m = q = 1$. By Theorem 1, $F_1, F_1^{pw} \Rightarrow c\chi^2(1)$, where $c = 12\int_0^1(s-1/2)^2\sigma^2(s)ds/\int_0^1\sigma^2(s)ds$. We call the scaling constant c as the inflation factor since it reflects the magnitude the nonstationarity in volatilities would inflate (or deflate) the values of the test statistics. We illustrate the behavior of c for a particular model of the volatility function σ. Let $\sigma^2(s)$ be the step function $\sigma(s)^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)\mathbf{1}_{\{s \ge \tau\}}$, $s \in (0, 1]$. This shows a shift in the variances from σ_0^2 to σ_1^2 at time $[T\tau]$. For this model, $c := c(\tau, \delta) = [\tau + \delta^2(1-\tau)]^{-1}[\delta^2 + 12(1-\delta^2)(\tau^3/3 + \tau/4 - \tau^2/2)]$, where $\delta = \sigma_1/\sigma_0$. Note that $c = 1$ (no inflation) when $\delta = 1$, i.e. the variance is constant. The effects of such a variance jump on F_1 and F_1^{pw} are clear from Fig. 1 which plots $c(\tau, \delta)$ versus τ for $\delta \in \{1/2, 1/5, 1/10, 2, 5, 10\}$. The effects are usually asymmetric for an upward and a downward variance shift of the same magnitude for a given break date. For example, a late upward variance shift ($\tau = 0.9, \delta = 10$) would inflate the values of the test statistics by a factor of 2.31 ($c = 2.31$). If the variances shift downwards with the same size ($\delta = 0.1$), the test statistics would be deflated by a factor of 0.84 ($c = 0.84$). This also implies that significant size distortions could be incurred if such nonstationary volatility is ignored.

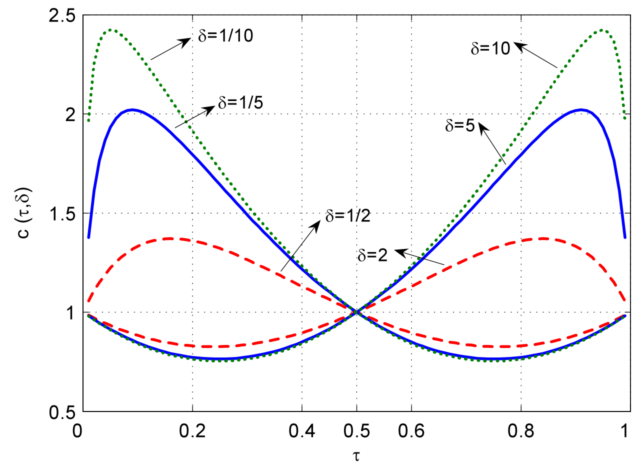


Fig. 1. The inflation factor $c(\tau, \delta)$ for the Class 1 test statistics under a variance break model.

3.2. The Class 2 tests

The Class 2 tests apply different asymptotics on the Class 1 tests. Define $\widehat{Q}_2 = \widehat{Q}_1(M = T)$ where $\widehat{Q}_1(M = T)$ stands for the variance estimator $\widehat{Q}_1$ where the truncation number M equal to T is used. The test statistic F_2 is defined by F/q where F is as in (4) with $\widehat{Q}_2$ plugged in. The prewhitened version of F_2, denoted as F_2^{pw}, is the same with F_2 except that $\widehat{Q}_2^{pw} = (T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2)^{-1}\widehat{A}(1)^{-1}\widehat{LRV}(\widehat{v}_t, M = T)\cdot\widehat{A}(1)^{-1\prime}$ is used instead of $\widehat{Q}_2$.

Theorem 2. (i). *Under Assumptions 1, 2 and H_0, as $T \to \infty$, $\widehat{Q}_2 \Rightarrow 24\Omega^{1/2}[\int_0^1 \widehat{V}_m^{\sigma}(r)\widehat{V}_m^{\sigma}(r)'dr]\Omega^{1/2\prime}$ where $\widehat{V}_m^{\sigma}(r) = W_m^{\sigma}(r) - rW_m^{\sigma}(1) - 12V_m^{\sigma}(1)\int_0^r(s-\frac{1}{2})ds$, and*

$$F_2 \Rightarrow [R\Omega^{1/2}V_m^{\sigma}(1)]' \times \left[\frac{1}{6}R\Omega^{1/2}\left(\int_0^1 \widehat{V}_m^{\sigma}(r)\widehat{V}_m^{\sigma}(r)'dr\right)\Omega^{1/2\prime}R'\right]^{-1} \times [R\Omega^{1/2}V_m^{\sigma}(1)]/q.$$

(ii). *Under the same assumptions,* (i) *is also true for $\widehat{Q}_2^{pw}$ and F_2^{pw} as $T \to \infty$.*

Remark 3. The Class 2 tests were recommended by VF to use under unknown serial correlation in the errors. They have pivotal nonstandard limit distribution under asymptotic stationarity. Compared to the standard chi-square limit distribution theory (as that of the Class 1 tests), the nonstandard limit theory accounts

[3] The mathematical proofs of the main results of the paper are available from Xu (2009).

for the sampling variability of the asymptotic variance estimate in a better way and thus yields more accurate distributional approximation in finite samples (Jansson, 2004; Kiefer and Vogelsang, 2005; Sun et al., 2008). However, as shown in Theorem 2, when there are structural changes in volatilities, F_2 and F_2^{pw} are not pivotal involving the unknown volatility function $\sigma(\cdot)$ in a complicated way. The reasons are two-folded. The first one is due to the construction of the test statistics, which, like the Class 1 tests, is based on estimation of the LRV of u_t instead of Q (the LRV of $\widetilde{t}u_t$). The second is due to the presence of nonstationary volatility which invalidates the standard invariance principle (see Lemma 1).

Remark 4. Theorem 2 can be extended under general fixed-b asymptotics with a general kernel function (Kiefer and Vogelsang, 2005) in which the truncation number M is a fixed portion of the sample size T.

Remark 5. Kiefer and Vogelsang (2002) showed that the quantity $\widehat{LRV}(\widehat{u}_t, M = T)$ involved in $\widehat{Q}_2$ has a simple form. Let $\widehat{S}_t = \sum_{j=1}^{t}\widehat{u}_j$, then $\widehat{LRV}(\widehat{u}_t, M = T) = (2/T)\sum_{t=1}^{T}(\widehat{S}_t\widehat{S}_t'/T)$. Similarly, in computing $\widehat{Q}_2^{pw}$, $\widehat{LRV}(\widehat{v}_t, M = T) = (2/T)\sum_{t=p+1}^{T}(\widetilde{S}_t\widetilde{S}_t'/T)$ where $\widetilde{S}_t = \sum_{j=p+1}^{t}\widehat{v}_j$.

3.3. The Class 3 tests

The tests of Classes 3 and 4 use robust standard errors in the construction of the test statistics. Define $\widehat{Q}_3 = (T^{-3}\sum_{t=1}^{T}\widetilde{t}^2)^{-2}\widehat{LRV}(\widetilde{t}\widehat{u}_t/T, M)$, and the test statistic F_3 as in (4) with variance estimator $\widehat{Q}_3$ plugged in. The prewhitened version of F_3, F_3^{pw}, uses the variance estimator $\widehat{Q}_3^{pw} = (T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2)^{-2}\widehat{A}(1)^{-1}\cdot\widehat{LRV}(\widetilde{t}\widehat{v}_t/T, M = 0)\widehat{A}(1)'^{-1}$ instead of $\widehat{Q}_3$, where $\widehat{LRV}(\widetilde{t}\widehat{v}_t/T, M = 0) = T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2\widehat{v}_t\widehat{v}_t'$. In constructing $\widehat{Q}_3^{pw}$, the term $\widehat{LRV}(\widetilde{t}\widehat{v}_t/T, M = 0)$ handles the nonstationary volatility while the serial correlation in the data is accounted for by the two terms of prewhitening VAR coefficients $\widehat{A}(1)^{-1}$. So it is different from the traditional prewhitened LRV estimator (Andrews and Monahan, 1992) in which prewhitening applies directly to $\widetilde{t}\widehat{u}_t/T$.[4]

Theorem 3. (i). *Under Assumptions 1, 2 and H_0, when $M/T \to 0$ as $T \to \infty$, $\widehat{Q}_3 \xrightarrow{p} Q$, $F_3 \Rightarrow \chi^2(q)$.*
(ii). *Under the same assumptions,* (i) *is also true for $\widehat{Q}_3^{pw}$ and F_3^{pw} as $T \to \infty$.*

Remark 6. The Class 3 test statistics have pivotal asymptotic distribution under nonstationary volatility. The construction of F_3^{pw} can be alternatively motivated as follows. First suppose that the autoregressive coefficients $A_1, \ldots, A_p$ are known. Pre-multiplying (1) by $A(L)$ gives $A(L)y_t = A(L)\mu + A(L)\beta\cdot t + v_t$ or

$$y_t^\star = \mu^\star + \beta^\star t + v_t, \tag{5}$$

where $y_t^\star = A(L)y_t$, $\mu^\star = (A_1 + 2A_2 + \cdots + pA_p)\mu + A(1)\mu$, $\beta^\star = A(1)\beta$. The hypothesis H_0 is correspondingly transformed as $H_0^\star : RA(1)^{-1}\beta^\star = r$. Since the errors in the transformed model (5) are uncorrelated, we can construct the (infeasible) robust Wald test statistic by using White's heteroskedasticity-corrected standard error (White, 1980) as $\widetilde{W} = T^{3/2}(RA(1)^{-1}\widetilde{\beta}^\star - r)'[(T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2)^{-2}RA(1)^{-1}(T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2\widetilde{v}_t^\star\widetilde{v}_t^{\star\prime})\cdot A(1)'^{-1}R']^{-1}T^{3/2}(RA(1)^{-1}\widetilde{\beta}^\star - r)$, where $\widetilde{\beta}^\star$ and $\widetilde{v}_t^\star$ are the OLS estimate and the residuals of the regression model (5). The feasible version of $\widetilde{W}$ is formed by substituting $A_1, \ldots, A_p$ by $\widehat{A}_1, \ldots, \widehat{A}_p$. Let $\widehat{y}_t^\star = \widehat{A}(L)y_t$. The feasible estimator of $\beta^\star$, denoted by $\widehat{\beta}^\star$, is obtained by regressing $\widehat{y}_t^\star$ on 1 and t, with the residuals $\widehat{v}_t^\star$. Then the feasible test statistic is defined as $\widehat{W} = T^{3/2}(R\widehat{A}(1)^{-1}\widehat{\beta}^\star - r)'[(T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2)^{-2}R\widehat{A}(1)^{-1}(T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2\widehat{v}_t^\star\widehat{v}_t^{\star\prime})\cdot\widehat{A}(1)'^{-1}R']^{-1}T^{3/2}(R\widehat{A}(1)^{-1}\widehat{\beta}^\star - r)$. Simple algebra shows that $\widehat{A}(1)^{-1}\widehat{\beta}^\star = \widehat{\beta}$, $\widehat{v}_t^\star = \widehat{v}_t$, and thus the statistic $\widehat{W}$ is identical to F_3^{pw}.

3.4. The Class 4 tests

The Class 4 tests are obtained by applying the fixed-b asymptotics on the Class 3 tests. Define $\widehat{Q}_4 = \widehat{Q}_3(M = T)$ where $\widehat{Q}_3(M = T)$ stands for the variance estimator $\widehat{Q}_3$ with the truncation number M equal to T. The test statistic F_4 is defined by F/q where F is as in (4) with $\widehat{Q}_4$ plugged in. The prewhitened version of F_4, F_4^{pw}, is the same with F_4 except that $\widehat{Q}_4^{pw} = (T^{-3}\sum_{t=p+1}^{T}\widetilde{t}^2)^{-2}\widehat{A}(1)^{-1}\widehat{LRV}(\widetilde{t}\widehat{v}_t/T, M = T)\cdot\widehat{A}(1)'^{-1}$ is used instead of $\widehat{Q}_4$.

Theorem 4. (i). *Under Assumptions 1, 2 and H_0, as $T \to \infty$, $\widehat{Q}_4 \Rightarrow 2\cdot 12^2\Omega^{1/2}[\int_0^1\widetilde{V}_m^\sigma(r)\widetilde{V}_m^\sigma(r)'dr]\Omega^{1/2\prime}$, where $\widetilde{V}_m^\sigma(r) = V_m^\sigma(r) - \int_0^r(s - 1/2)ds\cdot W_m^\sigma(1) - 12V_m^\sigma(1)\int_0^r(s - 1/2)^2ds$ and*

$$F_4 \Rightarrow V_q^\sigma(1)'\left[2\int_0^1\widetilde{V}_q^\sigma(r)\widetilde{V}_q^\sigma(r)'dr\right]^{-1}V_q^\sigma(1)/q.$$

(ii). *Under the same assumptions,* (i) *is also true for $\widehat{Q}_4^{pw}$ and F_4^{pw} as $T \to \infty$.*

Remark 7. Like the Class 2 tests, nonstationary volatility affects the limit distribution of the Class 4 tests by introducing an unknown volatility function. When the volatility function is constant, they have pivotal nonstandard limit distribution and the critical values are tabulated by VF.

4. Bootstrapping the multivariate trend tests

We have shown that the Class 3 tests have the correct sizes asymptotically robust to nonstationary volatility. In this section we apply the bootstrap method to improve their finite-sample performance. The bootstrap data are generated in two steps. In the first step, we resample $\{\widehat{v}_t, t = p+1, \ldots, T\}$ to obtain the bootstrap innovations $\{v_t^*, t = p+1, \ldots, T\}$. Then the bootstrap residuals of the model (1) are obtained recursively by the equation $u_t^* = \widehat{A}_1u_{t-1}^* + \cdots + \widehat{A}_pu_{t-p}^* + v_t^*$ for $t = p+1, \ldots, T$, where $\widehat{A}_1, \ldots, \widehat{A}_p$ are the OLS estimators studied in Lemma 2. We set $u_1^* = \cdots = u_p^* = 0$. In the second step, the bootstrap time series are generated as $y_t^* = \widehat{\mu} + \widehat{\beta}t + u_t^*$ for $t = p+1, \ldots, T$.

We use two resampling schemes in the first step. The first one is the conventional i.i.d. bootstrap which samples $\widehat{v}_t$ randomly with replacement. The second one is the wild bootstrap (Wu, 1986) obtained as $v_t^* = \widehat{v}_tz_t$, where z_t is a sequence of standard normal random variables. Note that the multivariate wild bootstrap scheme adopted here multiplies the same scalar to each element of $\widehat{v}_t$ and was first used by Cavaliere et al. (2010). The wild bootstrap seems more natural in our setting to account for the heteroskedasticity appeared in the data; see also Gonçalves and Kilian (2004, 2007), Cavaliere and Taylor (2008) and Xu (2008) for recent applications to heterogeneous data in the univariate time series environment.

Having the bootstrap data generated above, for each iteration the bootstrap version of the test statistic F_3^{pw}, denoted as $F_3^{pw,*}$,

[4] This comment on prewhitening in a non-standard way also applies to F_4^{pw} below.

Table 1
Empirical rejection rates of various tests and two bootstrap tests [i.i.d. bootstrap (iidB) and wild bootstrap (WB)] under the null hypothesis where there is a simultaneous variance break ($q = 1$, nominal size: 0.05).

T	$(\tau, \delta) = (0.9, 10)$			$(\tau, \delta) = (0.9, 1)$			$(\tau, \delta) = (0.9, 0.1)$		
	100	200	400	100	200	400	100	200	400
$\rho = 0$									
F_1	0.230	0.202	0.198	0.053	0.060	0.054	0.034	0.039	0.035
F_1^{pw}	0.242	0.226	0.217	0.064	0.057	0.060	0.041	0.035	0.040
F_2	0.010	0.013	0.010	0.050	0.048	0.060	0.027	0.026	0.034
F_2^{pw}	0.012	0.012	0.010	0.036	0.037	0.053	0.019	0.019	0.027
F_3	0.054	0.055	0.049	0.062	0.061	0.055	0.057	0.056	0.049
F_3^{pw}	0.087	0.066	0.065	0.072	0.060	0.059	0.069	0.055	0.059
F_4	0.001	0.002	0.000	0.053	0.048	0.058	0.031	0.031	0.033
F_4^{pw}	0.003	0.002	0.001	0.046	0.044	0.057	0.032	0.028	0.035
$F_3^{pw,*}$ (iidB)	0.044	0.062	0.045	0.045	0.045	0.059	0.036	0.052	0.051
$F_3^{pw,*}$ (WB)	0.039	0.056	0.045	0.050	0.041	0.053	0.055	0.053	0.053
$\rho = 0.8$									
F_1	0.444	0.359	0.333	0.197	0.157	0.116	0.180	0.119	0.085
F_1^{pw}	0.188	0.185	0.200	0.112	0.081	0.063	0.102	0.067	0.053
F_2	0.057	0.031	0.026	0.103	0.075	0.059	0.088	0.060	0.043
F_2^{pw}	0.032	0.020	0.018	0.048	0.039	0.041	0.034	0.030	0.027
F_3	0.221	0.126	0.096	0.257	0.189	0.131	0.249	0.163	0.136
F_3^{pw}	0.060	0.034	0.032	0.119	0.083	0.065	0.133	0.095	0.075
F_4	0.019	0.008	0.005	0.096	0.068	0.057	0.088	0.065	0.044
F_4^{pw}	0.023	0.008	0.004	0.062	0.048	0.049	0.046	0.032	0.030
$F_3^{pw,*}$ (iidB)	0.021	0.017	0.026	0.056	0.052	0.052	0.058	0.055	0.056
$F_3^{pw,*}$ (WB)	0.023	0.030	0.046	0.055	0.055	0.051	0.057	0.059	0.055

can be constructed.[5] Denote $\widehat{\mu}^*$, $\widehat{\beta}^*$ as the OLS estimators obtained by regressing y_t^* on 1 and t, and the residuals as $\widehat{u}_t^* = y_t^* - \widehat{\mu}^* - \widehat{\beta}^* t$. Let $\widehat{A}_1^*, \ldots, \widehat{A}_p^*$ be the OLS estimators of the VAR slopes obtained by regressing $\widehat{u}_t^*$ on $\widehat{u}_{t-1}^*, \ldots, \widehat{u}_{t-p}^*$ (for $t = 2p+1, \ldots, T$), and the residuals be $\widehat{v}_t^* = \widehat{u}_t^* - \widehat{A}_1^*\widehat{u}_{t-1}^* - \cdots - \widehat{A}_p^*\widehat{u}_{t-p}^*$. Then $F_3^{pw,*}$ is defined as $F_3^{pw,*} = T^{3/2}(R\widehat{\beta}^* - R\widehat{\beta})'(R\widehat{Q}_3^{pw,*}R')^{-1} \cdot T^{3/2}(R\widehat{\beta}^* - R\widehat{\beta})$ where $\widehat{Q}_3^{pw,*} = (T^{-3}\sum_{t=2p+1}^{T}\widetilde{t}^2)^{-2}[\widehat{A}^*(1)]^{-1}(T^{-3}\sum_{t=2p+1}^{T}\widetilde{t}^2\widehat{v}_t^*\widehat{v}_t^{*\prime})[\widehat{A}^*(1)]^{-1\prime}$ and $\widehat{A}^*(L) = I_m - \widehat{A}_1^*L - \cdots - \widehat{A}_p^*L^p$. The inference decisions are then made by comparing F_3^{pw} with the bootstrap critical values, which are the quantiles of the empirical distribution of $F_3^{pw,*}$ over a number of bootstrap replications. The follow theorem shows that both the i.i.d. bootstrap and the wild bootstrap are asymptotically valid.

Theorem 5. *Let P^* be the probability measure induced by the i.i.d. bootstrap or the wild bootstrap procedure. Under Assumptions 1, 2 and H_0, $\sup_{x\in\mathbb{R}} |P^*(F_3^{pw,*} \leq x) - P(F_3^{pw} \leq x)| \xrightarrow{p} 0$, as $T \to \infty$.*

The wild bootstrap can be also applied to other test statistics analyzed in Section 3 to recover the limit distribution. Here we only focus on the asymptotically *pivotal* statistic F_3^{pw} following the recommendation of Horowitz (2001). The i.i.d. bootstrap is generally invalid when applied to the asymptotically non-pivotal statistics under nonstationary volatility.

5. Monte Carlo experiments

In this section we present some evidence on the finite-sample behaviors of the multivariate deterministic trend tests studied above via Monte Carlo experiments.

The designs. We generate T observations on three time series following (1) with $\mu = \beta = (0, 0, 0)'$ and innovations $u_t = \mathrm{diag}(\rho, \rho, \rho)u_{t-1} + v_t$, where $v_t = \sigma_t \varepsilon_t$,

$$\sigma_t = \mathrm{diag}(\sigma_{1t}, \sigma_{1t}, \sigma_{1t}), \quad \sigma_{1t} = \sigma\left(\frac{t}{T}\right),$$
$$\sigma(r)^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)\mathbf{1}_{\{r \geq \tau\}}, \quad r \in (0, 1] \tag{6}$$

and $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t})'$ is i.i.d. normally distributed such that $\mathrm{E}\varepsilon_t = 0$, $\mathrm{Var}(\varepsilon_{1t}) = \mathrm{Var}(\varepsilon_{2t}) = \mathrm{Var}(\varepsilon_{3t}) = 1$, $\mathrm{E}\varepsilon_{1t}\varepsilon_{2t} = \mathrm{E}\varepsilon_{2t}\varepsilon_{3t} = 0.6$ and $\mathrm{E}\varepsilon_{1t}\varepsilon_{3t} = 0.2$. The specification in (6) shows there is a simultaneous variance break for the three time series at the time $[T\tau]$ from σ_0^2 to σ_1^2. The magnitude of the variance shifting is determined by $\delta = \sigma_1/\sigma_0$ and we choose $\delta \in \{0.1, 1, 10\}$. We also consider the benchmark case $\delta = 1$ where there is no variance break for any component time series (corresponding to the case considered by VF), which is contrasted to identify the effects on various tests of variance break. Without loss of generality we set $\sigma_0^2 = 1$. We concentrate on the case $\tau = 0.9$ which corresponds to a variance break occurred near the end of the sample. We choose $a \in \{0, 0.8\}$ to check the sensitivity of the tests to the persistency of the time series. A group of sample sizes including $T = 100, 200, 400$ are considered to verify the asymptotic theories derived earlier.

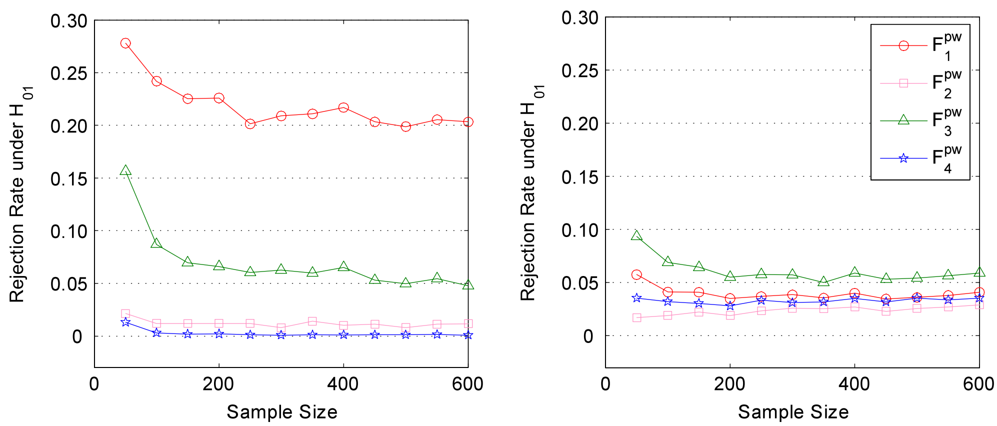
We test two hypotheses on the multivariate trend coefficients ($q = 1$ and 3 respectively): $H_{01} : \beta_1 = \beta_2$ and $H_{02} : \beta_1 = \beta_2 = \beta_3 = 0$. We consider the eight tests analyzed in Section 3 and two bootstrap tests in Section 4. For F_1 and F_3 the bandwidths M are selected by the data-dependent method as in Andrews (1991). The nominal size is set as 5% and empirical rejection rates are computed over 5000 replications. The number of bootstrap replications is 999.

The results. We report the empirical rejection rates of the tests under the null hypotheses in Tables 1 and 2 for $q = 1$ and 3 respectively. The effects of a late variance break on the trend coefficients tests can be summarized as follows. First, Class 1 tests over-reject the null hypothesis seriously when the variance break is upward ($\delta = 10$) and under-reject mildly when the variance break is downward ($\delta = 0.1$). This confirms the asymptotic theory for Class 1 tests (c.f. Section 3.1 and Fig. 1). Second, Classes 2 and 4 tests (which use the critical values under fixed-b asymptotics) under-reject when the variance break is late, no matter it is upward or downward, with the relatively larger under-rejection effect by an upward break. The size distortions of Classes 1, 2 and 3 tests do not disappear when the sample size increases. This can be seen in

[5] Since the bootstrap schemes rely on the parametric VAR specification of u_t, we only consider them for the prewhitened tests.

Table 2
Empirical rejection rates of various tests and two bootstrap tests [i.i.d. bootstrap (iidB) and wild bootstrap (WB)] under the null hypothesis where there is a simultaneous variance break ($q = 3$, nominal size: 0.05).

T	$(\tau, \delta) = (0.9, 10)$			$(\tau, \delta) = (0.9, 1)$			$(\tau, \delta) = (0.9, 0.1)$		
	100	200	400	100	200	400	100	200	400
$\rho = 0$									
F_1	0.338	0.372	0.350	0.067	0.057	0.052	0.037	0.026	0.025
F_1^{pw}	0.491	0.417	0.379	0.102	0.064	0.061	0.053	0.037	0.033
F_2	0.004	0.003	0.003	0.060	0.050	0.051	0.011	0.015	0.013
F_2^{pw}	0.008	0.004	0.002	0.042	0.035	0.043	0.009	0.011	0.010
F_3	0.027	0.051	0.044	0.087	0.068	0.057	0.073	0.055	0.057
F_3^{pw}	0.163	0.086	0.063	0.122	0.070	0.065	0.104	0.076	0.060
F_4	0.000	0.000	0.000	0.061	0.050	0.051	0.013	0.014	0.012
F_4^{pw}	0.004	0.000	0.001	0.065	0.049	0.052	0.023	0.016	0.013
$F_3^{pw,*}$ (iidB)	0.080	0.069	0.060	0.049	0.049	0.038	0.038	0.045	0.049
$F_3^{pw,*}$ (WB)	0.032	0.060	0.064	0.056	0.052	0.042	0.060	0.055	0.055
$\rho = 0.8$									
F_1	0.599	0.444	0.415	0.459	0.294	0.208	0.403	0.244	0.123
F_1^{pw}	0.514	0.402	0.404	0.284	0.179	0.105	0.236	0.117	0.063
F_2	0.121	0.067	0.026	0.215	0.132	0.086	0.163	0.077	0.038
F_2^{pw}	0.093	0.041	0.015	0.089	0.059	0.044	0.056	0.025	0.014
F_3	0.432	0.221	0.109	0.579	0.389	0.281	0.536	0.355	0.196
F_3^{pw}	0.245	0.082	0.043	0.307	0.190	0.108	0.323	0.175	0.107
F_4	0.065	0.024	0.009	0.213	0.133	0.088	0.159	0.075	0.034
F_4^{pw}	0.075	0.024	0.006	0.122	0.079	0.056	0.054	0.024	0.013
$F_3^{pw,*}$ (iidB)	0.043	0.030	0.020	0.085	0.059	0.052	0.069	0.048	0.052
$F_3^{pw,*}$ (WB)	0.043	0.048	0.039	0.086	0.064	0.055	0.096	0.058	0.058



(a) $(\rho, \tau, \delta) = (0, 0.9, 10)$. (b) $(\rho, \tau, \delta) = (0, 0.9, 0.1)$.

Fig. 2. The rejection rates of the tests of the four classes under the null hypothesis H_{01} for an upward [Panel (a), $\delta = 10$] and a downward [Panel (b), $\delta = 0.1$] variance break when the break is late ($\tau = 0.9$). Nominal size: 0.05.

Fig. 2 where we plot the rejection rates of various tests when the sample size increases from $T = 50$ to $T = 600$.[6] Third, Class 3 tests have reasonable sizes in both cases of $\delta = 10$ and 0.1 and are quite robust to the variance break. Fig. 2 shows that the size distortions disappear when the sample size increases, although some large over-rejections are seen for small sample sizes. This is consistent with the asymptotic validity of Class 3 tests. Fourth, the sizes of tests are generally inflated (especially for Classes 1 and 3 tests) when there is stronger serial correlation ($\rho = 0.8$) and/or the number of restrictions to be tested is larger ($q = 3$). Note that in such cases, the robust tests F_3 and F_3^{pw} also suffer from large size distortions. Fifth, both residual-based i.i.d. and wild bootstrap reduce the size distortions of the robust test F_3^{pw} dramatically, especially in the cases when the size distortions are large (e.g. $q = 3$ and/or $\rho = 0.8$).[7]

In the unreported simulation experiments we also find that the effects of variance break are less significant when only some component time series experience variance breaks while others do not. We also consider the powers of the bootstrap tests $F_3^{pw,*}$ (iidB and WB) which are reported in Table 3. The powers are computed as the rejection rates when the coefficients in the data generating process are $(\beta_1, \beta_2) \in \{(0.005, 0), (0.01, 0)\}$ for the null H_{01} and $(\beta_1, \beta_2, \beta_3) \in \{(0.005, 0, 0), (0.01, 0, 0)\}$ for the null H_{02}. Both bootstrap tests are consistent under variance break. They have the reduced (increased) powers when the late variance break is upward (downward, respectively) when compared to the benchmark case of no variance break.

6. Conclusion and possible extensions

In this paper, we study testing of linear restrictions on the multivariate trend slopes when the innovations follow a

6 To separate out the effects of serial correlation and many restrictions (and focus on the effects of nonstationary volatility), in Fig. 2 we only consider the prewhitened tests and the case when $\rho = 0$ and $q = 1$.

7 We also consider a variant of the test F_3^{pw} in simulations with a nonzero bandwidth (selected by the data-dependent method) in the LRV estimator $\widehat{LRV}(\widetilde{t}\widehat{v}_t/T, M)$. This modification might be useful in practice to handle the serial correlation in $\widehat{v}_t$ (e.g. due to the estimation error in the VAR parameter estimates). We find it performs very closely to F_3^{pw} under our simulation designs.

Table 3
Powers of the bootstrap tests $F_3^{pw,*}$ [i.i.d. bootstrap and wild bootstrap (WB)] when $\rho = 0$ (nominal size: 0.05).

	$\beta_1 \setminus T$	$(\tau, \delta) = (0.9, 10)$			$(\tau, \delta) = (0.9, 1)$			$(\tau, \delta) = (0.9, 0.1)$		
		100	200	400	100	200	400	100	200	400
$q = 1$										
iidB	0.005	0.084	0.205	0.744	0.331	0.997	1.000	0.428	1.000	1.000
WB		0.072	0.200	0.740	0.342	0.997	1.000	0.441	1.000	1.000
iidB	0.010	0.107	0.462	0.998	0.857	1.000	1.000	0.970	1.000	1.000
WB		0.100	0.466	0.997	0.862	1.000	1.000	0.961	1.000	1.000
$q = 3$										
iidB	0.005	0.112	0.210	0.759	0.271	1.000	1.000	0.327	1.000	1.000
WB		0.064	0.191	0.756	0.293	1.000	1.000	0.384	1.000	1.000
iidB	0.010	0.199	0.545	0.998	0.832	1.000	1.000	0.925	1.000	1.000
WB		0.138	0.510	0.997	0.848	1.000	1.000	0.940	1.000	1.000

finite-order stable vector autoregressive process with possible but unknown nonstationary variances. We show that the conventional tests considered in Vogelsang and Franses (2005a) are generally invalid, involving the volatility function as nuisance parameters in a complicated way. Robust tests built on heteroskedasticity correction and two residual-based bootstrap schemes are suggested, and are shown to be asymptotically valid under general nonstationary volatility. Monte Carlo simulations confirm the asymptotic theory developed and illustrate the finite-sample behaviors of various trend tests studied.

Several extensions of the current work are possible. First, to focus on the effects of nonstationary volatility, we parameterize the structure of the weak serial correlation in the data. This can be generalized, e.g. by using a VARMA (vector autoregressive and moving average) model, or even be avoided via nonparametric specification.

Second, the stable assumption on the autoregressive coefficients used in this paper precludes application to time series with strong dependence. Developing tests that are robust to both persistence and nonstationary volatility would be a natural and useful extension of the current paper. Consider the univariate case, i.e. $m = 1$. Following Harvey et al. (2007), a test statistic robust to both persistence and nonstationary volatility can be constructed as a weighted average $\Xi = \omega F_3^{pw} + (1 - \omega)G$, where G is a valid test statistic on the trend coefficient β when the data are persistent. More precisely, suppose $A(L)$ in (2) contains a unit root and other roots outside the unit circle, i.e. $A(L) = (1 - L)(1 - \overline{A}_1 L - \cdots - \overline{A}_{p-1} L^{p-1})$. In this case, β is estimated based on differenced data, i.e. $\widehat{\beta}^d = \sum_{t=2}^{T} \Delta y_t/(T - 1)$, and G is of the form $T(R\widehat{\beta}^d - r)'(R \cdot \widehat{LRV}(\Delta u_t) \cdot R')^{-1}(R\widehat{\beta}^d - r)$ where $\widehat{LRV}(\Delta u_t)$ is an estimator of the long run variance of Δu_t. The weight ω is constructed such that it converges to one (thereby Ξ is asymptotically equivalent to F_3^{pw}) when $A(L)$ contains all stable roots, and converges to zero (thereby Ξ is asymptotically equivalent to G) when $A(L)$ contains a unit root. Harvey et al. (2007) chose ω as a function of test statistics for unit root and stationarity respectively. Note that Ξ formed above extends Harvey et al. (2007) (see also Vogelsang, 1998; Bunzel and Vogelsang, 2005; Perron and Yabu, 2009) by building firewalls for the trend test guarding against not only persistence of time series but also nonstationary volatility. Robust testing of this sort is much more complicated in the multivariate case and co-movement of component time series has to be taken into consideration.

Third, in this paper we are only concerned about the effects of nonstationary volatility on the inference of the trend coefficients. The effects on point estimation and possible efficiency gains over OLS need further investigation. All these extensions are nontrivial and left for future research.

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References

Andrews, D.W.K., 1991. Heteroskedasticity and autocorrelation consistent covariance matrix estimation. Econometrica 59, 817–858.
Andrews, D.W.K., Monahan, C.J., 1992. An improved heteroskedasticity and autocorrelation consistent covariance matrix estimator. Econometrica 60, 953–966.
Bacigál, T, 2005. Testing for common deterministic trends in geodetic data. Journal of Electrical Engineering 56, 115–118.
Beare, B., 2008. Unit root testing with unstable volatility, Working Paper, University of Oxford.
Blanchard, O., Simon, J., 2001. The long and large decline in US output volatility. Brookings Papers on Economic Activity 32, 135–174.
Bunzel, H., Vogelsang, T.J., 2005. Powerful trend function tests that are robust to strong serial correlation with an application to the Prebish–Singer hypothesis. Journal of Business and Economic Statistics 23, 381–394.
Busetti, F., Taylor, A.M.R., 2003. Testing against stochastic trend in the presence of variance shifts. Journal of Business and Economic Statistics 21, 510–531.
Canjels, E., Watson, M.W., 1997. Estimating deterministic trends in the presence of serially correlated errors. Review of Economics and Statistics 79, 184–200.
Carlino, G.A., Mills, L.O., 1993. Are US regional incomes converging? Journal of Monetary Economics 32, 335–346.
Carrieri, F., Errunza, V., Sarkissian, S., 2004. The dynamics of geographic versus sectoral diversification: is there a link to the real economy? Working Paper, McGill University.
Cavaliere, G., Rahbek, A., Taylor, A.M.R., 2010. Testing for co-integration in vector autoregressions with non-stationary volatility. Journal of Econometrics 158 (1), 7–24.
Cavaliere, G., Taylor, A.M.R., 2007. Testing for unit roots in time series models with non-stationary volatility. Journal of Econometrics 140, 919–947.
Cavaliere, G., Taylor, A.M.R., 2008. Bootstrap unit root tests for time series with non-stationary volatility. Econometric Theory 24, 43–71.
Chung, H., Park, J.Y., 2007. Nonstationary nonlinear heteroskedasticity in regression. Journal of Econometrics 137, 230–259.
Davidson, J., 1994. Stochastic Limit Theory: An Introduction for Econometricians. Oxford University Press.
Eun, C.S., Lee, J., 2006. Mean–variance convergence around the world. Working Paper, Georgia Institute of Technology.
Fomby, T., Vogelsang, T., 2003. Tests of common deterministic trend slopes applied to quarterly global temperature data. In: Fomby, T., Hill, R. (Eds.), Advances in Econometrics, vol. 17. Elsevier Science, pp. 29–43.
Gonçalves, S., Kilian, L., 2004. Bootstrapping autoregression with conditional heteroskedasticity of unknown form. Journal of Econometrics 123, 89–120.
Gonçalves, S., Kilian, L., 2007. Asymptotic and bootstrap inference for AR(∞) processes with conditional heteroskedasticity. Econometric Reviews 26, 609–641.
Harvey, D.I., Leybourne, S.J., Taylor, A.M.R., 2007. A simple, robust and powerful test of the trend hypothesis. Journal of Econometrics 141, 1302–1330.
Hobijn, B., Franses, P.H., 2000. Asymptotically perfect and relative convergence of productivity. Journal of Applied Econometrics 15, 59–81.
Horowitz, J., 2001. The bootstrap. In: Heckman, J., Leamer, E. (Eds.), Handbook of Econometrics, vol. 5. Elsevier Science, pp. 3159–3228.

Jansson, M., 2004. The error rejection probability of simple autocorrelation robust tests. Econometrica 72, 937–946.

Kiefer, N.M., Vogelsang, T.J., 2002. Heteroskedasticity–autocorrelation robust standard errors using the Bartlett kernel without truncation. Econometrica 70, 2093–2095.

Kiefer, N.M., Vogelsang, T.J., 2005. A new asymptotic theory for heteroskedasticity autocorrelation robust tests. Econometric Theory 21, 1130–1164.

Kiefer, N.M., Vogelsang, T.J., Bunzel, H., 2000. Simple robust testing of regression hypotheses. Econometrica 68, 695–714.

Kim, C.-J., Nelson, C.R., 1999. Has the US economy become more stable? A Bayesian approach based on a Markov-switching model of the business cycle. Review of Economics and Statistics 81, 608–616.

Loretan, M., Phillips, P.C.B., 1994. Testing the covariance stationarity of heavy-tailed time series: an overview of the theory with applications to several financial datasets. Journal of Empirical Finance 1, 211–248.

McConnell, M.M., Perez Quiros, G., 2000. Output fluctuations in the United States: what has changed since the early 1980s? American Economic Review 90, 1464–1476.

Perron, P., Yabu, T., 2009. Estimating deterministic trends with an integrated or stationary noise component. Journal of Econometrics 151, 56–69.

Phillips, P.C.B., Xu, K.-L., 2006. Inference in autoregression under heteroskedasticity. Journal of Time Series Analysis 27, 289–308.

Roy, A., Falk, B., Fuller, A., 2004. Testing for trend in the presence of autoregressive error. Journal of the American Statistical Association 99, 1082–1091.

Sensier, M., van Dijk, D., 2004. Testing for volatility changes in US macroeconomic time series. Review of Economics and Statistics 86, 833–839.

Sun, Y., Phillips, P.C.B., Jin, S., 2008. Optimal bandwidth selection in heteroskedasticity–autocorrelation robust testing. Econometrica 76, 175–194.

Vogelsang, T.J., 1998. Trend function hypothesis testing in the presence of serial correlation parameters. Econometrica 65, 123–148.

Vogelsang, T.J., Franses, P.H., 2005a. Testing for common deterministic trend slopes. Journal of Econometrics 126, 1–24.

Vogelsang, T.J., Franses, P.H., 2005b. Are winters getting warmer? Environmental Modeling and Software 20, 1449–1455.

White, H., 1980. A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. Econometrica 48, 817–838.

Wu, C.F.J., 1986. Jackknife, bootstrap and other resampling methods in regression analysis. Annals of Statistics 14, 1261–1295.

Xu, K.-L., 2008. Bootstrapping autoregression under nonstationary volatility. Econometrics Journal 11, 1–26.

Xu, K.-L., 2009. Robustifying multivariate trend tests to nonstationary volatility. Working Paper, University of Alberta School of Business.

Xu, K.-L., Phillips, P.C.B., 2008. Adaptive estimation of autoregressive models with time-varying variances. Journal of Econometrics 142, 265–280.