Multiscale comparison of nonparametric trend curves

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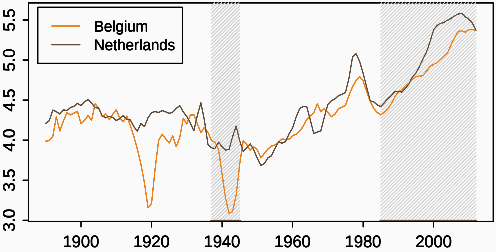
Introduction

Motivation

Aim of the paper

To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.



Research question: Out of many given intervals, how to find those where the trends are significantly different?

Motivation

Why is it relevant?

Finding systematic differences between trends = basis for further research.

What can we do if we find significant differences?

We can use the estimated differences as a distance measure for clustering

$\Rightarrow$ discover underlying group structure.

Model

Model

We observe a panel of n time series $\mathcal{Z}_i = \{(Y_{it}, \boldsymbol{X}_{it}) : 1 \leq t \leq T\}$ of length T, where $Y_{it} \in \mathbb{R}$ and $\boldsymbol{X}_{it} \in \mathbb{R}^d$. We assume that n is fixed.

We consider the following model:

$$Y_{it} = m_i\Big(\frac{t}{T}\Big) + \boldsymbol{\beta}_i^T \boldsymbol{X}_{it} + \alpha_i + \varepsilon_{it},$$

where

- m_i are unknown trend functions on $[0, 1]$;
- $\boldsymbol{\beta}_i$ is $d \times 1$ vector of unknown parameters;
- α_i are so-called fixed effect error terms;
- $\{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

Model, part 2

$$Y_{it} = m_i\left(\frac{t}{T}\right) + \boldsymbol{\beta}_i^T \boldsymbol{X}_{it} + \alpha_i + \varepsilon_{it},$$

If we knew α_i and $\boldsymbol{\beta}_i$, then the model becomes much simpler:

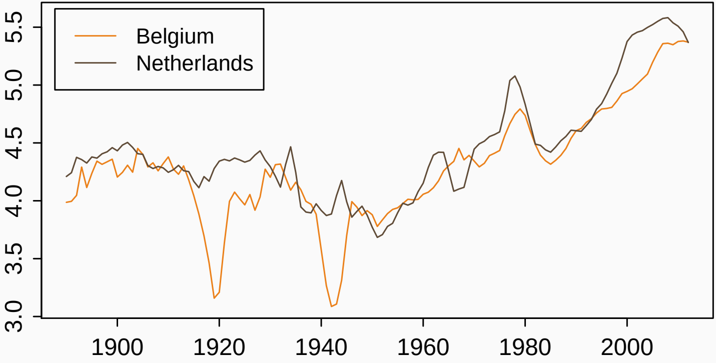
$$\begin{aligned} Y_{it} - \alpha_i - \boldsymbol{\beta}_i^\top \boldsymbol{X}_{it} &=: Y_{it}^\circ \\ &= m_i\left(\frac{t}{T}\right) + \varepsilon_{it}. \end{aligned}$$

In reality the variables Y_{it}° are **not** observed.

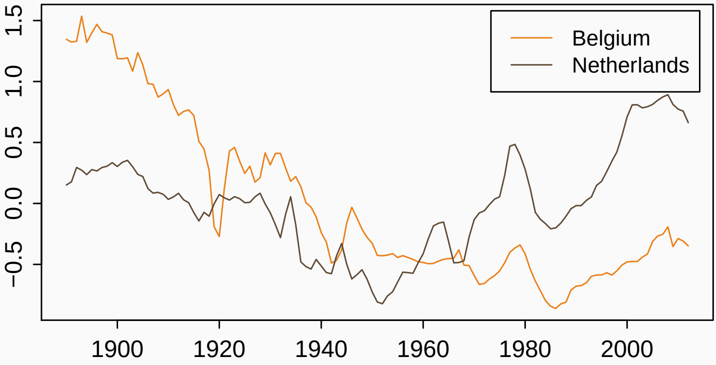
But given $\widehat{\alpha}_i$ and $\widehat{\boldsymbol{\beta}}_i$, we consider an augmented time series

$$\widehat{Y}_{it} := Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} = (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \boldsymbol{X}_{it} + m_i\left(\frac{t}{T}\right) + \left(\alpha_i - \widehat{\alpha}_i\right) + \varepsilon_{it}.$$

Original time series: Belgium and Netherlands



Augmented time series: Belgium and Netherlands



Testing procedure

Testing problem

$$H_0 : m_1 = m_2 = \ldots = m_n$$

Question: if we reject the global null, how to locate the differences between the trends?

Consider a grid $\mathcal{G}_T = \{(u, h) : [u - h, u + h] \subseteq [0, 1]\}$ of location-bandwidth parameters. For each pair (i, j) and for each interval $[u - h, u + h]$ we consider the null hypothesis

$$H_0^{[i,j]}(u, h) : m_i(w) = m_j(w) \text{ for all } w \in [u - h, u + h].$$

Test statistic

For a given location $u \in [0,1]$ and bandwidth h and a given pair (i,j) we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

where $w_{t,T}(u,h)$ are appropriate weights.

The kernel averages $\widehat{\psi}_{ij,T}(u,h)$ measure the distance between two trend curves m_i and m_j on $[u-h, u+h]$.

Test statistic, part 2

Instead with working directly with $\widehat{\psi}_{ij,T}(u,h)$, we replace them by

$$\widehat{\psi}^{0}_{ij,T}(u,h) = \left| \frac{\widehat{\psi}_{ij,T}(u,h)}{\left(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\right)^{1/2}} \right| - \lambda(h),$$

where

- $\widehat{\sigma}_i^2$ is an appropriate estimator of the long-run variance σ_i^2;
- $\lambda(h) = \sqrt{2\log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

Test statistic, part 3

To test the global null, we aggregate the individual test statistics for all (i, j) and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\widehat{\Psi}_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \widehat{\psi}^0_{ij,T}(u, h).$$

Main theoretical result

Under certain conditions and under the null, $\widehat{\psi}^0_{ij,T}(u, h)$ and $\widehat{\Psi}_{n,T}$ can be approximated by the corresponding Gaussian versions of the test statistics.

Gaussian test statistics

Gaussian version of the individual test statistics:

$$\phi^0_{ij,T}(u,h) = \left| \frac{\phi_{ij,T}(u,h)}{\left(\sigma_i^2 + \sigma_j^2\right)^{1/2}} \right| - \lambda(h),$$

where

- $\phi_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h) \left\{ \sigma_i (Z_{it} - \bar{Z}_i) - \sigma_j (Z_{jt} - \bar{Z}_j) \right\}$;
- Z_{it} are independent (across i and t) standard normal RVs and $\bar{Z}_i$ is the empirical average of $Z_{i1}, \ldots, Z_{iT}$.

Aggregated Gaussian test statistics:

$$\Phi_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \phi^0_{ij,T}(u,h).$$

Clustering

Dissimilarity measure

Underlying group structure: there exist groups of time series $G_1, \ldots, G_N$ with $\{1, \ldots, n\} = \dot{\bigcup}_{\ell=1}^N G_\ell$ such that for each $1 \le \ell \le N$,

$$m_i = f_\ell \quad \text{for all } i \in G_\ell,$$

where f_ℓ are group-specific trend functions.

Dissimilarity measure between two time series i and j:

$$\max_{(u,h) \in \mathcal{G}_T} \hat{\psi}^0_{ij,T}(u,h).$$

Dissimilarity measure between two sets of time series S and S':

$$\widehat{\Delta}(S, S') = \max_{\substack{i \in S, \\ j \in S'}} \Big(\max_{(u,h) \in \mathcal{G}_T} \hat{\psi}^0_{ij,T}(u,h) \Big).$$

Clustering algorithm

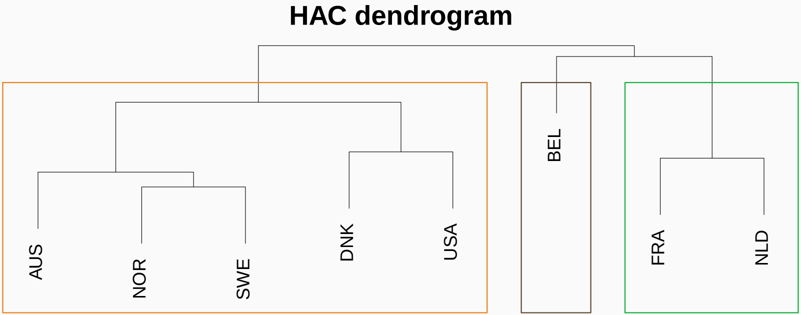
1. For each pair of time series (i, j), calculate the dissimilarity measure $\max_{(u,h)\in\mathcal{G}_T} \hat{\psi}^0_{ij,T}(u, h)$.
2. Using the dissimilarity measure and the complete linkage method, perform HAC. The result is a tree of nested partitions: $\{\widehat{G}^{[r]}_1, \ldots\ldots, \widehat{G}^{[r]}_{n-r}\}$ for $r = 1, \ldots, n-1$.
3. Estimate the number of clusters $\widehat{N}$ using the $(1-\alpha)$-quantile $q_{n,T}(\alpha)$ of the Gaussian test statistics $\Phi_{n,T}$:
$$\widehat{N} = \min\Big\{ r = 1, 2, \ldots \,\Big|\, \max_{1\le\ell\le r} \widehat{\Delta}\big(\widehat{G}^{[n-r]}_\ell, \widehat{G}^{[n-r]}_\ell\big) \le q_{n,T}(\alpha) \Big\}.$$

Proposition

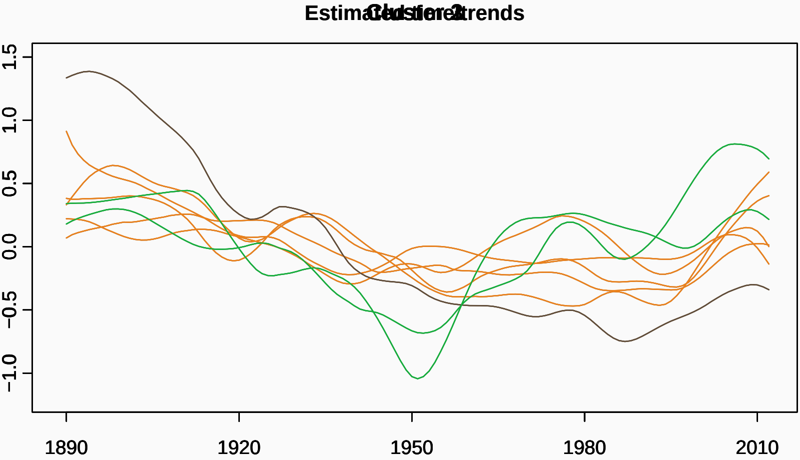
Under certain assumptions, we have
$$\mathbb{P}\Big(\big\{\widehat{G}_1, \ldots, \widehat{G}_{\widehat{N}}\big\} = \big\{G_1, \ldots, G_N\big\}\Big) \ge (1-\alpha) + o(1)$$
$$\text{and} \quad \mathbb{P}\big(\widehat{N} = N\big) \ge (1-\alpha) + o(1).$$

Illustration

Clustering results



Clustering results



Discussion

We can claim, with confidence of at least 95%, that the null hypothesis is violated for all intervals (and all pairs of time series) for which our test rejects the null.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Furthermore, if we reject the null, we can use the calculated test statistics as a distance measure between two time series on an interval.

Further possible extensions:

- introduce scaling factor in the trend function;
- include the dependence between covariates and error terms.

Thank you!

Model, part 3

1. We estimate $\boldsymbol{\beta}_i$:

$$\widehat{\boldsymbol{\beta}}_i = \Big(\sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta \boldsymbol{X}_{it}^\top \Big)^{-1} \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta Y_{it}$$

Theorem

Under certain regularity assumptions, $\widehat{\boldsymbol{\beta}}_i$ is a consistent estimator of $\boldsymbol{\beta}_i$ with the property $\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i = O_P(T^{-1/2})$.

2. We estimate the fixed effects α_i:

$$\widehat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} \big(Y_{it} - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} \big)$$

We then work with the augmented time series $\widehat{Y}_{it} = Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it}$.

Test statistic

For a given location $u \in [0,1]$ and bandwidth h and a given pair (i,j) we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

where

$$w_{t,T}(u,h) = \frac{\Lambda_{t,T}(u,h)}{\{\sum_{t=1}^{T} \Lambda_{t,T}^2(u,h)\}^{1/2}},$$

$$\Lambda_{t,T}(u,h) = K\Big(\frac{t/T - u}{h}\Big)\Big[S_{T,2}(u,h) - S_{T,1}(u,h)\Big(\frac{t/T - u}{h}\Big)\Big],$$

$$S_{T,\ell}(u,h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T - u}{h}\Big)\Big(\frac{t/T - u}{h}\Big)^{\ell}$$

for $\ell = 1, 2$ and K is a kernel function.

Assumptions

$\mathcal{C}1$ For all i it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.

$\mathcal{C}2$ For each i the variables ε_{it} are weakly dependent. Details

$\mathcal{C}3$ For each i we have that $\boldsymbol{X}_{it}$ is stationary and causal with all the necessary moments and no asymptotic multicollinearity.

$\mathcal{C}4$ For each i the variables $\boldsymbol{X}_{it}$ are weakly dependent. Details

$\mathcal{C}5$ $\boldsymbol{X}_{it}$ (elementwise) and ε_{is} are uncorrelated for each t, s.

$\mathcal{C}6$ All of the variables in the model are short-range dependent. Details

Assumptions, part 2

$\mathcal{C}7$ Standard assumptions on the kernel function K.

$\mathcal{C}8$ $|\mathcal{G}_T| = O(T^\theta)$ for some arbitrarily large but fixed constant $\theta > 0$.

$$\mathcal{G}_T = \Big\{(u,h) : u = t/T \text{ for some } 1 \le t \le T \text{ and } h \in [h_{\min}, h_{\max}] \\ \text{with } h = t/T \text{ for some } 1 \le t \le T\Big\},$$

$\mathcal{C}9$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

$\mathcal{C}10$ Assume that $\sigma_i^2 = \sigma_j^2$ for all i,j and $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(\sqrt{h_{\min}}/\log T)$.

Strategy of the proof

- Introduce $\widehat{\Phi}_{n,T}$ that is close in distribution to $\widehat{\Psi}_{n,T}$ under the null.
- Using strong approximation theory for dependent processes as derived in Berkes et al. (2014), replace $\widehat{\Phi}_{n,T}$ by $\widetilde{\Phi}_{n,T}$ with the same distribution and the property that

$$\left|\widetilde{\Phi}_{n,T} - \Phi_{n,T}\right| = o_p(\delta_T),$$

  where δ_T goes to 0 as $T \to \infty$ sufficiently fast.
- Using the anti-concentration results for Gaussian random vectors (Chernozhukov et al. 2015), prove that $\Phi_{n,T}$ does not concentrate too strongly in small regions of the form $[x - \delta_T, x + \delta_T]$, i.e.

$$\sup_{x \in \mathbb{R}} \mathbb{P}\big(|\Phi_{n,T} - x| \leq \delta_T\big) = o(1).$$

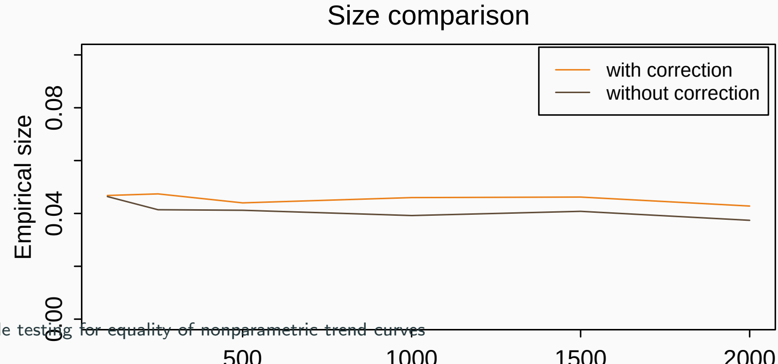
- Show that

$$\sup_{x \in \mathbb{R}} \left|\mathbb{P}(\widetilde{\Phi}_{n,T} \leq x) - \mathbb{P}(\Phi_{n,T} \leq x)\right| = o(1).$$

Idea behind $\lambda(h)$

Dümbgen and Spokoiny (2001): the critical values for testing the 'local' null hypothesis depend on the scale of the testing problem, i.e. the length h of the time interval.

Introduction of a scale-dependent parameter helps us balance the significance of hypotheses between the time intervals of different lengths h_k:



Idea behind the additive correction

Consider the uncorrected Gaussian statistic

$$\Phi^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk}|$$

and let the family of intervals be

$$\mathcal{F} = \Big\{[(m-1)h_l, mh_l] \text{ for } 1 \leq m \leq 1/h_l, 1 \leq l \leq L\Big\}$$

Then we can rewrite the uncorrected test statistic as

$$\Phi^{\text{uncor}} = \max_{i,j} \max_{\substack{1 \leq l \leq L, \\ 1 \leq m \leq 1/h_l}} \left| \frac{1}{\sqrt{2Th_l}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in [(m-1)h_l, mh_l]\Big)(Z_{it} - Z_{jt}) \right|$$

$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l. Go back

Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\boldsymbol{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\boldsymbol{L}, t) = ||\boldsymbol{L}(\mathcal{F}_t) - \boldsymbol{L}(\mathcal{F}_t')||_q,$$

where $\mathcal{F}_t = (\ldots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}_t' = (\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of $\mathcal{F}_t$ with ϵ_0' being an i.i.d. copy of ϵ_0.

Intuitively, $\delta_q(\boldsymbol{L}, t)$ measures the dependency of $\boldsymbol{L}(\mathcal{F}_t)$ on ϵ_0, i.e., how replacing ϵ_0 by an i.i.d. copy while keeping all other innovations in place affects the output $\boldsymbol{L}(\mathcal{F}_t)$.

Technical assumptions

$\mathcal{C}1'$ The variables ε_{it} are independent across i and allow for the representation $\varepsilon_{it} = G_i(\ldots, \eta_{it-1}, \eta_{it})$, where η_{it} are i.i.d. random variables across t and $G_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function..

$\mathcal{C}1''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each i it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q$. Go back

Technical assumptions, part 2

- $\mathcal{C}3'$ $\boldsymbol{X}_{it}$ allow for the representation $\boldsymbol{X}_{it} = \boldsymbol{H}_i(\ldots, u_{it-1}, u_{it})$ with u_{it} being i.i.d. random variables and $\boldsymbol{H}_i := (H_{i1}, H_{i2}, \ldots, H_{id})^\top : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^d$ being a measurable function such that $\boldsymbol{H}_i(\mathcal{U}_{it})$ is well defined.
- $\mathcal{C}3''$ Let $\boldsymbol{N}_i$ be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being kl-th entry. We assume that the smallest eigenvalue of $\boldsymbol{N}_i$ is strictly bigger than 0.
- $\mathcal{C}3'''$ Let $\mathbb{E}[\boldsymbol{H}_i(\mathcal{U}_{i0})] = 0$ and $||\boldsymbol{H}_i(\mathcal{U}_{it})||_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where θ will be introduced further.
- $\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) < \infty$ for q' from Assumption $\mathcal{C}3'''$.
- $\mathcal{C}4''$ For each i it holds that $\sum_{s=t}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) = O(t^{-\alpha})$ for q' from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. Go back

Technical assumptions, part 3

$\mathcal{C}6$ Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Denote $\mathcal{I}_{it} = (\ldots, \zeta_{i,t-1}, \zeta_{i,t})$, $\mathcal{J}_{it} = (\ldots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$, $\mathcal{U}_{it} = (\ldots, u_{it-1}, u_{it})$, and $\mathrm{U}_i(\mathcal{I}_{it}) = \boldsymbol{H}_i(\mathcal{U}_{it}) G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(\mathrm{U}_i, s) < \infty$. Go back

Graphical representation

How to represent the results of the test?

We can plot all of the intervals where we reject the local null.

But what if there are too many?

An interval $[u - h, u + h]$ is called **minimal** if the corresponding local null $H_0^{[i,j]}(u, h)$ is rejected and there is no other interval $[u' - h', u' + h']$ such that we reject $H_0^{[i,j]}(u', h')$ and $[u' - h', u' + h'] \subset [u - h, u + h]$.