# Multiscale comparison of nonparametric trend curves


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# Introduction

# Motivation

**Aim of the paper**

To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.

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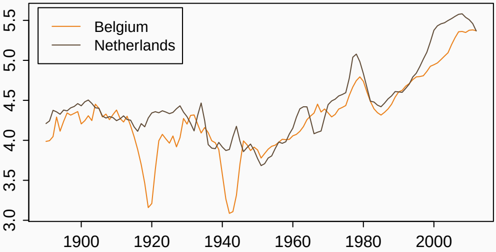

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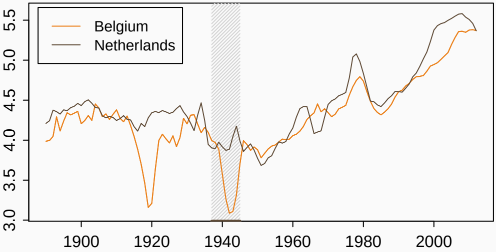

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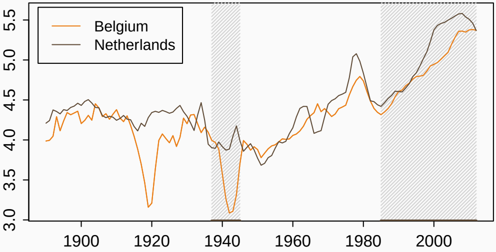

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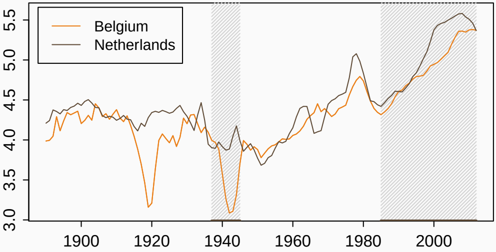


**Research question:** Out of many given intervals, how to find those where the trends are significantly different?

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Finding systematic differences between trends = basis for further research.

**What can we do if we find significant differences?**

We can use the estimated differences as a distance measure for clustering

$\Rightarrow$ discover underlying group structure.

# Model

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- $\{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

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But given $\widehat{\alpha}_i$ and $\widehat{\boldsymbol{\beta}}_i$, we consider an augmented time series

$$\widehat{Y}_{it} := Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} = (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \boldsymbol{X}_{it} + m_i\Big(\frac{t}{T}\Big) + \big(\alpha_i - \widehat{\alpha}_i\big) + \varepsilon_{it}.$$

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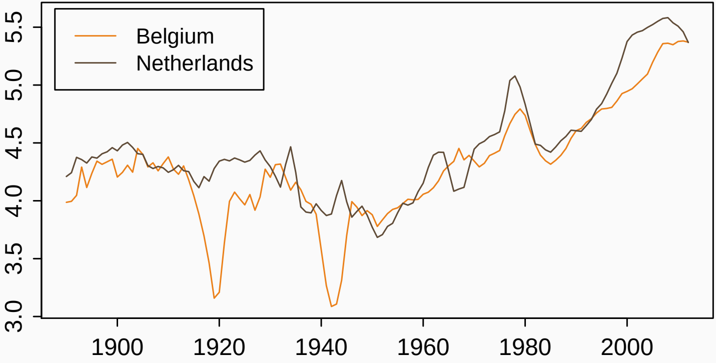
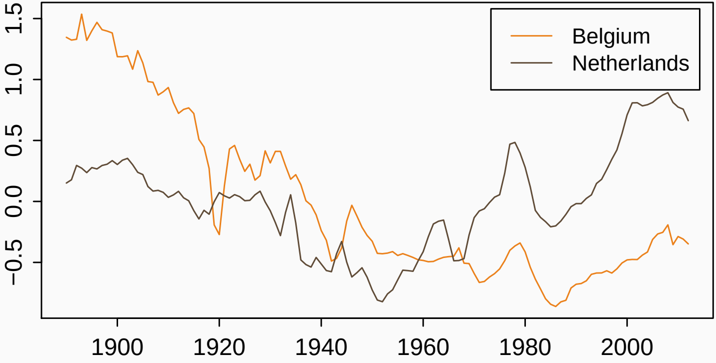


# Augmented time series: Belgium and Netherlands



# Testing procedure

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# Test statistic

For a given location $u \in [0,1]$ and bandwidth $h$ and a given pair $(i,j)$ we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

where $w_{t,T}(u,h)$ are appropriate weights.

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The kernel averages $\widehat{\psi}_{ij,T}(u,h)$ measure the distance between two trend curves $m_i$ and $m_j$ on $[u-h, u+h]$.

# Test statistic, part 2

Instead with working directly with $\widehat{\psi}_{ij,T}(u,h)$, we replace them by

$$\widehat{\psi}^0_{ij,T}(u,h) = \left| \frac{\widehat{\psi}_{ij,T}(u,h)}{\left(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\right)^{1/2}} \right| - \lambda(h),$$

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- $\widehat{\sigma}_i^2$ is an appropriate estimator of the long-run variance $\sigma_i^2$;
- $\lambda(h) = \sqrt{2\log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

# Test statistic, part 3

To test the global null, we aggregate the individual test statistics for all $(i, j)$ and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\widehat{\Psi}_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \widehat{\psi}^0_{ij,T}(u, h).$$

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**Main theoretical result**

Under certain conditions and under the null, $\widehat{\psi}^0_{ij,T}(u, h)$ and $\widehat{\Psi}_{n,T}$ can be approximated by the corresponding Gaussian versions of the test statistics.

# Gaussian test statistics

Gaussian version of the individual test statistics:

$$\phi^0_{ij,T}(u,h) = \left| \frac{\phi_{ij,T}(u,h)}{\left(\sigma_i^2 + \sigma_j^2\right)^{1/2}} \right| - \lambda(h),$$

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- $\phi_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h) \left\{ \sigma_i (Z_{it} - \bar{Z}_i) - \sigma_j (Z_{jt} - \bar{Z}_j) \right\}$;
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Aggregated Gaussian test statistics:

$$\Phi_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \phi^0_{ij,T}(u,h).$$

# Clustering

# Dissimilarity measure

**Underlying group structure:** there exist groups of time series $G_1, \ldots, G_N$ with $\{1, \ldots, n\} = \dot{\bigcup}_{\ell=1}^{N} G_\ell$ such that for each $1 \leq \ell \leq N$,

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**Dissimilarity measure between two sets of time series $S$ and $S'$:**

$$\widehat{\Delta}(S, S') = \max_{\substack{i \in S, \\ j \in S'}} \Big( \max_{(u,h) \in \mathcal{G}_T} \hat{\psi}^0_{ij,T}(u, h) \Big).$$

# Clustering algorithm

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$$\widehat{N} = \min\Big\{ r = 1, 2, \ldots \Big| \max_{1\le\ell\le r} \widehat{\Delta}\big(\widehat{G}_\ell^{[n-r]}, \widehat{G}_\ell^{[n-r]}\big) \le q_{n,T}(\alpha) \Big\}.$$

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**Proposition**

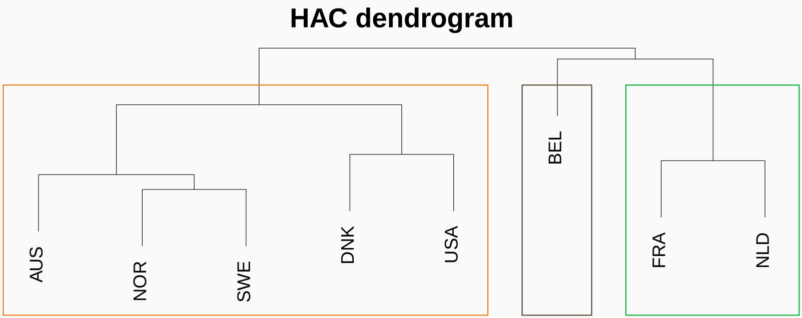
Under certain assumptions, we have

$$\mathbb{P}\Big(\big\{\widehat{G}_1, \ldots, \widehat{G}_{\widehat{N}}\big\} = \big\{G_1, \ldots, G_N\big\}\Big) \ge (1-\alpha) + o(1)$$

$$\text{and} \quad \mathbb{P}\big(\widehat{N} = N\big) \ge (1-\alpha) + o(1).$$

# Illustration

# Clustering results



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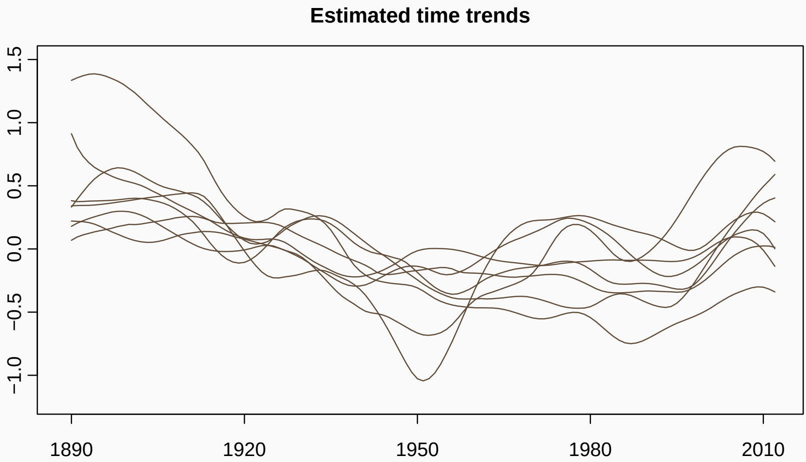

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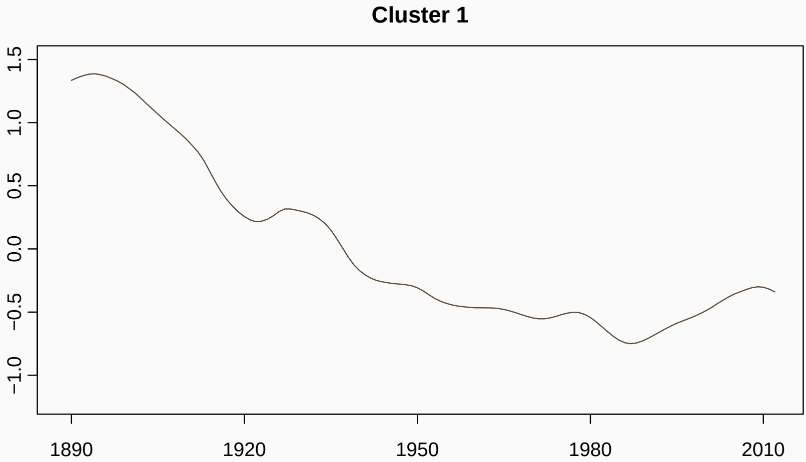

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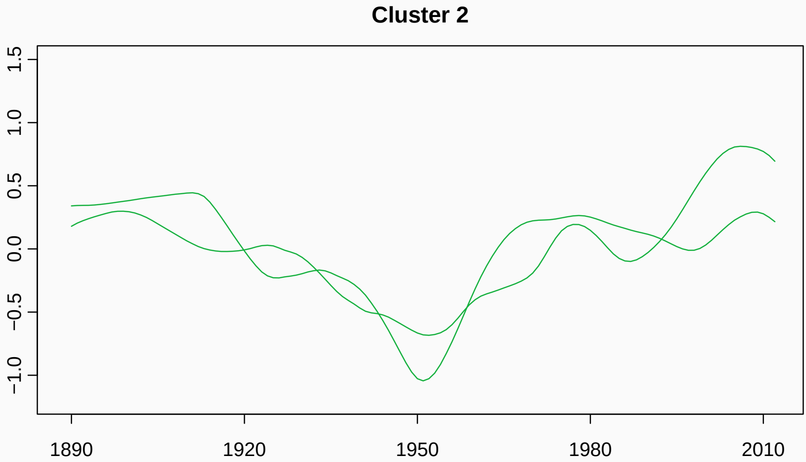

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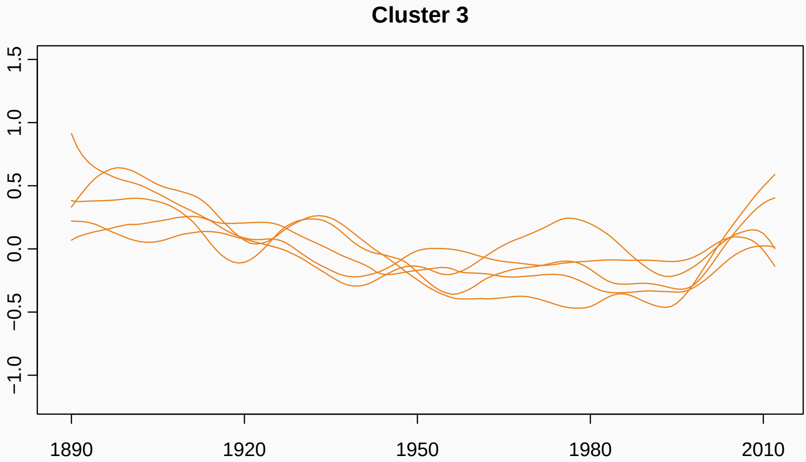

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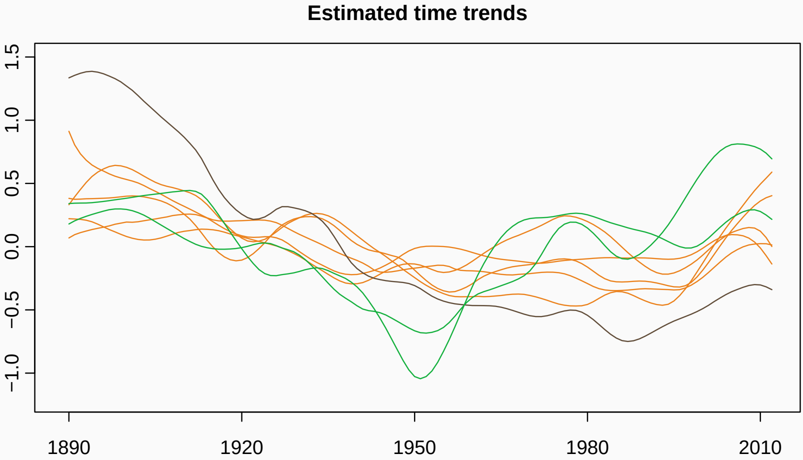

# Discussion

We can claim, with confidence of at least 95%, that the null hypothesis is violated for all intervals (and all pairs of time series) for which our test rejects the null.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Furthermore, if we reject the null, we can use the calculated test statistics as a distance measure between two time series on an interval.

Further possible extensions:

- introduce scaling factor in the trend function;
- include the dependence between covariates and error terms.

**Thank you!**

# Model, part 3

1. We estimate $\boldsymbol{\beta}_i$:

$$\widehat{\boldsymbol{\beta}}_i = \Big( \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta \boldsymbol{X}_{it}^\top \Big)^{-1} \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta Y_{it}$$

**Theorem**

Under certain regularity assumptions, $\widehat{\boldsymbol{\beta}}_i$ is a consistent estimator of $\boldsymbol{\beta}_i$ with the property $\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i = O_P(T^{-1/2})$.

2. We estimate the fixed effects $\alpha_i$:

$$\widehat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} \big( Y_{it} - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} \big)$$

We then work with the augmented time series $\widehat{Y}_{it} = Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it}$.

# Test statistic

For a given location $u \in [0,1]$ and bandwidth $h$ and a given pair $(i,j)$ we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

where

$$w_{t,T}(u,h) = \frac{\Lambda_{t,T}(u,h)}{\{\sum_{t=1}^{T} \Lambda_{t,T}^2(u,h)\}^{1/2}},$$

$$\Lambda_{t,T}(u,h) = K\Big(\frac{t/T-u}{h}\Big)\Big[S_{T,2}(u,h) - S_{T,1}(u,h)\Big(\frac{t/T-u}{h}\Big)\Big],$$

$$S_{T,\ell}(u,h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T-u}{h}\Big)\Big(\frac{t/T-u}{h}\Big)^{\ell}$$

for $\ell = 1,2$ and $K$ is a kernel function.

# Assumptions

$\mathcal{C}1$ For all $i$ it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.

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- $\mathcal{C}1$ For all $i$ it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.
- $\mathcal{C}2$ For each $i$ the variables $\varepsilon_{it}$ are weakly dependent. Details
- $\mathcal{C}3$ For each $i$ we have that $\boldsymbol{X}_{it}$ is stationary and causal with all the necessary moments and no asymptotic multicollinearity.

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$\mathcal{C}9$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

$\mathcal{C}10$ Assume that $\sigma_i^2 = \sigma_j^2$ for all $i, j$ and $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(\sqrt{h_{\min}}/\log T)$.

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- Show that

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# Idea behind $\lambda(h)$

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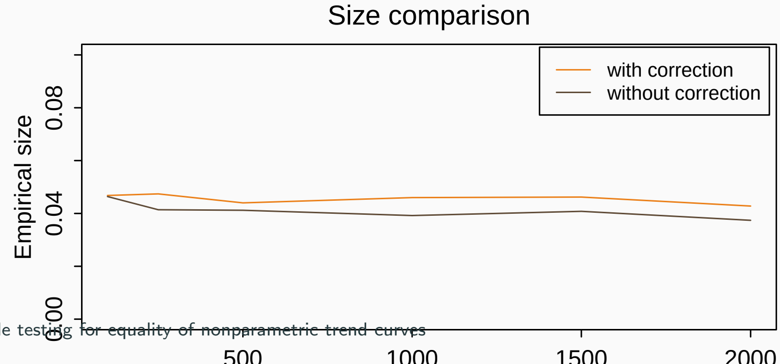
Introduction of a scale-dependent parameter helps us balance the significance of hypotheses between the time intervals of different lengths $h_k$:



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Then we can rewrite the uncorrected test statistic as

$$\Phi^{\text{uncor}} = \max_{i,j} \max_{\substack{1 \leq l \leq L, \\ 1 \leq m \leq 1/h_l}} \Big| \frac{1}{\sqrt{2Th_l}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in [(m-1)h_l, mh_l]\Big)(Z_{it} - Z_{jt})\Big|$$

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$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of $\Phi^{\text{uncor}}$ is dominated by the elements with small bandwidths $h_l$. Go back

# Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\boldsymbol{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\boldsymbol{L}, t) = ||\boldsymbol{L}(\mathcal{F}_t) - \boldsymbol{L}(\mathcal{F}_t')||_q,$$

where $\mathcal{F}_t = (\ldots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}_t' = (\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of $\mathcal{F}_t$ with $\epsilon_0'$ being an i.i.d. copy of $\epsilon_0$.

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Intuitively, $\delta_q(\boldsymbol{L}, t)$ measures the dependency of $\boldsymbol{L}(\mathcal{F}_t)$ on $\epsilon_0$, i.e., how replacing $\epsilon_0$ by an i.i.d. copy while keeping all other innovations in place affects the output $\boldsymbol{L}(\mathcal{F}_t)$.

# Technical assumptions

$\mathcal{C}1'$ The variables $\varepsilon_{it}$ are independent across $i$ and allow for the representation $\varepsilon_{it} = G_i(\ldots, \eta_{it-1}, \eta_{it})$, where $\eta_{it}$ are i.i.d. random variables across $t$ and $G_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function..

$\mathcal{C}1''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each $i$ it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q$. Go back

# Technical assumptions, part 2

- $\mathcal{C}3'$ $\boldsymbol{X}_{it}$ allow for the representation $\boldsymbol{X}_{it} = \boldsymbol{H}_i(\ldots, u_{it-1}, u_{it})$ with $u_{it}$ being i.i.d. random variables and $\boldsymbol{H}_i := (H_{i1}, H_{i2}, \ldots, H_{id})^\top : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^d$ being a measurable function such that $\boldsymbol{H}_i(\mathcal{U}_{it})$ is well defined.
- $\mathcal{C}3''$ Let $\boldsymbol{N}_i$ be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being $kl$-th entry. We assume that the smallest eigenvalue of $\boldsymbol{N}_i$ is strictly bigger than 0.
- $\mathcal{C}3'''$ Let $\mathbb{E}[\boldsymbol{H}_i(\mathcal{U}_{i0})] = 0$ and $||\boldsymbol{H}_i(\mathcal{U}_{it})||_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where $\theta$ will be introduced further.
- $\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) < \infty$ for $q'$ from Assumption $\mathcal{C}3'''$.
- $\mathcal{C}4''$ For each $i$ it holds that $\sum_{s=t}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) = O(t^{-\alpha})$ for $q'$ from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. Go back

# Technical assumptions, part 3

$\mathcal{C}6$ Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Denote $\mathcal{I}_{it} = (\ldots, \zeta_{i,t-1}, \zeta_{i,t})$, $\mathcal{J}_{it} = (\ldots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$, $\mathcal{U}_{it} = (\ldots, u_{it-1}, u_{it})$, and $\mathsf{U}_i(\mathcal{I}_{it}) = \boldsymbol{H}_i(\mathcal{U}_{it}) G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(\mathsf{U}_i, s) < \infty$. Go back

# Graphical representation

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But what if there are too many?

An interval $[u-h, u+h]$ is called **minimal** if the corresponding local null $H_0^{[i,j]}(u,h)$ is rejected and there is no other interval $[u'-h', u'+h']$ such that we reject $H_0^{[i,j]}(u',h')$ and $[u'-h', u'+h'] \subset [u-h, u+h]$.