# Multiscale testing for equality of nonparametric trend curves

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Workshop on panel data

Tinbergen Institute & University of Amsterdam

# Table of contents

# Introduction

# Motivation

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To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.

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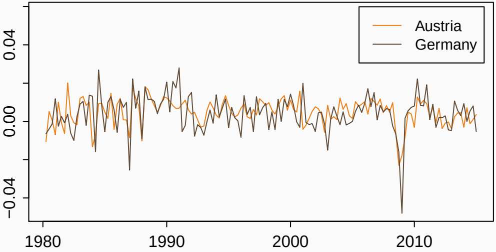

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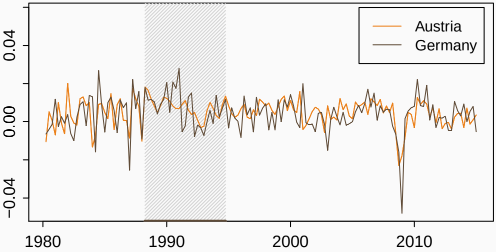

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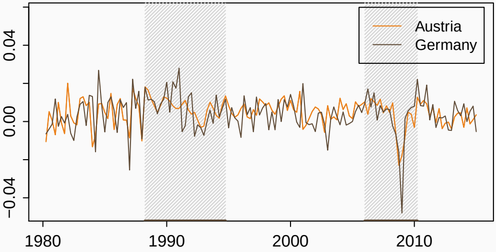

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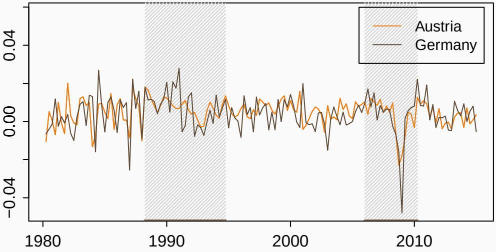


**Research question:** Out of many given intervals, how to find those where the trends are significantly different?

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Finding systematic differences between trends = basis for further research.

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**Is it limited to a particular application?**

No! Our method = general method for comparing nonparametric trends

$\Rightarrow$ new statistical test for equality of nonparametric trend curves.

# Literature

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We observe a panel of $n$ time series $\mathcal{Z}_i = \{(Y_{it}, \boldsymbol{X}_{it}) : 1 \leq t \leq T\}$ of length $T$, where $Y_{it} \in \mathbb{R}$ and $\boldsymbol{X}_{it} \in \mathbb{R}^d$.

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- $\alpha_i$ are so-called fixed effect error terms;
- $\mathcal{E}_i = \{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

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But given $\widehat{\alpha}_i$ and $\widehat{\boldsymbol{\beta}}_i$, we can consider

$$\widehat{Y}_{it} := Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} = (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \boldsymbol{X}_{it} + m_i\Big(\frac{t}{T}\Big) + \big(\alpha_i - \widehat{\alpha}_i\big) + \varepsilon_{it}.$$

# Model, part 3

1. We estimate $\boldsymbol{\beta}_i$:

$$\widehat{\boldsymbol{\beta}}_i = \Big( \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta \boldsymbol{X}_{it}^\top \Big)^{-1} \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta Y_{it}$$

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**Theorem**

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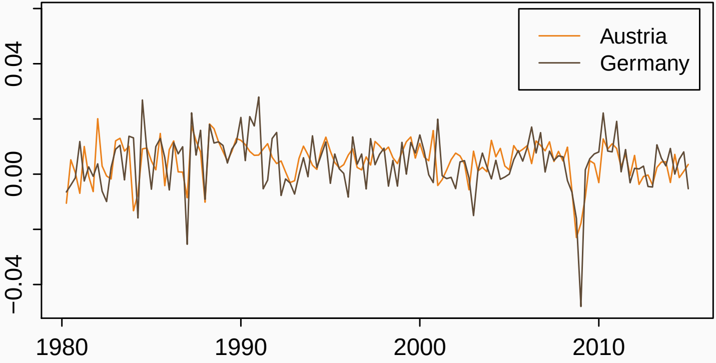
We then work with the augmented time series $\widehat{Y}_{it} = Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it}$.

# Original time series: Austria and Germany



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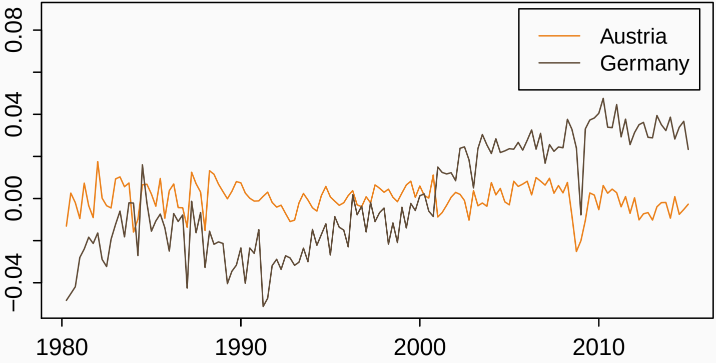

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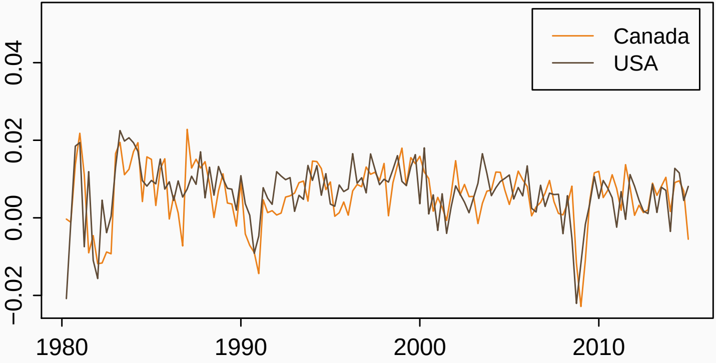

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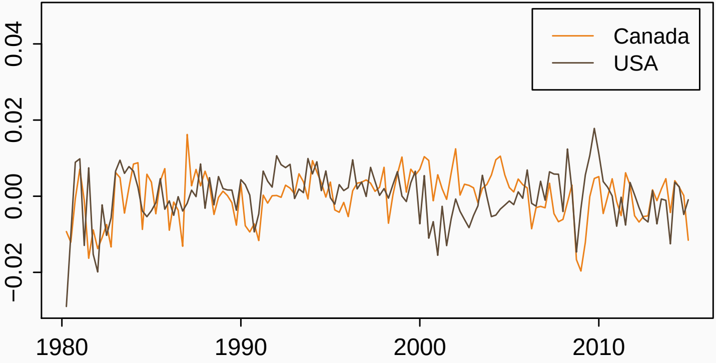

# Testing procedure

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$$H_0^{[i,j]}(u, h) : m_i(w) = m_j(w) \text{ for all } w \in [u - h, u + h].$$

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Then the global null $H_0 : m_1 = m_2 = \ldots = m_n$ can be reformulated as

$$H_0 : \text{The hypotheses } H_0^{[i,j]}(u, h) \text{ hold true for all intervals } [u - h, u + h], (u, h) \in \mathcal{G}_T, \text{ and for all } 1 \leq i < j \leq n.$$

# Test statistic

For a given location $u \in [0, 1]$ and bandwidth $h$ and a given pair $(i, j)$ we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u, h) = \sum_{t=1}^{T} w_{t,T}(u, h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

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where

$$w_{t,T}(u,h) = \frac{\Lambda_{t,T}(u,h)}{\{\sum_{t=1}^{T} \Lambda_{t,T}^2(u,h)\}^{1/2}},$$

$$\Lambda_{t,T}(u,h) = K\Big(\frac{t/T-u}{h}\Big)\Big[S_{T,2}(u,h) - S_{T,1}(u,h)\Big(\frac{t/T-u}{h}\Big)\Big],$$

$$S_{T,\ell}(u,h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T-u}{h}\Big)\Big(\frac{t/T-u}{h}\Big)^{\ell}$$

for $\ell = 1,2$ and $K$ is a kernel function.

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$$\widehat{\psi}^0_{ij,T}(u,h) = \left\{ \left| \frac{\widehat{\psi}_{ij,T}(u,h)}{\left(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\right)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\widehat{\sigma}_i^2$ is an appropriate estimator of the long-run variance $\sigma_i^2$;
- $\lambda(h) = \sqrt{2 \log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

# Test statistic, part 3

To test the global null, we aggregate the individual test statistics for all $(i, j)$ and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\widehat{\Psi}_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \widehat{\psi}^0_{ij,T}(u, h).$$

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**Main theoretical result**

Under certain conditions and under the null, $\widehat{\psi}^0_{ij,T}(u, h)$ and $\widehat{\Psi}_{n,T}$ can be approximated by the corresponding Gaussian versions of the test statistics.

# Test procedure

Gaussian version of the individual test statistics:

$$\phi^0_{ij,T}(u,h) = \max_{(u,h)\in\mathcal{G}_T} \left\{ \left| \frac{\phi_{ij,T}(u,h)}{\left(\sigma_i^2 + \sigma_j^2\right)^{1/2}} \right| - \lambda(h) \right\},$$

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Aggregated Gaussian test statistics:

$$\Phi_{n,T} = \max_{1\le i<j\le n} \max_{(u,h)\in\mathcal{G}_T} \phi^0_{ij,T}(u,h).$$

# Test procedure, part 2

1. Consider the Gaussian test statistic

$$\Phi_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \phi^0_{ij,T}(u,h),$$

where $\phi^0_{ij,T}$ are weighted averages of the differences of standard normal random variables.

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3. Perform the test for the global hypothesis $H_0$: reject $H_0$ if $\widehat{\Psi}_{n,T} > q_{n,T}(\alpha)$.
4. For each $i, j$, and each $(u,h) \in \mathcal{G}_T$, carry out the test for the local null hypothesis $H_0^{[i,j]}(u,h)$: reject $H_0^{[i,j]}(u,h)$ if $\widehat{\psi}^0_{ij,T}(u,h) > q_{n,T}(\alpha)$.

# Theoretical properties

# Assumptions

$\mathcal{C}1$ For all $i$ it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.

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$\mathcal{C}2$ For each $i$ the variables $\varepsilon_{it}$ are weakly dependent. Details

$\mathcal{C}3$ For each $i$ we have that $\boldsymbol{X}_{it}$ is stationary and causal with all the necessary moments and no asymptotic multicollinearity.

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$\mathcal{C}9$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

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# Theoretical properties

**Proposition**

*Under* $\mathcal{C}1 - \mathcal{C}10$ *and under the null, it holds that*

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**Proposition**

*Consider a sequence of functions $m_i = m_{i,T}$, $m_j = m_{j,T}$ such that*

$$\exists\,(u,h) \in \mathcal{G}_T : m_i(w) - m_j(w) \geq c_T \sqrt{\log T/(Th)}\ \forall w \in [u-h, u+h],$$

*and $c_T \to \infty$. Then under $\mathcal{C}1 - \mathcal{C}10$, it holds that*

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# Illustration

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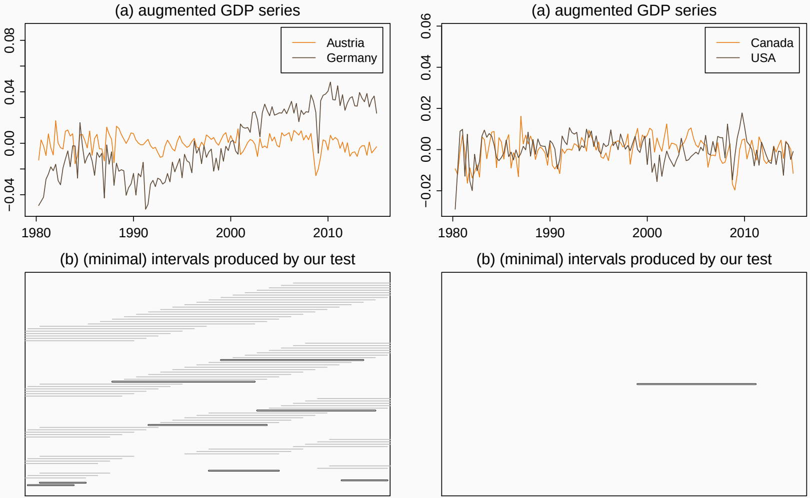
But what if there are too many?

An interval $[u-h, u+h]$ is called **minimal** if the corresponding local null $H_0^{[i,j]}(u,h)$ is rejected and there is no other interval $[u'-h', u'+h']$ such that we reject $H_0^{[i,j]}(u',h')$ and $[u'-h', u'+h'] \subset [u-h, u+h]$.

# Application results



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**Thank you!**

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- Show that

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# Idea behind $\lambda(h)$

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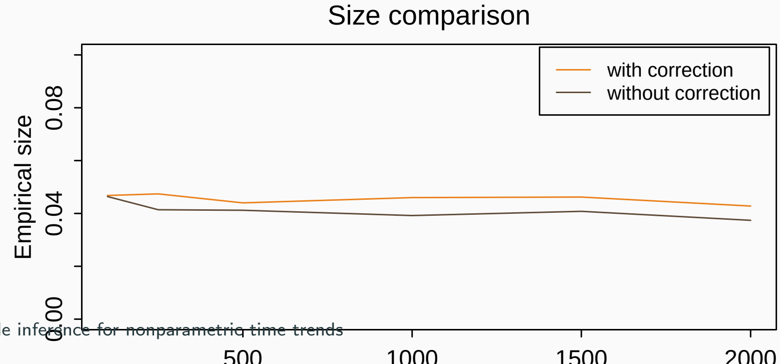
Introduction of a scale-dependent parameter helps us balance the significance of hypotheses between the time intervals of different lengths $h_k$:



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$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of $\Phi^{\text{uncor}}$ is dominated by the elements with small bandwidths $h_l$. Go back

# Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\boldsymbol{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\boldsymbol{L}, t) = ||\boldsymbol{L}(\mathcal{F}_t) - \boldsymbol{L}(\mathcal{F}_t')||_q,$$

where $\mathcal{F}_t = (\ldots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}_t' = (\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of $\mathcal{F}_t$ with $\epsilon_0'$ being an i.i.d. copy of $\epsilon_0$.

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Intuitively, $\delta_q(\boldsymbol{L}, t)$ measures the dependency of $\boldsymbol{L}(\mathcal{F}_t)$ on $\epsilon_0$, i.e., how replacing $\epsilon_0$ by an i.i.d. copy while keeping all other innovations in place affects the output $\boldsymbol{L}(\mathcal{F}_t)$.

# Technical assumptions

$\mathcal{C}1'$ The variables $\varepsilon_{it}$ are independent across $i$ and allow for the representation $\varepsilon_{it} = G_i(\ldots, \eta_{it-1}, \eta_{it})$, where $\eta_{it}$ are i.i.d. random variables across $t$ and $G_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function..

$\mathcal{C}1''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each $i$ it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q$. Go back

# Technical assumptions, part 2

- $\mathcal{C}3'$ $\boldsymbol{X}_{it}$ allow for the representation $\boldsymbol{X}_{it} = \boldsymbol{H}_i(\ldots, u_{it-1}, u_{it})$ with $u_{it}$ being i.i.d. random variables and $\boldsymbol{H}_i := (H_{i1}, H_{i2}, \ldots, H_{id})^\top : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^d$ being a measurable function such that $\boldsymbol{H}_i(\mathcal{U}_{it})$ is well defined.
- $\mathcal{C}3''$ Let $\boldsymbol{N}_i$ be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being $kl$-th entry. We assume that the smallest eigenvalue of $\boldsymbol{N}_i$ is strictly bigger than 0.
- $\mathcal{C}3'''$ Let $\mathbb{E}[\boldsymbol{H}_i(\mathcal{U}_{i0})] = 0$ and $||\boldsymbol{H}_i(\mathcal{U}_{it})||_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where $\theta$ will be introduced further.
- $\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) < \infty$ for $q'$ from Assumption $\mathcal{C}3'''$.
- $\mathcal{C}4''$ For each $i$ it holds that $\sum_{s=t}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) = O(t^{-\alpha})$ for $q'$ from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. Go back

# Technical assumptions, part 3

$\mathcal{C}6$ Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Denote $\mathcal{I}_{it} = (\ldots, \zeta_{i,t-1}, \zeta_{i,t})$, $\mathcal{J}_{it} = (\ldots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$, $\mathcal{U}_{it} = (\ldots, u_{it-1}, u_{it})$, and $\mathrm{U}_i(\mathcal{I}_{it}) = \boldsymbol{H}_i(\mathcal{U}_{it}) G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(\mathrm{U}_i, s) < \infty$. Go back