Proposition A.1. *There exists a sequence of random numbers* $\{\gamma_{n,T}\}_T$*, that converges to* 0 *as* $T \to \infty$*, such that*

$$\mathbb{P}\Big(\big|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\big| > \gamma_{n,T}\Big) = o(1). \tag{1}$$

Proof of Proposition A.1. Straightforward calculations yield that

$$\begin{aligned}
\big|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\big| &\leq \max_{1\leq i<j\leq n} \max_{(u,h)\in\mathcal{G}_T} \left(\left| \frac{\widehat{\widehat{\phi}}_{ij,T}(u,h)}{(\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2)^{1/2}} - \frac{\widehat{\widehat{\phi}}_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} \right| + \left| \frac{\widehat{\widehat{\phi}}_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} - \frac{\widehat{\phi}_{ij,T}(u,h)}{\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{1/2}} \right| \right) \leq \\
&\leq \max_{1\leq i<j\leq n} \left(\left| \{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \right| \max_{(u,h)\in\mathcal{G}_T} \left| \widehat{\widehat{\phi}}_{ij,T}(u,h) \right| \right) + \\
&\quad + \max_{1\leq i<j\leq n} \left(\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \max_{(u,h)\in\mathcal{G}_T} \left| \widehat{\widehat{\phi}}_{ij,T}(u,h) - \widehat{\phi}_{ij,T}(u,h) \right| \right).
\end{aligned}$$

First, consider the maximum of the kernel averages $\left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right|$:

$$\begin{aligned}
\max_{1\leq i<j\leq n} \max_{(u,h)\in\mathcal{G}_T} \left|\widehat{\widehat{\phi}}_{ij,T}(u,h)\right| &= \max_{1\leq i<j\leq n} \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^T w_{t,T}(u,h) \big\{ (\varepsilon_{it} - \bar{\varepsilon}_i) - (\varepsilon_{jt} - \bar{\varepsilon}_j) \big\} \right| \leq \\
&\leq 2 \max_{1\leq i\leq n} \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^T w_{t,T}(u,h)(\varepsilon_{it} - \bar{\varepsilon}_i) \right|
\end{aligned}$$

Then, consider the difference of the kernel averages:

$$\begin{aligned}
\left|\widehat{\widehat{\phi}}_{ij,T}(u,h) - \widehat{\phi}_{ij,T}(u,h)\right| &= \Big| \sum_{t=1}^T w_{t,T}(u,h) \big\{ (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top (\mathbf{X}_{it} - \bar{\mathbf{X}}_i) - (\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j)^\top (\mathbf{X}_{jt} - \bar{\mathbf{X}}_j) \big\} \Big| \leq \\
&\leq \Big| \sum_{t=1}^T w_{t,T}(u,h) (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top (\mathbf{X}_{it} - \bar{\mathbf{X}}_i) \Big| + \Big| \sum_{t=1}^T w_{t,T}(u,h) (\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j)^\top (\mathbf{X}_{jt} - \bar{\mathbf{X}}_j) \Big| = \\
&= |\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top \Big| \sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{it} - \bar{\mathbf{X}}_i) \Big| + |\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j|^\top \Big| \sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{jt} - \bar{\mathbf{X}}_j) \Big| = \\
&= |\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top \Big| \sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{it} - \bar{\mathbf{X}}_i) \Big| + |\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j|^\top \Big| \sum_{t=1}^T w_{t,T}(u,h)(\mathbf{X}_{jt} - \bar{\mathbf{X}}_j) \Big| \leq \\
&\leq |\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top \Big| \sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{it} \Big| + \big| (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \bar{\mathbf{X}}_i \big| \Big| \sum_{t=1}^T w_{t,T}(u,h) \Big| + \\
&\quad + |\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j|^\top \Big| \sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{jt} \Big| + \big| (\boldsymbol{\beta}_j - \widehat{\boldsymbol{\beta}}_j)^\top \bar{\mathbf{X}}_j \big| \Big| \sum_{t=1}^T w_{t,T}(u,h) \Big|
\end{aligned}$$

Hence,

$$
\begin{aligned}
\big|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\big| \leq & 2 \max_{1\leq i<j\leq n} \big|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\big| \max_{1\leq i\leq n} \max_{(u,h)\in\mathcal{G}_T} \left|\sum_{t=1}^T w_{t,T}(u,h)(\varepsilon_{it} - \bar{\varepsilon}_i)\right| + \\
& + 2 \max_{1\leq i<j\leq n} \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \max_{1\leq i\leq n} \left(|\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i|^\top \max_{(u,h)\in\mathcal{G}_T} \Big|\sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{it}\Big|\right) + \\
& + 2 \max_{1\leq i<j\leq n} \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \max_{1\leq i\leq n} \big|(\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \bar{\mathbf{X}}_i\big| \max_{(u,h)\in\mathcal{G}_T} \Big|\sum_{t=1}^T w_{t,T}(u,h)\Big|
\end{aligned} \tag{2}
$$

We start by evaluating the second summand in (2).
First, by our assumptions $\widehat{\sigma}_i^2 = \sigma_i^2 + o_P(\rho_T)$. Moreover, for all $i \in \{1, \ldots, n\}$ we know $\sigma_i^2 \neq 0$. Hence, $\max_{1\leq i<j\leq n}\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} = O_P(1)$.
Then, by Theorem **??**, we know that $|\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i| = O_P(1/\sqrt{T})$.
Now consider the term $\big|\sum_{t=1}^T w_{t,T}(u,h)\mathbf{X}_{it}\big|$. Without loss of generality, we can regard the covariates $\mathbf{X}_{it}$ to be scalars X_{it}, not vectors. The proof in case of vectors proceeds analogously.
By construction the weights $w_{t,T}(u,h)$ are not equal to 0 if and only if $T(u-h) \leq t \leq T(u+h)$. We can use this fact to rewrite

$$
\Big|\sum_{t=1}^T w_{t,T}(u,h)X_{it}\Big| = \Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\Big|.
$$

We want to show that

$$
\max_{(u,h)\in\mathcal{G}_T} \Big|\sum_{t=1}^T w_{t,T}(u,h)X_{it}\Big| = \max_{(u,h)\in\mathcal{G}_T} \Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\Big| = o_P(\sqrt{T}). \tag{3}
$$

Note that

$$
\begin{aligned}
\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}^2(u,h) = \sum_{t=1}^T w_{t,T}^2(u,h) = \\
&= \sum_{t=1}^T \frac{K^2\Big(\frac{\frac{t}{T}-u}{h}\Big)\Big[S_{T,2}(u,h) - \Big(\frac{\frac{t}{T}-u}{h}\Big)S_{T,1}(u,h)\Big]^2}{\Big\{\sum_{s=1}^T K^2\Big(\frac{\frac{s}{T}-u}{h}\Big)\Big[S_{T,2}(u,h) - \Big(\frac{\frac{s}{T}-u}{h}\Big)S_{T,1}(u,h)\Big]^2\Big\}} = \\
&= 1.
\end{aligned}
$$

Denoting by $D_{T,u,h}$ the number of integers between $\lfloor T(u-h)\rfloor$ and $\lceil T(u+h)\rceil$ incl. (with obvious bounds $2Th \leq D_{T,u,h} \leq 2Th + 2$), we can normalize the weights as follows:

$$
\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} \big(\sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)\big)^2 = D_{T,u,h}.
$$

According to Theorem ?? (Theorem 2(ii) in ?), if we denote the weights from the theorem as $a_t = \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)$, we can bound the following probability:

$$
\begin{aligned}
&\mathbb{P}\left(\Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)X_{it}\Big| \geq x\right) \leq \\
&\leq C_1 \frac{\big(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} |\sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)|^{q'}\big)||X_{i\cdot}||_{q',\alpha}^{q'}}{x^{q'}} + C_2 \exp\left(-\frac{C_3 x^2}{D_{T,u,h}||X_{i\cdot}||_{2,\alpha}^2}\right) = \\
&= C_1 \frac{(\sqrt{D_{T,u,h}})^{q'}\big(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} |w_{t,T}(u,h)|^{q'}\big)||X_{i\cdot}||_{q',\alpha}^{q'}}{x^{q'}} + C_2 \exp\left(-\frac{C_3 x^2}{D_{T,u,h}||X_{i\cdot}||_{2,\alpha}^2}\right)
\end{aligned}
$$

We want to prove (3). For that, take any $\delta > 0$:

$$
\begin{aligned}
&\mathbb{P}\left(\frac{\max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\Big|}{\sqrt{T}} \geq \delta\right) = \\
&= \mathbb{P}\left(\max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\Big| \geq \delta\sqrt{T}\right) \leq \\
&\overset{\text{Boole's inequality}}{\leq} \sum_{(u,h)\in\mathcal{G}_T} \mathbb{P}\left(\Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\Big| \geq \delta\sqrt{T}\right) = \\
&\overset{\text{``normalisation''}}{=} \sum_{(u,h)\in\mathcal{G}_T} \mathbb{P}\left(\Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)X_{it}\Big| \geq \delta\sqrt{D_{T,u,h}T}\right) \leq \\
&\overset{\text{Wu's Theorem}}{\leq} \sum_{(u,h)\in\mathcal{G}_T}\left[C_1 \frac{(\sqrt{D_{T,u,h}})^{q'}\big(\sum |w_{t,T}(u,h)|^{q'}\big)||X_{i\cdot}||_{q',\alpha}^{q'}}{\big(\delta\sqrt{D_{T,u,h}T}\big)^{q'}} + C_2\exp\left(-\frac{C_3\big(\delta\sqrt{D_{T,u,h}T}\big)^2}{D_{T,u,h}||X_{i\cdot}||_{2,\alpha}^2}\right)\right] = \\
&\overset{\text{simplification}}{=} \sum_{(u,h)\in\mathcal{G}_T}\left[C_1 \frac{\big(\sum |w_{t,T}(u,h)|^{q'}\big)||X_{i\cdot}||_{q',\alpha}^{q'}}{\big(\delta\sqrt{T}\big)^{q'}} + C_2\exp\left(-\frac{C_3\delta^2 T}{||X_{i\cdot}||_{2,\alpha}^2}\right)\right] \leq \\
&\leq C_1 \frac{T^\theta ||X_{i\cdot}||_{q',\alpha}^{q'}}{T^{q'/2}\cdot\delta^{q'}} \max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} |w_{t,T}(u,h)|^{q'}\right) + C_2 T^\theta \exp\left(-\frac{C_3\delta^2 T}{||X_{i\cdot}||_{2,\alpha}^2}\right) = \\
&= C\frac{T^{\theta - q'/2}}{\delta^{q'}} + CT^\theta\exp\big(-CT\delta^2\big).
\end{aligned}
$$

where the symbol C denotes a universal real constant that does not depend neither on T nor on δ and that takes a different value on each occurrence. Here in the last equality we used the following facts:

1. $||X_{i\cdot}||_{q',\alpha}^{q'} = \sup_{t\geq 0}(t+1)^\alpha \sum_{s=t}^\infty \delta_{q'}(H_i, s) < \infty$ holds true since $\sum_{s=t}^\infty \delta_{q'}(H_i,s) = O(t^{-\alpha})$ by Assumption ??;

2. $\max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} |w_{t,T}(u,h)|^{q'}\right) < \infty$ because for every $x \in [0,1]$ we have $0 \leq |x|^{q'/2} \leq x \leq 1$. Thus, since $\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}^2(u,h) = 1$, we have

$0 \le w_{t,T}^2(u,h) \le 1$ for all t and

$$0 \le |w_{t,T}(u,h)|^{q'} = |w_{t,T}^2(u,h)|^{q'/2} \le w_{t,T}^2(u,h) \le 1.$$

This leads us to a bound:

$$\max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} |w_{t,T}(u,h)|^{q'}\right) \le \max_{(u,h)\in\mathcal{G}_T}\left(\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} |w_{t,T}(u,h)|^{2}\right) = 1 < \infty.$$

3. $||X_{i\cdot}||_{2,\alpha}^2 < \infty$ (follows from 1).

By Assumption **??**, $\theta - q'/2 < 0$ and the term on the RHS of the above inequality is converging to 0 as $T \to \infty$ for any fixed $\delta > 0$. Hence,

$$\max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)X_{it}\Big| = o_P(\sqrt{T}).$$

Combining the results above, we get the following:

$$\begin{aligned}&2\max_{1\le i<j\le n}\{\widehat{\sigma}_i^2+\widehat{\sigma}_j^2\}^{-1/2}\max_{1\le i\le n}\left(|\boldsymbol{\beta}_i-\widehat{\boldsymbol{\beta}}_i|^\top \max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=1}^{T} w_{t,T}(u,h)\mathbf{X}_{it}\Big|\right) = \\ &= O_P(1)\cdot O_P(1/\sqrt{T})\cdot o_P(\sqrt{T}) = o_P(1).\end{aligned} \tag{4}$$

Now, consider the third summand in (2).
Similarly as before, $\max_{1\le i<j\le n}\{\widehat{\sigma}_i^2+\widehat{\sigma}_j^2\}^{-1/2} = O_P(1)$ and $|\boldsymbol{\beta}_i-\widehat{\boldsymbol{\beta}}_i| = O_P(1/\sqrt{T})$.
Then, by Proposition **??** $\bar{\mathbf{X}}_i = o_P(1)$.
Finally, consider the local linear kernel weights $w_{t,T}(u,h)$ defined in (**??**). Again, by construction the weights $w_{t,T}(u,h)$ are not equal to 0 if and only if $T(u-h)\le t\le T(u+h)$. We can use this fact to bound $\max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=1}^T w_{t,T}(u,h)\Big|$ using the Cauchy-Schwarz inequality:

$$\begin{aligned}\Big|\sum_{t=1}^{T} w_{t,T}(u,h)\Big| &= \left|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)\right| = \left|\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)\cdot 1\right| \le \\ &\le \sqrt{\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}^2(u,h)}\sqrt{\sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} 1^2} = \\ &= \sqrt{1}\cdot\sqrt{D_{T,u,h}} \le \sqrt{2Th+2} \le \sqrt{2Th_{\max}+2} \le \sqrt{T+2}.\end{aligned}$$

Hence, $\max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=1}^T w_{t,T}(u,h)\Big| = O(\sqrt{T})$.
Combining the results above, we get the following:

$$\begin{aligned}&2\max_{1\le i<j\le n}\{\widehat{\sigma}_i^2+\widehat{\sigma}_j^2\}^{-1/2}\max_{1\le i\le n}\big|(\boldsymbol{\beta}_i-\widehat{\boldsymbol{\beta}}_i)^\top\bar{\mathbf{X}}_i\big|\max_{(u,h)\in\mathcal{G}_T}\Big|\sum_{t=1}^{T} w_{t,T}(u,h)\Big| = \\ &= O_P(1)\cdot O_P(1/\sqrt{T})\cdot o_P(1)\cdot O(\sqrt{T}) = o_P(1).\end{aligned} \tag{5}$$

Lastly, we look at the first summand in (2). Since $\widehat{\widehat{\sigma}}_i^2 = \sigma_i^2 + o_P(\rho_T)$ and $\widehat{\sigma}_i^2 = \sigma_i^2 + o_P(\rho_T)$ by our assumptions, we have that

$$\max_{1 \le i < j \le n} \left| \{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2} \right| = o_P(\rho_T).$$

Then since $\left|\sum_{t=1}^T w_{t,T}(u,h)(\varepsilon_{it} - \bar{\varepsilon}_i)\right| \le \left|\sum_{t=1}^T w_{t,T}(u,h)\varepsilon_{it}\right| + \left|\sum_{t=1}^T w_{t,T}(u,h)\bar{\varepsilon}_i\right|$ we evaluate

$$\max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \left| \sum_{t=1}^T w_{t,T}(u,h)\varepsilon_{it} \right| = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \left| \sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} w_{t,T}(u,h)\varepsilon_{it} \right|.$$

and

$$\max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \left| \sum_{t=1}^T w_{t,T}(u,h)\bar{\varepsilon}_i \right| = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \left| \bar{\varepsilon}_i \sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} w_{t,T}(u,h) \right|.$$

separately. We proceed in the same way as before.
According to Theorem **??** (Theorem 2(ii) in **?**), if we denote the weights from the theorem as $a_t = \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)$, we can bound the following probability:

$$\begin{aligned}
&\mathbb{P}\left(\Big| \sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} \sqrt{D_{T,u,h}} \cdot w_{t,T}(u,h)\varepsilon_{it} \Big| \ge x \right) \le \\
&\le C_4 \frac{(\sqrt{D_{T,u,h}})^q \big(\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} |w_{t,T}(u,h)|^q \big) ||\varepsilon_{i\cdot}||_{q,\tau_q}^q}{x^q} + C_5 \exp\left(-\frac{C_6 x^2}{D_{T,u,h} ||\varepsilon_{i\cdot}||_{2,\tau_q}^2} \right)
\end{aligned}$$

and for any $\delta > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{\max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} w_{t,T}(u,h)\varepsilon_{it} \right|}{\sqrt{T}} \ge \delta \right) \le \\
&\le C_4 \frac{T^\theta ||\varepsilon_{i\cdot}||_{q,\tau_q}^q}{T^{q/2} \cdot \delta^q} \max_{(u,h)\in\mathcal{G}_T} \left(\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} |w_{t,T}(u,h)|^q \right) + C_5 T^\theta \exp\left(-\frac{C_6 \delta^2 T}{||\varepsilon_{i\cdot}||_{2,\tau_q}^2} \right) = \\
&= C \frac{T^{\theta - q/2}}{\delta^q} + C T^\theta \exp\left(-C T \delta^2 \right).
\end{aligned}$$

where the symbol C denotes a universal real constant that does not depend neither on T nor on δ and that takes a different value on each occurrence. Here in the last equality we used the following facts:

1. $||\varepsilon_{i\cdot}||_{q,\tau_q}^q = \sup_{t \ge 0}(t+1)^{\tau_q} \sum_{s=t}^\infty \delta_q(G_i, s) = \sup_{t \ge 0}(t+1)^{\tau_q}\Theta_{i,t,q} < \infty$ holds true since $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q > 1/2 - 1/q$ by Assumption **??**;
2. $\max_{(u,h)\in\mathcal{G}_T} \left(\sum_{t=\lfloor T(u-h) \rfloor}^{\lceil T(u+h) \rceil} |w_{t,T}(u,h)|^q \right) \le 1 < \infty$ as before.

3. $||\varepsilon_{i\cdot}||^2_{2,\tau_q} < \infty$ (follows from 1).

By Assumption ??, $\theta - q/2 < 0$ and the term on the RHS of the above inequality is converging to 0 as $T \to \infty$ for any fixed $\delta > 0$. Hence,

$$\max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h)\varepsilon_{it} \right| = o_P(\sqrt{T}). \tag{6}$$

Now, consider

$$\max_{1\le i<j\le n} \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^{T} w_{t,T}(u,h)\bar{\varepsilon}_i \right| \le \max_{1\le i<j\le n} |\bar{\varepsilon}_i| \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=\lfloor T(u-h)\rfloor}^{\lceil T(u+h)\rceil} w_{t,T}(u,h) \right|.$$

Similarly as before, $\max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^{T} w_{t,T}(u,h) \right| = O(\sqrt{T})$. Then, by Proposition ?? $\bar{\varepsilon}_i = o_P(1)$.
Combining (6) and (**??**), we get the following:

$$\begin{aligned}
&\max_{1\le i<j\le n} \left(\left|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\right| \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^{T} w_{t,T}(u,h)(\varepsilon_{it} - \bar{\varepsilon}_i) \right| \right) \le \\
&\le \max_{1\le i<j\le n} \left|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\right| \max_{1\le i<j\le n} \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^{T} w_{t,T}(u,h)\varepsilon_{it} \right| + \\
&\quad + \max_{1\le i<j\le n} \left|\{\widehat{\widehat{\sigma}}_i^2 + \widehat{\widehat{\sigma}}_j^2\}^{-1/2} - \{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{-1/2}\right| \max_{1\le i<j\le n} |\bar{\varepsilon}_i| \max_{(u,h)\in\mathcal{G}_T} \left| \sum_{t=1}^{T} w_{t,T}(u,h) \right| = \\
&= o_P(\rho_T) \cdot o_P(\sqrt{T}) + o_P(\rho_T) \cdot o_P(1) \cdot O(\sqrt{T}) = \\
&= o_P(\rho_T\sqrt{T}).
\end{aligned} \tag{7}$$

Plugging (4), (5) and (7) in (2), we get that $\left|\widehat{\widehat{\Phi}}_{n,T} - \widehat{\Phi}_{n,T}\right| = o_P(1)$ and the statement of the theorem follows. $\square$