Multiscale Inference in Nonparametric regression with Time Series Errors

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Introduction

Motivation

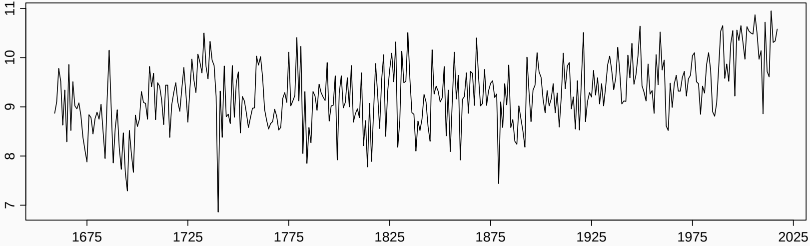

Figure 1: Yearly mean temperature in Central England from 1659 to 2017

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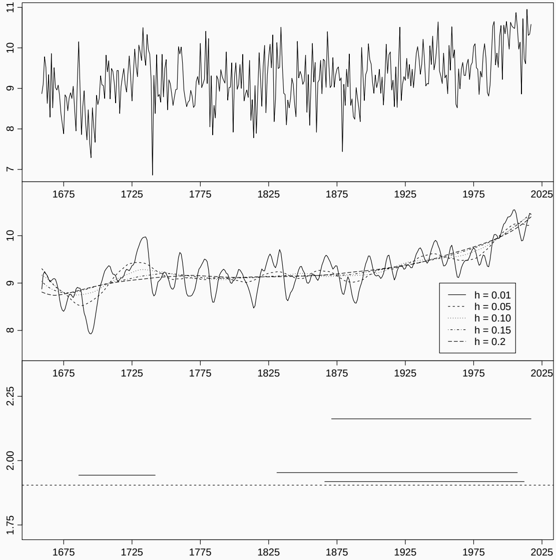
11
10
9
8
7
1675
1725
1775
1825
1875
1925
1975
2025
10
9
8
h = 0.01
h = 0.05
h = 0.10
h = 0.15
h = 0.2
1675
1725
1775
1825
1875
1925
1975
2025
2.25
2.00
1.75
1675
1725
1775
1825
1875
1925
1975
2025

Model

Model

We observe a single time series $\{Y_t : 1 \leq t \leq T\}$ of length T. The observations come from the following model:

$$Y_t = m\Big(\frac{t}{T}\Big) + \varepsilon_t$$

- m is an unknown trend function on $[0, 1]$;
- $\{\varepsilon_t : 1 \leq t \leq T\}$ is a zero-mean stationary error process.

Literature

Multiscale approaches for independent data

- SiZer method (Chaudhuri and Marron, 1999, 2000)
- Testing monotonicity of the trend function (Hall and Heckman, 2000)
- Testing qualitative hypotheses (Dümbgen and Spokoiny, 2001)

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Multiscale methods for dependent data

- Extensions to SiZer method (Park et al. 2004, 2009, Rondonotti et al. 2007)

The multiscale method

Testing

Testing problem:

$$\begin{aligned} H_0 :& \quad m' = 0 \\ H_1 :& \quad m' \neq 0 \end{aligned}$$

Test Statistic

For a given location $u \in [0, 1]$ and bandwidth h we construct the kernel averages

$$\widehat{\psi}_T(u, h) = \sum_{t=1}^{T} w_{t,T}(u, h) Y_t,$$

where

$$w_{t,T}(u, h) = \frac{\Lambda_{t,T}(u, h)}{\{\sum_{t=1}^{T} \Lambda_{t,T}(u, h)^2\}^{1/2}},$$

$$\Lambda_{t,T}(u, h) = K\Big(\frac{t/T - u}{h}\Big)\Big[S_{T,0}(u, h)\Big(\frac{t/T - u}{h}\Big) - S_{T,1}(u, h)\Big]$$

$$S_{T,\ell}(u, h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T - u}{h}\Big)\Big(\frac{t/T - u}{h}\Big)^{\ell}$$

for $\ell = 0, 1, 2$ and K is a kernel function.

Test Statistic

Test statistic is defined as follows

$$\widehat{\Psi}_T = \max_{(u,h) \in \mathcal{G}_T} \left\{ \left| \frac{\widehat{\psi}_T(u,h)}{\widehat{\sigma}} \right| - \lambda(h) \right\},$$

where

- $\lambda(h) = \sqrt{2 \log\{1/(2h)\}}$ is an additive correction term; Idea

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- $\mathcal{G}_T$ is the set of points (u, h) that are taken into consideration;
- $\widehat{\sigma}^2$ is an appropriate estimator of the long-run variance σ^2.

Test procedure

Gaussian version of the test statistic:

$$\Phi_T = \max_{(u,h)\in\mathcal{G}_T} \left\{ \left| \frac{\phi_T(u,h)}{\sigma} \right| - \lambda(h) \right\},$$

where

- $\phi_T(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\,\sigma Z_t$;
- Z_t are independent standard normal random variables;
- $q_T(\alpha)$ is $(1-\alpha)$ quantile of Φ_T.

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Test procedure

For a given significance level $\alpha \in (0,1)$, we reject H_0 if $\widehat{\Psi}_T > q_T(\alpha)$.

Theoretical properties

Assumptions

$\mathcal{C}1$ The variables ε_t are weakly dependent.

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$\mathcal{C}5$ $|\mathcal{G}_T| = O(T^\theta)$ for some arbitrarily large but fixed constant $\theta > 0$.

$$\mathcal{G}_T = \Big\{(u,h) : u = t/T \text{ for some } 1 \leq t \leq T \text{ and } h \in [h_{\min}, h_{\max}] \\ \text{with } h = t/T \text{ for some } 1 \leq t \leq T\Big\},$$

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$\mathcal{C}5$ $|\mathcal{G}_T| = O(T^\theta)$ for some arbitrarily large but fixed constant $\theta > 0$.

$\mathcal{C}6$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} = o(1)$.

Theoretical properties

Proposition

Under our assumptions and under $H_0 : m' = 0$ it holds that

$$\mathrm{P}\big(\widehat{\Psi}_T \le q_T(\alpha)\big) = (1-\alpha) + o(1).$$

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Proposition

Under our assumptions and under local alternatives, we have

$$\mathrm{P}\big(\widehat{\Psi}_T \le q_T(\alpha)\big) = o(1).$$

Strategy of the proof

- Replace the statistic $\widehat{\Psi}_T$ under $H_0 : m = 0$ by a statistic $\widetilde{\Phi}_T$ with the same distribution and the property that

$$\left|\widetilde{\Phi}_T - \Phi_T\right| = o_p(\delta_T),$$

where $\delta_T = o(1)$. To do so, we make use of strong approximation theory for dependent processes as derived in Berkes et al. (2014)

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- Using the anti-concentration results for Gaussian random vectors (Chernozhukov et al. 2015), prove that Φ_T does not concentrate too strongly in small regions of the form $[x - \delta_T, x + \delta_T]$, i.e.

$$\sup_{x \in \mathbb{R}} \mathrm{P}\big(|\Phi_T - x| \le \delta_T\big) = o(1).$$

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$$\sup_{x \in \mathbb{R}} \mathrm{P}\Big(|\Phi_T - x| \leq \delta_T\Big) = o(1).$$

- Show that

$$\sup_{x \in \mathbb{R}} \left|\mathrm{P}(\widetilde{\Phi}_T \leq x) - \mathrm{P}(\Phi_T \leq x)\right| = o(1).$$

Theoretical properties

Define

$$\Pi_T^+ = \left\{ I_{u,h} = [u-h, u+h] : (u,h) \in \mathcal{A}_T^+ \text{ and } I_{u,h} \subseteq [0,1] \right\}$$

with

$$\mathcal{A}_T^+ = \left\{ (u,h) \in \mathcal{G}_T : \frac{\widehat{\psi}_T(u,h)}{\widehat{\sigma}} > q_T(\alpha) + \lambda(h) \right\}$$

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$$\mathcal{A}_T^+ = \left\{ (u,h) \in \mathcal{G}_T : \frac{\widehat{\psi}_T(u,h)}{\widehat{\sigma}} > q_T(\alpha) + \lambda(h) \right\}$$
$$\mathcal{A}_T^- = \left\{ (u,h) \in \mathcal{G}_T : -\frac{\widehat{\psi}_T(u,h)}{\widehat{\sigma}} > q_T(\alpha) + \lambda(h) \right\}$$

Theoretical properties

Proposition

Under our assumptions, for events $E_T^+ = \left\{ \forall I_{u,h} \in \Pi_T^+ : m'(v) > 0 \text{ for some } v \in I_{u,h} \right\}$ *it holds that*

$$\mathrm{P}\left(E_T^+\right) \geq (1 - \alpha) + o(1)$$

Theoretical properties

Proposition

Under our assumptions, for events
$E_T^+ = \left\{ \forall I_{u,h} \in \Pi_T^+ : m'(v) > 0 \textit{ for some } v \in I_{u,h} \right\}$ *and*
$E_T^- = \left\{ \forall I_{u,h} \in \Pi_T^- : m'(v) < 0 \textit{ for some } v \in I_{u,h} \right\}$ *it holds that*

$$\mathrm{P}\left(E_T^+\right) \geq (1-\alpha) + o(1)$$
$$\mathrm{P}\left(E_T^-\right) \geq (1-\alpha) + o(1)$$

Graphical representation

Minimal intervals

An interval $I_{u,h} \in \Pi_T^+$ is called **minimal** if there is no other interval $I_{u',h'} \in \Pi_T^+$ with $I_{u',h'} \subset I_{u,h}$.

Graphical representation

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Define

$$\Pi_T^{min,+} = \text{ set of minimal intervals from } \Pi_T^+,$$
$$E_T^{min,+} = \Big\{ \forall I_{u,h} \in \Pi_T^{min,+} : m'(v) > 0 \text{ for some } v \in I_{u,h} \Big\}$$

Graphical representation

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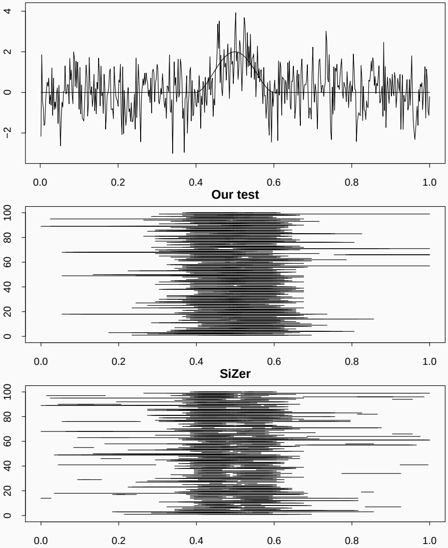
Define

$$\begin{aligned} \Pi_T^{min,+} &= \text{ set of minimal intervals from } \Pi_T^+, \\ E_T^{min,+} &= \Big\{ \forall I_{u,h} \in \Pi_T^{min,+} : m'(v) > 0 \text{ for some } v \in I_{u,h} \Big\} \end{aligned}$$

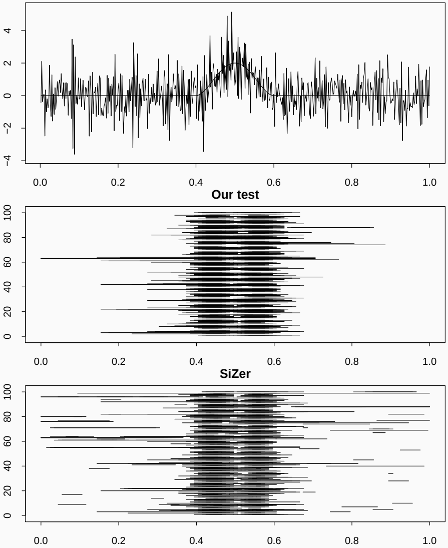
Since $E_T^{min,+} = E_T^+$, we have

$$\mathrm{P}\big(E_T^{min,+}\big) \geq (1-\alpha) + o(1).$$

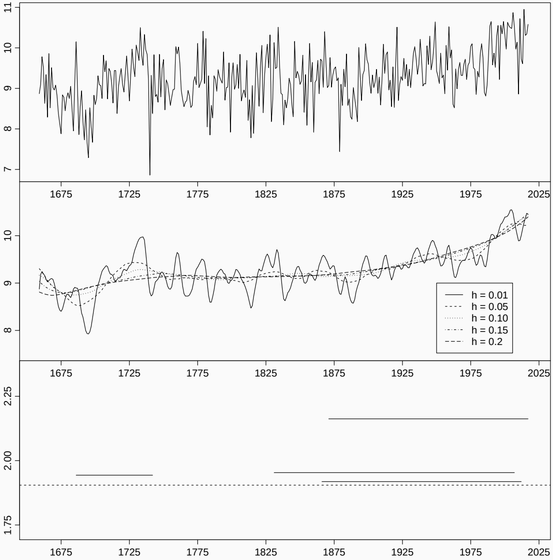
Graphical representation, $a_1 = 0.25$



Graphical representation, $a_1 = -0.5$



Application



Conclusion

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We developed multiscale methods to test qualitative hypotheses about nonparametric time trends:

- whether the trend is present at all;
- whether the trend function is constant;
- in which time regions there is an upward/downward movement in the trend.

We derived asymptotic theory for the proposed tests.

As an application of our method, we analyzed the behavior of the yearly mean temperature in Central England from 1659 to 2017.

Thank you!

Long-run error variance estimator

Setting

Estimate the long-run error variance $\sigma^2 = \sum_{\ell=-\infty}^{\infty} \mathrm{Cov}(\varepsilon_0, \varepsilon_\ell)$ of the error terms $\{\varepsilon_t\}$ in the model

$$Y_t = m\Big(\frac{t}{T}\Big) + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a stationary and causal AR(p) process of the form

$$\varepsilon_t = \sum_{j=1}^{p} a_j \varepsilon_{t-j} + \eta_t.$$

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- $\boldsymbol{a} = (a_1, \ldots, a_p)$ is a vector of the unknown parameters;
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- p is known.

Estimator, first stage

Yule-Walker equations yield

$$\boldsymbol{\Gamma}_q \boldsymbol{a} = \boldsymbol{\gamma}_q + \nu^2 \boldsymbol{c}_q,$$

where

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- $\boldsymbol{\gamma}_q = (\gamma_q(1), \dots, \gamma_q(p))^\top$ with $\gamma_q(\ell) = \mathrm{Cov}(\Delta_q \varepsilon_t, \, \Delta_q \varepsilon_{t-\ell})$;

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- and $\boldsymbol{\Gamma}_q$ is the $p \times p$ covariance matrix $\boldsymbol{\Gamma}_q = (\gamma_q(i-j) : 1 \leq i, j \leq p)$.

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Note

$\mathbf{\Gamma}_q \boldsymbol{a} \approx \boldsymbol{\gamma}_q$ for large values of q.

We construct the first-stage estimator by

$$\widetilde{\boldsymbol{a}}_q = \widehat{\mathbf{\Gamma}}_q^{-1} \widehat{\boldsymbol{\gamma}}_q,$$

where $\widehat{\mathbf{\Gamma}}_q$ and $\widehat{\boldsymbol{\gamma}}_q$ are constructed from the sample autocovariances $\widehat{\gamma}_q(\ell) = (T-q)^{-1} \sum_{t=q+\ell+1}^{T} \Delta_q Y_{t,T} \Delta_q Y_{t-\ell,T}$.

Estimator, second stage

Problem

If the trend m is pronounced, the estimator $\widetilde{\boldsymbol{a}}_q$ will have a strong bias.

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Solution:

- Compute estimators $\widetilde{c}_k$ of c_k based on $\widetilde{\boldsymbol{a}}_q$.

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If the trend m is pronounced, the estimator $\widetilde{\boldsymbol{a}}_q$ will have a strong bias.

Solution:

- Compute estimators $\widetilde{c}_k$ of c_k based on $\widetilde{\boldsymbol{a}}_q$.
- Estimate the innovation variance ν^2 by $\widetilde{\nu}^2 = (2T)^{-1} \sum_{t=p+1}^{T} \widetilde{r}_{t,T}^2$, where $\widetilde{r}_{t,T} = \Delta_1 Y_{t,T} - \sum_{j=1}^{p} \widetilde{a}_j \Delta_1 Y_{t-j,T}$.

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- Estimate $\boldsymbol{a}$ by

$$\widehat{\boldsymbol{a}}_r = \widehat{\boldsymbol{\Gamma}}_r^{-1} (\widehat{\boldsymbol{\gamma}}_r + \widetilde{\nu}^2 \widetilde{\boldsymbol{c}}_r).$$

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- Estimate $\boldsymbol{a}$ by

$$\widehat{\boldsymbol{a}}_r = \widehat{\boldsymbol{\Gamma}}_r^{-1} (\widehat{\boldsymbol{\gamma}}_r + \widetilde{\nu}^2 \widetilde{\boldsymbol{c}}_r).$$

- Average the estimators $\widehat{\boldsymbol{a}}_r$: $\widehat{\boldsymbol{a}} = \frac{1}{\bar{r}} \sum_{r=1}^{\bar{r}} \widehat{\boldsymbol{a}}_r$.

Estimator, second stage

Problem

If the trend m is pronounced, the estimator $\widetilde{\boldsymbol{a}}_q$ will have a strong bias.

Solution:

- Compute estimators $\widetilde{c}_k$ of c_k based on $\widetilde{\boldsymbol{a}}_q$.
- Estimate the innovation variance ν^2 by $\widetilde{\nu}^2 = (2T)^{-1} \sum_{t=p+1}^{T} \widetilde{r}_{t,T}^2$, where $\widetilde{r}_{t,T} = \Delta_1 Y_{t,T} - \sum_{j=1}^{p} \widetilde{a}_j \Delta_1 Y_{t-j,T}$.
- Estimate $\boldsymbol{a}$ by
$$\widehat{\boldsymbol{a}}_r = \widehat{\boldsymbol{\Gamma}}_r^{-1} (\widehat{\boldsymbol{\gamma}}_r + \widetilde{\nu}^2 \widetilde{\boldsymbol{c}}_r).$$
- Average the estimators $\widehat{\boldsymbol{a}}_r$: $\widehat{\boldsymbol{a}} = \frac{1}{\bar{r}} \sum\limits_{r=1}^{\bar{r}} \widehat{\boldsymbol{a}}_r$.
- Estimate the long-run variance σ^2 by
$$\widehat{\sigma}^2 = \frac{\widehat{\nu}^2}{(1 - \sum_{j=1}^{p} \widehat{a}_j)^2}.$$

Motivation for the estimator

If $\{\varepsilon_t\}$ is an AR(p) process, then the time series $\{\Delta_q \varepsilon_t\}$ of the differences $\Delta_q \varepsilon_t = \varepsilon_t - \varepsilon_{t-q}$ is an ARMA(p, q) process of the form

$$\Delta_q \varepsilon_t - \sum_{j=1}^{p} a_j \Delta_q \varepsilon_{t-j} = \eta_t - \eta_{t-q}.$$

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$$\Delta_q \varepsilon_t - \sum_{j=1}^{p} a_j \Delta_q \varepsilon_{t-j} = \eta_t - \eta_{t-q}.$$

Then $\Delta_q Y_{t,T} = Y_{t,T} - Y_{t-q,T}$ is approximately an ARMA(p, q) process.

Theoretical properties of the estimator

Performance:

- Our estimator $\widehat{\boldsymbol{a}}$ produces accurate estimation results even when the AR polynomial $A(z) = 1 - \sum_{j=1}^{p} a_j z^j$ has a root close to the unit circle.

Theoretical properties of the estimator

Performance:

- Our estimator $\widehat{\boldsymbol{a}}$ produces accurate estimation results even when the AR polynomial $A(z) = 1 - \sum_{j=1}^{p} a_j z^j$ has a root close to the unit circle.
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Proposition

Our estimators $\widetilde{\boldsymbol{a}}_q$, $\widehat{\boldsymbol{a}}$ and $\widehat{\sigma}^2$ are $\sqrt{T}$-consistent.

Testing for equality of the time trends

Model

We observe n time series $\mathcal{Y}_i = \{Y_{it} : 1 \leq t \leq T\}$ of length T for $1 \leq i \leq n$

$$Y_{it} = m_i\Big(\frac{t}{T}\Big) + \varepsilon_{it}$$

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where

- m_i is an unknown trend function on $[0, 1]$, that are Lipschitz continuous and normalized such that $\int_0^1 m_i(u)du = 0$;
- $\mathcal{E}_i = \{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary error process;
- $\mathcal{E}_i$ are independent across i.

Test Statistic

For a given location $u \in [0, 1]$, bandwidth h and a pair of time series i and j we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u, h) = \sum_{t=1}^{T} w_{t,T}(u, h)(Y_{it} - Y_{jt}),$$

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where

$$w_{t,T}(u,h) = \frac{\Lambda_{t,T}(u,h)}{\{\sum_{t=1}^{T} \Lambda_{t,T}^2(u,h)\}^{1/2}},$$

$$\Lambda_{t,T}(u,h) = K\Big(\frac{t/T-u}{h}\Big)\Big[S_{T,2}(u,h) - S_{T,1}(u,h)\Big(\frac{t/T-u}{h}\Big)\Big],$$

$$S_{T,\ell}(u,h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T-u}{h}\Big)\Big(\frac{t/T-u}{h}\Big)^{\ell}$$

for $\ell = 0,1,2$ and K is a kernel function.

Test Statistic

Our multiscale statistic is defined as follows

$$\widehat{\Psi}_{n,T} = \max_{1 \le i < j \le n} \widehat{\Psi}_{ij,T},$$

$$\widehat{\Psi}_{ij,T} = \max_{(u,h) \in \mathcal{G}_T} \Big\{ \Big| \frac{\widehat{\psi}_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} \Big| - \lambda(h) \Big\},$$

where

- $\lambda(h) = \sqrt{2 \log\{1/(2h)\}}$ is an additive correction term;
- $\mathcal{G}_T$ is the set of points (u, h) that are taken into consideration;
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Gaussian version of the test statistic:

$$\Phi_{n,T} = \max_{1 \le i < j \le n} \Phi_{ij,T},$$

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$$\phi_{ij,T}(u,h) = \sum\nolimits_{t=1}^{T} w_{t,T}(u,h) \Big\{ \widehat{\sigma}_i \Big(Z_{it} - \frac{1}{T} \sum_{t=1}^{T} Z_{it} \Big) - \widehat{\sigma}_j \Big(Z_{jt} - \frac{1}{T} \sum_{t=1}^{T} Z_{jt} \Big) \Big\};$$

Z_t are independent standard normal random variables;

$q_{n,T}(\alpha)$ is $(1-\alpha)$ quantile of $\Phi_{n,T}$.

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Test procedure

For a given significance level $\alpha \in (0,1)$, we reject H_0 if $\widehat{\Psi}_{n,T} > q_{n,T}(\alpha)$.

Theoretical properties

Proposition

Supose that $\mathcal{E}_i$ are independent across i and satisfy $\mathcal{C}1-\mathcal{C}2$ for each i. Under our remaining assumptions and under $H_0 : m_1 = m_2 = \ldots = m_n$ it holds that

$$\mathrm{P}\big(\widehat{\Psi}_{n,T} \le q_{n,T}(\alpha)\big) = (1-\alpha) + o(1).$$

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Proposition

Let the conditions of previous proposition be satisfied. Under local alternatives we have

$$\mathrm{P}\big(\widehat{\Psi}_{n,T} \le q_{n,T}(\alpha)\big) = o(1).$$

Clustering, group structure

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- For any $\ell \neq \ell'$, the trends $g_{\ell,T}$ and $g_{\ell',T}$ differ in the following sense: There exists $(u, h) \in \mathcal{G}_T$ with $[u - h, u + h] \subseteq [0, 1]$ such that $g_{\ell,T}(w) - g_{\ell',T}(w) \geq c_T \sqrt{\log T/(Th)}$ for all $w \in [u - h, u + h]$ or $g_{\ell',T}(w) - g_{\ell,T}(w) \geq c_T \sqrt{\log T/(Th)}$ for all $w \in [u - h, u + h]$, where $0 < c_T \to \infty$.

Clustering, algorithm

Dissimilarity measure between two sets of time series S and S':

$$\widehat{\Delta}(S, S') = \max_{\substack{i \in S, \\ j \in S'}} \widehat{\Psi}_{ij,T}.$$

Clustering algorithm

Step 0 *(Initialization):* Let $\widehat{G}_i^{[0]} = \{i\}$ denote the i-th singleton cluster for $1 \leq i \leq n$ and define $\{\widehat{G}_1^{[0]}, \dots, \widehat{G}_n^{[0]}\}$ to be the initial partition of time series into clusters.

Step r (Iteration): Let $\widehat{G}_1^{[r-1]}, \dots, \widehat{G}_{n-(r-1)}^{[r-1]}$ be the $n-(r-1)$ clusters from the previous step. Determine the pair of clusters $\widehat{G}_\ell^{[r-1]}$ and $\widehat{G}_{\ell'}^{[r-1]}$ for which

$$\widehat{\Delta}(\widehat{G}_\ell^{[r-1]}, \widehat{G}_{\ell'}^{[r-1]}) = \min_{1 \leq k < k' \leq n-(r-1)} \widehat{\Delta}(\widehat{G}_k^{[r-1]}, \widehat{G}_{k'}^{[r-1]})$$

and merge them into a new cluster.

Clustering, theoretical properties

The estimator of the number of groups is

$$\widehat{N} = \min\Big\{ r = 1, 2, \ldots \,\Big|\, \max_{1 \le \ell \le r} \widehat{\Delta}\big(\widehat{G}_{\ell}^{[n-r]}\big) \le q_{n,T}(\alpha) \Big\}.$$

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Proposition

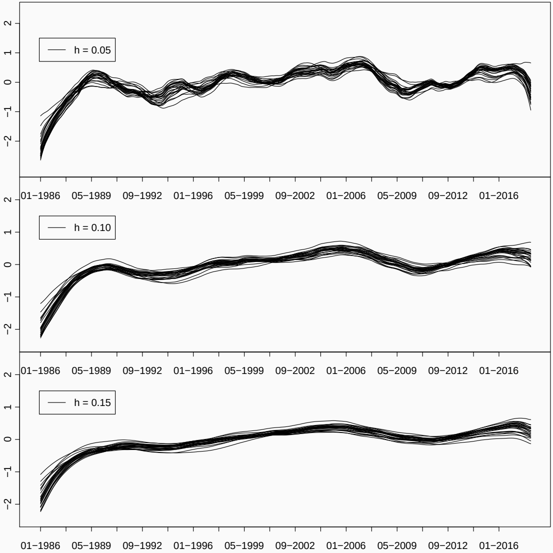
Let the conditions of previous propositions be satisfied. Then

$$\mathrm{P}\Big(\big\{\widehat{G}_1, \ldots, \widehat{G}_{\widehat{N}}\big\} = \big\{G_1, \ldots, G_N\big\}\Big) \ge (1-\alpha) + o(1)$$

and

$$\mathrm{P}\big(\widehat{N} = N\big) \ge (1-\alpha) + o(1).$$

Testing for equality of different temperature time trends



Idea behind the additive correction

Consider the uncorrected statistic

$$\widehat{\Psi}_{T,\text{uncorrected}} = \max_{(u,h)\in\mathcal{G}_T} \left| \frac{\widehat{\psi}_T(u,h)}{\widehat{\sigma}} \right|$$

under the null hypothesis $H_0 : m = 0$ and under simplifying assumptions:

- the errors ε_i are i.i.d. normally distributed;
- $\widehat{\sigma} = \sigma$;
- $\mathcal{G}_T = \{(u_k, h_l) | u_k = (2k-1)h_l \text{ for } 1 \leq k \leq 1/2h_l, 1 \leq l \leq L\}$.

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