Multiscale Testing for Equality of Nonparametric Trend Curves

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Proof of Theorem ??. Define $\Delta m_{it} = m_i\left(\frac{t}{T}\right) - m_i\left(\frac{t-1}{T}\right)$.
Recall the differencing estimator $\widehat{\boldsymbol{\beta}}_i$:

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_i &= \Big(\sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta \mathbf{X}_{it}^\top\Big)^{-1} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta Y_{it} = \\
&= \Big(\sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta \mathbf{X}_{it}^\top\Big)^{-1} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Big(\Delta \mathbf{X}_{it}^\top \boldsymbol{\beta}_i + \Delta\varepsilon_{it} + \Delta m_{it}\Big) = \\
&= \boldsymbol{\beta}_i + \Big(\sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta \mathbf{X}_{it}^\top\Big)^{-1} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta\varepsilon_{it} + \Big(\sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta \mathbf{X}_{it}^\top\Big)^{-1} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta m_{it}.
\end{aligned}$$

This leads to

$$\begin{aligned}
\big|\sqrt{T}(\widehat{\boldsymbol{\beta}}_i - \boldsymbol{\beta}_i)\big| \leq & \bigg|\Big(\frac{1}{T}\sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta \mathbf{X}_{it}^\top\Big)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta\varepsilon_{it}\bigg| + \\
& + \bigg|\Big(\frac{1}{T}\sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta \mathbf{X}_{it}^\top\Big)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta m_{it}\bigg|.
\end{aligned} \tag{0.1}$$

First, we take a closer look at the second summand in (0.1). By the assumption in Theorem ??, $m_i(\cdot)$ is Lipschitz continuous, that is, $|\Delta m_{it}| = \big|m_i\left(\frac{t}{T}\right) - m_i\left(\frac{t-1}{T}\right)\big| \leq C\frac{1}{T}$ for all $t \in \{1, \dots, T\}$ and some constant $C > 0$. Hence,

$$\begin{aligned}
\bigg|\frac{1}{\sqrt{T}} \sum_{t=1}^T \Delta \mathbf{X}_{it} \Delta m_{it}\bigg| &= \bigg|\frac{1}{\sqrt{T}} \sum_{t=1}^T \Delta \mathbf{H}_i(\mathcal{U}_{it}) \Delta m_{it}\bigg| \leq \\
&\leq \frac{1}{\sqrt{T}} \sum_{t=1}^T |\Delta \mathbf{H}_i(\mathcal{U}_{it})| \cdot |\Delta m_{it}| \leq \\
&\leq \frac{1}{\sqrt{T}} \sum_{t=1}^T |\Delta \mathbf{H}_i(\mathcal{U}_{it})| \cdot |\Delta m_{it}| \leq \frac{C}{\sqrt{T}} \cdot \frac{1}{T} \sum_{t=1}^T |\Delta \mathbf{H}_i(\mathcal{U}_{it})|.
\end{aligned} \tag{0.2}$$

Now, in order to show that the whole vector $\frac{1}{T}\sum_{t=1}^T |\Delta \mathbf{H}_i(\mathcal{U}_{it})|$ is $O_P(1)$, we will do that for every element $\frac{1}{T}\sum_{t=1}^T |\Delta H_{ij}(\mathcal{U}_{it})|$ of this vector separately.

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Fix $j \in 1, \dots, d$. By Chebyshev's inequality we have

$$\mathbb{P}\left(\frac{1}{T}\sum_{t=1}^{T}|\Delta H_{ij}(\mathcal{U}_{it})| > a\right) \leq \frac{\mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}|\Delta H_{ij}(\mathcal{U}_{it})|\right)^2\right]}{a^2} \tag{0.3}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}|\Delta H_{ij}(\mathcal{U}_{it})|\right)^2\right] &= \frac{1}{T^2}\mathbb{E}\left[\left(\sum_{t=1}^{T}|\Delta H_{ij}(\mathcal{U}_{it})|\right)^2\right] = \\ = \frac{1}{T^2}\sum_{t=1}^{T}\mathbb{E}\left[\Delta H_{ij}^2(\mathcal{U}_{it})\right] &+ \frac{1}{T^2}\sum_{t=1,s=1,t\neq s}^{T}\mathbb{E}\big[\,|\Delta H_{ij}(\mathcal{U}_{it})\Delta H_{ij}(\mathcal{U}_{is})|\,\big].\end{aligned} \tag{0.4}$$

Note that by the Cauchy-Scwarz inequality for all t and s we have

$$0 \leq \mathbb{E}\big[\,|H_{ij}(\mathcal{U}_{it})H_{ij}(\mathcal{U}_{is})|\,\big] \leq \sqrt{\mathbb{E}\big[H_{ij}^2(\mathcal{U}_{it})\big]}\sqrt{\mathbb{E}\big[H_{ij}^2(\mathcal{U}_{is})\big]} = \mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right]$$

and

$$0 \leq \big|\mathbb{E}\big[H_{ij}(\mathcal{U}_{it})H_{ij}(\mathcal{U}_{is})\big]\big| \leq \mathbb{E}\big[\,|H_{ij}(\mathcal{U}_{it})H_{ij}(\mathcal{U}_{is})|\,\big] \leq \mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right]. \tag{0.5}$$

Hence,

$$\begin{aligned}0 \leq \mathbb{E}\left[\Delta H_{ij}^2(\mathcal{U}_{it})\right] &= \mathbb{E}\left[(H_{ij}(\mathcal{U}_{it}) - H_{ij}(\mathcal{U}_{it-1}))^2\right] = \\ &= \mathbb{E}\left[H_{ij}^2(\mathcal{U}_{it})\right] - 2\mathbb{E}\left[H_{ij}(\mathcal{U}_{it})H_{ij}(\mathcal{U}_{it-1})\right] + \mathbb{E}\left[H_{ij}^2(\mathcal{U}_{it-1})\right] \leq \\ &\leq \mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right] + 2\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right] + \mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right] = \\ &= 4\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right]\end{aligned}$$

and the first summand in (0.4) can be bounded by $\frac{4}{T}\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right]$.
Now to the second summand in (0.4):

$$\begin{aligned}0 &\leq \mathbb{E}\big[\,|\Delta H_{ij}(\mathcal{U}_{it})\Delta H_{ij}(\mathcal{U}_{is})|\,\big] = \mathbb{E}\Big[\big|\left(H_{ij}(\mathcal{U}_{it}) - H_{ij}(\mathcal{U}_{it-1})\right)\left(H_{ij}(\mathcal{U}_{is}) - H_{ij}(\mathcal{U}_{is-1})\right)\big|\Big] \leq \\ &\leq \mathbb{E}\big[\,|H_{ij}(\mathcal{U}_{it})H_{ij}(\mathcal{U}_{is})|\,\big] + \mathbb{E}\big[\,|H_{ij}(\mathcal{U}_{it-1})H_{ij}(\mathcal{U}_{is})|\,\big] + \\ &\quad + \mathbb{E}\big[\,|H_{ij}(\mathcal{U}_{it})H_{ij}(\mathcal{U}_{is-1})|\,\big] + \mathbb{E}\big[\,|H_{ij}(\mathcal{U}_{it-1})H_{ij}(\mathcal{U}_{is-1})|\,\big] \leq \\ &\leq 4\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right],\end{aligned}$$

where in the last inequality we used (0.5). This means that the second summand in (0.4) can be bounded by $\frac{4T(T-1)}{T^2}\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right] = \frac{4(T-1)}{T}\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right]$.
Plugging these bounds in (0.4), we get

$$\mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}|\Delta H_{ij}(\mathcal{U}_{it})|\right)^2\right] \leq \frac{4}{T}\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right] + \frac{4(T-1)}{T}\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right] = 4\mathbb{E}\left[H_{ij}^2(\mathcal{U}_{i0})\right],$$

which together with (0.3) leads to $\frac{1}{T}\sum_{t=1}^{T}\left|\Delta H_{ij}(\mathcal{U}_{it})\right| = O_P(1)$. Since it holds for each $j \in \{1,\dots,d\}$, we can establish that

$$\frac{1}{T}\sum_{t=1}^{T}\left|\Delta \mathbf{H}_i(\mathcal{U}_{it})\right| = \frac{1}{T}\sum_{t=1}^{T}\left|\Delta \mathbf{X}_{it}\right| = O_P(1). \tag{0.6}$$

Similarly, by Proposition **??** and Chebyshev's inequality, we have that for each $j,k \in \{1,\dots,d\}$

$$\left|\frac{1}{T}\sum_{t=1}^{T}\Delta H_{ij}(\mathcal{U}_{it})\Delta H_{ik}(\mathcal{U}_{it})\right| = O_P(1),$$

which leads to

$$\left|\frac{1}{T}\sum_{t=1}^{T}\Delta \mathbf{H}_i(\mathcal{U}_{it})\Delta \mathbf{H}_i(\mathcal{U}_{it})^\top\right| = \left|\frac{1}{T}\sum_{t=1}^{T}\Delta \mathbf{X}_{it}\Delta \mathbf{X}_{it}^\top\right| = O_P(1),$$

where $|A|$ with A being a matrix is any matrix norm.
By Assumption **??**, we know that $\mathbb{E}[\Delta \mathbf{X}_{it}\Delta \mathbf{X}_{it}^\top] = \mathbb{E}[\Delta \mathbf{X}_{i0}\Delta \mathbf{X}_{i0}^\top]$ is invertible, thus,

$$\left|\left(\frac{1}{T}\sum_{t=1}^{T}\Delta \mathbf{X}_{it}\Delta \mathbf{X}_{it}^\top\right)^{-1}\right| = O_P(1). \tag{0.7}$$

Plugging (0.6) into (0.2) and combining it with (0.7), we get that the second summand in (0.1) is $O_P(1/\sqrt{T})$.
Furthermore, we can apply the Proposition **??** together with (0.7) to get that the first summand in (**??**) is $O_P(1)$. And the statement of the theorem follows. $\square$