Simultaneous statistical inference for epidemic trends: the case of COVID-19

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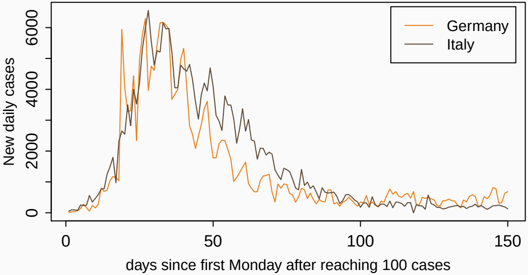
Introduction

Motivation

Research question: How do outbreak patterns of COVID-19 compare across countries?



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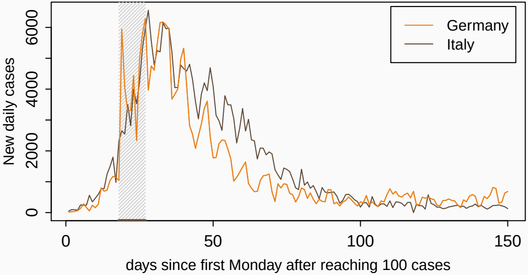

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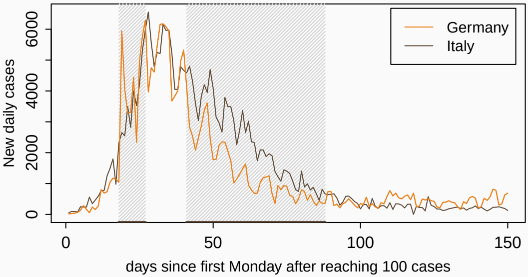

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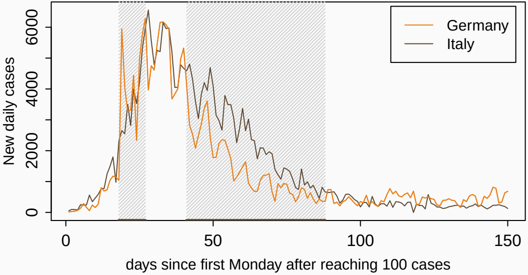

Aim of the paper

To develop new inference methods that allow to *identify* and *locate* differences between epidemic time trends.

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Literature

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Studies of COVID-19:

- Time series analysis: Gu et al. (2020), Li and Linton (2020).

Model

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Since $\lambda_i(t/T) = \mathbb{E}[X_{it}] = \mathrm{Var}(X_{it})$, we can rewrite X_{it} as

$$X_{it} = \lambda_i\Big(\frac{t}{T}\Big) + u_{it} \quad \text{with} \quad u_{it} = \sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it}$$

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In applications the variance can be larger than the mean $\Rightarrow$ quasi-Poisson models.

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- σ is the overdispersion parameter;
- η_{it} are error terms that are independent across i and t and have zero mean and unit variance.

Testing procedure

Testing problem

Let $\mathcal{F} := \{\mathcal{I}_k \subseteq [0,1] : 1 \leq k \leq K\}$ be a family of rescaled time intervals on $[0,1]$, and let $H_0^{(ijk)}$ be the hypothesis that the functions λ_i and λ_j are equal on an interval $\mathcal{I}_k$,

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Let $\mathcal{M}_0$ be the set of triplets (i,j,k) for which $H_0^{(ijk)}$ holds true. Then, FWER is

$$\text{FWER}(\alpha) = \mathrm{P}\Big(\exists (i,j,k) \in \mathcal{M}_0 : \text{we reject } H_0^{(ijk)}\Big)$$

Test statistic

For the given interval $\mathcal{I}_k$ and a pair of time series i and j we calculate

$$\hat{s}_{ijk} = \frac{1}{Th_k} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it} - X_{jt}),$$

where h_k is the length of $\mathcal{I}_k$.

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$$\begin{aligned}\hat{s}_{ijk} =& \frac{1}{Th_k} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Bigg(\lambda_i\Big(\frac{t}{T}\Big) - \lambda_j\Big(\frac{t}{T}\Big)\Bigg) \\ & + \frac{\sigma}{Th_k} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Bigg(\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it} - \sqrt{\lambda_j\Big(\frac{t}{T}\Big)}\eta_{jt}\Bigg)\end{aligned}$$

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Applying a law of large numbers, we get that

$$\hat{s}_{ijk} = \frac{1}{Th_k} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big(\lambda_i\Big(\frac{t}{T}\Big) - \lambda_j\Big(\frac{t}{T}\Big)\Big) + o_P(1)$$

Test statistic, part 2

Under certain assumptions,

$$\mathrm{Var}(\hat{s}_{ijk}) = \frac{\sigma^2}{T^2 h_k^2} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big\{\lambda_i\Big(\frac{t}{T}\Big) + \lambda_j\Big(\frac{t}{T}\Big)\Big\}$$

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In order to normalize the variance of the statistic $\hat{s}_{ijk}$, we scale it by an estimator of its variance:

$$\widehat{\mathrm{Var}(\hat{s}_{ijk})} = \frac{\hat{\sigma}^2}{T^2 h_k^2} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big) (X_{it} + X_{jt}),$$

with $\hat{\sigma}^2$ being an appropriate estimator of σ^2. Details

Test statistic, part 2

Test statistic for the hypothesis $H_0^{(ijk)}$ is defined as

$$\widehat{\psi}_{ijk} = \frac{\sum_{t=1}^{T} 1\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} - X_{jt})}{\hat{\sigma}\big\{\sum_{t=1}^{T} 1\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\big\}^{1/2}}$$

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Under certain conditions and under the null, $\widehat{\psi}_{ijk}$ can be approximated by a Gaussian version of the test statistic:

$$\phi_{ijk} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(Z_{it} - Z_{jt}),$$

where Z_{it} are independent standard normal random variables.

Critical values

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- More modern approach: $c_{ijk}(\alpha)$ depend on the length h_k of the time interval (Dümbgen and Spokoiny (2001)):

$$c_{ijk}(\alpha) = c(\alpha, h_k) := b_k + q(\alpha)/a_k,$$

where a_k and b_k are scale-dependent constants and $q(\alpha)$ is chosen such that we control FWER. Details

Critical values, part 2

We want to control FWER:

$$\begin{aligned}
\mathrm{FWER}(\alpha) &= \mathrm{P}\Big(\exists(i,j,k) \in \mathcal{M}_0 : |\widehat{\psi}_{ijk}| > c_{ijk}(\alpha)\Big) \\
&= 1 - \mathrm{P}\Big(\forall(i,j,k) \in \mathcal{M}_0 : |\widehat{\psi}_{ijk}| \leq c_{ijk}(\alpha)\Big) \\
&= 1 - \mathrm{P}\Big(\forall(i,j,k) \in \mathcal{M}_0 : a_k\big(|\hat{\psi}_{ijk}| - b_k\big) \leq q(\alpha)\Big) \\
&= 1 - \mathrm{P}\Big(\max_{(i,j,k)\in\mathcal{M}_0} a_k\big(|\hat{\psi}_{ijk}| - b_k\big) \leq q(\alpha)\Big) \\
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&\le \alpha
\end{aligned}
$$

Hence, we choose $q(\alpha)$ as the $(1-\alpha)$-quantile of the statistic

$$
\hat{\Psi} = \max_{(i,j,k)} a_k\big(|\hat{\psi}^0_{ijk}| - b_k\big),
$$

where $\hat{\psi}^0_{ijk}$ is equal to $\hat{\psi}_{ijk}$ under the null.

Test procedure

1. Consider the Gaussian test statistic

$$\Phi = \max_{(i,j,k)} a_k \big(|\phi_{ijk}| - b_k \big),$$

   where a_k and b_k are scale-dependent constants and ϕ_{ijk} are weighted averages of the differences of standard normal random variables.

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Test procedure

For the given significance level $\alpha \in (0,1)$ and for each (i,j,k), reject $H_0^{(ijk)}$ if $|\widehat{\psi}_{ijk}| > c_{\text{Gauss}}(\alpha, h_k)$.

Theoretical properties

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous: $|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

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- $\mathcal{C}4$ $\mathbb{E}[\eta_{it}] = 0$, $\mathbb{E}[\eta_{it}^2] = 1$ and $\mathbb{E}[|\eta_{it}|^\theta] \leq C_\theta < \infty$ for some $\theta > 4$.

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$\mathcal{C}5$ $h_{\max} = o(1/\log T)$ and $h_{\min} \geq CT^{-b}$ for some $b \in (0,1)$.

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- $\mathcal{C}6$ $p := \{\#(i, j, k)\} = O(T^{(\theta/2)(1-b)-(1+\delta)})$ for some small $\delta > 0$.

Theoretical properties

Proposition

Let $\mathcal{M}_0$ be the set of triplets (i, j, k) for which $H_0^{(ijk)}$ holds true. Then under $\mathcal{C}1 - \mathcal{C}6$, it holds that

$$\mathrm{P}\Big(\forall(i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk}| \leq c_{\mathsf{Gauss}}(\alpha, h_k)\Big) \geq 1 - \alpha + o(1)$$

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Proposition

Consider a sequence of functions $\lambda_i = \lambda_{i,T}$, $\lambda_j = \lambda_{j,T}$ such that

$$\exists \mathcal{I}_k : \lambda_i(w) - \lambda_j(w) \ge c_T \sqrt{\log T/(Th_k)} \; \forall w \in \mathcal{I}_k, \tag{1}$$

and $c_T \to \infty$ faster than $\frac{\sqrt{\log T}\sqrt{\log \log T}}{\log \log \log T}$.

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and $c_T \to \infty$ faster than $\frac{\sqrt{\log T}\sqrt{\log\log T}}{\log\log\log T}$. Let $\mathcal{M}_1$ be the set of triplets (i, j, k) for which (1) holds true. Then under $\mathcal{C}1 - \mathcal{C}6$, it holds that

$$\mathrm{P}\Big(\forall (i,j,k) \in \mathcal{M}_1 : |\hat{\psi}_{ijk}| > c_{\mathsf{Gauss}}(\alpha, h_k)\Big) = 1 - o(1)$$

Application

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An interval $\mathcal{I}_k \in \mathcal{F}_{\text{reject}}(i,j)$ is called **minimal** if there is no other interval $\mathcal{I}_{k'} \in \mathcal{F}_{\text{reject}}(i,j)$ with $\mathcal{I}_{k'} \subset \mathcal{I}_k$. The set of minimal intervals is denoted $\mathcal{F}^{\text{min}}_{\text{reject}}(i,j)$.

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An interval $\mathcal{I}_k \in \mathcal{F}_{\text{reject}}(i,j)$ is called **minimal** if there is no other interval $\mathcal{I}_{k'} \in \mathcal{F}_{\text{reject}}(i,j)$ with $\mathcal{I}_{k'} \subset \mathcal{I}_k$. The set of minimal intervals is denoted $\mathcal{F}^{\min}_{\text{reject}}(i,j)$.

We can make very similar confidence statement about the set of minimal intervals as well:

$$\mathrm{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : \mathcal{I}_k \notin \mathcal{F}^{\min}_{\text{reject}}(i,j)\Big) \geq 1 - \alpha + o(1)$$

Application setting

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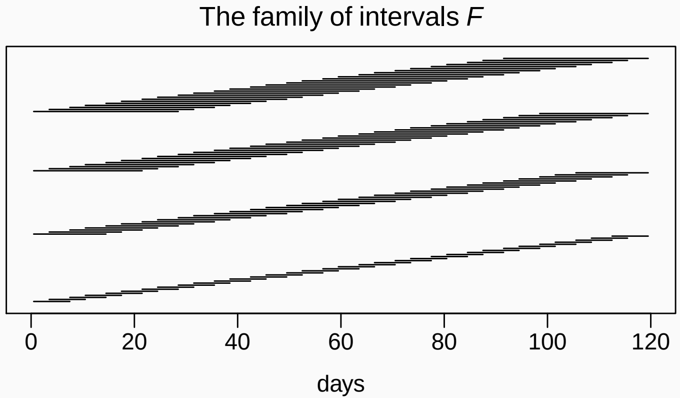
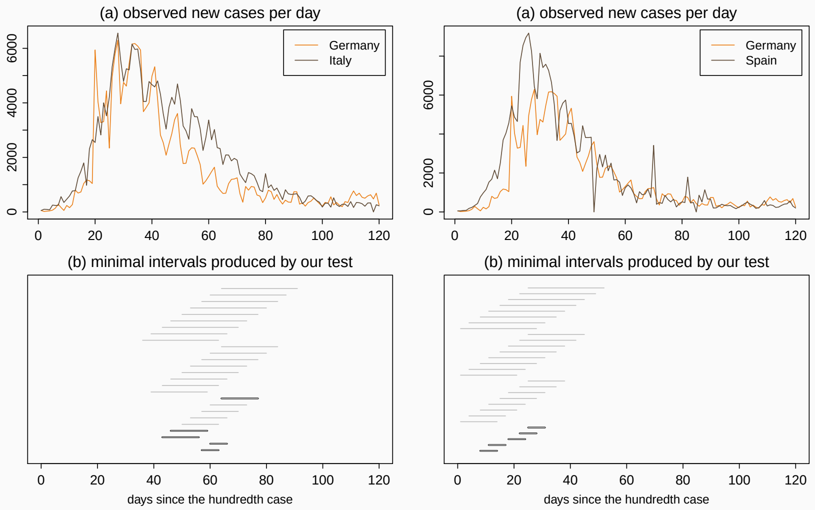
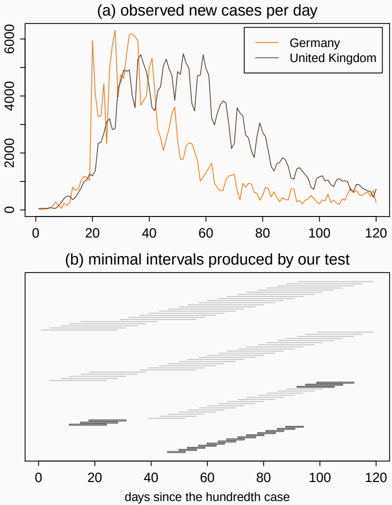
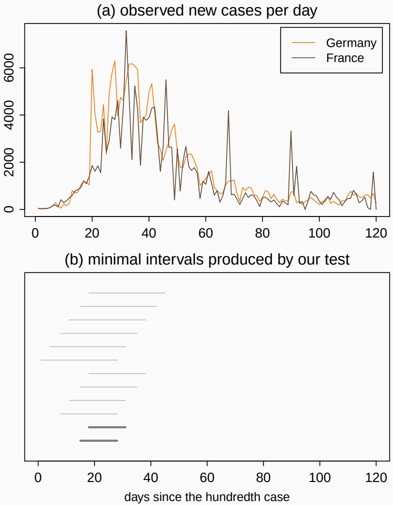


Application results



Application results, part 2





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- introduce scaling factor in the trend function, that allow for adjusting for the size of the country (population, density, testing regimes, etc.);
- connect with data-driven techniques such as machine learning;
- cluster the countries based on the trends they exhibit.

Thank you!

Simulation results for the size of the test

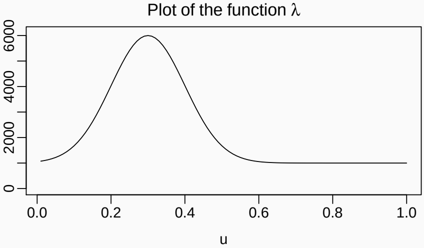


Table 1: Size of the multiscale test

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.011	0.047	0.093	0.010	0.044	0.087	0.008	0.037	0.075
$T = 250$	0.009	0.047	0.091	0.009	0.046	0.087	0.008	0.035	0.069
$T = 500$	0.010	0.044	0.083	0.008	0.048	0.093	0.007	0.035	0.077

Simulation results for the power of the test

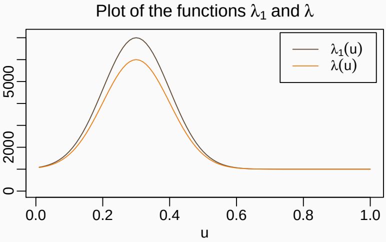
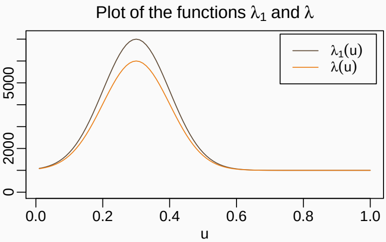


Table 2: Power of the multiscale test for scenario A

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.335	0.518	0.597	0.306	0.474	0.545	0.212	0.352	0.418
$T = 250$	0.615	0.790	0.836	0.580	0.764	0.800	0.470	0.648	0.705
$T = 500$	0.736	0.905	0.917	0.738	0.884	0.890	0.636	0.799	0.830

Simulation results for the power of the test



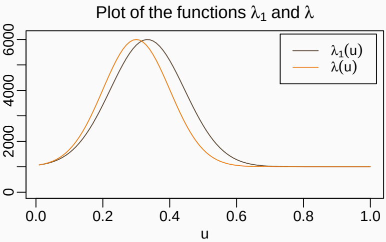


Table 3: Power of the multiscale test for scenario B

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.824	0.910	0.903	0.812	0.893	0.890	0.738	0.847	0.857
$T = 250$	0.991	0.972	0.941	0.991	0.960	0.920	0.991	0.965	0.933
$T = 500$	0.997	0.973	0.949	0.995	0.961	0.923	0.996	0.969	0.932

Estimator of σ^2

We estimate the overdispersion paramter σ^2 by

$$\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \hat{\sigma}_i^2 \text{ and } \hat{\sigma}_i^2 = \frac{\sum_{t=2}^{T} (X_{it} - X_{it-1})^2}{2 \sum_{t=1}^{T} X_{it}}$$

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We assume that λ_i is Lipschitz continuous. Then

$$X_{it} - X_{it-1} = \sigma\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}(\eta_{it} - \eta_{it-1}) + r_{it},$$

where $|r_{it}| \leq C(1 + |\eta_{it-1}|)/T$ with a sufficiently large C.

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$$\frac{1}{T}\sum_{t=2}^{T}(X_{it} - X_{it-1})^2 = 2\sigma^2\Big\{\frac{1}{T}\sum_{t=2}^{T}\lambda_i(t/T)\Big\} + o_p(1)$$

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Together with

$$\frac{1}{T}\sum_{t=1}^{T} X_{it} = \frac{1}{T}\sum_{t=1}^{T}\lambda_i(t/T) + o_p(1),$$

we get that $\hat{\sigma}_i^2 = \sigma^2 + o_p(1)$ for any i and thus $\hat{\sigma}^2 = \sigma^2 + o_p(1)$. Go back

Notation

In order to proceed with the proof, we will need the following notation:

$$\widehat{\psi}_{ijk,T} = \frac{\sum_{t=1}^{T} 1\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} - X_{jt})}{\hat{\sigma}\big\{\sum_{t=1}^{T} 1\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\big\}^{1/2}}$$

$$\hat{\psi}^0_{ijk,T} = \frac{\sum_{t=1}^{T} 1\big(\frac{t}{T} \in \mathcal{I}_k\big)\,\sigma\overline{\lambda}_{ij}^{1/2}\big(\frac{t}{T}\big)(\eta_{it} - \eta_{jt})}{\hat{\sigma}\{\sum_{t=1}^{T} 1\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\}^{1/2}} \qquad \hat{\Psi}^0_T = \max_{(i,j,k)} a_k\big(|\hat{\psi}^0_{ijk,T}| - b_k\big)$$

$$\psi^0_{ijk,T} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(\eta_{it} - \eta_{jt}) \qquad \Psi_T = \max_{(i,j,k)} a_k\big(|\psi^0_{ijk,T}| - b_k\big)$$

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4. It can be shown that $\mathrm{P}(\Phi_T \leq q_{T,\mathsf{Gauss}}(\alpha)) = 1 - \alpha$. From this and (2), it immediately follows that

$$\mathrm{P}\big(\hat{\Psi}_T^0 \leq q_{T,\mathsf{Gauss}}(\alpha)\big) = 1 - \alpha + o(1),$$

which in turn implies the desired statement.

Idea behind a_k and b_k

A more modern approach of constructing the individual critical values $c_{ijk}(\alpha)$ (Dümbgen and Spokoiny (2001)): let them depend on the scale of the testing problem, i.e. the length h_k of the time interval.

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$$c_{ijk}(\alpha) = c(\alpha, h_k) := b_k + q(\alpha)/a_k,$$

where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants and $q_T(\alpha)$ is chosen such that we control FWER. Go back

Idea behind the additive correction

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$$\Phi^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk}|$$

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$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l. Go back