

Multiscale testing for equality of nonparametric trend curves

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Introduction

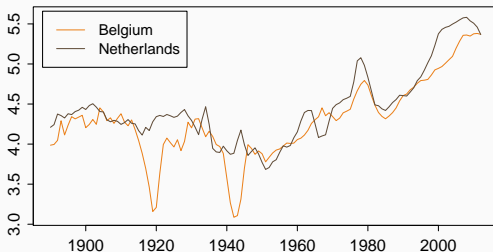
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To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.

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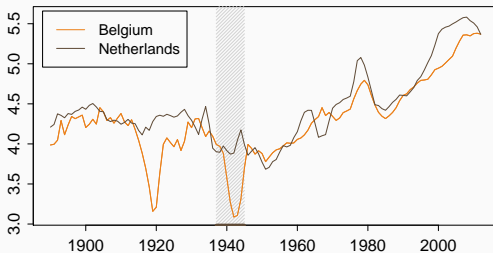
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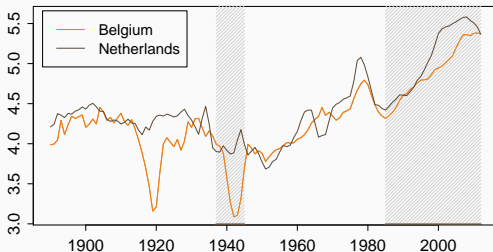
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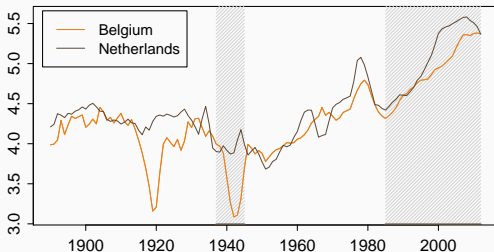
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To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.



Research question: Out of many given intervals, how to find those where the trends are significantly different?

Why is it relevant?

Finding systematic differences between trends = basis for further research.

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Testing many hypotheses at the same time = multiple testing problem
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Is it limited to a particular application?

No! Our method = general method for comparing nonparametric trends
⇒ new statistical test for equality of nonparametric trend curves.

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- Park et al. (2009), Degras et al. (2012), Zhang et al. (2012), Hidalgo and Lee (2014), Chen and Wu (2019).

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Model

We observe a panel of n time series $\mathcal{Z}_i = \{(Y_{it}, \mathbf{X}_{it}) : 1 \leq t \leq T\}$ of length T , where $Y_{it} \in \mathbb{R}$ and $\mathbf{X}_{it} \in \mathbb{R}^d$.

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- β_i is $d \times 1$ vector of unknown parameters;
- α_i are so-called fixed effect error terms;
- $\mathcal{E}_i = \{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

$$Y_{it} = m_i\left(\frac{t}{T}\right) + \beta_i^T \mathbf{X}_{it} + \alpha_i + \varepsilon_{it},$$

$$Y_{it} = m_i\left(\frac{t}{T}\right) + \beta_i^\top \mathbf{X}_{it} + \alpha_i + \varepsilon_{it},$$

If we knew α_i and β_i , then the model becomes much simpler:

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Model, part 2

$$Y_{it} = m_i\left(\frac{t}{T}\right) + \beta_i^\top \mathbf{X}_{it} + \alpha_i + \varepsilon_{it},$$

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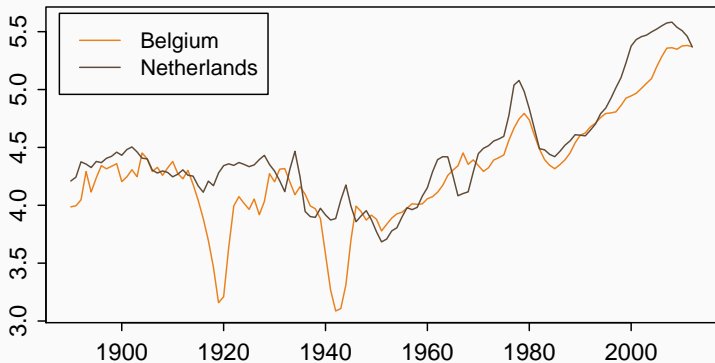
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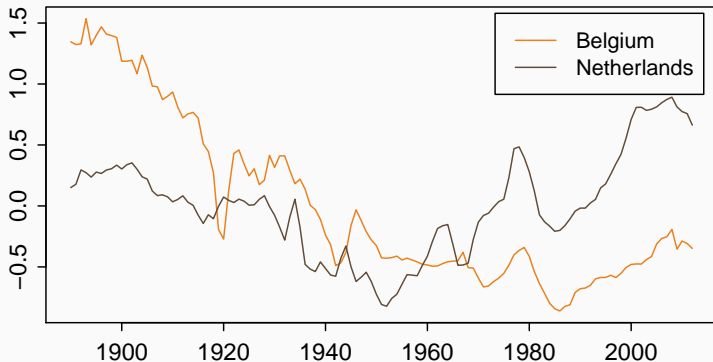
But given $\hat{\alpha}_i$ and $\hat{\beta}_i$, we consider an **augmented time series**

$$\hat{Y}_{it} := Y_{it} - \hat{\alpha}_i - \hat{\beta}_i^\top \mathbf{X}_{it} = (\beta_i - \hat{\beta}_i)^\top \mathbf{X}_{it} + m_i\left(\frac{t}{T}\right) + (\alpha_i - \hat{\alpha}_i) + \varepsilon_{it}.$$

Original time series: Belgium and Netherlands



Augmented time series: Belgium and Netherlands



Testing procedure

$$H_0 : m_1 = m_2 = \dots = m_n$$

Testing problem

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$$H_0^{[i,j]}(u, h) : m_i(w) = m_j(w) \text{ for all } w \in [u - h, u + h].$$

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Then the global null $H_0 : m_1 = m_2 = \dots = m_n$ can be reformulated as

$$H_0 : \text{The hypotheses } H_0^{[i,j]}(u, h) \text{ hold true for all intervals } [u - h, u + h], (u, h) \in \mathcal{G}_T, \text{ and for all } 1 \leq i < j \leq n.$$

For a given location $u \in [0, 1]$ and bandwidth h and a given pair (i, j) we construct the kernel averages

$$\hat{\psi}_{ij,T}(u, h) = \sum_{t=1}^T w_{t,T}(u, h) (\hat{Y}_{it} - \hat{Y}_{jt}),$$

where $w_{t,T}(u, h)$ are appropriate weights.

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The kernel averages $\hat{\psi}_{ij,T}(u, h)$ measure the distance between two trend curves m_i and m_j on $[u - h, u + h]$.

Instead with working directly with $\hat{\psi}_{ij,\tau}(u, h)$, we replace them by

$$\hat{\psi}_{ij,\tau}^0(u, h) = \left\{ \left| \frac{\hat{\psi}_{ij,\tau}(u, h)}{(\hat{\sigma}_i^2 + \hat{\sigma}_j^2)^{1/2}} \right| - \lambda(h) \right\},$$

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- $\hat{\sigma}_i^2$ is an appropriate estimator of the long-run variance σ_i^2 ;
- $\lambda(h) = \sqrt{2 \log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

To test the global null, we aggregate the individual test statistics for all (i, j) and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\hat{\Psi}_{n,T} = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \hat{\psi}_{ij,T}^0(u, h).$$

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Main theoretical result

Under certain conditions and under the null, $\widehat{\psi}_{ij,T}^0(u, h)$ and $\widehat{\Psi}_{n,T}$ can be approximated by the corresponding Gaussian versions of the test statistics.

Gaussian version of the individual test statistics:

$$\phi_{ij,T}^0(u, h) = \max_{(u,h) \in \mathcal{G}_T} \left\{ \left| \frac{\phi_{ij,T}(u, h)}{(\sigma_i^2 + \sigma_j^2)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\phi_{ij,T}(u, h) = \sum_{t=1}^T w_{t,T}(u, h) \{ \sigma_i(Z_{it} - \bar{Z}_i) - \sigma_j(Z_{jt} - \bar{Z}_j) \};$

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Aggregated Gaussian test statistics:

$$\Phi_{n,T} = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \phi_{ij,T}^0(u, h).$$

1. Consider the Gaussian test statistic

$$\Phi_{n,T} = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \phi_{ij,T}^0(u,h),$$

where $\phi_{ij,T}^0$ are weighted averages of the differences of standard normal random variables.

Test procedure, part 2

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3. Perform the test for the global hypothesis H_0 : reject H_0 if $\hat{\Psi}_{n,T} > q_{n,T}(\alpha)$.
4. For each i, j , and each $(u, h) \in \mathcal{G}_T$, carry out the test for the local null hypothesis $H_0^{[i,j]}(u, h)$: reject $H_0^{[i,j]}(u, h)$ if $\hat{\psi}_{ij,T}^0(u, h) > q_{n,T}(\alpha)$.

Proposition

Under certain assumptions and under the null, it holds that

$$\mathbb{P}\left(\widehat{\Psi}_{n,T} \leq q_{n,T}(\alpha)\right) = 1 - \alpha + o(1)$$

Theoretical properties

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Corollary

$$FWER(\alpha) \leq \alpha$$

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Corollary

$$FWER(\alpha) \leq \alpha$$

Proposition

Consider a sequence of functions $m_i = m_{i,T}$, $m_j = m_{j,T}$ such that

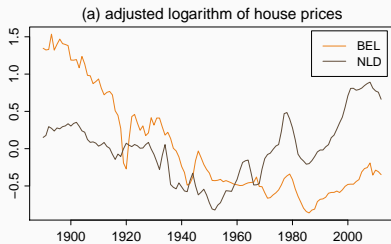
$$\exists (u, h) \in \mathcal{G}_T : m_i(w) - m_j(w) \geq c_T \sqrt{\log T / (Th)} \quad \forall w \in [u - h, u + h],$$

and $c_T \rightarrow \infty$. Then under our assumptions, it holds that

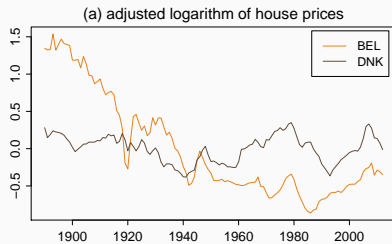
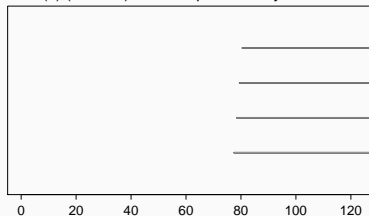
$$\mathbb{P}\left(\widehat{\Psi}_{n,T} \leq q_{n,T}(\alpha)\right) = o(1)$$

Illustration

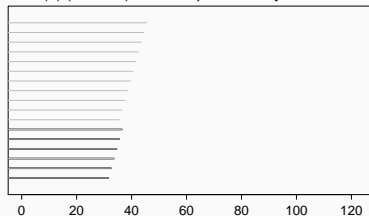
Application results



(b) (minimal) intervals produced by our test



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Further possible extensions:

- cluster the time series based on the trends they exhibit;

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Further possible extensions:

- cluster the time series based on the trends they exhibit;
- introduce scaling factor in the trend function;
- include the dependence between covariates and error terms.

Thank you!

Model, part 3

1. We estimate β_i :

$$\hat{\beta}_i = \left(\sum_{t=2}^T \Delta \mathbf{x}_{it} \Delta \mathbf{x}_{it}^\top \right)^{-1} \sum_{t=2}^T \Delta \mathbf{x}_{it} \Delta Y_{it}$$

Theorem

Under certain regularity assumptions, $\hat{\beta}_i$ is a consistent estimator of β_i with the property $\beta_i - \hat{\beta}_i = O_P(T^{-1/2})$.

2. We estimate the fixed effects α_i :

$$\hat{\alpha}_i = \frac{1}{T} \sum_{t=1}^T (Y_{it} - \hat{\beta}_i^\top \mathbf{x}_{it})$$

We then work with the augmented time series $\hat{Y}_{it} = Y_{it} - \hat{\alpha}_i - \hat{\beta}_i^\top \mathbf{x}_{it}$.

Test statistic

For a given location $u \in [0, 1]$ and bandwidth h and a given pair (i, j) we construct the kernel averages

$$\hat{\psi}_{ij,T}(u, h) = \sum_{t=1}^T w_{t,T}(u, h) (\hat{Y}_{it} - \hat{Y}_{jt}),$$

where

$$w_{t,T}(u, h) = \frac{\Lambda_{t,T}(u, h)}{\{\sum_{t=1}^T \Lambda_{t,T}^2(u, h)\}^{1/2}},$$

$$\Lambda_{t,T}(u, h) = K\left(\frac{t/T - u}{h}\right) \left[S_{T,2}(u, h) - S_{T,1}(u, h) \left(\frac{t/T - u}{h}\right) \right],$$

$$S_{T,\ell}(u, h) = \frac{1}{Th} \sum_{t=1}^T K\left(\frac{t/T - u}{h}\right) \left(\frac{t/T - u}{h}\right)^\ell$$

for $\ell = 1, 2$ and K is a kernel function.

Assumptions

$\mathcal{C}1$ For all i it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.

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$\mathcal{C}2$ For each i the variables ε_{it} are weakly dependent. [Details](#)

Assumptions

- $\mathcal{C}1$ For all i it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.
- $\mathcal{C}2$ For each i the variables ε_{it} are weakly dependent. [Details](#)
- $\mathcal{C}3$ For each i we have that \mathbf{X}_{it} is stationary and causal with all the necessary moments and no asymptotic multicollinearity.

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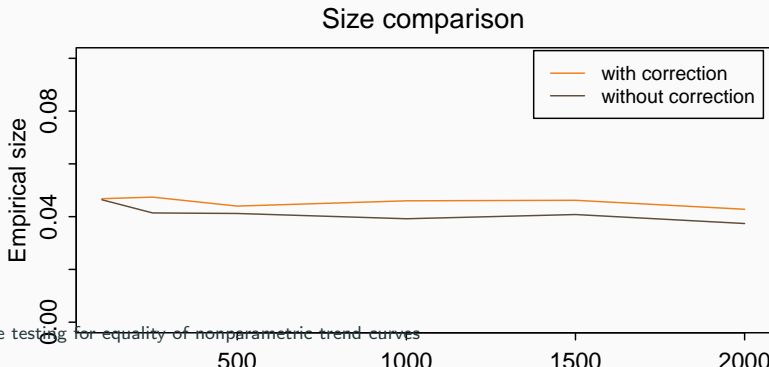
Idea behind $\lambda(h)$

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Introduction of a scale-dependent parameter helps us balance the significance of hypotheses between the time intervals of different lengths h_k :



Idea behind the additive correction

Consider the uncorrected Gaussian statistic

$$\Phi^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk}|$$

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$\Rightarrow \max_m \dots = \sqrt{2 \log(1/h_l)} + o_P(1) \rightarrow \infty$ as $h \rightarrow 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l . [Go back](#)

Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\mathbf{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\mathbf{L}, t) = \|\mathbf{L}(\mathcal{F}_t) - \mathbf{L}(\mathcal{F}'_t)\|_q,$$

where $\mathcal{F}_t = (\dots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \dots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}'_t = (\dots, \epsilon_{-1}, \epsilon'_0, \epsilon_1, \dots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of \mathcal{F}_t with ϵ'_0 being an i.i.d. copy of ϵ_0 .

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Intuitively, $\delta_q(\mathbf{L}, t)$ measures the dependency of $\mathbf{L}(\mathcal{F}_t)$ on ϵ_0 , i.e., how replacing ϵ_0 by an i.i.d. copy while keeping all other innovations in place affects the output $\mathbf{L}(\mathcal{F}_t)$.

Technical assumptions

- $\mathcal{C}1'$ The variables ε_{it} are independent across i and allow for the representation $\varepsilon_{it} = G_i(\dots, \eta_{it-1}, \eta_{it})$, where η_{it} are i.i.d. random variables across t and $G_i : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ is a measurable function..
- $\mathcal{C}1''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each i it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q - 2)\sqrt{q^2 + 20q + 4}\}/8q$. [Go back](#)

Technical assumptions, part 2

$\mathcal{C}3'$ \mathbf{X}_{it} allow for the representation $\mathbf{X}_{it} = \mathbf{H}_i(\dots, u_{it-1}, u_{it})$ with u_{it} being i.i.d. random variables and $\mathbf{H}_i := (H_{i1}, H_{i2}, \dots, H_{id})^\top : \mathbb{R}^Z \rightarrow \mathbb{R}^d$ being a measurable function such that $\mathbf{H}_i(\mathcal{U}_{it})$ is well defined.

$\mathcal{C}3''$ Let \mathbf{N}_i be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being kl -th entry. We assume that the smallest eigenvalue of \mathbf{N}_i is strictly bigger than 0.

$\mathcal{C}3'''$ Let $\mathbb{E}[\mathbf{H}_i(\mathcal{U}_{i0})] = 0$ and $\|\mathbf{H}_i(\mathcal{U}_{it})\|_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where θ will be introduced further.

$\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(\mathbf{H}_i, s) < \infty$ for q' from Assumption $\mathcal{C}3'''$.

$\mathcal{C}4''$ For each i it holds that $\sum_{s=t}^{\infty} \delta_{q'}(\mathbf{H}_i, s) = O(t^{-\alpha})$ for q' from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. [Go back](#)

Technical assumptions, part 3

C6 Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Denote $\mathcal{I}_{it} = (\dots, \zeta_{i,t-1}, \zeta_{i,t})$, $\mathcal{J}_{it} = (\dots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$, $\mathcal{U}_{it} = (\dots, u_{it-1}, u_{it})$, and $U_i(\mathcal{I}_{it}) = \mathbf{H}_i(\mathcal{U}_{it})G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(U_i, s) < \infty$. [Go back](#)

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An interval $[u - h, u + h]$ is called **minimal** if the corresponding local null $H_0^{[i,j]}(u, h)$ is rejected and there is no other interval $[u' - h', u' + h']$ such that we reject $H_0^{[i,j]}(u', h')$ and $[u' - h', u' + h'] \subset [u - h, u + h]$.