Supplement to
“Multiscale Inference for Nonparametric Time Trends”

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In this supplement, we provide the proofs that are omitted in the paper. Specifically, we derive the theoretical results from Section 4 and prove Proposition A.3. We employ the same notation as summarized at the beginning of the Appendix in the paper.

Proof of Proposition A.3

The proof makes use of the following three lemmas, which correspond to Lemmas 5–7 in Chernozhukov et al. (2015).

Lemma S.1. *Let $(W_1, \ldots, W_p)^\top$ be a (not necessarily centred) Gaussian random vector in $\mathbb{R}^p$ with $\text{Var}(W_j) = 1$ for all $1 \le j \le p$. Suppose that $\text{Corr}(W_j, W_k) < 1$ whenever $j \ne k$. Then the distribution of $\max_{1 \le j \le p} W_j$ is absolutely continuous with respect to Lebesgue measure and a version of the density is given by*

$$f(x) = f_0(x) \sum_{j=1}^{p} e^{\mathbb{E}[W_j]x - \mathbb{E}[W_j]^2/2} \, \mathbb{P}\big(W_k \le x \textit{ for all } k \ne j \,\big|\, W_j = x\big).$$

Lemma S.2. *Let $(W_0, W_1, \ldots, W_p)^\top$ be a (not necessarily centred) Gaussian random vector with $\text{Var}(W_j) = 1$ for all $0 \le j \le p$. Suppose that $\mathbb{E}[W_0] \ge 0$. Then the map*

$$x \mapsto e^{\mathbb{E}[W_0]x - \mathbb{E}[W_0]^2/2} \, \mathbb{P}\big(W_j \le x \textit{ for } 1 \le j \le p \,\big|\, W_0 = x\big)$$

is non-decreasing on $\mathbb{R}$.

Lemma S.3. *Let $(X_1, \ldots, X_p)^\top$ be a centred Gaussian random vector in $\mathbb{R}^p$ with $\max_{1 \le j \le p} \mathbb{E}[X_j^2] \le \sigma^2$ for some $\sigma^2 > 0$. Then for any $r > 0$,*

$$\mathbb{P}\Big(\max_{1 \le j \le p} X_j \ge \mathbb{E}\Big[\max_{1 \le j \le p} X_j\Big] + r\Big) \le e^{-r^2/(2\sigma^2)}.$$

The proof of Lemmas S.1 and S.2 can be found in Chernozhukov et al. (2015). Lemma S.3 is a standard result on Gaussian concentration whose proof is given e.g. in Ledoux (2001); see Theorem 7.1 therein. We now closely follow the arguments for the proof of Theorem 3 in Chernozhukov et al. (2015). The proof splits up into three steps.

Step 1. Pick any $x \geq 0$ and set

$$W_j = \frac{X_j - x}{\sigma_j} + \frac{\overline{\mu} + x}{\underline{\sigma}}.$$

By construction, $\mathbb{E}[W_j] \geq 0$ and $\mathrm{Var}(W_j) = 1$. Defining $Z = \max_{1 \leq j \leq p} W_j$, it holds that

$$\begin{aligned}\mathbb{P}\Big(\Big|\max_{1\leq j\leq p} X_j - x\Big| \leq \delta\Big) &\leq \mathbb{P}\Big(\Big|\max_{1\leq j\leq p} \frac{X_j - x}{\sigma_j}\Big| \leq \frac{\delta}{\underline{\sigma}}\Big) \\ &\leq \sup_{y\in\mathbb{R}} \mathbb{P}\Big(\Big|\max_{1\leq j\leq p} \frac{X_j - x}{\sigma_j} + \frac{\overline{\mu}+x}{\underline{\sigma}} - y\Big| \leq \frac{\delta}{\underline{\sigma}}\Big) \\ &= \sup_{y\in\mathbb{R}} \mathbb{P}\Big(|Z - y| \leq \frac{\delta}{\underline{\sigma}}\Big).\end{aligned}$$

Step 2. We now bound the density of Z. Without loss of generality, we assume that $\mathrm{Corr}(W_j, W_k) < 1$ for $k \neq j$. The marginal distribution of W_j is $N(\nu_j, 1)$ with $\nu_j = \mathbb{E}[W_j] = (\mu_j/\sigma_j + \overline{\mu}/\underline{\sigma}) + (x/\underline{\sigma} - x/\sigma_j) \geq 0$. Hence, by Lemmas S.1 and S.2, the random variable Z has a density of the form

$$f_p(z) = f_0(z) G_p(z), \tag{S.1}$$

where the map $z \mapsto G_p(z)$ is non-decreasing. Define $\overline{Z} = \max_{1\leq j\leq p}(W_j - \mathbb{E}[W_j])$ and set $\overline{z} = 2\overline{\mu}/\underline{\sigma} + x(1/\underline{\sigma} - 1/\overline{\sigma})$ such that $\mathbb{E}[W_j] \leq \overline{z}$ for any $1 \leq j \leq p$. With these definitions at hand, we obtain that

$$\begin{aligned}\int_z^\infty f_0(u)du\, G_p(z) &\leq \int_z^\infty f_0(u) G_p(u) du = \mathbb{P}(Z > z) \\ &\leq P(\overline{Z} > z - \overline{z}) \leq \exp\Big(-\frac{(z - \overline{z} - \mathbb{E}[\overline{Z}])_+^2}{2}\Big),\end{aligned}$$

where the last inequality follows from Lemma S.3. Since $W_j - \mathbb{E}[W_j] = (X_j - \mu_j)/\sigma_j$, it holds that

$$\mathbb{E}[\overline{Z}] = \mathbb{E}\Big[\max_{1\leq j\leq p}\Big\{\frac{X_j - \mu_j}{\sigma_j}\Big\}\Big] =: a_p.$$

Hence, for every $z \in \mathbb{R}$,

$$G_p(z) \leq \frac{1}{1 - F_0(z)} \exp\Big(-\frac{(z - \overline{z} - a_p)_+^2}{2}\Big). \tag{S.2}$$

Mill's inequality states that for $z > 0$,

$$z \leq \frac{f_0(z)}{1 - F_0(z)} \leq z\frac{1 + z^2}{z^2}.$$

Since $(1 + z^2)/z^2 \leq 2$ for $z \geq 1$ and $f_0(z)/\{1 - F_0(z)\} \leq 1.53 \leq 2$ for $z \in (-\infty, 1)$, we can infer that

$$\frac{f_0(z)}{1-F_0(z)} \leq 2(z \vee 1) \quad \text{for any } z \in \mathbb{R}.$$

This together with (S.1) and (S.2) yields that

$$f_p(z) \leq 2(z \vee 1) \exp\Big(- \frac{(z - \overline{z} - a_p)_+^2}{2}\Big) \quad \text{for any } z \in \mathbb{R}.$$

Step 3. By Step 2, we get that for any $y \in \mathbb{R}$ and $u > 0$,

$$\mathbb{P}(|Z - y| \leq u) = \int_{y-u}^{y+u} f_p(z) dz \leq 2u \max_{z \in [y-u, y+u]} f_p(z) \leq 4u(\overline{z} + a_p + 1),$$

where the last inequality follows from the fact that the map $z \mapsto z e^{-(z-a)^2/2}$ (with $a > 0$) is non-increasing on $[a+1, \infty)$. Combining this bound with Step 1, we further obtain that for any $x \geq 0$ and $\delta > 0$,

$$\mathbb{P}\Big(\Big| \max_{1 \leq j \leq p} X_j - x \Big| \leq \delta \Big) \leq 4\delta \Big\{ \frac{2\overline{\mu}}{\underline{\sigma}} + |x| \Big(\frac{1}{\underline{\sigma}} - \frac{1}{\overline{\sigma}} \Big) + a_p + 1 \Big\} / \underline{\sigma}. \tag{S.3}$$

This inequality also holds for $x < 0$ by an analogous argument, and hence for all $x \in \mathbb{R}$. Now let $0 < \delta \leq \underline{\sigma}$ and define $b_p = \mathbb{E} \max_{1 \leq j \leq p} \{X_j - \mu_j\}$. For any $|x| \leq \delta + \overline{\mu} + b_p + \overline{\sigma}\sqrt{2 \log(\underline{\sigma}/\delta)}$, (S.3) yields that

$$\begin{aligned} \mathbb{P}\Big(\Big| \max_{1 \leq j \leq p} X_j - x \Big| \leq \delta \Big) &\leq \frac{4\delta}{\underline{\sigma}} \Big\{ \overline{\mu} \Big(\frac{3}{\underline{\sigma}} - \frac{1}{\overline{\sigma}} \Big) + a_p + \Big(\frac{1}{\underline{\sigma}} - \frac{1}{\overline{\sigma}} \Big) b_p \\ &\qquad + \Big(\frac{\overline{\sigma}}{\underline{\sigma}} - 1 \Big) \sqrt{2 \log\Big(\frac{\underline{\sigma}}{\delta} \Big)} + 2 - \frac{\underline{\sigma}}{\overline{\sigma}} \Big\} \\ &\leq C\delta \big\{ \overline{\mu} + a_p + b_p + \sqrt{1 \vee \log(\underline{\sigma}/\delta)} \big\} \end{aligned} \tag{S.4}$$

with a sufficiently large constant $C > 0$ that depends only on $\underline{\sigma}$ and $\overline{\sigma}$. For $|x| \geq \delta + \overline{\mu} + b_p + \overline{\sigma}\sqrt{2 \log(\underline{\sigma}/\delta)}$, we obtain that

$$\mathbb{P}\Big(\Big| \max_{1 \leq j \leq p} X_j - x \Big| \leq \delta \Big) \leq \frac{\delta}{\underline{\sigma}}, \tag{S.5}$$

which can be seen as follows: If $x > \delta + \overline{\mu}$, then $|\max_j X_j - x| \leq \delta$ implies that $|x| - \delta \leq \max_j X_j \leq \max_j \{X_j - \mu_j\} + \overline{\mu}$ and thus $\max_j \{X_j - \mu_j\} \geq |x| - \delta - \overline{\mu}$. Hence, it holds that

$$\mathbb{P}\Big(\Big| \max_{1 \leq j \leq p} X_j - x \Big| \leq \delta \Big) \leq \mathbb{P}\Big(\max_{1 \leq j \leq p} \{X_j - \mu_j\} \geq |x| - \delta - \overline{\mu} \Big). \tag{S.6}$$

If $x < -(\delta + \overline{\mu})$, then $|\max_j X_j - x| \leq \delta$ implies that $\max_j \{X_j - \mu_j\} \leq -|x| + \delta + \overline{\mu}$.

Hence, in this case,

$$\mathbb{P}\Big(\Big|\max_{1\le j\le p} X_j - x\Big| \le \delta\Big) \le \mathbb{P}\Big(\max_{1\le j\le p}\big\{X_j-\mu_j\big\} \le -|x|+\delta+\overline{\mu}\Big)$$
$$\le \mathbb{P}\Big(\max_{1\le j\le p}\big\{X_j-\mu_j\big\} \ge |x|-\delta-\overline{\mu}\Big), \qquad \text{(S.7)}$$

where the last inequality follows from the fact that for centred Gaussian random variables V_j and $v>0$, $\mathbb{P}(\max_j V_j \le -v) \le \mathbb{P}(V_1 \le -v) = P(V_1 \ge v) \le \mathbb{P}(\max_j V_j \ge v)$. With (S.6) and (S.7), we obtain that for any $|x| \ge \delta + \overline{\mu} + b_p + \overline{\sigma}\sqrt{2\log(\underline{\sigma}/\delta)}$,

$$\mathbb{P}\Big(\Big|\max_{1\le j\le p} X_j - x\Big| \le \delta\Big) \le \mathbb{P}\Big(\max_{1\le j\le p}\big\{X_j-\mu_j\big\} \ge |x|-\delta-\overline{\mu}\Big)$$
$$\le \mathbb{P}\Big(\max_{1\le j\le p}\big\{X_j-\mu_j\big\} \ge \mathbb{E}\Big[\max_{1\le j\le p}\big\{X_j-\mu_j\big\}\Big] + \overline{\sigma}\sqrt{2\log(\underline{\sigma}/\delta)}\Big) \le \frac{\delta}{\underline{\sigma}},$$

the last inequality following from Lemma S.3. To sum up, we have established that for any $0<\delta\le\underline{\sigma}$ and any $x\in\mathbb{R}$,

$$\mathbb{P}\Big(\Big|\max_{1\le j\le p} X_j - x\Big| \le \delta\Big) \le C\delta\big\{\overline{\mu} + a_p + b_p + \sqrt{1\vee\log(\underline{\sigma}/\delta)}\big\} \qquad \text{(S.8)}$$

with some constant $C>0$ that does only depend on $\underline{\sigma}$ and $\overline{\sigma}$. For $\delta > \underline{\sigma}$, (S.8) trivially follows upon setting $C \ge 1/\underline{\sigma}$. This completes the proof.

References

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