Simultaneous statistical inference for epidemic trends: the case of COVID-19

Marina Khismatullina Michael Vogt

01/10/2020

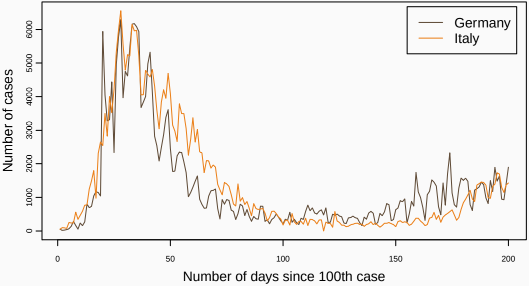
Table of contents

Introduction

Motivation

Research question:

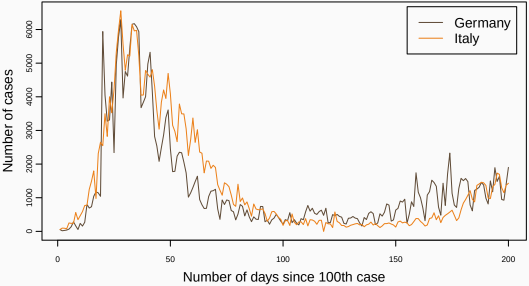
How do outbreak patterns of COVID-19 compare across countries?



Motivation

Research question:

How do outbreak patterns of COVID-19 compare across countries?



Aim of the paper

To develop new inference methods that allow to *identify* and *locate* differences between time trends.

Model

Model

We observe n time series $\mathcal{X}_i = \{X_{it} : 1 \leq t \leq T\}$ of length T:

$$X_{it} = \lambda_i\Big(\frac{t}{T}\Big) + \sigma\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it},$$

Model

We observe n time series $\mathcal{X}_i = \{X_{it} : 1 \leq t \leq T\}$ of length T:

$$X_{it} = \lambda_i\Big(\frac{t}{T}\Big) + \sigma\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it},$$

where

- λ_i are unknown trend functions on $[0, 1]$;
- σ is the overdispersion parameter;
- η_{it} are error terms that are independent across i and t and have zero mean and unit variance.

Literature

Comparison of deterministic trends:

- Park et al. (2009), Degras et al. (2012), Zhang et al. (2012), Hidalgo and Lee (2014), Chen and Wu (2019).

Literature

Comparison of deterministic trends:

- Park et al. (2009), Degras et al. (2012), Zhang et al. (2012), Hidalgo and Lee (2014), Chen and Wu (2019).

Studies of COVID-19:

- SEIR models: Yang et al. (2020), Wu et al. (2020), De Brouwer et al. (2020).
- Time series analysis: Gu et al. (2020), Li and Linton (2020).
- Dong et al. (2020).

Testing

Testing problem

Let $\mathcal{F} = \{\mathcal{I}_k \subseteq [0,1] : 1 \leq k \leq K\}$ be a family of intervals on $[0,1]$, and for a given interval $\mathcal{I}_k$ we want to test whether the functions λ_i and λ_j are the same on this interval. Formally, the testing problem is

$$H_0^{(ijk)} : \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

Testing problem

Let $\mathcal{F} = \{\mathcal{I}_k \subseteq [0,1] : 1 \leq k \leq K\}$ be a family of intervals on $[0,1]$, and for a given interval $\mathcal{I}_k$ we want to test whether the functions λ_i and λ_j are the same on this interval. Formally, the testing problem is

$$H_0^{(ijk)} : \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

We want to test these hypothesis $H_0^{(ijk)}$ simultaneously for all pairs of countries i and j and all intervals $\mathcal{I}_k$ in the family $\mathcal{F}$.

Test statistic

For the given interval $\mathcal{I}_k$ and a pair of time series i and j we calculate

$$\hat{s}_{ijk,T} = \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it} - X_{jt}),$$

where h_k is the length of $\mathcal{I}_k$.

Test statistic

For the given interval $\mathcal{I}_k$ and a pair of time series i and j we calculate

$$\hat{s}_{ijk,T} = \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it} - X_{jt}),$$

where h_k is the length of $\mathcal{I}_k$. The statistic $\hat{s}_{ijk,T}$ estimates the average distance between λ_i and λ_j on $\mathcal{I}_k$.

Test statistic

For the given interval $\mathcal{I}_k$ and a pair of time series i and j we calculate

$$\hat{s}_{ijk,T} = \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it} - X_{jt}),$$

where h_k is the length of $\mathcal{I}_k$. The statistic $\hat{s}_{ijk,T}$ estimates the average distance between λ_i and λ_j on $\mathcal{I}_k$. Under certain assumptions,

$$\mathrm{Var}(\hat{s}_{ijk,T}) = \frac{\sigma^2}{T^2 h_k^2} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big\{\lambda_i\Big(\frac{t}{T}\Big) + \lambda_j\Big(\frac{t}{T}\Big)\Big\}.$$

Test statistic

For the given interval $\mathcal{I}_k$ and a pair of time series i and j we calculate

$$\hat{s}_{ijk,T} = \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it} - X_{jt}),$$

where h_k is the length of $\mathcal{I}_k$. The statistic $\hat{s}_{ijk,T}$ estimates the average distance between λ_i and λ_j on $\mathcal{I}_k$. Under certain assumptions,

$$\mathrm{Var}(\hat{s}_{ijk,T}) = \frac{\sigma^2}{T^2 h_k^2} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big\{\lambda_i\Big(\frac{t}{T}\Big) + \lambda_j\Big(\frac{t}{T}\Big)\Big\}.$$

In order to normalize the variance of the statistic $\hat{s}_{ijk,T}$, we scale it by an estimator of its variance:

$$\widehat{\mathrm{Var}(\hat{s}_{ijk,T})} = \frac{\hat{\sigma}^2}{T^2 h_k^2} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it} + X_{jt}),$$

with $\hat{\sigma}^2 = n^{-1} \sum_{i=1}^{n} \hat{\sigma}_i^2$ and $\hat{\sigma}_i^2 = \frac{\sum_{t=2}^{T} (X_{it} - X_{it-1})^2}{2 \sum_{t=1}^{T} X_{it}}$. Idea

Test statistic

Test statistic for the hypothesis $H_0^{(ijk)}$ is defined as

$$\widehat{\psi}_{ijk,T} = \frac{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} - X_{jt})}{\hat{\sigma}\big\{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\big\}^{1/2}}.$$

Test statistic

Test statistic for the hypothesis $H_0^{(ijk)}$ is defined as

$$\widehat{\psi}_{ijk,T} = \frac{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} - X_{jt})}{\hat{\sigma}\big\{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\big\}^{1/2}}.$$

Under certain conditions and under the null, $\widehat{\psi}_{ijk,T}$ can be approximated by the Gaussian version of the test statistic:

$$\phi_{ijk,T} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(Z_{it} - Z_{jt}),$$

where Z_{it} are independent standard normal random variables.

Test procedure

$$H_0^{(ijk)}: \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

Test procedure

$$H_0^{(ijk)}: \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

How do determine critical values?

Test procedure

$$H_0^{(ijk)}: \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

How do determine critical values?

1. Consider the Gaussian test statistic $\Phi_T = \max_{(i,j,k)} a_k\big(|\phi_{ijk,T}| - b_k\big)$, where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants.

Test procedure

$$H_0^{(ijk)}: \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

How do determine critical values?

1. Consider the Gaussian test statistic $\Phi_T = \max_{(i,j,k)} a_k \big(|\phi_{ijk,T}| - b_k\big)$, where $a_k = \{\log(e/h_k)\}^{1/2} / \log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants.
2. Denote by $q_{T,\text{Gauss}}(\alpha)$ a $(1-\alpha)$-quantile of Φ_T, which can be computed by Monte Carlo simulations.

Test procedure

$$H_0^{(ijk)}: \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

How do determine critical values?

1. Consider the Gaussian test statistic $\Phi_T = \max_{(i,j,k)} a_k\big(|\phi_{ijk,T}| - b_k\big)$, where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants.
2. Denote by $q_{T,\text{Gauss}}(\alpha)$ a $(1-\alpha)$-quantile of Φ_T, which can be computed by Monte Carlo simulations.
3. Adjust the quantile $q_{T,\text{Gauss}}(\alpha)$ by the scale-dependent constants: $c_{T,\text{Gauss}}(\alpha, h_k) = b_k + q_{T,\text{Gauss}}(\alpha)/a_k$. Idea

Test procedure

$$H_0^{(ijk)}: \quad \lambda_i(w) = \lambda_j(w) \text{ for all } w \in \mathcal{I}_k.$$

How do determine critical values?

1. Consider the Gaussian test statistic $\Phi_T = \max_{(i,j,k)} a_k\big(|\phi_{ijk,T}| - b_k\big)$, where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants.
2. Denote by $q_{T,\text{Gauss}}(\alpha)$ a $(1-\alpha)$-quantile of Φ_T, which can be computed by Monte Carlo simulations.
3. Adjust the quantile $q_{T,\text{Gauss}}(\alpha)$ by the scale-dependent constants: $c_{T,\text{Gauss}}(\alpha, h_k) = b_k + q_{T,\text{Gauss}}(\alpha)/a_k$. Idea

Test procedure

For the given significance level $\alpha \in (0,1)$ and for each (i,j,k), reject $H_0^{(ijk)}$ if $|\widehat{\psi}_{ijk,T}| > c_{T,\text{Gauss}}(\alpha, h_k)$.

Theoretical properties

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous: $|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous: $|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

$\mathcal{C}2$ $0 < \lambda_{\min} \leq \lambda_i(w) \leq \lambda_{\max} < \infty$ for all $w \in [0, 1]$ and all i.

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous:
$|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

$\mathcal{C}2$ $0 < \lambda_{\text{min}} \leq \lambda_i(w) \leq \lambda_{\text{max}} < \infty$ for all $w \in [0, 1]$ and all i.

$\mathcal{C}3$ η_{it} are independent both across i and t.

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous:
$|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

$\mathcal{C}2$ $0 < \lambda_{\min} \leq \lambda_i(w) \leq \lambda_{\max} < \infty$ for all $w \in [0, 1]$ and all i.

$\mathcal{C}3$ η_{it} are independent both across i and t.

$\mathcal{C}4$ $\mathbb{E}[\eta_{it}] = 0$, $\mathbb{E}[\eta_{it}^2] = 1$ and $\mathbb{E}[|\eta_{it}|^\theta] \leq C_\theta < \infty$ for some $\theta > 4$.

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous:
$|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

$\mathcal{C}2$ $0 < \lambda_{\min} \leq \lambda_i(w) \leq \lambda_{\max} < \infty$ for all $w \in [0, 1]$ and all i.

$\mathcal{C}3$ η_{it} are independent both across i and t.

$\mathcal{C}4$ $\mathbb{E}[\eta_{it}] = 0$, $\mathbb{E}[\eta_{it}^2] = 1$ and $\mathbb{E}[|\eta_{it}|^\theta] \leq C_\theta < \infty$ for some $\theta > 4$.

$\mathcal{C}5$ $h_{\max} = o(1/\log T)$ and $h_{\min} \geq CT^{-b}$ for some $b \in (0, 1)$.

Assumptions

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous:
$|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

$\mathcal{C}2$ $0 < \lambda_{\text{min}} \leq \lambda_i(w) \leq \lambda_{\text{max}} < \infty$ for all $w \in [0, 1]$ and all i.

$\mathcal{C}3$ η_{it} are independent both across i and t.

$\mathcal{C}4$ $\mathbb{E}[\eta_{it}] = 0$, $\mathbb{E}[\eta_{it}^2] = 1$ and $\mathbb{E}[|\eta_{it}|^\theta] \leq C_\theta < \infty$ for some $\theta > 4$.

$\mathcal{C}5$ $h_{\text{max}} = o(1/\log T)$ and $h_{\text{min}} \geq CT^{-b}$ for some $b \in (0, 1)$.

$\mathcal{C}6$ $p := \{\#(i, j, k)\} = O(T^{(\theta/2)(1-b)-(1+\delta)})$ for some small $\delta > 0$.

Theoretical properties

Proposition

Denote $\mathcal{M}_0$ the set of triplets (i, j, k) where $H_0^{(ijk)}$ holds true. Then under $\mathcal{C}1 - \mathcal{C}6$, it holds that

$$\mathrm{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk,T}| \leq c_{T,\mathrm{Gauss}}(\alpha, h_k)\Big) \geq 1 - \alpha + o(1)$$

Theoretical properties

Proposition

Denote $\mathcal{M}_0$ the set of triplets (i, j, k) where $H_0^{(ijk)}$ holds true. Then under $\mathcal{C}1 - \mathcal{C}6$, it holds that

$$\mathrm{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk,T}| \le c_{T,\text{Gauss}}(\alpha, h_k)\Big) \ge 1 - \alpha + o(1)$$

Proposition

Consider any sequence of functions $\lambda_i = \lambda_{i,T}$, $\lambda_j = \lambda_{j,T}$ with the following property: There exists an interval $\mathcal{I}_k$ such that $\lambda_{i,T}(w) - \lambda_{j,T}(w) \ge c_T \sqrt{T \log T / h_k}$ for all $w \in \mathcal{I}_k$, and $c_T \to \infty$. Then under $\mathcal{C}1 - \mathcal{C}6$, it holds that

$$\mathrm{P}\big(|\hat{\psi}_{ijk,T}| \le c_{T,\text{Gauss}}(\alpha, h_k)\big) = o(1).$$

Notation

In order to proceed with the proof, we will need the following notation:

$$\widehat{\psi}_{ijk,T} = \frac{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} - X_{jt})}{\hat{\sigma}\big\{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\big\}^{1/2}},$$

$$\hat{\psi}^0_{ijk,T} = \frac{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)\, \sigma \overline{\lambda}_{ij}^{1/2}\big(\frac{t}{T}\big)(\eta_{it} - \eta_{jt})}{\hat{\sigma}\{\sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)(X_{it} + X_{jt})\}^{1/2}} \qquad \hat{\Psi}^0_T = \max_{(i,j,k)} a_k\big(|\hat{\psi}^0_{ijk,T}| - b_k\big),$$

$$\psi^0_{ijk,T} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(\eta_{it} - \eta_{jt}) \qquad \Psi_T = \max_{(i,j,k)} a_k\big(|\psi^0_{ijk,T}| - b_k\big),$$

$$\phi_{ijk,T} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(Z_{it} - Z_{jt}) \qquad \Phi_T = \max_{(i,j,k)} a_k\big(|\phi_{ijk,T}| - b_k\big).$$

Strategy of the proof

1. We prove that $\left|\hat{\Psi}^0_T - \Psi_T\right| = o_p(r_T)$, where $\{r_T\}$ is some null sequence.
2. With the help of results from Chernozhukov et al. (2017), we prove

$$\sup_{q \in \mathbf{R}} \left| \mathrm{P}\big(\Psi_T \leq q\big) - \mathrm{P}\big(\Phi_T \leq q\big) \right| = o(1).$$

3. By using these two results, we now show that

$$\sup_{q \in \mathbb{R}} \left| \mathrm{P}\big(\hat{\Psi}^0_T \leq q\big) - \mathrm{P}\big(\Phi_T \leq q\big) \right| = o(1). \tag{1}$$

4. $\mathrm{P}(\Phi_T \leq q_{T,\mathsf{Gauss}}(\alpha)) = 1 - \alpha$ by definition of the quantile $q_{T,\mathsf{Gauss}}(\alpha)$. From this and (1), it immediately follows that

$$\mathrm{P}\big(\hat{\Psi}^0_T \leq q_{T,\mathsf{Gauss}}(\alpha)\big) = 1 - \alpha + o(1),$$

which in turn implies the desired statement.

Graphical representation

Minimal intervals

An interval $\mathcal{I}_k \in \mathcal{F}_{\text{reject}}(i, j)$ is called **minimal** if there is no other interval $\mathcal{I}_{k'} \in \mathcal{F}_{\text{reject}}(i, j)$ with $\mathcal{I}_{k'} \subset \mathcal{I}_k$. The set of minimal intervals is denoted $\mathcal{F}^{\text{min}}_{\text{reject}}(i, j)$.

Graphical representation

Minimal intervals

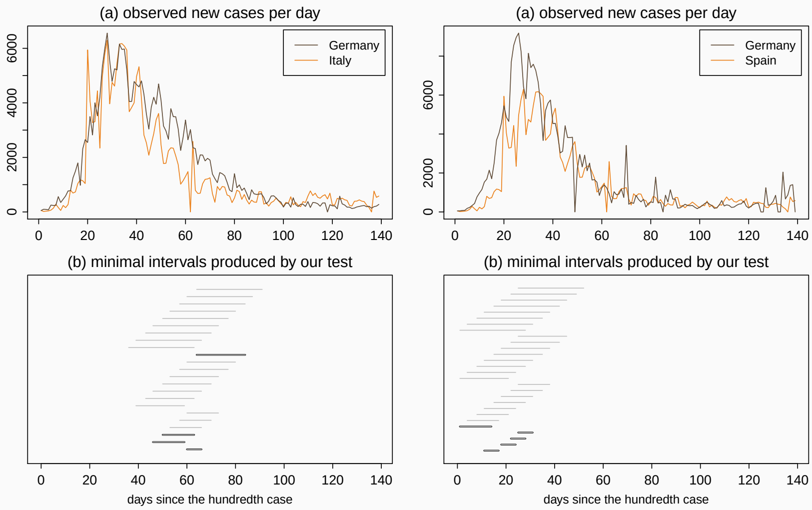
An interval $\mathcal{I}_k \in \mathcal{F}_{\text{reject}}(i,j)$ is called **minimal** if there is no other interval $\mathcal{I}_{k'} \in \mathcal{F}_{\text{reject}}(i,j)$ with $\mathcal{I}_{k'} \subset \mathcal{I}_k$. The set of minimal intervals is denoted $\mathcal{F}^{\text{min}}_{\text{reject}}(i,j)$.

We can make very similar confidence statement about the set of minimal intervals as well:

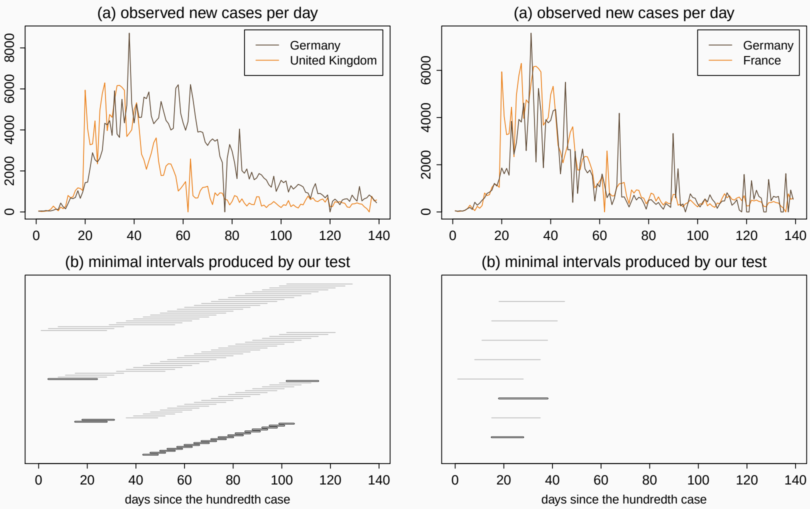
$$\mathrm{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : \mathcal{I}_k \notin \mathcal{F}^{\text{min}}_{\text{reject}}(i,j)\Big) \geq 1 - \alpha + o(1).$$

Application

Application results



Application results, part 2



Discussion

We can claim, with confidence of about 95%, that we are able to detect all of the differences between the trend functions in different countries.

Discussion

We can claim, with confidence of about 95%, that we are able to detect all of the differences between the trend functions in different countries.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Discussion

We can claim, with confidence of about 95%, that we are able to detect all of the differences between the trend functions in different countries.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Further possible extensions:

- introduce scaling factor in the trend function, that allow for adjusting for the size of the country (population, density, testing regimes, etc.);

Discussion

We can claim, with confidence of about 95%, that we are able to detect all of the differences between the trend functions in different countries.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Further possible extensions:

- introduce scaling factor in the trend function, that allow for adjusting for the size of the country (population, density, testing regimes, etc.);
- connect with data-driven techniques such as machine learning;

Discussion

We can claim, with confidence of about 95%, that we are able to detect all of the differences between the trend functions in different countries.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Further possible extensions:

- introduce scaling factor in the trend function, that allow for adjusting for the size of the country (population, density, testing regimes, etc.);
- connect with data-driven techniques such as machine learning;
- cluster the countries based on the trends they exhibit;

Discussion

We can claim, with confidence of about 95%, that we are able to detect all of the differences between the trend functions in different countries.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Further possible extensions:

- introduce scaling factor in the trend function, that allow for adjusting for the size of the country (population, density, testing regimes, etc.);
- connect with data-driven techniques such as machine learning;
- cluster the countries based on the trends they exhibit;
- build in policy changes.

Thank you!

Simulation results for the size of the test

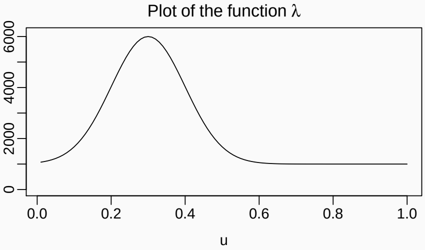


Table 1: Size of the multiscale test

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.011	0.047	0.093	0.010	0.044	0.087	0.008	0.037	0.075
$T = 250$	0.009	0.047	0.091	0.009	0.046	0.087	0.008	0.035	0.069
$T = 500$	0.010	0.044	0.083	0.008	0.048	0.093	0.007	0.035	0.077

Simulation results for the power of the test

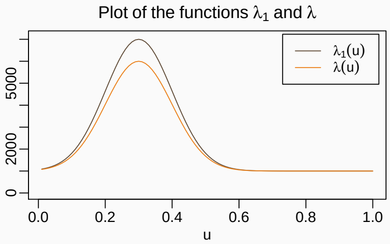
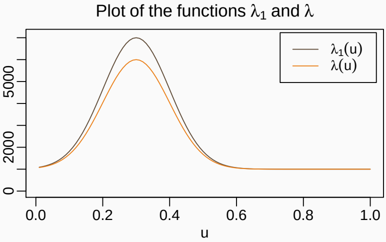


Table 2: Power of the multiscale test for scenario A

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.335	0.518	0.597	0.306	0.474	0.545	0.212	0.352	0.418
$T = 250$	0.615	0.790	0.836	0.580	0.764	0.800	0.470	0.648	0.705
$T = 500$	0.736	0.905	0.917	0.738	0.884	0.890	0.636	0.799	0.830

Simulation results for the power of the test



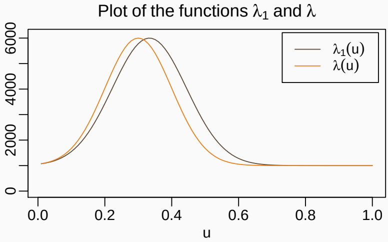


Table 3: Power of the multiscale test for scenario B

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.824	0.910	0.903	0.812	0.893	0.890	0.738	0.847	0.857
$T = 250$	0.991	0.972	0.941	0.991	0.960	0.920	0.991	0.965	0.933
$T = 500$	0.997	0.973	0.949	0.995	0.961	0.923	0.996	0.969	0.932

Idea behind $\hat{\sigma}$

We assume that λ_i is Lipschitz continuous. Then

$$X_{it} - X_{it-1} = \sigma \sqrt{\lambda_i\Big(\frac{t}{T}\Big)}(\eta_{it} - \eta_{it-1}) + r_{it},$$

where $|r_{it}| \leq C(1 + |\eta_{it-1}|)/T$ with a sufficiently large C.

Idea behind $\hat{\sigma}$

We assume that λ_i is Lipschitz continuous. Then

$$X_{it} - X_{it-1} = \sigma \sqrt{\lambda_i\Big(\frac{t}{T}\Big)} (\eta_{it} - \eta_{it-1}) + r_{it},$$

where $|r_{it}| \leq C(1 + |\eta_{it-1}|)/T$ with a sufficiently large C. Hence,

$$\frac{1}{T} \sum_{t=2}^{T} (X_{it} - X_{it-1})^2 = 2\sigma^2 \Big\{ \frac{1}{T} \sum_{t=2}^{T} \lambda_i(t/T) \Big\} + o_p(1).$$

Idea behind $\hat{\sigma}$

We assume that λ_i is Lipschitz continuous. Then

$$X_{it} - X_{it-1} = \sigma\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}(\eta_{it} - \eta_{it-1}) + r_{it},$$

where $|r_{it}| \leq C(1 + |\eta_{it-1}|)/T$ with a sufficiently large C. Hence,

$$\frac{1}{T}\sum_{t=2}^{T}(X_{it} - X_{it-1})^2 = 2\sigma^2\Big\{\frac{1}{T}\sum_{t=2}^{T}\lambda_i(t/T)\Big\} + o_p(1).$$

Together with

$$\frac{1}{T}\sum_{t=1}^{T}X_{it} = \frac{1}{T}\sum_{t=1}^{T}\lambda_i(t/T) + o_p(1),$$

we get that $\hat{\sigma}_i^2 = \sigma^2 + o_p(1)$ for any i and thus $\hat{\sigma}^2 = \sigma^2 + o_p(1)$.

Go back

Idea behind a_k and b_k

How to construct critical values $c_{ijk,T}(\alpha)$?

Idea behind a_k and b_k

How to construct critical values $c_{ijk,T}(\alpha)$?

- Traditional approach: $c_T(\alpha) = c_{ijk,T}(\alpha)$ for all (i, j, k).

Idea behind a_k and b_k

How to construct critical values $c_{ijk,T}(\alpha)$?

- Traditional approach: $c_T(\alpha) = c_{ijk,T}(\alpha)$ for all (i,j,k).
- A more modern approach: $c_{ijk,T}(\alpha)$ depend on the length h_k of the time interval (Dümbgen and Spokoiny (2001)). In our context:

$$c_{ijk,T}(\alpha) = c_T(\alpha, h_k) := b_k + q_T(\alpha)/a_k,$$

where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants and $q_T(\alpha)$ is the $(1-\alpha)$-quantile of the statistic

$$\hat{\Psi}_T = \max_{(i,j,k)} a_k\big(|\hat{\psi}^0_{ijk,T}| - b_k\big)$$

in order to ensure control of the FWER at level α.

Go back

Idea behind the additive correction

Consider the uncorrected Gaussian statistic

$$\Phi_T^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk,T}|$$

Idea behind the additive correction

Consider the uncorrected Gaussian statistic

$$\Phi_T^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk,T}|$$

and let the family of intervals be

$$\mathcal{F} = \big\{[(m-1)h_l, mh_l] \text{ for } 1 \leq m \leq 1/h_l, 1 \leq l \leq L\big\}.$$

Idea behind the additive correction

Consider the uncorrected Gaussian statistic

$$\Phi_T^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk,T}|$$

and let the family of intervals be

$$\mathcal{F} = \big\{[(m-1)h_l, mh_l] \text{ for } 1 \leq m \leq 1/h_l, 1 \leq l \leq L\big\}.$$

Then we can rewrite the uncorrected test statistic as

$$\Phi_T^{\text{uncor}} = \max_{i,j} \max_{\substack{1 \leq l \leq L, \\ 1 \leq m \leq 1/h_l}} \Big| \frac{1}{\sqrt{2Th_l}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in [(m-1)h_l, mh_l]\Big)(Z_{it} - Z_{jt})\Big|$$

Idea behind the additive correction

Consider the uncorrected Gaussian statistic

$$\Phi_T^{\text{uncor}} = \max_{(i,j,k)} |\phi_{ijk,T}|$$

and let the family of intervals be

$$\mathcal{F} = \big\{[(m-1)h_l, mh_l] \text{ for } 1 \le m \le 1/h_l, 1 \le l \le L\big\}.$$

Then we can rewrite the uncorrected test statistic as

$$\Phi_T^{\text{uncor}} = \max_{i,j} \max_{\substack{1 \le l \le L, \\ 1 \le m \le 1/h_l}} \left| \frac{1}{\sqrt{2Th_l}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in [(m-1)h_l, mh_l]\Big)(Z_{it} - Z_{jt}) \right|$$

$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of Φ_T^{uncor} is dominated by the elements with small bandwidths h_l. Go back