1 Nonparametric inference for the time-varying regression model

Consider the time-varying regression model

$$Y_t = m_t(X_t) + e_t, \quad t = 1, \dots, T, \tag{1}$$

where Y_t, X_t and e_t are the responses, the predictors and the errors, respectively, and $m_t(\cdot) = m(\cdot, t/T)$ is a time-varying regression function. Suppose that X_t has compact support $\mathcal{X} \in \mathbb{R}$. Here $m : \mathcal{X} \times [0,1] \to \mathbb{R}$ is a smooth function, and t/T, $t = 1, \dots, T$, represents the time rescaled to the unit interval.
Let $K_X(\cdot)$ be a (potentially in the future d-dimensional) kernel function, and let $K_T(\cdot)$ be a temporal kernel function. Assumptions on the kernel functions are provided in Assumption (C3) and (C4).
Consider two bandwidths h_x and h_t, a point $(u, s) \in \mathbb{R} \times [0,1]$ and the corresponding kernel average

$$\widehat{\psi}_{h_x,h_t}(u,s) = \sum_{t=1}^{T} w_{t,h_x,h_t}(u,s) Y_t,$$

where $w_{t,h_x,h_t}(u,s)$ is a kernel weight. In order to avoid boundary issues, we work with a local linear weighting scheme. We in particular set

$$w_{t,h_x,h_t}(u,s) = \frac{\Lambda_{t,h_t}(s) K_X\left(\frac{X_t - u}{h_x}\right)}{\sum_{t=1}^{T} \Lambda_{t,h_t}(s) K_X\left(\frac{X_t - u}{h_x}\right)}, \tag{2}$$

where

$$\Lambda_{t,h_t}(s) = K_T\Big(\frac{\frac{t}{T} - s}{h_t}\Big)\Big[S_2(s) - \Big(\frac{\frac{t}{T} - s}{h_t}\Big) S_1(s)\Big],$$

and $S_\ell(s) = (Th_t)^{-1} \sum_{t=1}^{T} K_T(\frac{\frac{t}{T}-s}{h_t})(\frac{\frac{t}{T}-s}{h_t})^\ell$ for $\ell = 0, 1, 2$.
The kernel average $\widehat{\psi}_{h_x,h_t}(u,s)$ is nothing else than a rescaled local linear estimator of the function $m(\cdot,\cdot)$.
To allow nonstationary and dependent observations, we assume that the covariates X_t have the following properties.

(C1) The variables X_t allow for the representation $X_t = H(t/T; \mathcal{G}_t)$, where $\mathcal{G}_t = (\dots, \xi_{t-1}, \xi_t)$, the random variables ξ_t are i.i.d. and $H : [0,1] \times \mathbb{R}^\infty \to \mathcal{X}$ is a measurable function such that $H(t/T; \mathcal{G}_t)$ is well-defined for each t.

(C2) The value of $\mathbb{E}[H^2(t/T; \mathcal{G}_0)]$ is bounded away from zero and infinity on $[0,1]$.

For the error process, we assume that

$$e_t = \sigma_t(X_t)\eta_t = \sigma(X_t, t/T)\eta_t,$$

where for now we consider i.i.d. η_t.
In order for the theory to work, we need the following assumptions:

(C3) The kernel K_X is non-negative, symmetric about zero and integrates to one. Moreover, it has compact support $[-1, 1]$ and is Lipschitz continuous, that is, $|K_X(v) - K_X(w)| \leq C|v - w|$ for any $v, w \in \mathbb{R}$ and some constant $C > 0$.

(C4) The kernel K_T is non-negative, symmetric about zero and integrates to one. Moreover, it has compact support $[-1, 1]$ and is Lipschitz continuous, that is, $|K_T(v) - K_T(w)| \leq C|v - w|$ for any $v, w \in \mathbb{R}$ and some constant $C > 0$.