Multiscale testing for equality of nonparametric trend curves

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Introduction

Motivation

Aim of the paper

To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.

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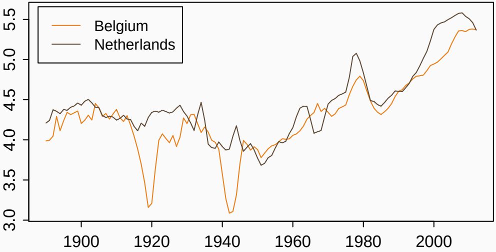

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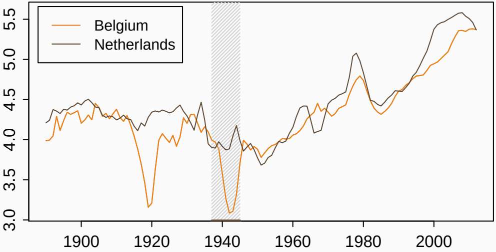

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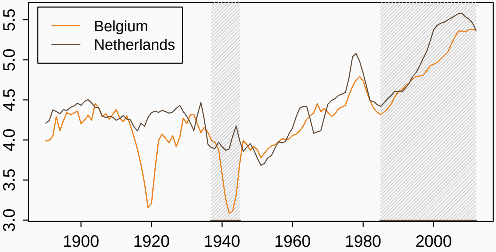

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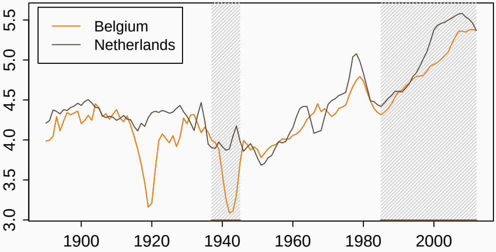

Research question: Out of many given intervals, how to find those where the trends are significantly different?

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Finding systematic differences between trends = basis for further research.

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Is it limited to a particular application?

No! Our method = general method for comparing nonparametric trends

$\Rightarrow$ new statistical test for equality of nonparametric trend curves.

Literature

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Model

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We observe a panel of n time series $\mathcal{Z}_i = \{(Y_{it}, \boldsymbol{X}_{it}) : 1 \leq t \leq T\}$ of length T, where $Y_{it} \in \mathbb{R}$ and $\boldsymbol{X}_{it} \in \mathbb{R}^d$.

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$$Y_{it} = m_i\Big(\frac{t}{T}\Big) + \boldsymbol{\beta}_i^T \boldsymbol{X}_{it} + \alpha_i + \varepsilon_{it},$$

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- α_i are so-called fixed effect error terms;
- $\mathcal{E}_i = \{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

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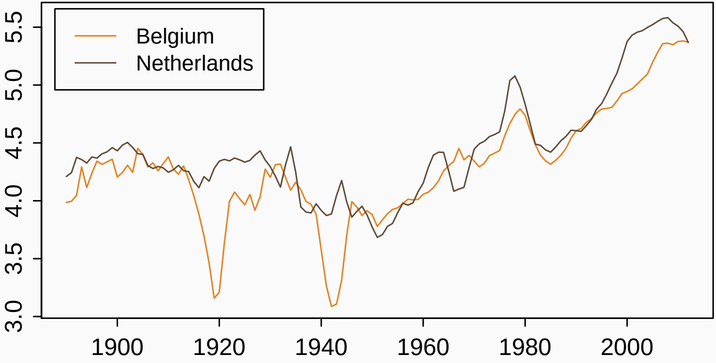
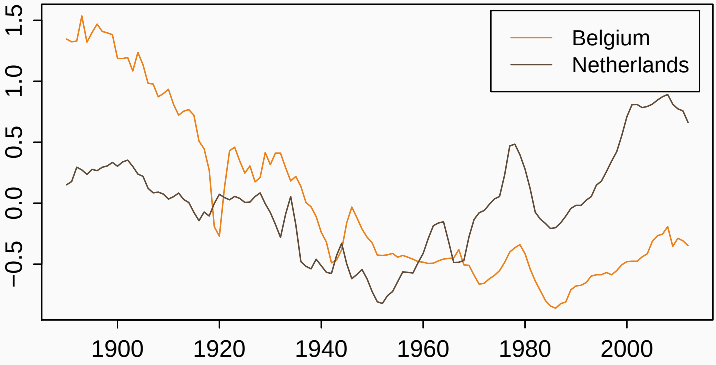
Original time series: Belgium and Netherlands



Augmented time series: Belgium and Netherlands



Testing procedure

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$$H_0^{[i,j]}(u, h) : m_i(w) = m_j(w) \text{ for all } w \in [u - h, u + h].$$

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Then the global null $H_0 : m_1 = m_2 = \ldots = m_n$ can be reformulated as

$$H_0 : \text{The hypotheses } H_0^{[i,j]}(u, h) \text{ hold true for all intervals}$$
$$[u - h, u + h], (u, h) \in \mathcal{G}_T, \text{ and for all } 1 \leq i < j \leq n.$$

Test statistic

For a given location $u \in [0,1]$ and bandwidth h and a given pair (i,j) we construct the kernel averages

$$\widehat{\psi}_{ij}(u,h) = \sum_{t=1}^{T} w_t(u,h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

where $w_t(u,h)$ are appropriate weights.

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The kernel averages $\widehat{\psi}_{ij}(u,h)$ measure the distance between two trend curves m_i and m_j on $[u-h, u+h]$.

Test statistic, part 2

Instead with working directly with $\widehat{\psi}_{ij}(u,h)$, we replace them by

$$\widehat{\psi}^0_{ij}(u,h) = \left\{ \left| \frac{\widehat{\psi}_{ij}(u,h)}{\left(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\right)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\widehat{\sigma}_i^2$ is an appropriate estimator of the long-run variance σ_i^2;
- $\lambda(h) = \sqrt{2\log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

Test statistic, part 3

To test the global null, we aggregate the individual test statistics for all (i, j) and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\widehat{\Psi}_T = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \widehat{\psi}^0_{ij}(u, h).$$

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Main theoretical result

Under certain conditions and under the null, $\widehat{\psi}^0_{ij}(u, h)$ and $\widehat{\Psi}_T$ can be approximated by the corresponding Gaussian versions of the test statistics.

Test procedure

Gaussian version of the individual test statistics:

$$\phi_{ij}^0(u,h) = \max_{(u,h)\in\mathcal{G}_T} \left\{ \left| \frac{\phi_{ij}(u,h)}{\left(\sigma_i^2 + \sigma_j^2\right)^{1/2}} \right| - \lambda(h) \right\},$$

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Aggregated Gaussian test statistics:

$$\Phi_T = \max_{1\le i<j\le n} \max_{(u,h)\in\mathcal{G}_T} \phi_{ij}^0(u,h).$$

Test procedure, part 2

1. Consider the Gaussian test statistic

$$\Phi_T = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \phi_{ij}^0(u, h),$$

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3. Perform the test for the global hypothesis H_0: reject H_0 if $\widehat{\Psi}_T > q_T(\alpha)$.
4. For each i, j, and each $(u,h) \in \mathcal{G}_T$, carry out the test for the local null hypothesis $H_0^{[i,j]}(u,h)$: reject $H_0^{[i,j]}(u,h)$ if $\widehat{\psi}_{ij}^0(u,h) > q_T(\alpha)$.

Theoretical properties

Proposition

Under certain assumptions and under the null, it holds that

$$\mathbb{P}\Big(\widehat{\Psi}_T \leq q_T(\alpha)\Big) = 1 - \alpha + o(1)$$

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$$FWER(\alpha) \leq \alpha$$

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Corollary

$$\mathit{FWER}(\alpha) \leq \alpha$$

Proposition

Consider a sequence of functions $m_i = m_{i,T}$, $m_j = m_{j,T}$ *such that*

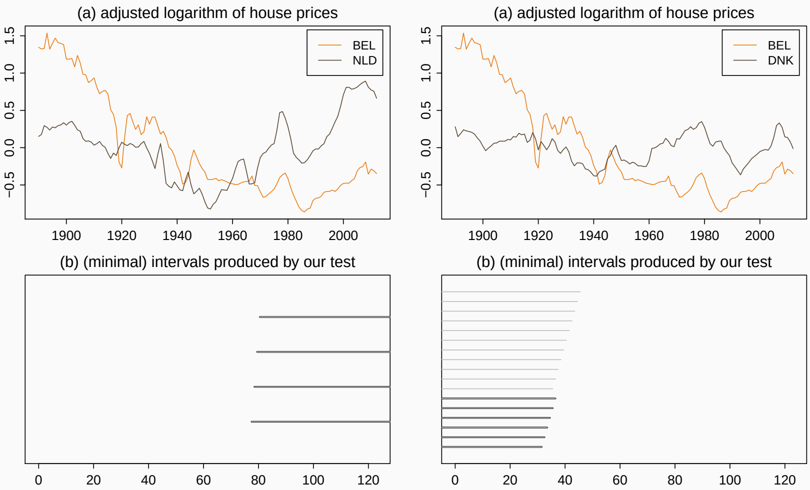
$$\exists\, (u,h) \in \mathcal{G}_T : m_i(w) - m_j(w) \geq c_T \sqrt{\log T/(Th)}\ \forall w \in [u-h, u+h],$$

and $c_T \to \infty$. *Then under our assumptions, it holds that*

$$\mathbb{P}\Big(\widehat{\Psi}_T \leq q_T(\alpha)\Big) = o(1)$$

Illustration

Application results



Discussion

We can claim, with confidence of at least 95%, that the null hypothesis is violated for all intervals (and all pairs of time series) for which our test rejects the null.

However, we can not say anything about the causes of such differences. This question requires further (probably not purely statistical) analysis.

Further possible extensions:

- cluster the time series based on the trends they exhibit;
- introduce scaling factor in the trend function;
- include the dependence between covariates and error terms.

Thank you!

- R package: https://github.com/marina-khi/MSinference
- Khismatullina M., Vogt M. "Multiscale Comparison of Nonparametric Trend Curves", https://arxiv.org/abs/2209.10841

Model, part 3

1. We estimate $\boldsymbol{\beta}_i$:

$$\widehat{\boldsymbol{\beta}}_i = \Big(\sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta \boldsymbol{X}_{it}^\top \Big)^{-1} \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta Y_{it}$$

Theorem

Under certain regularity assumptions, $\widehat{\boldsymbol{\beta}}_i$ is a consistent estimator of $\boldsymbol{\beta}_i$ with the property $\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i = O_P(T^{-1/2})$.

2. We estimate the fixed effects α_i:

$$\widehat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} \big(Y_{it} - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} \big)$$

We then work with the augmented time series $\widehat{Y}_{it} = Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it}$.

Test statistic

For a given location $u \in [0,1]$ and bandwidth h and a given pair (i,j) we construct the kernel averages

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where

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$$\Lambda_{t,T}(u,h) = K\Big(\frac{t/T-u}{h}\Big)\Big[S_{T,2}(u,h) - S_{T,1}(u,h)\Big(\frac{t/T-u}{h}\Big)\Big],$$

$$S_{T,\ell}(u,h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T-u}{h}\Big)\Big(\frac{t/T-u}{h}\Big)^{\ell}$$

for $\ell = 1, 2$ and K is a kernel function.

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- $\mathcal{C}6$ All of the variables in the model are short-range dependent. Details

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$\mathcal{C}10$ Assume that $\sigma_i^2 = \sigma_j^2$ for all i,j and $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(\sqrt{h_{\min}}/\log T)$.

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- Show that

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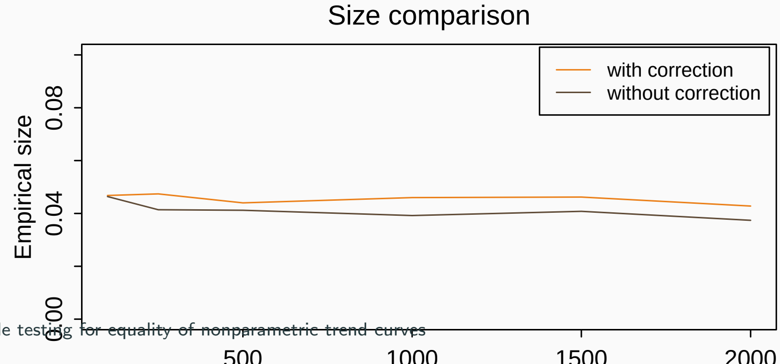
Introduction of a scale-dependent parameter helps us balance the significance of hypotheses between the time intervals of different lengths h_k:



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Then we can rewrite the uncorrected test statistic as

$$\Phi^{\text{uncor}} = \max_{i,j} \max_{\substack{1 \leq l \leq L, \\ 1 \leq m \leq 1/h_l}} \Big| \frac{1}{\sqrt{2Th_l}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in [(m-1)h_l, mh_l]\Big)(Z_{it} - Z_{jt}) \Big|$$

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$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l. Go back

Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\boldsymbol{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\boldsymbol{L}, t) = ||\boldsymbol{L}(\mathcal{F}_t) - \boldsymbol{L}(\mathcal{F}_t')||_q,$$

where $\mathcal{F}_t = (\ldots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}_t' = (\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of $\mathcal{F}_t$ with ϵ_0' being an i.i.d. copy of ϵ_0.

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Intuitively, $\delta_q(\boldsymbol{L}, t)$ measures the dependency of $\boldsymbol{L}(\mathcal{F}_t)$ on ϵ_0, i.e., how replacing ϵ_0 by an i.i.d. copy while keeping all other innovations in place affects the output $\boldsymbol{L}(\mathcal{F}_t)$.

Technical assumptions

$\mathcal{C}1'$ The variables ε_{it} are independent across i and allow for the representation $\varepsilon_{it} = G_i(\ldots, \eta_{it-1}, \eta_{it})$, where η_{it} are i.i.d. random variables across t and $G_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function..

$\mathcal{C}1''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each i it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q$. Go back

Technical assumptions, part 2

$\mathcal{C}3'$ $\boldsymbol{X}_{it}$ allow for the representation $\boldsymbol{X}_{it} = \boldsymbol{H}_i(\ldots, u_{it-1}, u_{it})$ with u_{it} being i.i.d. random variables and $\boldsymbol{H}_i := (H_{i1}, H_{i2}, \ldots, H_{id})^\top : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^d$ being a measurable function such that $\boldsymbol{H}_i(\mathcal{U}_{it})$ is well defined.

$\mathcal{C}3''$ Let $\boldsymbol{N}_i$ be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being kl-th entry. We assume that the smallest eigenvalue of $\boldsymbol{N}_i$ is strictly bigger than 0.

$\mathcal{C}3'''$ Let $\mathbb{E}[\boldsymbol{H}_i(\mathcal{U}_{i0})] = \boldsymbol{0}$ and $||\boldsymbol{H}_i(\mathcal{U}_{it})||_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where θ will be introduced further.

$\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) < \infty$ for q' from Assumption $\mathcal{C}3'''$.

$\mathcal{C}4''$ For each i it holds that $\sum_{s=t}^{\infty} \delta_{q'}(\boldsymbol{H}_i, s) = O(t^{-\alpha})$ for q' from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. Go back

Technical assumptions, part 3

$\mathcal{C}6$ Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Denote $\mathcal{I}_{it} = (\ldots, \zeta_{i,t-1}, \zeta_{i,t})$, $\mathcal{J}_{it} = (\ldots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$, $\mathcal{U}_{it} = (\ldots, u_{it-1}, u_{it})$, and $\mathbf{U}_i(\mathcal{I}_{it}) = \boldsymbol{H}_i(\mathcal{U}_{it}) G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(\mathbf{U}_i, s) < \infty$. Go back

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But what if there are too many?

An interval $[u-h, u+h]$ is called **minimal** if the corresponding local null $H_0^{[i,j]}(u,h)$ is rejected and there is no other interval $[u'-h', u'+h']$ such that we reject $H_0^{[i,j]}(u',h')$ and $[u'-h', u'+h'] \subset [u-h, u+h]$.