Multiscale inference for nonparametric time trends

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HCM Symposium

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Introduction

Motivation

Aim of the paper

To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.

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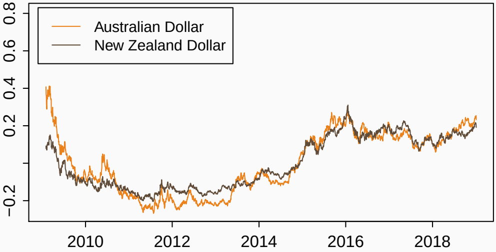

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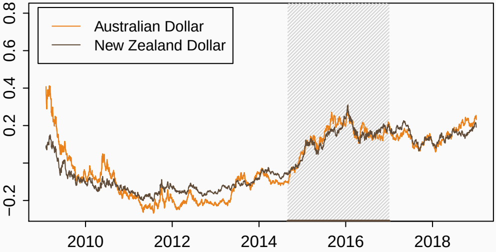

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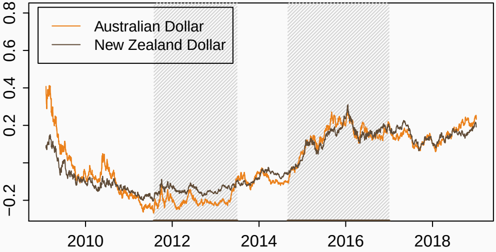

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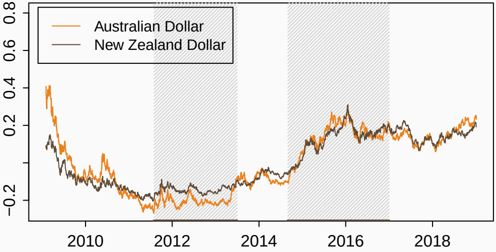

Research question: Out of many given intervals, how to find those where the trends are significantly different?

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Finding systematic differences between trends = basis for further research.

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Is it limited to one application?

No! Our method = general method for comparing nonparametric trends

$\Rightarrow$ new statistical test for equality of nonparametric trend curves.

Literature

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Model

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We observe a panel of n time series $\mathcal{Z}_i = \{(Y_{it}, \boldsymbol{X}_{it}) : 1 \leq t \leq T\}$ of length T, where $Y_{it} \in \mathbb{R}$ and $\boldsymbol{X}_{it} \in \mathbb{R}^d$.

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$$Y_{it} = m_i\Big(\frac{t}{T}\Big) + \boldsymbol{\beta}_i^T \boldsymbol{X}_{it} + \alpha_i + \varepsilon_{it},$$

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- $\boldsymbol{\beta}_i$ is $d \times 1$ vector of unknown parameters;
- α_i are so-called fixed effect error terms;
- $\mathcal{E}_i = \{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

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If we knew α_i and $\boldsymbol{\beta}_i$, then the model becomes much simpler:

$$\begin{aligned} Y_{it} - \alpha_i - \boldsymbol{\beta}_i^\top \boldsymbol{X}_{it} &=: Y_{it}^\circ \\ &= m_i\left(\frac{t}{T}\right) + \varepsilon_{it}. \end{aligned}$$

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But given $\widehat{\alpha}_i$ and $\widehat{\boldsymbol{\beta}}_i$, we can consider

$$\widehat{Y}_{it} := Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it} = (\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i)^\top \boldsymbol{X}_{it} + m_i\Big(\frac{t}{T}\Big) + \big(\alpha_i - \widehat{\alpha}_i\big) + \varepsilon_{it}.$$

Model, part 3

1. We estimate $\boldsymbol{\beta}_i$:

$$\widehat{\boldsymbol{\beta}}_i = \Big(\sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta \boldsymbol{X}_{it}^\top \Big)^{-1} \sum_{t=2}^{T} \Delta \boldsymbol{X}_{it} \Delta Y_{it}$$

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Theorem

Under certain regularity assumptions, $\widehat{\boldsymbol{\beta}}_i$ is a consistent estimator of $\boldsymbol{\beta}_i$ with the property $\boldsymbol{\beta}_i - \widehat{\boldsymbol{\beta}}_i = O_P(T^{-1/2})$.

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We then work with the augmented time series $\widehat{Y}_{it} = Y_{it} - \widehat{\alpha}_i - \widehat{\boldsymbol{\beta}}_i^\top \boldsymbol{X}_{it}$.

Testing procedure

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$$H_0 : m_1 = m_2 = \ldots = m_n$$

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$$H_0^{[i,j]}(u,h) : m_i(w) = m_j(w) \text{ for all } w \in [u-h, u+h].$$

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Then the global null $H_0 : m_1 = m_2 = \ldots = m_n$ can be reformulated as

$$H_0 : \text{The hypotheses } H_0^{[i,j]}(u, h) \text{ hold true for all intervals } [u - h, u + h], (u, h) \in \mathcal{G}_T, \text{ and for all } 1 \leq i < j \leq n.$$

Test statistic

For a given location $u \in [0,1]$ and bandwidth h and a given pair (i,j) we construct the kernel averages

$$\widehat{\psi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\big(\widehat{Y}_{it} - \widehat{Y}_{jt}\big),$$

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where

$$w_{t,T}(u,h) = \frac{\Lambda_{t,T}(u,h)}{\{\sum_{t=1}^{T} \Lambda_{t,T}^2(u,h)\}^{1/2}},$$

$$\Lambda_{t,T}(u,h) = K\Big(\frac{t/T-u}{h}\Big)\Big[S_{T,2}(u,h) - S_{T,1}(u,h)\Big(\frac{t/T-u}{h}\Big)\Big],$$

$$S_{T,\ell}(u,h) = \frac{1}{Th}\sum_{t=1}^{T} K\Big(\frac{t/T-u}{h}\Big)\Big(\frac{t/T-u}{h}\Big)^{\ell}$$

for $\ell = 1,2$ and K is a kernel function.

Test statistic, part 2

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Instead with working directly with $\widehat{\psi}_{ij,T}(u,h)$, we replace them by

$$\widehat{\psi}^0_{ij,T}(u,h) = \left\{ \left| \frac{\widehat{\psi}_{ij,T}(u,h)}{\left(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\right)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\widehat{\sigma}_i^2$ is an appropriate estimator of the long-run variance σ_i^2;
- $\lambda(h) = \sqrt{2 \log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

Test statistic, part 3

To test the global null, we aggregate the individual test statistics for all (i, j) and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\widehat{\Psi}_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \widehat{\psi}^0_{ij,T}(u, h).$$

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Note

Under certain conditions and under the null, $\widehat{\psi}^0_{ij,T}(u, h)$ and $\widehat{\Psi}_{n,T}$ can be approximated by the corresponding Gaussian versions of the test statistics.

Test procedure

Gaussian version of the individual test statistics:

$$\phi^0_{ij,T}(u,h) = \max_{(u,h)\in\mathcal{G}_T} \left\{ \left| \frac{\phi_T(u,h)}{\left(\sigma_i^2 + \sigma_j^2\right)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\phi_T(u,h) = \sum_{t=1}^T w_{t,T}(u,h) \left\{ \sigma_i (Z_{it} - \bar{Z}_i) - \sigma_j (Z_{jt} - \bar{Z}_j) \right\}$;

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Aggregated Gaussian test statistics:

$$\Phi_{n,T} = \max_{1\le i<j\le n} \max_{(u,h)\in\mathcal{G}_T} \phi^0_{ij,T}(u,h).$$

Test procedure, part 2

1. Consider the Gaussian test statistic

$$\Phi_{n,T} = \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \phi^0_{ij,T}(u,h),$$

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3. Perform the test for the global hypothesis H_0: reject H_0 if $\widehat{\Psi}_{n,T} > q_{n,T}(\alpha)$.
4. For each i, j, and each $(u,h) \in \mathcal{G}_T$, carry out the test for the local null hypothesis $H_0^{[i,j]}(u,h)$: reject $H_0^{[i,j]}(u,h)$ if $\widehat{\psi}^0_{ij,T}(u,h) > q_{n,T}(\alpha)$.

Theoretical properties

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$\mathcal{C}9$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

Assumptions, part 2

$\mathcal{C}7$ Standard assumptions on the kernel function K.

$\mathcal{C}8$ $|\mathcal{G}_T| = O(T^\theta)$ for some arbitrarily large but fixed constant $\theta > 0$.

$$\mathcal{G}_T = \Big\{(u,h) : u = t/T \text{ for some } 1 \leq t \leq T \text{ and } h \in [h_{\min}, h_{\max}] \\ \text{with } h = t/T \text{ for some } 1 \leq t \leq T\Big\},$$

$\mathcal{C}9$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

$\mathcal{C}10$ Assume that $\sigma_i^2 = \sigma_j^2$ for all i, j and $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(\sqrt{h_{\min}}/\log T)$.

Theoretical properties

Proposition

Under $\mathcal{C}1 - \mathcal{C}10$ and under the null, it holds that

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$$FWER(\alpha) \leq \alpha$$

Proposition

Consider a sequence of functions $m_i = m_{i,T}$, $m_j = m_{j,T}$ such that

$$\exists\,(u,h) \in \mathcal{G}_T : m_i(w) - m_j(w) \geq c_T \sqrt{\log T/(Th)}\ \forall w \in [u-h, u+h],$$

and $c_T \to \infty$. Then under $\mathcal{C}1 - \mathcal{C}10$, it holds that

$$\mathrm{P}\Big(\widehat{\Psi}_{n,T} \leq q_{n,T}(\alpha)\Big) = 1 - o(1)$$

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- Introduce $\widehat{\Phi}_{n,T}$ that is close in distribution to $\widehat{\Psi}_{n,T}$ under the null.

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- Show that

$$\sup_{x \in \mathbb{R}} \left|\mathrm{P}(\widetilde{\Phi}_{n,T} \leq x) - \mathrm{P}(\Phi_{n,T} \leq x)\right| = o(1).$$

Illustration

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$$\mathrm{P}\Big(\forall\, i, j \in \{1, \ldots, n\}, (u,h) \in \mathcal{G}_T \text{ such that } H_0^{[i,j]}(u,h) \text{ is true} : |\hat{\psi}^0_{ij,T}(u,h)| \le q_{n,T}(\alpha)\Big) \ge 1 - \alpha + o(1)$$

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Minimal intervals

An interval $[u-h, u+h] \in \mathcal{S}^{[i,j]}$ is called **minimal** if there is no other interval $[u'-h', u'+h'] \in \mathcal{S}^{[i,j]}$ with $[u'-h', u'+h'] \subset [u-h, u+h]$.

Discussion

We can claim, with confidence of about 95%, that the null hypothesis is violated for all intervals (and all pairs of time series) for which our test rejects the null.

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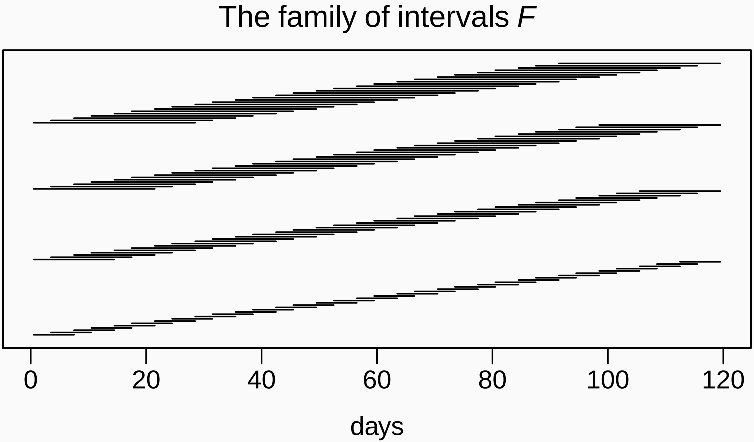
- introduce scaling factor in the trend function, that will allow to adjust for the size of the country (population, density, testing regimes, etc.);
- include the dependence between covariates and error terms;
- cluster the time series based on the trends they exhibit.

Thank you!

Family of time intervals



Idea behind a_k and b_k

Dümbgen and Spokoiny (2001): the critical values $c_{ijk}(\alpha)$ depend on the scale of the testing problem, i.e. the length h_k of the time interval.

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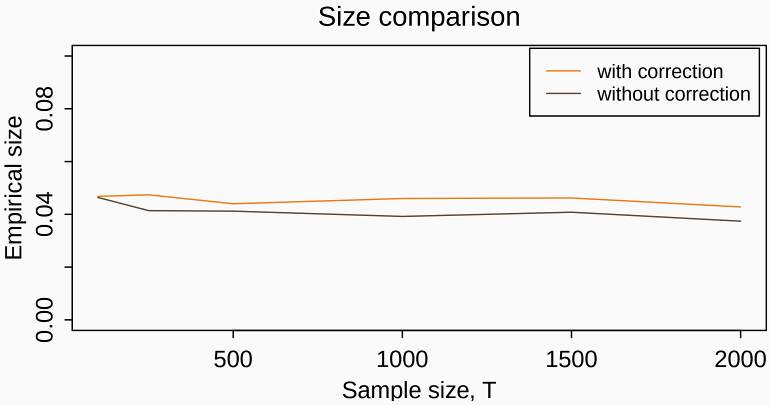
Specifically,

$$c_{ijk}(\alpha) = c(\alpha, h_k) := b_k + q(\alpha)/a_k,$$

where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants and $q(\alpha)$ is chosen such that we control FWER.

Idea behind a_k and b_k, part 2

This choice of scale-dependent constants helps us balance the significance of hypotheses between the time intervals of different lengths h_k:



Go back

Idea behind the additive correction

Consider the uncorrected Gaussian statistic

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Then we can rewrite the uncorrected test statistic as

$$\Phi^{\text{uncor}} = \max_{i,j} \max_{\substack{1 \le l \le L, \\ 1 \le m \le 1/h_l}} \Big| \frac{1}{\sqrt{2Th_l}} \sum_{t=1}^{T} 1\Big(\frac{t}{T} \in [(m-1)h_l, mh_l]\Big)(Z_{it} - Z_{jt}) \Big|$$

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$\Rightarrow \quad \max_m \ldots = \sqrt{2\log(1/h_l)} + o_P(1) \to \infty$ as $h \to 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l. Go back

Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\boldsymbol{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\boldsymbol{L}, t) = ||\boldsymbol{L}(\mathcal{F}_t) - \boldsymbol{L}(\mathcal{F}_t')||_q,$$

where $\mathcal{F}_t = (\ldots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}_t' = (\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of $\mathcal{F}_t$ with ϵ_0' being an i.i.d. copy of ϵ_0.

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Intuitively, $\delta_q(\boldsymbol{L}, t)$ measures the dependency of $\boldsymbol{L}(\mathcal{F}_t)$ on ϵ_0, i.e., how replacing ϵ_0 by an i.i.d. copy while keeping all other innovations in place affects the output $\boldsymbol{L}(\mathcal{F}_t)$.

Technical assumptions

$\mathcal{C}1'$ The variables ε_{it} allow for the representation $\varepsilon_{it} = G_i(\ldots, \eta_{it-1}, \eta_{it})$, where η_{it} are i.i.d. random variables across t and $G_i : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is a measurable function. Denote $\mathcal{J}_{it} = (\ldots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$.

$\mathcal{C}1'''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each i it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q$. Go back

Technical assumptions, part 2

$\mathcal{C}3'$ $\boldsymbol{X}_{it}$ allow for the representation $\boldsymbol{X}_{it} = \boldsymbol{H}_i(\ldots, u_{it-1}, u_{it})$ with u_{it} being i.i.d. random variables and $\boldsymbol{H}_i := (H_{i1}, H_{i2}, \ldots, H_{id})^\top : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^d$ being a measurable function such that $\boldsymbol{H}_i(\mathcal{U}_{it})$ is well defined. Denote $\mathcal{U}_{it} = (\ldots, u_{it-1}, u_{it})$.

$\mathcal{C}3''$ Let $\boldsymbol{N}_i$ be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being kl-th entry. We assume that the smallest eigenvalue of $\boldsymbol{N}_i$ is strictly bigger than 0.

$\mathcal{C}3'''$ Let $\mathbb{E}[\mathrm{H}_i(\mathcal{U}_{i0})] = 0$ and $||\mathrm{H}_i(\mathcal{U}_{it})||_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where θ will be introduced further.

$\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(\mathrm{H}_i, s) < \infty$ for q' from Assumption $\mathcal{C}3'''$.

$\mathcal{C}4''$ For each i it holds that $\sum_{s=t}^{\infty} \delta_{q'}(\mathrm{H}_i, s) = O(t^{-\alpha})$ for q' from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. Go back

Technical assumptions, part 3

$\mathcal{C}6$ Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Define $\mathcal{I}_{it} = (\ldots, \zeta_{i,t-1}, \zeta_{i,t})$ and $\mathsf{U}_i(\mathcal{I}_{it}) = \mathsf{H}_i(\mathcal{U}_{it}) G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(\mathsf{U}_i, s) < \infty$. Go back