Clustering of the epidemic time trends: the case of COVID-19

We consider the following nonparametric regression equation:

$$X_{it} = c_i \lambda_i\Big(\frac{t}{T}\Big) + \varepsilon_{it} \quad \text{with} \quad \varepsilon_{it} = \sigma \sqrt{\lambda_i\Big(\frac{t}{T}\Big)} \eta_{it},$$

where c_i is the country-specific scaling parameter that accounts for the size of the country or population density. We introduce this additional parameter in order to be able to compare countries that differ substantially in terms of the population, i.e. Luxembourg and Russia. In what follows, we present a method that allows researchers to test the hypothesis that the time trends of new COVID-19 cases in different countries are the same up to some scaling parameter and to cluster the countries based on the differences.

For the identification purposes, we need to assume that for each $i \in \mathcal{C}$ we have $\int_0^1 \lambda_i(u)du = 1$. Only then we are able to estimate the scaling parameter c_i. Thus, the testing procedure is as follows.

Step 1

First, we estimate the scaling parameter:

$$\begin{aligned}
\widehat{c}_i &= \frac{1}{T}\sum_{t=1}^{T} X_{it} \\
&= c_i \frac{1}{T}\sum_{t=1}^{T} \lambda_i\Big(\frac{t}{T}\Big) + \sigma \frac{1}{T}\sum_{t=1}^{T} \sqrt{\lambda_i\Big(\frac{t}{T}\Big)} \eta_{it} \\
&= c_i \frac{1}{T}\sum_{t=1}^{T} \lambda_i\Big(\frac{t}{T}\Big) + o_P(1) \\
&= c_i + o_P(1),
\end{aligned}$$

where in the last inequality we used the normalization $\int_0^1 \lambda_i(u)du = 1$. Hence, for any fixed $i \in \mathcal{C}$, $\widehat{c}_i$ is a consistent estimator of c_i.

Step 2

Instead of working with X_{it}, we consider the following variables:

$$\begin{aligned}
X^*_{it} &= \frac{X_{it}}{\frac{1}{T}\sum_{t=1}^{T} X_{it}} \\
&= \frac{c_i}{\widehat{c}_i} \lambda_i\Big(\frac{t}{T}\Big) + \frac{\sigma}{\widehat{c}_i} \sqrt{\lambda_i\Big(\frac{t}{T}\Big)} \eta_{it}.
\end{aligned}$$

A statistic to test the hypothesis $H_0^{(ijk)}$ for a given triple (i, j, k) is then constructed as follows. We work with the following quantity

$$\hat{s}_{ijk,T} = \frac{1}{\sqrt{Th_k}} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it}^* - X_{jt}^*).$$

Then

$$\begin{aligned}\frac{\hat{s}_{ijk,T}}{\sqrt{Th_k}} =& \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it}^* - X_{jt}^*) \\ =& \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big(\lambda_i\Big(\frac{t}{T}\Big) - \lambda_j\Big(\frac{t}{T}\Big)\Big) + R_1 + R_2,\end{aligned}$$

where

$$\begin{aligned}R_1 &= \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big(\Big(\frac{c_i}{\widehat{c}_i} - 1\Big)\lambda_i\Big(\frac{t}{T}\Big) - \Big(\frac{c_j}{\widehat{c}_j} - 1\Big)\lambda_j\Big(\frac{t}{T}\Big)\Big), \\ R_2 &= \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big(\frac{\sigma}{\widehat{c}_i}\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it} - \frac{\sigma}{\widehat{c}_j}\sqrt{\lambda_j\Big(\frac{t}{T}\Big)}\eta_{jt}\Big).\end{aligned}$$

Since $\widehat{c}_i = c_i + o_P(1)$ and $0 \le \sum_{t=1}^{T} \mathbf{1}\big(\frac{t}{T} \in \mathcal{I}_k\big)\lambda_i\big(\frac{t}{T}\big) \le h_k \lambda_{max}$, we have

$$\begin{aligned}|R_1| &\le \Big|\frac{c_i}{\widehat{c}_i} - 1\Big| \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\lambda_i\Big(\frac{t}{T}\Big) + \Big|\frac{c_j}{\widehat{c}_j} - 1\Big| \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\lambda_j\Big(\frac{t}{T}\Big), \\ &\le o_P(1) \cdot \frac{\lambda_{max}}{T} + o_P(1) \cdot \frac{\lambda_{max}}{T} = o_P\Big(\frac{1}{T}\Big). \qquad (0.1)\end{aligned}$$

Furthermore, applying the law of large numbers, we get:

$$\frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it} = o_P(1).$$

Hence, if we uniformly bound the scaling parameters away from 0, i.e. $\exists\, c_{min}$ such that for all $i \in \mathcal{C}$ we have $0 < c_{min} \le c_i$, we can use the fact that $\frac{\sigma}{\widehat{c}_i} = O_P(1)$ to get that

$$\begin{aligned}R_2 &= \frac{\sigma}{\widehat{c}_i} \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it} - \frac{\sigma}{\widehat{c}_j} \frac{1}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\sqrt{\lambda_j\Big(\frac{t}{T}\Big)}\eta_{jt} \\ &= o_P(1). \qquad (0.2)\end{aligned}$$

Combining (0.1) and (0.2) together, we get $\hat{s}_{ijk,T}/\sqrt{Th_k} = (Th_k)^{-1} \sum_{t=1}^{T} \mathbf{1}(t/T \in \mathcal{I}_k)\{\lambda_i(t/T)-$

$\lambda_j(t/T)\} + o_p(1)$ for any fixed pair of countries (i,j). Hence, the statistic $\hat{s}_{ijk,T}/\sqrt{Th_k}$ estimates the average distance between the functions λ_i and λ_j on the interval $\mathcal{I}_k$. The variance of $\hat{s}_{ijk,T}$ can not be easily calculated:

$$\begin{aligned}
\mathrm{Var}(\hat{s}_{ijk,T}) =& \frac{1}{Th_k}\mathrm{Var}\Big(\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)(X^*_{it}-X^*_{jt})\Big)\\
=& \frac{1}{Th_k}\mathrm{Var}\Big(\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)X^*_{it}\Big)+\frac{1}{Th_k}\mathrm{Var}\Big(\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)X^*_{jt}\Big)\\
=& \frac{1}{Th_k}\mathrm{Var}\Big(\frac{\sum_{t=1}^{T}\mathbf{1}(\frac{t}{T}\in\mathcal{I}_k)X_{it}}{\frac{1}{T}\sum_{t=1}^{T}X_{it}}\Big)+\frac{1}{Th_k}\mathrm{Var}\Big(\frac{\sum_{t=1}^{T}\mathbf{1}(\frac{t}{T}\in\mathcal{I}_k)X_{jt}}{\frac{1}{T}\sum_{t=1}^{T}X_{jt}}\Big),
\end{aligned}$$

hence, we "normalize" $\hat{s}_{ijk,T}$ intuitively by dividing it by the following value:

$$\hat{\nu}^2_{ijk,T} = \frac{\hat{\sigma}^2}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\{X^*_{it}+X^*_{jt}\}.$$

Instead look at the bootstrap!

Normalizing the statistic $\hat{s}_{ijk,T}$ by the estimator $\hat{\nu}_{ijk,T}$ yields the expression

$$\hat{\psi}_{ijk,T} := \frac{\hat{s}_{ijk,T}}{\hat{\nu}_{ijk,T}} = \frac{\sum_{t=1}^{T}\mathbf{1}(\frac{t}{T}\in\mathcal{I}_k)(X_{it}-X_{jt})}{\hat{\sigma}\{\sum_{t=1}^{T}\mathbf{1}(\frac{t}{T}\in\mathcal{I}_k)(X^*_{it}+X^*_{jt})\}^{1/2}},$$

which serves as our test statistic of the hypothesis $H_0^{(ijk)}$. For later reference, we additionally introduce the statistic

$$\hat{\psi}^0_{ijk,T} = \frac{\sum_{t=1}^{T}\mathbf{1}(\frac{t}{T}\in\mathcal{I}_k)\Big((\frac{c_i}{\hat{c}_i}-\frac{c_j}{\hat{c}_j})\overline{\lambda}_{ij}+\overline{\lambda}_{ij}^{1/2}(\frac{t}{T})(\frac{\sigma}{\hat{c}_i}\eta_{it}-\frac{\sigma}{\hat{c}_j}\eta_{jt})\Big)}{\hat{\sigma}\{\sum_{t=1}^{T}\mathbf{1}(\frac{t}{T}\in\mathcal{I}_k)(X^*_{it}+X^*_{jt})\}^{1/2}}$$

with $\overline{\lambda}_{ij}(u) = \{\lambda_i(u)+\lambda_j(u)\}/2$, which is identical to $\hat{\psi}_{ijk,T}$ under $H_0^{(ijk)}$.

0.1 Construction of the test

Our multiscale test is carried out as follows: For a given significance level $\alpha\in(0,1)$ and each $(i,j,k)\in\mathcal{M}$, we reject $H_0^{(ijk)}$ if

$$|\hat{\psi}_{ijk,T}| > c_{ijk,T}(\alpha),$$

where $c_{ijk,T}(\alpha)$ is the critical value for the (i,j,k)-th test problem. The critical values $c_{ijk,T}(\alpha)$ are chosen such that the familywise error rate (FWER) is controlled at level α, which is defined as the probability of wrongly rejecting $H_0^{(ijk)}$ for at least one (i,j,k). More formally speaking, for a given significance level $\alpha \in (0,1)$, the FWER is

$$\begin{aligned} \text{FWER}(\alpha) &= \mathbb{P}\Big(\exists (i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk,T}| > c_{ijk,T}(\alpha)\Big) \\ &= 1 - \mathbb{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk,T}| \le c_{ijk,T}(\alpha)\Big) \\ &= 1 - \mathbb{P}\Big(\max_{(i,j,k)\in\mathcal{M}_0} |\hat{\psi}_{ijk,T}| \le c_{ijk,T}(\alpha)\Big), \end{aligned}$$

where $\mathcal{M}_0 \subseteq \mathcal{M}$ is the set of triples (i,j,k) for which $H_0^{(ijk)}$ holds true. As before, the critical values are chosen as

$$c_{ijk,T}(\alpha) = c_T(\alpha, h_k) := b_k + q_T(\alpha)/a_k,$$

where $a_k = \{\log(e/h_k)\}^{1/2}/\log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$ are scale-dependent constants and the quantity $q_T(\alpha)$ is determined by the following consideration: Since

$$\begin{aligned} \text{FWER}(\alpha) &= \mathbb{P}\Big(\exists (i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk,T}| > c_T(\alpha, h_k)\Big) \\ &= 1 - \mathbb{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk,T}| \le c_T(\alpha, h_k)\Big) \\ &= 1 - \mathbb{P}\Big(\forall (i,j,k) \in \mathcal{M}_0 : a_k\big(|\hat{\psi}_{ijk,T}| - b_k\big) \le q_T(\alpha)\Big) \\ &= 1 - \mathbb{P}\Big(\max_{(i,j,k)\in\mathcal{M}_0} a_k\big(|\hat{\psi}_{ijk,T}| - b_k\big) \le q_T(\alpha)\Big), \end{aligned}$$

we need to choose the quantity $q_T(\alpha)$ as the $(1-\alpha)$-quantile of the statistic

$$\hat{\Psi}_T = \max_{(i,j,k)\in\mathcal{M}} a_k\big(|\hat{\psi}^0_{ijk,T}| - b_k\big)$$

in order to ensure control of the FWER at level α. As the quantiles $q_T(\alpha)$ are not known in practice, we cannot compute the critical values $c_T(\alpha, h_k)$ exactly in practice but need to approximate them. This can be achieved as follows: Under appropriate regularity conditions, it can be shown that ???

$$\hat{\psi}^0_{ijk,T} = \frac{\sum_{t=1}^T \mathbf{1}(\frac{t}{T} \in \mathcal{I}_k)\Big(\big(\frac{c_i}{\hat{c}_i} - \frac{c_j}{\hat{c}_j}\big)\overline{\lambda}_{ij} + \overline{\lambda}_{ij}^{1/2}(\frac{t}{T})(\frac{\sigma}{\hat{c}_i}\eta_{it} - \frac{\sigma}{\hat{c}_j}\eta_{jt})\Big)}{\hat{\sigma}\{\sum_{t=1}^T \mathbf{1}(\frac{t}{T} \in \mathcal{I}_k)(X^*_{it} + X^*_{jt})\}^{1/2}}$$

$$\approx \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big) \Big\{ \frac{\eta_{it}}{\hat{c}_i} - \frac{\eta_{jt}}{\hat{c}_j} \Big\}.$$

A Gaussian version of the statistic displayed in the final line above is given by

$$\phi_{ijk,T} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big) \Big\{ \frac{Z_{it}}{c_i} - \frac{Z_{jt}}{c_j} \Big\},$$

where Z_{it} are independent standard normal random variables for $1 \le t \le T$ and $1 \le i \le n$. Hence, the statistic

$$\Phi_T = \max_{(i,j,k) \in \mathcal{M}} a_k \big(|\phi_{ijk,T}| - b_k \big)$$

can be regarded as a Gaussian version of the statistic $\hat{\Psi}_T$.

To summarize, we propose the following procedure to simultaneously test the hypothesis $H_0^{(ijk)}$ for all $(i,j,k) \in \mathcal{M}$ at the significance level $\alpha \in (0,1)$:

$$\text{For each } (i,j,k) \in \mathcal{M}, \text{ reject } H_0^{(ijk)} \text{ if } |\hat{\psi}_{ijk,T}| > c_{T,\text{Gauss}}(\alpha, h_k), \tag{0.3}$$

where $c_{T,\text{Gauss}}(\alpha, h_k) = b_k + q_{T,\text{Gauss}}(\alpha)/a_k$ with $a_k = \{\log(e/h_k)\}^{1/2} / \log\log(e^e/h_k)$ and $b_k = \sqrt{2\log(1/h_k)}$.

All of the above can probably be proven using the same methods as before.

0.2 Multiplier bootstrap

Now we have the problem that c_i and c_j are unknown in real life. This means that the quantile $q_T(\alpha)$ of Φ_T are not known and can not be approximated by usual Monte Carlo simulations. We need to find another way of approximating them, for example, multiplier bootstrap from Chernozhukov et al. (2017).

First, we rewrite the statistics Φ_T as follows. Define

$$W_t^{(ijk)} = W_{t,T}^{(ijk)} := \sqrt{\frac{T}{2Th_k}} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big) \Big\{ \frac{Z_{it}}{c_i} - \frac{Z_{jt}}{c_j} \Big\}$$

with Z_{it} i.i.d. standard normal and let $\boldsymbol{W}_t = (W_t^{(ijk)} : (i,j,k) \in \mathcal{M})$. The vector $\boldsymbol{W}_t$ is a Gaussian vector with zero mean $\mathbb{E}[\boldsymbol{W}_t] = \mathbf{0}$. With this notation, we get that $\phi_{ijk,T} = T^{-1/2} \sum_{t=1}^{T} W_t^{(ijk)}$ and

$$\Phi_T = \max_{(i,j,k) \in \mathcal{M}} a_k \big(|\phi_{ijk,T}| - b_k \big)$$

$$= \max_{(i,j,k)\in\mathcal{M}} a_k \Big\{ \Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T W_t^{(ijk)} \Big| - b_k \Big\}.$$

For any $q \in \mathbb{R}$, it holds that

$$\begin{aligned}
\mathbb{P}\big(\Phi_T \le q\big) &= \mathbb{P}\Big(\max_{(i,j,k)\in\mathcal{M}} a_k \Big\{ \Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T W_t^{(ijk)} \Big| - b_k \Big\} \le q \Big) \\
&= \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T W_t^{(ijk)} \Big| \le c_{ijk}(q) \text{ for all } (i,j,k) \in \mathcal{M} \Big) \\
&= \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T \boldsymbol{W}_t \Big| \le \boldsymbol{c}(q) \Big),
\end{aligned}$$

where $\boldsymbol{c}(q) = (c_{ijk}(q) : (i,j,k) \in \mathcal{M})$ is the $\mathbb{R}^p$-vector with the entries $c_{ijk}(q) = q/a_k + b_k$, we use the notation $|v| = (|v_1|, \dots, |v_p|)^\top$ for vectors $v \in \mathbb{R}^p$ and the inequality $v \le w$ is to be understood componentwise for $v, w \in \mathbb{R}^p$.

Then, consider a sequence of i.i.d. standard normal random variables $e_1, \dots, e_T$, that are independent of η_{it} and Z_{it} for all i and t. Define

$$V_t^{(ijk)} = V_{t,T}^{(ijk)} := \sqrt{\frac{T}{2Th_k}} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big) \Big\{ \frac{\eta_{it}}{\hat{c}_i} - \frac{\eta_{jt}}{\hat{c}_j} \Big\}$$

for $(i,j,k) \in \mathcal{M}$ and let $\boldsymbol{V}_t = (V_t^{(ijk)} : (i,j,k) \in \mathcal{M})$ be the p-dimensional random vector with the entries $V_t^{(ijk)}$. Additionally, define

$$\bar{V}_t^{(ijk)} = \bar{V}_{t,T}^{(ijk)} := \frac{1}{T} \sum_{t=1}^T \sqrt{\frac{T}{2Th_k}} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big) \Big\{ \frac{\eta_{it}}{\hat{c}_i} - \frac{\eta_{jt}}{\hat{c}_j} \Big\}$$

and let $\bar{\boldsymbol{V}}_t = (\bar{V}_t^{(ijk)} : (i,j,k) \in \mathcal{M})$ be the p-dimensional random vector with the entries $\bar{V}_t^{(ijk)}$. We consider the following conditional probability:

$$\begin{aligned}
&\mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T e_t (\boldsymbol{V}_t - \bar{\boldsymbol{V}}_t) \Big| \le \boldsymbol{c}(q) \Big| \{V_t^{(ijk)}\} \Big) = \\
&= \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T e_t (V_t^{(ijk)} - \bar{V}_t^{(ijk)}) \Big| \le c_{ijk}(q) \text{ for all } (i,j,k) \in \mathcal{M} \Big| \{V_t^{(ijk)}\} \Big) \\
&= \mathbb{P}\Big(\max_{(i,j,k)\in\mathcal{M}} a_k \Big\{ \Big| \frac{1}{\sqrt{T}} \sum_{t=1}^T e_t (V_t^{(ijk)} - \bar{V}_t^{(ijk)}) \Big| - b_k \Big\} \le q \Big| \{V_t^{(ijk)}\} \Big)
\end{aligned}$$

Need to check the same mean and variance! In particular, $\mathbb{E}[\boldsymbol{W}_t] = \mathbb{E}[\boldsymbol{V}_t] = 0$ and

$\mathbb{E}[\boldsymbol{W}_t \boldsymbol{W}_t^\top] = \mathbb{E}[\boldsymbol{V}_t \boldsymbol{V}_t^\top]$.

With this notation at hand, we can make use of Corollary 4.2 from Chernozhukov et al. (2017). In our context, this proposition can be stated as follows:

Proposition A.1. *Let $\alpha \in (0, e^{-1})$ be a constant and assume that*

(a) $T^{-1} \sum_{t=1}^{T} \mathbb{E}(V_t^{(ijk)})^2 \geq \delta > 0$ *for all* $(i,j,k) \in \mathcal{M}$.

(b) $T^{-1} \sum_{t=1}^{T} \mathbb{E}[|V_t^{(ijk)}|^{2+r}] \leq B_T^r$ *for all* $(i,j,k) \in \mathcal{M}$ *and* $r = 1,2$, *where* $B_T \geq 1$ *are constants that may tend to infinity as* $T \to \infty$.

(c) $\mathbb{E}[\{\max_{(i,j,k) \in \mathcal{M}} |V_t^{(ijk)}|/B_T\}^\theta] \leq 2$ *for all* t *and some* $\theta > 4$.

Then we have with probability at least $1 - \alpha$,

$$\sup_{\boldsymbol{c} \in \mathbb{R}^p} \Big| \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_t (\boldsymbol{V}_t - \bar{\boldsymbol{V}}_t) \Big| \leq \boldsymbol{c}(q) \Big| \big\{ V_t^{(ijk)} \big\} \Big) - \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{W}_t \Big| \leq \boldsymbol{c} \Big) \Big|$$
$$\leq C \Big\{ \Big(\frac{B_T^2 \log^5(pT) \log^2(1/\alpha)}{T} \Big)^{1/6} + \Big(\frac{B_T^2 \log^3(pT)}{\alpha^{2/\theta} T^{1-2/\theta}} \Big)^{1/3} \Big\}, \tag{0.4}$$

where C *depends only on* δ *and* θ.

Need to verify that assumptions (a)–(c) are satisfied for sufficiently large T! where B_T can be chosen as $B_T = C p^{1/\theta} h_{\min}^{-1/2}$ with C sufficiently large. Moreover, it can be shown that the right-hand side of (0.4) is $o(1)$ for this choice of B_T. Hence, Proposition A.1 yields that

$$\sup_{\boldsymbol{c} \in \mathbb{R}^p} \Big| \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_t (\boldsymbol{V}_t - \bar{\boldsymbol{V}}_t) \Big| \leq \boldsymbol{c}(q) \Big| \big\{ V_t^{(ijk)} \big\} \Big) - \mathbb{P}\Big(\Big| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{W}_t \Big| \leq \boldsymbol{c} \Big) \Big| = o(1),$$

which in turn implies (**??**).

References

Chernozhukov, V., Chetverikov, D. and Kato, K. (2017). Central limit theorems and bootstrap in high dimensions. *Annals of Probability*, **45** 2309–2352.