

Multiscale inference for nonparametric time trends

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August 24, 2021

HCM Symposium

Table of contents

1. Introduction
2. Model
3. Testing procedure
4. Theoretical properties
5. Illustration

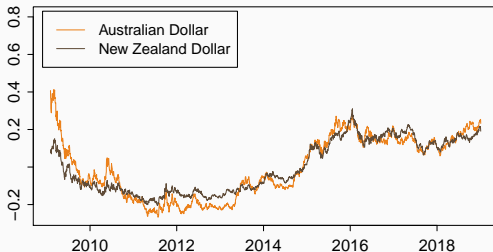
Introduction

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To develop new inference methods that allow to *identify* and *locate* differences between nonparametric trend curves with dependent errors.

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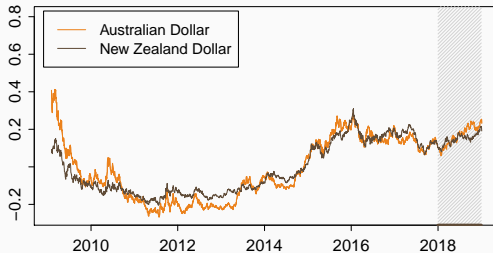
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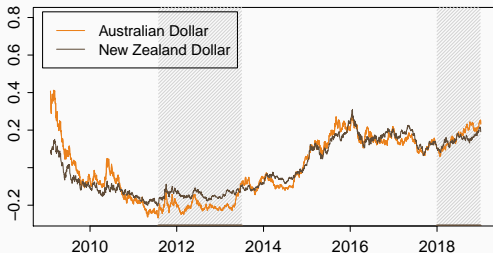
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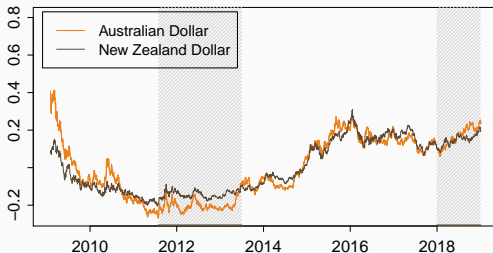
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Research question: Out of many given intervals, how to find those where the trends are significantly different?

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Is it limited to one application?

No! Our method = general method for comparing nonparametric trends
⇒ new statistical test for equality of nonparametric trend curves.

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Model

We observe a panel of n time series $\mathcal{Z}_i = \{(Y_{it}, \mathbf{X}_{it}) : 1 \leq t \leq T\}$ of length T , where $Y_{it} \in \mathbb{R}$ and $\mathbf{X}_{it} \in \mathbb{R}^d$.

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- α_i are so-called fixed effect error terms;
- $\mathcal{E}_i = \{\varepsilon_{it} : 1 \leq t \leq T\}$ is a zero-mean stationary and causal error process.

$$Y_{it} = m_i\left(\frac{t}{T}\right) + \beta_i^T \mathbf{X}_{it} + \alpha_i + \varepsilon_{it},$$

$$Y_{it} = m_i\left(\frac{t}{T}\right) + \beta_i^\top \mathbf{X}_{it} + \alpha_i + \varepsilon_{it},$$

If we knew α_i and β_i , then the model becomes much simpler:

$$\begin{aligned} Y_{it} - \alpha_i - \beta_i^\top \mathbf{X}_{it} &=: Y_{it}^\circ \\ &= m_i\left(\frac{t}{T}\right) + \varepsilon_{it}. \end{aligned}$$

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But given $\hat{\alpha}_i$ and $\hat{\beta}_i$, we can consider

$$\hat{Y}_{it} := Y_{it} - \hat{\alpha}_i - \hat{\beta}_i^\top \mathbf{X}_{it} = (\beta_i - \hat{\beta}_i)^\top \mathbf{X}_{it} + m_i\left(\frac{t}{T}\right) + (\alpha_i - \hat{\alpha}_i) + \varepsilon_{it}.$$

Model, part 3

1. We estimate β_i :

$$\hat{\beta}_i = \left(\sum_{t=2}^T \Delta \mathbf{x}_{it} \Delta \mathbf{x}_{it}^\top \right)^{-1} \sum_{t=2}^T \Delta \mathbf{x}_{it} \Delta Y_{it}$$

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We then work with the augmented time series $\hat{Y}_{it} = Y_{it} - \hat{\alpha}_i - \hat{\beta}_i^\top \mathbf{x}_{it}$.

Testing procedure

$$H_0 : m_1 = m_2 = \dots = m_n$$

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$$H_0^{[i,j]}(u, h) : m_i(w) = m_j(w) \text{ for all } w \in [u - h, u + h].$$

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Then the global null $H_0 : m_1 = m_2 = \dots = m_n$ can be reformulated as

$$H_0 : \text{The hypotheses } H_0^{[i,j]}(u, h) \text{ hold true for all intervals } [u - h, u + h], (u, h) \in \mathcal{G}_T, \text{ and for all } 1 \leq i < j \leq n.$$

For a given location $u \in [0, 1]$ and bandwidth h and a given pair (i, j) we construct the kernel averages

$$\hat{\psi}_{ij,T}(u, h) = \sum_{t=1}^T w_{t,T}(u, h) (\hat{Y}_{it} - \hat{Y}_{jt}),$$

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where

$$w_{t,T}(u, h) = \frac{\Lambda_{t,T}(u, h)}{\{\sum_{t=1}^T \Lambda_{t,T}^2(u, h)\}^{1/2}},$$

$$\Lambda_{t,T}(u, h) = K\left(\frac{t/T - u}{h}\right) \left[S_{T,2}(u, h) - S_{T,1}(u, h) \left(\frac{t/T - u}{h}\right) \right],$$

$$S_{T,\ell}(u, h) = \frac{1}{Th} \sum_{t=1}^T K\left(\frac{t/T - u}{h}\right) \left(\frac{t/T - u}{h}\right)^\ell$$

for $\ell = 1, 2$ and K is a kernel function.

The kernel averages $\hat{\psi}_{ij,\tau}(u, h)$ measure the distance between two trend curves m_i and m_j on $[u - h, u + h]$.

Test statistic, part 2

The kernel averages $\widehat{\psi}_{ij,\tau}(u, h)$ measure the distance between two trend curves m_i and m_j on $[u - h, u + h]$.

Instead with working directly with $\widehat{\psi}_{ij,\tau}(u, h)$, we replace them by

$$\widehat{\psi}_{ij,\tau}^0(u, h) = \left\{ \left| \frac{\widehat{\psi}_{ij,\tau}(u, h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\hat{\sigma}_i^2$ is an appropriate estimator of the long-run variance σ_i^2 ;
- $\lambda(h) = \sqrt{2 \log\{1/(2h)\}}$ is an additive correction term (Dümbgen and Spokoiny (2001)).

To test the global null, we aggregate the individual test statistics for all (i, j) and all location-bandwidth pairs $(u, h) \in \mathcal{G}_T$:

$$\hat{\Psi}_{n,T} = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \hat{\psi}_{ij,T}^0(u, h).$$

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Note

Under certain conditions and under the null, $\widehat{\psi}_{ij,T}^0(u, h)$ and $\widehat{\Psi}_{n,T}$ can be approximated by the corresponding Gaussian versions of the test statistics.

Gaussian version of the individual test statistics:

$$\phi_{ij,T}^0(u, h) = \max_{(u,h) \in \mathcal{G}_T} \left\{ \left| \frac{\phi_T(u, h)}{(\sigma_i^2 + \sigma_j^2)^{1/2}} \right| - \lambda(h) \right\},$$

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where

- $\phi_T(u, h) = \sum_{t=1}^T w_{t,T}(u, h) \{ \sigma_i(Z_{it} - \bar{Z}_i) - \sigma_j(Z_{jt} - \bar{Z}_j) \};$

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Aggregated Gaussian test statistics:

$$\Phi_{n,T} = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \phi_{ij,T}^0(u, h).$$

Test procedure, part 2

1. Consider the Gaussian test statistic

$$\Phi_{n,T} = \max_{1 \leq i < j \leq n} \max_{(u,h) \in \mathcal{G}_T} \phi_{ij,T}^0(u,h),$$

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4. For each i, j , and each $(u, h) \in \mathcal{G}_T$, carry out the test for the local null hypothesis $H_0^{[i,j]}(u, h)$: reject $H_0^{[i,j]}(u, h)$ if $\hat{\psi}_{ij,T}^0(u, h) > q_{n,T}(\alpha)$.

Theoretical properties

$\mathcal{C}1$ For all i it holds that $\mathbb{E}[\varepsilon_{it}] = 0$ and $\|\varepsilon_{it}\|_q < \infty$ for some $q > 4$.

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- $\mathcal{C}6$ All of the variables in the model are short-range dependent. [Details](#)

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$\mathcal{C}9$ $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

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C9 $h_{\min} \gg T^{-(1-\frac{2}{q})} \log T$ and $h_{\max} < 1/2$.

C10 Assume that $\sigma_i^2 = \sigma_j^2$ for all i, j and $\hat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(\sqrt{h_{\min}}/\log T)$.

Proposition

Under C1 – C10 and under the null, it holds that

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Corollary

$$FWER(\alpha) \leq \alpha$$

Theoretical properties

Proposition

Under $\mathcal{C}1 - \mathcal{C}10$ and under the null, it holds that

$$\mathbb{P}\left(\widehat{\Psi}_{n,T} \leq q_{n,T}(\alpha)\right) = 1 - \alpha + o(1)$$

Corollary

$$FWER(\alpha) \leq \alpha$$

Proposition

Consider a sequence of functions $m_i = m_{i,T}$, $m_j = m_{j,T}$ such that

$$\exists (u, h) \in \mathcal{G}_T : m_i(w) - m_j(w) \geq c_T \sqrt{\log T / (Th)} \quad \forall w \in [u - h, u + h],$$

and $c_T \rightarrow \infty$. Then under $\mathcal{C}1 - \mathcal{C}10$, it holds that

$$\mathbb{P}\left(\widehat{\Psi}_{n,T} \leq q_{n,T}(\alpha)\right) = 1 - o(1)$$

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$$|\widetilde{\Phi}_{n,T} - \Phi_{n,T}| = o_p(\delta_T),$$

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- Using the anti-concentration results for Gaussian random vectors (Chernozhukov et al. 2015), prove that $\Phi_{n,T}$ does not concentrate too strongly in small regions of the form $[x - \delta_T, x + \delta_T]$, i.e.

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- Show that

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(\widetilde{\Phi}_{n,T} \leq x) - \mathbb{P}(\Phi_{n,T} \leq x)| = o(1).$$

Illustration

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For each pair of time series (i, j) , denote by $\mathcal{S}^{[i,j]}(\alpha)$ the set of intervals $[u - h, u + h]$ that consists of the intervals where we reject $H_0^{[i,j]}(u, h)$ at a significance level α .

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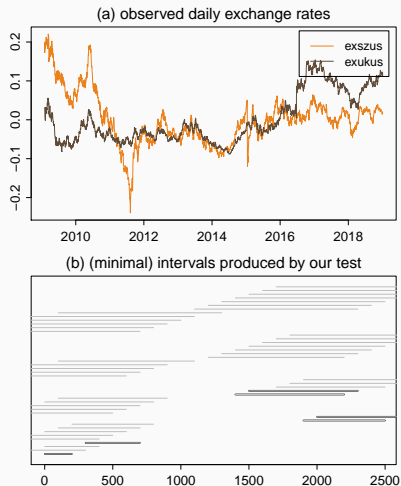
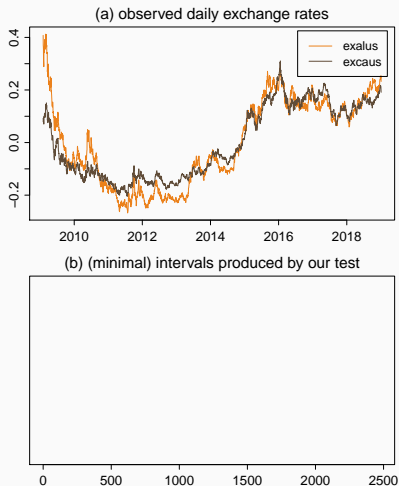
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Minimal intervals

An interval $[u - h, u + h] \in \mathcal{S}^{[i,j]}$ is called **minimal** if there is no other interval $[u' - h', u' + h'] \in \mathcal{S}^{[i,j]}$ with $[u' - h', u' + h'] \subset [u - h, u + h]$.

Application results



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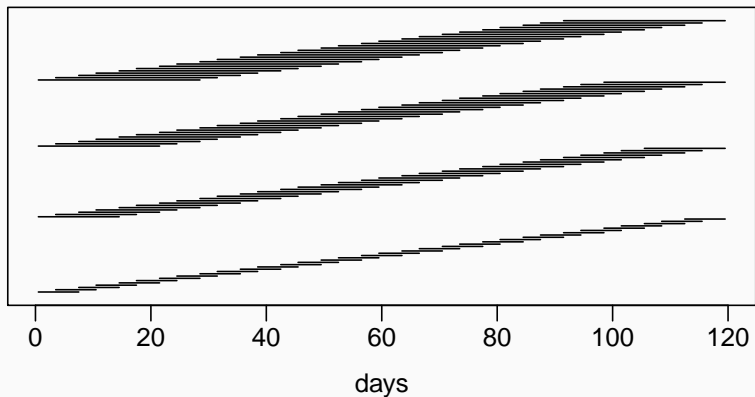
Further possible extensions:

- introduce scaling factor in the trend function;
- include the dependence between covariates and error terms;
- cluster the time series based on the trends they exhibit.

Thank you!

Family of time intervals

The family of intervals F



Idea behind a_k and b_k

Dümbgen and Spokoiny (2001): the critical values $c_{ijk}(\alpha)$ depend on the scale of the testing problem, i.e. the length h_k of the time interval.

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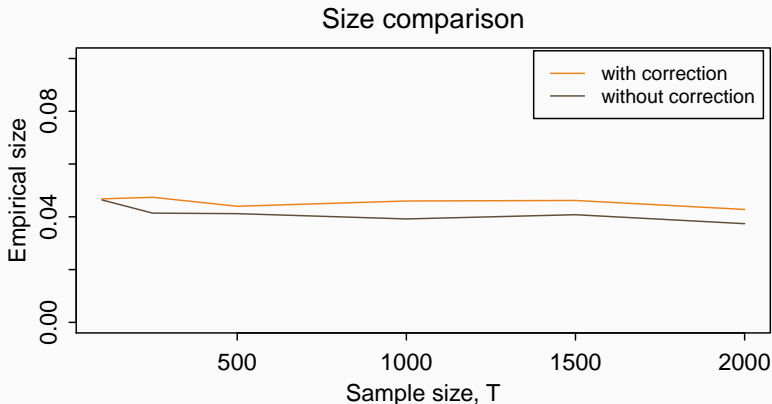
Specifically,

$$c_{ijk}(\alpha) = c(\alpha, h_k) := b_k + q(\alpha)/a_k,$$

where $a_k = \{\log(e/h_k)\}^{1/2} / \log \log(e^e/h_k)$ and $b_k = \sqrt{2 \log(1/h_k)}$ are scale-dependent constants and $q(\alpha)$ is chosen such that we control FWER.

Idea behind a_k and b_k , part 2

This choice of scale-dependent constants helps us balance the significance of hypotheses between the time intervals of different lengths h_k :



[Go back](#)

Idea behind the additive correction

Consider the uncorrected Gaussian statistic

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$\Rightarrow \max_m \dots = \sqrt{2 \log(1/h_l)} + o_P(1) \rightarrow \infty$ as $h \rightarrow 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l . [Go back](#)

Dependence measure

Following Wu (2005), we define the *physical dependence measure* for the process $\mathbf{L}(\mathcal{F}_t)$ as the following:

$$\delta_q(\mathbf{L}, t) = \|\mathbf{L}(\mathcal{F}_t) - \mathbf{L}(\mathcal{F}'_t)\|_q,$$

where $\mathcal{F}_t = (\dots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \dots, \epsilon_{t-1}, \epsilon_t)$ and $\mathcal{F}'_t = (\dots, \epsilon_{-1}, \epsilon'_0, \epsilon_1, \dots, \epsilon_{t-1}, \epsilon_t)$ is a coupled process of \mathcal{F}_t with ϵ'_0 being an i.i.d. copy of ϵ_0 .

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Intuitively, $\delta_q(\mathbf{L}, t)$ measures the dependency of $\mathbf{L}(\mathcal{F}_t)$ on ϵ_0 , i.e., how replacing ϵ_0 by an i.i.d. copy while keeping all other innovations in place affects the output $\mathbf{L}(\mathcal{F}_t)$.

Technical assumptions

$\mathcal{C1}'$ The variables ε_{it} allow for the representation $\varepsilon_{it} = G_i(\dots, \eta_{it-1}, \eta_{it})$, where η_{it} are i.i.d. random variables across t and $G_i : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ is a measurable function. Denote $\mathcal{J}_{it} = (\dots, \eta_{it-2}, \eta_{it-1}, \eta_{it})$.

$\mathcal{C1}'''$ Define $\Theta_{i,t,q} = \sum_{s \geq t} \delta_q(G_i, s)$ for $t \geq 0$. For each i it holds that $\Theta_{i,t,q} = O(t^{-\tau_q}(\log t)^{-A})$, where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q - 2)\sqrt{q^2 + 20q + 4}\}/8q$. [Go back](#)

Technical assumptions, part 2

$\mathcal{C}3'$ \mathbf{X}_{it} allow for the representation $\mathbf{X}_{it} = \mathbf{H}_i(\dots, u_{it-1}, u_{it})$ with u_{it} being i.i.d. random variables and $\mathbf{H}_i := (H_{i1}, H_{i2}, \dots, H_{id})^\top : \mathbb{R}^Z \rightarrow \mathbb{R}^d$ being a measurable function such that $\mathbf{H}_i(\mathcal{U}_{it})$ is well defined. Denote $\mathcal{U}_{it} = (\dots, u_{it-1}, u_{it})$.

$\mathcal{C}3''$ Let \mathbf{N}_i be the $d \times d$ matrix with $n_{i,kl} = \mathbb{E}[H_{ik}(\mathcal{U}_{i0})H_{il}(\mathcal{U}_{i0})]$ being kl -th entry. We assume that the smallest eigenvalue of \mathbf{N}_i is strictly bigger than 0.

$\mathcal{C}3'''$ Let $\mathbb{E}[H_i(\mathcal{U}_{i0})] = 0$ and $\|H_i(\mathcal{U}_{it})\|_{q'} < \infty$ for some $q' > \max\{2\theta, 4\}$, where θ will be introduced further.

$\mathcal{C}4'$ $\sum_{s=0}^{\infty} \delta_{q'}(H_i, s) < \infty$ for q' from Assumption $\mathcal{C}3'''$.

$\mathcal{C}4''$ For each i it holds that $\sum_{s=t}^{\infty} \delta_{q'}(H_i, s) = O(t^{-\alpha})$ for q' from Assumption $\mathcal{C}3'''$ and for some $\alpha > 1/2 - 1/q'$. [Go back](#)

Technical assumptions, part 3

C6 Let $\zeta_{i,t} = (u_{it}, \eta_{it})^\top$. Define $\mathcal{I}_{it} = (\dots, \zeta_{i,t-1}, \zeta_{i,t})$ and $U_i(\mathcal{I}_{it}) = H_i(\mathcal{U}_{it})G_i(\mathcal{J}_{it})$. With this notation at hand, we assume that $\sum_{s=0}^{\infty} \delta_2(U_i, s) < \infty$. [Go back](#)