Long-Run Variance Estimation

Model:

Let $\{\varepsilon_t\}$ be an AR(∞)-process of the form

$$\varepsilon_t = \sum_{j=1}^{\infty} a_j \varepsilon_{t-j} + \eta_t$$

with $\{\eta_t\}$ i.i.d. and $A(z) = 1 - \sum_{j=1}^{\infty} a_j z^j \neq 0$ for all $|z| \leq 1+\delta$ with some small $\delta > 0$.

Remarks:

(i.) The coefficients a_j decay to zero exp. fast, i.e., $|a_j| \leq C \xi^j$ for some $\xi \in (0,1)$ and $C < \infty$.

(ii.) $\{\varepsilon_t\}$ has an MA(∞)-representation of the form

$$\varepsilon_t = \sum_{j=0}^{\infty} c_j \eta_{t-j}.$$

The coefficients c_j decay exp. fast to zero, i.e., $|c_j| \leq C \xi^j$ for some $\xi \in (0,1)$ and $C < \infty$.

(iii) From (ii.), it follows that $\{\varepsilon_t\}$ is stationary and causal.

(iv.) If $a_j = 0$ for all $j > p^*$ and $a_{p^*} \neq 0$, then we have the special case of an AR(p^*)-process of finite order p^*.

(v.) The coefficients c_j can be computed iteratively from the following equations:

$$\underbrace{(1-a_1L-a_2L^2-\dots)}_{A(L)}\underbrace{(c_0+c_1L+c_2L^2+\dots)}_{C(L)}\eta_A = \eta_A,$$

that is,

$$c_0 = 1$$

$$c_1 - a_1 c_0 = 0$$

$$c_2 - a_1 c_1 - a_2 c_0 = 0$$

$$c_3 - a_1 c_2 - a_2 c_1 - a_3 c_0 = 0,$$

$$\vdots$$

that is,

$$c_j - \sum_{l=1}^{j} a_l c_{j-l} = 0 \quad \text{for } j \geq 1$$

with $c_l = 0$ for $l < 0$ and $a_l = 0$ for $l > p$.

Yule-Walker equations:

Consider the process $\{\Delta_q \varepsilon_t\}$ with $\Delta_q \varepsilon_t = \varepsilon_t - \varepsilon_{t-q}$.
$\{\Delta_q \varepsilon_t\}$ is an ARMA(∞, q)-process of the form

$$\Delta_q \varepsilon_t - \sum_{j=1}^{\infty} a_j \Delta_q \varepsilon_{t-j}$$
$$= A(L)\Delta_q \varepsilon_t = A(L)\varepsilon_t - A(L)\varepsilon_{t-q} = \eta_t - \eta_{t-q}.$$

The corresponding Yule-Walker equations are as follows:

$$\underbrace{E[\Delta_q \varepsilon_t \Delta_q \varepsilon_{t-l}]}_{=\gamma_q(l)} - \sum_{j=1}^{\infty} a_j \underbrace{E[\Delta_q \varepsilon_{t-j} \Delta_q \varepsilon_{t-l}]}_{=\gamma_q(l-j)}$$

$$= E[\eta_t \Delta_q \varepsilon_{t-l}] - E[\eta_{t-q} \Delta_q \varepsilon_{t-l}]$$
$$= E[\eta_t \varepsilon_{t-l}] - E[\eta_t \varepsilon_{t-l-q}]$$
$$- E[\eta_{t-q} \varepsilon_{t-l}] + E[\eta_{t-q} \varepsilon_{t-l-q}]$$
$$= 2 \cdot E[\eta_t \varepsilon_{t-l}] - E[\eta_t \varepsilon_{t-l-q}] - E[\eta_t \varepsilon_{t+l+q}]$$

with

$$E[\eta_t \varepsilon_{t+k}] = \sum_{j=0}^{\infty} c_j E[\eta_t \eta_{t+k-j}] = c_k \underset{E[\eta_0^2]}{\underset{\parallel}{\nu^2}}.$$

Hence,

$$\gamma_q(l) - \sum_{j=1}^{\infty} a_j \gamma_q(l-j) =: d_q(l) = \begin{cases} [2c_{|l|} - c_{|l|-q} - c_{|l|+q}]\nu^2 & , l = 0 \\ -c_{q-l}\nu^2 & , 1 \le l \le q \\ 0 & , l > q. \end{cases}$$

Combining these equations for $l = 1, \dots, p$, we get

$$\gamma_q - \Gamma_q a - r_q = d_q \text{ , i.e.}$$

$$\Gamma_q a = \gamma_q - d_q - r_q$$

with

$$\Gamma_q = \left(\gamma_q(i-j) : 1 \le i,j \le p \right)$$

$$\gamma_q = \left(\gamma_q(1), \dots, \gamma_q(p) \right)^T$$

$$d_q = \left(d_q(1), \dots, d_q(p) \right)^T$$

$$r_q = \left(\sum_{j=p+1}^{\infty} a_j \gamma_q(l-j) : l = 1, \dots, p \right)^T$$

$$a = (a_1, \dots, a_p)^T.$$

Note that $\|r_q\|_\infty \le C \sum_{j=p+1}^{\infty} |a_j| \le C \xi^p$

and thus $\|r_q\|_2 \le C \sqrt{p}\, \xi^p$.

Estimators:

Define

$$\hat{\gamma}_q(\ell) = \frac{1}{T-q} \sum_{t=q+\ell+1}^{T} \Delta_q Y_{t,T} \, \Delta_q Y_{t-\ell,T}$$

$$\hat{\gamma}_q^*(\ell) = \frac{1}{T-q} \sum_{t=q+\ell+1}^{T} \Delta_q \varepsilon_t \, \Delta_q \varepsilon_{t-\ell}.$$

We have

$$\hat{\gamma}_q(\ell) = \hat{\gamma}_q^*(\ell) + R_{q,1}(\ell) + R_{q,2}(\ell) + R_{q,3}(\ell)$$

with

$$R_{q,1}(\ell) = \frac{1}{T-q} \sum_{t=q+\ell+1}^{T} \overbrace{\Delta_q m_t}^{= m(t/T) - m(t-q/T)} \Delta_q \varepsilon_{t-\ell}$$

$$R_{q,2}(\ell) = \frac{1}{T-q} \sum_{t=q+\ell+1}^{T} \Delta_q \varepsilon_t \, \Delta_q m_{t-\ell}$$

$$R_{q,3}(\ell) = \frac{1}{T-q} \sum_{t=q+\ell+1}^{T} \Delta_q m_t \, \Delta_q m_{t-\ell}.$$

Write

$$\hat{\gamma}_q = (\hat{\gamma}_q(1), \ldots, \hat{\gamma}_q(p))^T$$

$$\hat{\Gamma}_q = (\hat{\gamma}_q(i-j) : 1 \le i,j \le p).$$

Define

$$\tilde{a}_q = \hat{\Gamma}_q^{-1} \hat{\gamma}_q = (\tilde{a}_1, \ldots, \tilde{a}_p)^T.$$

Moreover, let

$$\tilde{\nu}^2 = \frac{1}{2T} \sum_{t=p+2}^{T} \tilde{r}_{t,T}^2$$

with $\tilde{r}_{t,T} = \Delta_1 y_{t,T} - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 y_{t-j,T}$ and set

$$\tilde{c}^2 = \frac{\tilde{\nu}^2}{(1 - \sum_{j=1}^{p} \tilde{a}_j)^2}.$$

Finally, let the estimators $\tilde{c}_j$ $(j \geq 0)$ be defined by the recursions $\tilde{c}_0 = 1$ and

$$\tilde{c}_j = \sum_{\ell=1}^{j} \tilde{a}_\ell \tilde{c}_{j-\ell} \text{ for } j \geq 1.$$

Lemma 1: Let $1 \le p \ll \sqrt{T}$ and $1 \le q \ll \sqrt{T}$. Then for any $1 \le l \le p$,

$$E\left(\hat{\gamma}_q(l) - \gamma_q(l)\right)^2 \le C\left(\frac{1}{T-q} + \left(\frac{p}{T-q}\right)^2 + \left(\frac{q}{T}\right)^2 \right)$$

with a fixed constant C indep. of l, p, q, T.

Proof:

$$E\left(\hat{\gamma}_q(l) - \gamma_q(l)\right)^2$$
$$\le 16 \left\{ E\left(\hat{\gamma}^*_q(l) - \gamma_q(l)\right)^2 + \sum_{k=1}^{3} E R^2_{q,k}(l) \right\}.$$

We first have a close look at $E\left(\hat{\gamma}^*_q(l) - \gamma_q(l)\right)^2$:

$$\hat{\gamma}^*_q(l) - \gamma_q(l) = \frac{1}{T-q} \sum_{t=q+l+1}^{T} \left\{ \Delta_q \varepsilon_t \Delta_q \varepsilon_{t-l} - E \Delta_q \varepsilon_t \Delta_q \varepsilon_{t-l} \right\} - \left(1 - \frac{T-q-l}{T-q}\right) \gamma_q(l)$$

$$= \frac{1}{T-q} \sum_{t=q+l+1}^{T} \left\{ \varepsilon_t \varepsilon_{t-l} - E \varepsilon_t \varepsilon_{t-l} \right\}$$
$$- \frac{1}{T-q} \sum_{t=q+l+1}^{T} \left\{ \varepsilon_t \varepsilon_{t-l-q} - E \varepsilon_t \varepsilon_{t-l-q} \right\}$$
$$- \frac{1}{T-q} \sum_{t=q+l+1}^{T} \left\{ \varepsilon_{t-q} \varepsilon_{t-l} - E \varepsilon_{t-q} \varepsilon_{t-l} \right\}$$
$$+ \frac{1}{T-q} \sum_{t=q+l+1}^{T} \left\{ \varepsilon_{t-q} \varepsilon_{t-l-q} - E \varepsilon_{t-q} \varepsilon_{t-l-q} \right\}$$
$$- \left(1 - \frac{T-q-l}{T-q}\right) \gamma_q(l)$$

$$=: \Delta_{q,1}(l) + \ldots + \Delta_{q,4}(l) - \left(1 - \frac{T-q-l}{T-q}\right) \gamma_q(l).$$

and thus

$$E\left(\hat{\gamma}^*_q(l)-\gamma_q(l)\right)^2 \leq 25\left\{\sum_{k=1}^{4} E\Delta^2_{q,k}(l) + \underbrace{\left(1-\frac{T-q-l}{T-q}\right)^2 \gamma_q^2(l)}_{\leq C\left(P/_{T-q}\right)^2}\right\}.$$

It holds that

$$E\Delta^2_{q,1}(l) = \frac{1}{(T-q)^2}\sum_{t,t'=q+l+1}^{T} E\left[\varepsilon_t\varepsilon_{t-l}\varepsilon_{t'}\varepsilon_{t'-l}\right]$$

$$\underbrace{-\frac{1}{(T-q)^2}\sum_{t,t'=q+l+1}^{T} E\left[\varepsilon_t\varepsilon_{t-l}\right]E\left[\varepsilon_{t'}\varepsilon_{t'-l}\right]}_{=\frac{(T-q-l)^2}{(T-q)^2}\gamma_\varepsilon^2(l)}.$$

Since $\varepsilon_t = \sum_{k=0}^{\infty} c_k \eta_{t-k}$ and $\gamma_\varepsilon(l) = \left(\sum_{k=0}^{\infty} c_k c_{k+|l|}\right)\nu^2$,

$$E\left[\varepsilon_t\varepsilon_{t-l}\varepsilon_{t'}\varepsilon_{t'-l}\right]$$

$$= \sum_{k,k',j,j'} c_k c_{k'} c_j c_{j'} \underbrace{E\left[\eta_{t-k}\eta_{t-l-j}\eta_{t'-k'}\eta_{t'-l-j'}\right]}_{\substack{\neq 0 \text{ if} \\ (i.)\ k=l+j,\ k'=l+j' \\ (ii.)\ k'=k+(t'-t),\ j'=j+(t'-t) \\ (iii.)\ k'=(l+j)+(t'-t),\ k=(l+j')+(t-t') \\ (iv.)\ k=l+j,\ k'=k+(t'-t),\ j'=k'-l}}$$

$$= \Big(\sum_j c_j c_{j+\ell}\Big)^2 \nu^4 + \Big(\sum_k c_k c_{k+(t'-t)}\Big)^2 \nu^4$$

$$+ \Big(\sum_j c_j c_{(\ell+j)+(t'-t)} \sum_{j'} c_{j'} c_{(\ell+j')+(t-t')}\Big) \nu^4$$

$$+ \Big(\sum_j c_j c_{\ell+j} c_{(\ell+j)+(t'-t)} c_{(\ell+j)+(t'-t)-\ell}\Big) \underbrace{(E\eta_0^4 - 3\nu^4)}_{=K}$$

$$= \gamma_\varepsilon^2(\ell) + \gamma_\varepsilon^2(t'-t) + \gamma_\varepsilon(t'-t+\ell)\,\gamma_\varepsilon(t'-t-\ell)$$

$$+ \Big(\sum_j c_j c_{j+(t'-t)} c_{j+\ell} c_{j+\ell+(t'-t)}\Big) K.$$

Hence,

$$E\Delta_{q,1}^2(\ell) = \underbrace{\frac{1}{(T-q)^2}\sum_{t,t'=q+\ell+1}^{T} \gamma_\varepsilon^2(t'-t)}$$

$$= \frac{1}{(T-q)^2}\sum_r \underbrace{\#\{t'-t=r\}}_{\le T-q} \gamma_\varepsilon^2(r)$$

$$\le \frac{1}{T-q}\sum_{r=-\infty}^{\infty} \gamma_\varepsilon^2(r) \le C/T-q$$

$$+ \underbrace{\frac{1}{(T-q)^2}\sum_{t,t'=q+\ell+1}^{T} \gamma_\varepsilon(t'-t+\ell)\,\gamma_\varepsilon(t'-t-\ell)}$$

$$|\cdot| \le \frac{1}{(T-q)^2}\sum_r \#\{t'-t=r\}\,|\gamma_\varepsilon(r+\ell)\,\gamma_\varepsilon(r-\ell)|$$

$$\le \frac{\gamma_\varepsilon(0)}{T-q}\sum_{r=-\infty}^{\infty}|\gamma_\varepsilon(r)| \le C/T-q$$

$$+ \underbrace{\frac{1}{(T-q)^2} \sum_{t,t'=q+l+1}^{T} \left(\sum_j c_j \, c_{j+(t'-t)} \, c_{j+l} \, c_{j+l+(t'-t)} \right) \cdot K}$$

$$|\cdot| \le \frac{1}{(T-q)^2} \sum_r \#\{t'-t=r\} \sum_j |c_j \, c_{j+r} \, c_{j+l} \, c_{j+l+r}|$$

$$\le \frac{1}{T-q} \sum_j |c_j \, c_{j+l}| \sum_r |c_{j+r} \, c_{j+l+r}|$$

$$\le \frac{(\max_j |c_j|)^2}{T-q} \sum_j |c_j| \sum_r |c_r|$$

$$\le C/_{T-q},$$

which implies that

$$E \Delta^2_{q,k}(l) \le C/_{T-q} \qquad (*)$$

for $k=1$ with some fixed constant C that does not depend on l, p, q, T. Analogous calculations show that $(*)$ holds for $k=2,3,4$ as well. As a result, we get that

$$E\left(\hat{\gamma}^*_q(l) - \gamma_q(l)\right)^2 \le C\left(1/_{T-q} + \left(p/_{T-q} \right)^2 \right)$$

with some fixed constant independent of l, p, q, T.

Moreover,

$$ER^2_{q,1}(\ell) \le \underbrace{\left(\frac{1}{T-q}\sum_{t=q+\ell+1}^{T}\{\Delta_q m_t\}^2\right)}_{\le C\cdot(q/T)^2}\underbrace{\left(\frac{1}{T-q}\sum_{t=q+\ell+1}^{T}E\{\Delta_q \varepsilon_{t\ell}\}^2\right)}_{\le C}$$

$$\le C\left(q/T\right)^2$$

and analogously $ER^2_{q,2}(\ell) \le C\left(q/T\right)^2$ as well as $ER^2_{q,3}(\ell) \le C\left(q/T\right)^4$ with C indep. of ℓ, p, q, T.
Pulling everything together, we arrive at

$$E\left(\hat{\gamma}_q(\ell)-\gamma_q(\ell)\right)^2 \le C\left(1/T-q + \left(p/T-q\right)^2 + \left(q/T\right)^2\right)$$

with C indep. of ℓ, p, q, T. $\square$

Lemma 2: For $1 \le p \ll \sqrt{T}$ and $1 \le q \ll \sqrt{T}$,

$$E\|\hat{\gamma}_q - \gamma_q\|_2 \le C p^{1/2} \left(\frac{1}{T-q} + \left(\frac{p}{T-q}\right)^2 + \left(\frac{q}{T}\right)^2 \right)^{1/2}$$

$$E\|\hat{\Gamma}_q - \Gamma_q\|_2 \le C \cdot p \cdot \left(\frac{1}{T-q} + \left(\frac{p}{T-q}\right)^2 + \left(\frac{q}{T}\right)^2 \right)^{1/2}$$

with some fixed constant C indep. of p, q, T.

Proof:

$$\begin{aligned} E\|\hat{\gamma}_q - \gamma_q\|_2^2 &= E\left(\sum_{l=1}^{p} (\hat{\gamma}_q(l) - \gamma_q(l))^2 \right) \\ &= \sum_{l=1}^{p} E(\hat{\gamma}_q(l) - \gamma_q(l))^2 \\ &\le C \cdot p \left(\frac{1}{T-q} + \left(\frac{p}{T-q}\right)^2 + \left(\frac{q}{T}\right)^2 \right) \end{aligned}$$

by Lemma 1. Moreover, by Gershgorin's theorem,

$$\begin{aligned} \|\hat{\Gamma}_q - \Gamma_q\|_2 &\le \max_{1 \le i \le p} \left(\sum_{j=1}^{p} |\hat{\gamma}_q(i-j) - \gamma_q(i-j)| \right) \\ &\le \sum_{l=-p}^{p} |\hat{\gamma}_q(l) - \gamma_q(l)|, \end{aligned}$$

which immediately implies the second statement when combined with Lemma 1. $\square$

Lemma 3: Let $p \to \infty$ (in an arbitrary way) and spp that $(1+\delta)p \le q \ll \sqrt{T}$ for some small $\delta > 0$. Then

(i.) $\|\Gamma_q^{-1}\|_2 \le C$ for suff. large T with C indep. of p, q, T

(ii.) $\|\hat{\Gamma}_q^{-1} - \Gamma_q^{-1}\|_2 = O_p(p/\sqrt{T})$.

(iii) $\|\hat{\Gamma}_q^{-1}\|_2 = O_p(1)$.

Proof:

(i.) Consider $\Gamma_q = (\gamma_q(i-j) : 1 \le i,j \le p)$. Since $\gamma_q(\ell) = 2\gamma_\varepsilon(\ell) - \gamma_\varepsilon(q-\ell) - \gamma_\varepsilon(q+\ell)$ with $\gamma_\varepsilon(\ell) = Cov(\varepsilon_t, \varepsilon_{t-\ell})$, we have $\Gamma_q = 2\Gamma - R$, where $\Gamma = (\gamma_\varepsilon(i-j) : 1 \le i,j \le p)$ and $R = (\gamma_\varepsilon(q+(i-j)) + \gamma_\varepsilon(q-(i-j)) : 1 \le i,j \le p)$.

The spectral density f_ε of $\{\varepsilon_t\}$ is bounded away from zero and infinity, i.e., $0 < c_f \le f(w) \le C_f < \infty$ for all w and some constants c_f, C_f. This implies that the eigenvalues of Γ lie in some interval $[c_\Gamma, C_\Gamma]$ with constants $0 < c_\Gamma \le C_\Gamma < \infty$ indep. of p [Grenander & Szegö (1958), Toeplitz Forms and Their Applications, Sec. 5.2; Brockwell & Davis (1991), Prop. 4.5.3.]

From this, it immediately follows that $\|\Gamma^{-1}\|_2 \leq C$ with some C indep. of p.

For general matrices A and B, it holds that $A^{-1} - B^{-1} = (A^{-1} - B^{-1} + B^{-1})(B - A)B^{-1}$. Since $\|AB\|_2 \leq \|A\|_2 \|B\|_2$, this implies that

$$\|A^{-1} - B^{-1}\|_2 \leq \left(\|A^{-1} - B^{-1}\|_2 + \|B^{-1}\|_2\right) \|B - A\|_2 \, \|B^{-1}\|_2$$

and thus

$$\|A^{-1} - B^{-1}\|_2 \leq \frac{\|B^{-1}\|_2^2 \, \|B - A\|_2}{1 - \|B - A\|_2 \, \|B^{-1}\|_2}. \qquad (*)$$

Setting $A = \Gamma_q/2$ and $B = \Gamma$, we get that

$$\|(\Gamma_q/2)^{-1} - \Gamma^{-1}\|_2 \leq \frac{\|\Gamma^{-1}\|_2^2 \, \|\Gamma_q/2 - \Gamma\|_2}{1 - \|\Gamma_q/2 - \Gamma\|_2 \, \|\Gamma^{-1}\|_2} \leq C$$

for suff. large T with C indep. of p, q, T, since $\|\Gamma^{-1}\|_2 \leq C$ and

$$\|\Gamma_q/2 - \Gamma\|_2 = \|R/2\|_2 \leq \sum_{l=-p}^{p} |\gamma_\varepsilon(q+l) + \gamma_\varepsilon(q-l)|/2$$

$$\leq C p \cdot \xi^{q-p} \overset{(+)}{=} o(1).$$

From this, it is easy to conclude that $\|\Gamma_q^{-1}\|_2 \leq C$ with C indep. of p, q, T for suff. large T.

(ii.) With (*), we get that

$$\|\hat{\Gamma}_q^{-1} - \Gamma_q^{-1}\|_2 \leq \frac{\|\Gamma_q^{-1}\|_2^2 \, \|\hat{\Gamma}_q - \Gamma_q\|_2}{1 - \|\hat{\Gamma}_q - \Gamma_q\|_2 \, \|\Gamma_q^{-1}\|_2}$$

$$= O_p\left(P/\sqrt{T}\right),$$

since $\|\Gamma_q^{-1}\|_2 \leq C$ by (i.) and $\|\hat{\Gamma}_q - \Gamma_q\|_2 = O_p(P/\sqrt{T})$ by Lemma 2.

(iii.) follows directly from (i.) and (ii.). $\square$

(+): Use rule of l'Hopital: Let $f, g: \mathbb{R} \to \mathbb{R}$ diff. with $g' \neq 0$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \pm\infty$. Then $\lim_{x\to\infty} f(x)/g(x) = \lim_{x\to\infty} f'(x)/g'(x)$.

Here: $P \xi^{q-P} \leq P \cdot \xi^{\partial P} = P/\xi^{-\partial P}$

$(f(P) = P,\ g(P) = \xi^{-\partial P})$

$$\lim_{P\to\infty} \frac{P}{\xi^{-\partial P}} = \lim_{P\to\infty} \frac{1}{-\partial \xi^{-\partial P}} = -\frac{1}{\partial} \lim_{P\to\infty} \xi^{\partial P} = 0.$$

Lemma 4: Let $C \log T \le p \ll q$ and $q \ll \sqrt{T}$.
It holds that

$$\| \tilde{a}_q - a \|_2 = O_p\left(\sqrt{p/T}\right).$$

Proof:

$$\tilde{a}_q - a = \hat{\Gamma}_q^{-1} \hat{\gamma}_q - a$$
$$= \hat{\Gamma}_q^{-1} \left((\hat{\gamma}_q - \gamma_q) + (\gamma_q - \Gamma_q a) + (\Gamma_q - \hat{\Gamma}_q) a \right)$$

and thus

$$\| \tilde{a}_q - a \|_2 \le \underbrace{\| \hat{\Gamma}_q^{-1} \|_2}_{\substack{= O_p(1) \text{ by} \\ \text{Lemma 3}}} \Big\{ \underbrace{\| \hat{\gamma}_q - \gamma_q \|_2}_{\substack{= O_p(\sqrt{p/T}) \text{ by} \\ \text{Lemma 2}}} + \underbrace{\| \gamma_q - \Gamma_q a \|_2}_{\substack{= \| d_q + r_q \|_2 \\ \le C \sqrt{p} (\rho^{q-p} + \rho^p)}}$$
$$+ \underbrace{\| (\Gamma_q - \hat{\Gamma}_q) a \|_2}_{\overset{(*)}{=} O_p(\sqrt{p/T})} \Big\}$$

$$= O_p\left(\sqrt{p/T} + \sqrt{p} \rho^{q-p} + \sqrt{p} \rho^p \right).$$

It remains to prove $(*)$:

$$\|(\Gamma_q - \hat{\Gamma}_q) a\|_2^2$$

$$= \sum_{i=1}^{p} \left(\sum_{j=1}^{p} \left\{ \gamma_q(i-j) - \hat{\gamma}_q(i-j) \right\} a_j \right)^2$$

$$= \sum_{j,j'=1}^{p} \left(\sum_{i=1}^{p} \left\{ \gamma_q(i-j) - \hat{\gamma}_q(i-j) \right\} \left\{ \gamma_q(i-j') - \hat{\gamma}_q(i-j') \right\} \right) a_j a_{j'}$$

$$\leq \sum_{j,j'=1}^{p} |a_j a_{j'}| \left(\sum_{i=1}^{p} \left\{ \gamma_q(i-j) - \hat{\gamma}_q(i-j) \right\}^2 \right)^{1/2}$$

$$\times \left(\sum_{i=1}^{p} \left\{ \gamma_q(i-j') - \hat{\gamma}_q(i-j') \right\}^2 \right)^{1/2}$$

$$\leq \max_{1 \leq j \leq p} \sum_{i=1}^{p} \left\{ \gamma_q(i-j) - \hat{\gamma}_q(i-j) \right\}^2 \cdot \underbrace{\sum_{j,j'=1}^{p} |a_j a_{j'}|}_{\leq C}$$

$$\leq C \cdot \sum_{\ell=-p}^{p} \left\{ \gamma_q(\ell) - \hat{\gamma}_q(\ell) \right\}^2$$

$$= O_P(p/T)$$

by Lemma 1. $\square$

Lemma 5: Let $C \log T \leq p \ll \min\{T^{1/3}, q\}$ and $q \ll \sqrt{T}$. Then
$$\tilde{\nu}^2 = \nu^2 + O_p\left(\sqrt{p/T}\right).$$

Proof:

$$\tilde{r}_{t,T} = \left(\Delta_1 \varepsilon_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 \varepsilon_{t-j}\right) + \left(\Delta_1 m_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 m_{t-j}\right)$$

$$\begin{aligned}
\tilde{\nu}^2 &= \frac{1}{2T} \sum_{t=p+2}^{T} \left\{\Delta_1 \varepsilon_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 \varepsilon_{t-j}\right\}^2 \\
&\quad + \frac{1}{2T} \sum_{t=p+2}^{T} \left\{\Delta_1 m_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 m_{t-j}\right\}^2 \\
&\quad + \frac{1}{T} \sum_{t=p+2}^{T} \left\{\Delta_1 \varepsilon_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 \varepsilon_{t-j}\right\}\left\{\Delta_1 m_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 m_{t-j}\right\} \\
&=: \tilde{\nu}_1^2 + \tilde{\nu}_2^2 + \tilde{\nu}_3^2
\end{aligned}$$

$$\begin{aligned}
\tilde{\nu}_1^2 &= \frac{1}{2T} \sum_{t=p+2}^{T} \left\{\Delta_1 \varepsilon_t - \sum_{j=1}^{p} a_j \Delta_1 \varepsilon_{t-j}\right\}^2 \\
&\quad + \frac{1}{2T} \sum_{t=p+2}^{T} \left\{\sum_{j=1}^{p} (a_j - \tilde{a}_j) \Delta_1 \varepsilon_{t-j}\right\}^2 \\
&\quad + \frac{1}{T} \sum_{t=p+2}^{T} \left\{\Delta_1 \varepsilon_t - \sum_{j=1}^{p} a_j \Delta_1 \varepsilon_{t-j}\right\}\left\{\sum_{j=1}^{p} (a_j - \tilde{a}_j) \Delta_1 \varepsilon_{t-j}\right\} \\
&=: \tilde{\nu}_{1,A}^2 + \tilde{\nu}_{1,B}^2 + \tilde{\nu}_{1,C}^2.
\end{aligned}$$

$$\tilde{v}_{1,A}^2 = \frac{1}{2T}\sum_{t=p+2}^{T}\left(\eta_t-\eta_{t-1}\right)^2$$

$$+\underbrace{\frac{1}{2T}\sum_{t=p+2}^{T}\left(\sum_{j=p+1}^{\infty}a_j\Delta_1\varepsilon_{t-j}\right)^2}_{E[\cdot]\,\le\,\sum_{j,j'=p+1}^{\infty}|a_j||a_{j'}|\,E\left[|\Delta_1\varepsilon_{t-j}\Delta_1\varepsilon_{t-j'}|\right]\,\le\, C\cdot\rho^{2p}}$$

$$+\underbrace{\frac{1}{T}\sum_{t=p+2}^{T}\left(\eta_t-\eta_{t-1}\right)\left(\sum_{j=p+1}^{\infty}a_j\Delta_1\varepsilon_{t-j}\right)}_{E|\cdot|\,\le\, C\cdot\sum_{j=p+1}^{\infty}|a_j|\,\le\, C\cdot\rho^{p}}$$

$$= v^2+O_p\left(1/\sqrt{T}\right)+O_p\left(\rho^p\right)$$

$$\tilde{v}_{1,B}^2 \le \frac{1}{2T}\sum_{t=p+2}^{T}\left\{\sum_{j=1}^{p}\left(a_j-\tilde{a}_j\right)^2\right\}\left\{\sum_{j=1}^{p}\left(\Delta_1\varepsilon_{t-j}\right)^2\right\}$$

$$= \|\tilde{a}_q-a\|_2^2\cdot O_p(p)$$

$$= O_p\left(p^2/T\right).$$

$$\tilde{V}_{1,K}^2 = \frac{1}{T}\sum_{t=p+2}^{T}\left\{(u_t-u_{t-1})+\sum_{j=p+1}^{\infty}a_j\Delta_1\varepsilon_{t-j}\right\}$$

$$\times\left\{\sum_{j=1}^{p}(a_j-\tilde{a}_j)\Delta_1\varepsilon_{t-j}\right\}$$

$$=\sum_{j=1}^{p}(a_j-\tilde{a}_j)\left\{\frac{1}{T}\sum_{t=p+2}^{T}(u_t-u_{t-1})\Delta_1\varepsilon_{t-j}\right\}$$

$$+\underbrace{\frac{1}{T}\sum_{t=p+2}^{T}\left\{\sum_{j=p+1}^{\infty}a_j\Delta_1\varepsilon_{t-j}\right\}\left\{\sum_{j=1}^{p}(a_j-\tilde{a}_j)\Delta_1\varepsilon_{t-j}\right\}}$$

$$|\cdot| \le \underbrace{\left(\frac{1}{T}\sum_{t=p+2}^{T}\left\{\sum_{j=p+1}^{\infty}a_j\Delta_1\varepsilon_{t-j}\right\}^2\right)^{1/2}}_{=O_p(\varrho^p)}$$

$$\times\underbrace{\left(\frac{1}{T}\sum_{t=p+2}^{T}\left\{\sum_{j=1}^{p}(a_j-\tilde{a}_j)\Delta_1\varepsilon_{t-j}\right\}^2\right)^{1/2}}$$

$$\le \|a-\tilde{a}_q\|_2\cdot\left(\frac{1}{T}\sum_{t=p+2}^{T}\left\{\sum_{j=1}^{p}(\Delta_1\varepsilon_{t-j})^2\right\}\right)^{1/2}$$

$$=O_p\left(\sqrt{p/T}\right)\cdot O_p\left(\sqrt{p}\right)$$

$$=O_p\left(\varrho^p\cdot p/\sqrt{T}\right)$$

$$= \underbrace{(a_1 - \tilde{a}_1)}_{\substack{= O_p(1/\sqrt{T}) \text{ by} \\ \text{arguments similar} \\ \text{to those for Lemma 4}}} \underbrace{\left\{ \frac{1}{T} \sum_{t=p+2}^{T} (m_t - m_{t-1}) \Delta_1 \varepsilon_{t-1} \right\}}_{= O_p(1)}$$

$$+ \underbrace{\sum_{j=2}^{p} (a_j - \tilde{a}_j) \left\{ \frac{1}{T} \sum_{t=p+2}^{T} (m_t - m_{t-1}) \Delta_1 \varepsilon_{t-j} \right\}} + O_p\left(\rho^p \cdot p/\sqrt{T}\right)$$

$$|\cdot| \le \underbrace{\sum_{j=2}^{p} |a_j - \tilde{a}_j|}_{\substack{\le \sqrt{p} \|a - \tilde{a}_q\|_2 \\ = O_p(p/\sqrt{T})}} \cdot \underbrace{\max_{2 \le j \le p} \left| \frac{1}{T} \sum_{t=p+2}^{T} (m_t - m_{t-1}) \Delta_1 \varepsilon_{t-j} \right|}_{= O_p(p/\sqrt{T})}$$

$$= O_p\left(1/\sqrt{T} + p^2/T + \rho^p \cdot p/\sqrt{T}\right).$$

Hence,

$$\tilde{\nu}_1^2 = \nu^2 + O_p\left(\sqrt{p/T}\right).$$

(noting that $p^2/T \le \sqrt{p/T}$ for $p \le T^{1/3}$).

Moreover,

$$\tilde{\nu}_2^2 \le \frac{1}{2T} \sum_{t=p+2}^{T} \left\{ C/T \left(1 + \sum_{j=1}^{p} |\tilde{a}_j| \right) \right\}^2$$

$$\le C/T^2 \left(1 + \sum_{j=1}^{p} |\tilde{a}_j| \right)^2 = O_p\left(1/T^2\right)$$

and

$$|\tilde{\nu}_3^2| \le \left(\frac{1}{T} \sum_{t=p+2}^{T} \left\{ \Delta_1 \varepsilon_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 \varepsilon_{t-j} \right\}^2 \right)^{1/2}$$

$$\times \left(\frac{1}{T} \sum_{t=p+2}^{T} \left\{ \Delta_1 m_t - \sum_{j=1}^{p} \tilde{a}_j \Delta_1 m_{t-j} \right\}^2 \right)^{1/2}$$

$$= O_p(1/T).$$

Putting everything together, we arrive at

$$\tilde{\nu}^2 = \nu^2 + O_p\left(\sqrt{p/T}\right). \qquad \square$$

Lemma 6: Let $C \log T \leq p \ll \min\{T^{1/3}, q\}$ and $q \ll \sqrt{T}$. Then

$$\tilde{c}^2 = c^2 + O_p(p/\sqrt{T}).$$

Proof:

$$\left(1-\sum_{j=1}^{p} \tilde{a}_j\right)^2 = \left(1-\sum_{j=1}^{p} a_j + \sum_{j=1}^{p}(a_j-\tilde{a}_j)\right)^2$$

$$= \underbrace{\left(1-\sum_{j=1}^{p} a_j\right)^2}_{\to c^* \neq 0}$$

$$+ 2\cdot\left(1-\sum_{j=1}^{p} a_j\right)\underbrace{\left(\sum_{j=1}^{p}(a_j-\tilde{a}_j)\right)}_{\substack{|\cdot| \leq \sqrt{p}\,\|a-\tilde{a}_q\|_2 \\ = O_p(p/\sqrt{T})}}$$

$$+ \underbrace{\left(\sum_{j=1}^{p}(a_j-\tilde{a}_j)\right)^2}_{= O_p(p^2/T)}$$

$$= \underbrace{\left(1-\sum_{j=1}^{p} a_j\right)^2}_{\to c^* \neq 0} + O_p(p/\sqrt{T}).$$

Thus,

$$\tilde{\sigma}^2-\sigma^2 = \frac{\tilde{\nu}^2}{\left(1-\sum_{j=1}^{p}\tilde{a}_j\right)^2}-\frac{\nu^2}{\left(1-\sum_{j=1}^{\infty}a_j\right)^2}$$

$$= \underbrace{\frac{\tilde{\nu}^2-\nu^2}{\left(1-\sum_{j=1}^{p}\tilde{a}_j\right)^2}}_{=O_p\left(\sqrt{p/T}\right)} + \nu^2\underbrace{\left(\frac{1}{\left(1-\sum_{j=1}^{p}\tilde{a}_j\right)^2}-\frac{1}{\left(1-\sum_{j=1}^{\infty}a_j\right)^2}\right)}_{=\frac{\left(1-\sum_{j=1}^{\infty}a_j\right)^2-\left(1-\sum_{j=1}^{p}\tilde{a}_j\right)^2}{\left(1-\sum_{j=1}^{\infty}a_j\right)\left(1-\sum_{j=1}^{p}\tilde{a}_j\right)} = O_p\left(p/\sqrt{T}\right)}$$

$$= O_p\left(p/\sqrt{T}\right).$$

$\square$

Lemma 7: Let the estimators $\tilde{c}_j$ be defined by the recursions $\tilde{c}_0 = 1$ and

$$\tilde{c}_j = \sum_{\ell=1}^{j} \hat{a}_\ell \tilde{c}_{j-\ell} \quad \text{for } j \geq 1.$$

If $C \log T \leq p \ll \min\{T^{1/3}, q\}$ and $q \ll \sqrt{T}$, then

$$\max_{1 \leq j \leq p} |\tilde{c}_j - c_j| = O_p(p/\sqrt{T}).$$

Proof:

By assumption, $A(z) \neq 0$ for all $|z| < 1+\delta$. Hence, $A^{-1}(z)$ is analytic (complex. diff.) on $D = \{z \in \mathbb{C} : |z| < 1+\delta\}$ and has a unique Laurent series representation in the ring segment $R = \{z \in \mathbb{C} : 0 < \delta' < |z| < 1+\delta\}$ of the form $A^{-1}(z) = \sum_{j=-\infty}^{\infty} \phi_j z^j$ with

$$\phi_j = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{A^{-1}(\zeta)}{\zeta^{j+1}} \, d\zeta.$$

As

$$A^{-1}(z) = \sum_{j=1}^{\infty} c_j z^j \quad \text{for } |z| < 1+\delta,$$

the uniqueness of the Laurent series implies that

$$c_j = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{A^{-1}(\zeta)}{\zeta^{j+1}} \, d\zeta.$$

By Prop. 5.1.1 and Problems 7.11 and 8.3 in Brockwell & Davis, we have the following:

If $\hat{\gamma}_q(0) > 0$, then

$$\tilde{A}(z) = 1 - \sum_{j=1}^{p} \tilde{a}_j z^j \neq 0 \quad \forall |z| \leq 1,$$

which implies (since $\tilde{A}(z)$ cannot have more than p roots) that

$$\tilde{A}(z) = 1 - \sum_{j=1}^{p} \tilde{a}_j z^j \neq 0 \quad \forall |z| < 1 + \tilde{\delta},$$

where $\tilde{\delta} = \tilde{\delta}(\tilde{a}_1, \dots, \tilde{a}_p) > 0$ is random.

Since $\hat{\gamma}_q(0) > 0$ holds with prob. approaching 1, we obtain that

$$\tilde{A}(z) = 1 - \sum_{j=1}^{p} \tilde{a}_j z^j \neq 0 \quad \forall |z| < 1 + \tilde{\delta}$$

with $\tilde{\delta} > 0$ with prob. approaching 1.

By construction of $\tilde{c}_j$, it holds that

$$\tilde{A}^{-1}(z) = \sum_{j=0}^{\infty} \tilde{c}_j z^j \quad \text{for } |z| < 1 + \tilde{\delta}.$$

As before, we can argue that

$$\tilde{c}_j = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{\tilde{A}^{-1}(\xi)}{\xi^{j+1}} \, d\xi \quad \text{for } j \geq 0$$

with prob. approaching 1.

It holds that

$$\max_{1\le j\le p} |\tilde{c}_j - c_j|$$

$$= \max_{1\le j\le p} \left| \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\tilde{A}^{-1}(\zeta) - A^{-1}(\zeta)}{\zeta^{j+1}} \, d\zeta \right|$$

$$\le \underbrace{\max_{1\le j\le p} \frac{1}{2\pi} \int_{|\zeta|=1} \frac{|\tilde{A}^{-1}(\zeta) - A^{-1}(\zeta)|}{\underbrace{|\zeta|^{j+1}}_{=1}} \, d\zeta}_{\le \sup_{|\zeta|=1} |\tilde{A}^{-1}(\zeta) - A^{-1}(\zeta)| \cdot 2\pi}$$

$$\le \sup_{|\zeta|=1} |\tilde{A}^{-1}(\zeta) - A^{-1}(\zeta)|$$

$$= \sup_{|\zeta|=1} \left| \frac{1}{1-\sum_{j=1}^{p} \tilde{a}_j \zeta^j} - \frac{1}{1-\sum_{j=1}^{\infty} a_j \zeta^j} \right|$$

$$= \sup_{|\zeta|=1} \frac{\left|\sum_{j=1}^{p} (\tilde{a}_j - a_j)\zeta^j - \sum_{j=p+1}^{\infty} a_j \zeta^j\right|}{\left|1-\sum_{j=1}^{p} \tilde{a}_j \zeta^j\right| \underbrace{\left|1-\sum_{j=1}^{\infty} a_j \zeta^j\right|}_{\ge C^* > 0}}$$

$$\le \underbrace{\left(\sum_{j=1}^{p} |\tilde{a}_j - a_j| + \sum_{j=p+1}^{\infty} |a_j| \right)}_{= O_p\left(p/\sqrt{T} + \zeta^p\right)} \cdot \frac{1}{C^*} \cdot \underbrace{\sup_{|\zeta|=1} \frac{1}{\left|1-\sum_{j=1}^{p} \tilde{a}_j \zeta^j\right|}}_{=:(*)}$$

with

$$(*) = \sup_{|\varsigma| \leq 1} \frac{1}{\left|1 - \sum_{j=1}^{p} \tilde{a}_j \varsigma^j\right|}$$

$$= \sup_{|\varsigma| \leq 1} \frac{1}{\left|1 - \sum_{j=1}^{\infty} a_j \varsigma^j + \sum_{j=p+1}^{\infty} a_j \varsigma^j + \sum_{j=1}^{p} (a_j - \tilde{a}_j) \varsigma^j\right|}$$

$$\leq \sup_{|\varsigma| \leq 1} \frac{1}{\underbrace{\left|1 - \sum_{j=1}^{\infty} a_j \varsigma^j\right|}_{\geq c^* > 0} - \underbrace{\sum_{j=p+1}^{\infty} |a_j|}_{\substack{\leq c^*/2 \text{ for } p \\ \text{large enough}}} - \underbrace{\sum_{j=1}^{p} |a_j - \tilde{a}_j|}_{= O_p(p/\sqrt{T})}}$$

$$= O_p(1).$$

Hence,

$$\max_{1 \leq j \leq p} \left|\tilde{c}_j - c_j\right| = O_p\left(p/\sqrt{T} + \varsigma^p\right).$$

□

Lemma 8: Let $C\log T \le p \ll \min\{T^{1/3}, q\}$, $q \ll \sqrt{T}$ and $r = (1+\delta)p$ for some small $\delta > 0$. Then

$$\|\hat{a}_r - a\|_2 = O_p\left(p\sqrt{p/T}\right),$$

where $\hat{a}_r = \hat{\Gamma}_r^{-1}(\hat{\gamma}_r - \hat{d}_r)$.

Proof:

As in the proof of Lemma 4,

$$\|\hat{a}_r - a\|_2 \le \underbrace{\|\hat{\Gamma}_r^{-1}\|_2}_{=O_p(1)} \Big\{ \underbrace{\|\hat{\gamma}_r - \gamma_r\|_2}_{=O_p(\sqrt{p/T})} + \underbrace{\|\gamma_r - \Gamma_r a - \hat{d}_r\|_2}_{=:(*)}$$

$$+ \underbrace{\|(\Gamma_r - \hat{\Gamma}_r)a\|_2}_{=O_p(\sqrt{p/T})} \Big\}.$$

Noting that $\gamma_r - \Gamma_r a = d_r + r_r$,

$$(*) \le \underbrace{\|\hat{d}_r - d_r\|_2}_{\substack{=O_p(p\sqrt{p/T}) \\ \text{by Lemma 7}}} + \underbrace{\|r_r\|_2}_{\le C\sqrt{p}\rho^p} = O_p\left(p\sqrt{p/T} + \sqrt{p}\rho^p\right).$$

Hence,

$$\|\hat{a}_r - a\|_2 = O_p\left(p\sqrt{p/T}\right).$$

Repeating the arguments from Lemmas 5 and 6, we can finally prove the following: Let $C \log T \le p \ll \min\{T^{1/5}, q\}$, $q \ll \sqrt{T}$ and $r = (1+\delta)p$ for some small $\delta > 0$. Then

$$\hat{\nu}^2 = \nu^2 + O_p\left(p\sqrt{p/T}\right)$$

$$\hat{\sigma}^2 = \sigma^2 + O_p\left(p^2/\sqrt{T}\right).$$