1 Testing for equality of time trends

1.1 Construction of the test statistic

The i-th time series in the model satisfies the equation $Y_{it} = m_i(\frac{t}{T}) + \varepsilon_{it}$, where ε_{it} are zero-mean error terms and $\int_0^1 m_i(u)du = 0$ by normalization.

The stationary error process $\{\varepsilon_{it}\}$ for each i is assumed to have the following properties:

(C1) The variables ε_t allow for the representation $\varepsilon_t = G(\ldots, \eta_{t-1}, \eta_t)$, where η_t are i.i.d. random variables and G is a measurable function.

(C2) It holds that $\|\varepsilon_t\|_q < \infty$ for some $q > 4$, where $\|\varepsilon_t\|_q = (\mathbb{E}|\varepsilon_t|^q)^{1/q}$.

Following Wu (2005), we impose conditions on the dependence structure of the error process $\{\varepsilon_t\}$ in terms of the physical dependence measure $d_{t,q} = \|\varepsilon_t - \varepsilon_t'\|_q$, where $\varepsilon_t' = G(\ldots, \eta_{-1}, \eta_0', \eta_1, \ldots, \eta_{t-1}, \eta_t, \eta_{t+1}, \ldots)$ with $\{\eta_t'\}$ being an i.i.d. copy of $\{\eta_t\}$. In particular, we assume the following:

(C3) Define $\Theta_{t,q} = \sum_{|s| \geq t} d_{s,q}$ for $t \geq 0$. It holds that

$$\Theta_{t,q} = O\big(t^{-\tau_q}(\log t)^{-A}\big),$$

where $A > \frac{2}{3}(1/q + 1 + \tau_q)$ and $\tau_q = \{q^2 - 4 + (q-2)\sqrt{q^2 + 20q + 4}\}/8q$.

We further let $\widehat{\sigma}_i^2$ be an estimator of the long-run error variance $\sigma_i^2 = \sum_{\ell=-\infty}^{\infty} \mathrm{Cov}(\varepsilon_{i0}, \varepsilon_{i\ell})$. Throughout the section, we assume that $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(1/\log T)$. Furthermore, we need to impose certain bounds on the long-run error variances σ_i^2:

(C4) There exist constants $c > 0, C > 0$ such that $\forall i \in \{1, \ldots, n\}$ we have $c \leq \sigma_i^2 \leq C$.

In order to use the coupling method further, we split the whole sample into $S = \lceil T^{1/2} \rceil$ different blocks with the length of the blocks being $r = T/\lceil T^{1/2} \rceil \approx T^{1/2}$.

We are now ready to introduce the multiscale statistic for testing the hypothesis $H_0 : m_1 = m_2 = \ldots = m_n$. For any pair of time series i and j, we define the kernel averages

$$\widehat{\psi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)(Y_{it} - Y_{jt}),$$

where the kernel weights $w_{t,T}(u,h)$ are defined something about block-specific nature: assume that are kernel weights $w_{t,T}(u,h)$ are constant on each "block", i.e. $w_{t,T}(u,h) = u_{s,S}(u,h)$ for $t \in$ block s, $s \in \{1, \ldots, S\}$.

We aggregate the kernel averages $\widehat{\psi}_{ij,T}(u,h)$ for all $(u,h) \in \mathcal{G}_T$ by the multiscale statistic

$$\widehat{\Psi}_{ij,T} = \max_{(u,h)\in\mathcal{G}_T} \Big\{ \Big| \frac{\widehat{\psi}_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} \Big| - \lambda(h) \Big\},$$

where $\lambda(h) = \sqrt{2\log\{1/(2h)\}}$. The statistic $\widehat{\Psi}_{ij,T}$ can be interpreted as a distance measure between the two curves m_i and m_j. We finally define the multiscale statistic for testing the null hypothesis $H_0 : m_1 = m_2 = \ldots = m_n$ as

$$\widehat{\Psi}_{n,T} = \max_{1\le i<j\le n} \widehat{\Psi}_{ij,T},$$

that is, we define it as the maximal distance $\widehat{\Psi}_{ij,T}$ between any pair of curves m_i and m_j with $i \neq j$.

1.2 The test procedure

Let Z_{it} for $1 \le t \le T$ and $1 \le i \le n$ be independent standard normal random variables which are independent of the error terms ε_{it}. For each i and j, we introduce the Gaussian statistic

$$\Phi_{ij,T} = \max_{(u,h)\in\mathcal{G}_T} \Big\{ \Big| \frac{\phi_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} \Big| - \lambda(h) \Big\},$$

where

$$\phi_{ij,T}(u,h) = \sum\nolimits_{s=1}^{\lceil T^{1/2}\rceil} u_{s,\lceil T^{1/2}\rceil}(u,h) \, \{\widehat{\sigma}_i Z_{is} - \widehat{\sigma}_j Z_{js}\} = \sum\nolimits_{s=1}^{S} u_{s,S}(u,h) \, \{\widehat{\sigma}_i Z_{is} - \widehat{\sigma}_j Z_{js}\}.$$

Moreover, we define the statistic

$$\Phi_{n,T} = \max_{1\le i<j\le n} \Phi_{ij,T}$$

and denote its $(1-\alpha)$-quantile by $q_{n,T}(\alpha)$. Our multiscale test of the hypothesis $H_0 : m_1 = m_2 = \ldots = m_n$ is defined as follows: For a given significance level $\alpha \in (0,1)$, we reject H_0 if $\widehat{\Psi}_{n,T} > q_{n,T}(\alpha)$.

1.3 Theoretical properties of the test

To start with, we introduce the auxiliary statistic

$$\widehat{\Phi}_{n,T} = \max_{1\le i<j\le n} \widehat{\Phi}_{ij,T},$$

where

$$\widehat{\Phi}_{ij,T} = \max_{(u,h)\in\mathcal{G}_T} \Big\{ \Big| \frac{\widehat{\phi}_{ij,T}(u,h)}{\{\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2\}^{1/2}} \Big| - \lambda(h) \Big\}$$

and $\widehat{\phi}_{ij,T}(u,h) = \sum_{t=1}^{T} w_{t,T}(u,h)\{\varepsilon_{it} - \varepsilon_{jt}\}$.

Our first theoretical result characterizes the asymptotic behaviour of the statistic $\widehat{\Phi}_{n,T}$.

Theorem 1.1. *Suppose that the error processes $\mathcal{E}_i = \{\varepsilon_{it} : 1 \le t \le T\}$ are independent across i and satisfy (C1)–(C3) for each i. Moreover, let (C4) and standard assumptions on the kernel $K(\cdot)$ as well as usual assumptions on the set $\mathcal{G}_T$ be fulfilled and assume that $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$ with $\rho_T = o(1/\log T)$ for each i. Then*

$$\mathbb{P}\big(\widehat{\Phi}_{n,T} \le q_{n,T}(\alpha)\big) = (1-\alpha) + o(1).$$

Proof of Theorem 1.1

First of all, we let

$$\Phi^*_{n,T} = \max_{1 \le i < j \le n} \Phi^*_{ij,T} \quad \text{with} \quad \Phi^*_{ij,T} = \max_{(u,h) \in \mathcal{G}_T} \Big\{ \Big| \frac{\phi^*_{ij,T}(u,h)}{(\sigma_i^2 + \sigma_j^2)^{1/2}} \Big| - \lambda(h) \Big\},$$

where

$$\phi^*_{ij,T}(u,h) = \sum\nolimits_{s=1}^{\lceil T^{1/2} \rceil} u_{s,\lceil T^{1/2} \rceil}(u,h) \left\{ \sigma_i Z_{is} - \sigma_j Z_{js} \right\} = \sum\nolimits_{s=1}^{S} u_{s,S}(u,h) \left\{ \sigma_i Z_{is} - \sigma_j Z_{js} \right\}$$

with the same Z_{is} as in the Gaussian statistic $\Phi_{n,T}$.

It holds that

$$\big|\Phi_{n,T} - \Phi^*_{n,T}\big| = o_p\big(\rho_T \sqrt{\log S}\big) = o_p\big(\rho_T \sqrt{\log T}\big), \tag{1}$$

which is a consequence of the following facts: (i) the variables Z_{it} are i.i.d. standard normal, (ii) $|\mathcal{G}_T| = O(T^\theta)$ for some large but fixed constant θ, (iii) $\widehat{\sigma}_i^2 = \sigma_i^2 + o_p(\rho_T)$, (iv) $\max_{(u,h) \in \mathcal{G}_T} |\sum_{t=1}^S u_{s,S}(u,h)| \le CSh_{\max}$, where the constant C is independent of T, and (v) $n = T^\gamma$ for some large but fixed contant γ. Moreover, define

$$\Phi^\diamond_{n,T} = \max_{1 \le i < j \le n} \Phi^\diamond_{ij,T} \quad \text{with} \quad \Phi^\diamond_{ij,T} = \max_{(u,h) \in \mathcal{G}_T} \Big\{ \Big| \frac{\phi^*_{ij,T}(u,h)}{(\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{1/2}} \Big| - \lambda(h) \Big\}.$$

With this notation, we can write

$$\big|\widehat{\Phi}_{n,T} - \Phi^*_{n,T}\big| \le \big|\widehat{\Phi}_{n,T} - \Phi^\diamond_{n,T}\big| + \big|\Phi^\diamond_{n,T} - \Phi^*_{n,T}\big| = \big|\widehat{\Phi}_{n,T} - \Phi^\diamond_{n,T}\big| + o_p\big(\rho_T \sqrt{\log T}\big), \tag{2}$$

where the last equality follows by taking into account the same arguments as in (1).

Straightforward calculations yield that

$$\big|\widehat{\Phi}_{n,T} - \Phi^\diamond_{n,T}\big| \le \max_{1 \le i < j \le n} (\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{-1/2} \max_{1 \le i < j \le n} \max_{(u,h) \in \mathcal{G}_T} \big|\widehat{\phi}_{ij,T}(u,h) - \phi^*_{ij,T}(u,h)\big|.$$

Due to the Assumption (C4) we know that $\max_{1 \le i < j \le n} (\widehat{\sigma}_i^2 + \widehat{\sigma}_j^2)^{-1/2} = O_P(1)$.

Assuming that are kernel weights $w_{t,T}(u,h)$ are constant on each "block", i.e. $w_{t,T}(u,h) = u_{s,S}(u,h)$ for $t \in$ block s, $s \in \{1, \dots, S\}$, we have

$$\begin{aligned}\big|\widehat{\phi}_{ij,T}(u,h) - \phi^*_{ij,T}(u,h)\big| &= \Big| \sum_{t=1}^{T} w_{t,T}(u,h)\{\varepsilon_{it} - \varepsilon jt\} - \sum_{s=1}^{S} u_{s,S}(u,h)\{\sigma_i Z_{is} - \sigma_j Z_{js}\} \Big| = \\ &= \Big| \sum_{s=1}^{S} u_{s,S}(u,h) \Big[\sum_{t \in \text{ block } s} \{\varepsilon_{it} - \varepsilon_{jt}\} - \{\sigma_i Z_{is} - \sigma_j Z_{js}\} \Big] \Big|.\end{aligned}$$

Using summation by parts ($\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} A_i(b_i - b_{i+1}) + A_n b_n$ with $A_j = \sum_{j=1}^{i} a_j$), we further obtain that

$$
\begin{aligned}
\big|\widehat{\phi}_{ij,T}(u,h) - \phi^*_{ij,T}(u,h)\big| \leq & U_S(u,h) \max_{1\leq s\leq S} \Big| \sum_{o=1}^{s} \Big(\sum_{t\in \text{ block } o} \{\varepsilon_{it} - \varepsilon_{jt}\} - \{\sigma_i Z_{io} - \sigma_j Z_{jo}\} \Big) \Big| \leq \\
\leq & U_S(u,h) \max_{1\leq s\leq S} \Big| \sum_{o=1}^{s} \Big(\sum_{t\in \text{ block } o} \varepsilon_{it} - \sigma_i Z_{io} \Big) \Big| + \\
& + U_S(u,h) \max_{1\leq s\leq S} \Big| \sum_{o=1}^{s} \Big(\sum_{t\in \text{ block } o} \varepsilon_{jt} - \sigma_j Z_{jo} \Big) \Big|
\end{aligned}
\tag{3}
$$

where

$$
U_S(u,h) = \sum_{s=1}^{S-1} |u_{s+1,S}(u,h) - u_{s,S}(u,h)| + |u_{S,S}(u,h)|.
$$

Standard arguments show that $\max_{(u,h)\in\mathcal{G}_T} U_S(u,h) = O(1/\sqrt{S h_{\min}}) = O(1/\sqrt{\sqrt{T} h_{\min}})$.

Coupling

Fix $i \in \{1,\dots,n\}$. Let $S_{iT} = \sum_{t=1}^{T} \varepsilon_{it}$. Consider an auxiliary time process $X_{it} = \varepsilon_{i[t+1]}$, $t \in \mathbb{R}$. We clearly have $\int_0^T X_{iu} du = S_{iT} = \sum_{t=1}^{T} \varepsilon_{it}$.

Let us now define "subblocks" as follows:

$$
\begin{array}{lll}
V_{i1} = \int_0^{\log T} X_{iu} du & V'_{i1} = \int_{\log T}^{r-\log T} X_{iu} du & V''_{i1} = \int_{r-\log T}^{r} X_{iu} du \\
V_{i2} = \int_r^{r+\log T} X_{iu} du & V'_{i2} = \int_{r+\log T}^{2r-\log T} & V''_{i2} = \int_{2r-\log T}^{2r} X_{iu} du \\
\dots & \dots & \dots \\
V_{iS} = \int_{(S-1)r}^{(S-1)r+\log T} X_{iu} du & V'_{iS} = \int_{(S-1)r+\log T}^{Sr-\log T} & V''_{iS} = \int_{Sr-\log T}^{T} X_{iu} du
\end{array}
$$

with $S = \lceil T^{1/2} \rceil$ as before and $r = T/\lceil T^{1/2} \rceil$ are breaking points between blocks. The total number of "subblocks" is $3S$ and $V'_{i1},\dots,V'_{iS}$ lie in the interior of the bigger blocks, whereas $V_{i1},\dots,V_{iS}$ and $V''_{i1},\dots,V''_{iS}$ consist of the points close to the border of these bigger blocks.

Using recursively Bradley's lemma we may

- define independent random variables $W_{i1},\dots,W_{iS}$ such that $\mathbb{P}_{W_{i1}} = \mathbb{P}_{V_{i1}},\dots,\mathbb{P}_{W_{iS}} = \mathbb{P}_{V_{iS}}$ and

$$
\mathbb{P}\big(|W_{is} - V_{is}| > \xi\big) \leq 11 \Big(\frac{\|V_{is} + c\|_q}{\xi} \Big)^{\frac{q}{2q+1}} \alpha\big([r - \log T]\big)^{\frac{2q}{2q+1}}; \tag{4}
$$

- define independent random variables $W'_{i1},\dots,W'_{iS}$ such that $\mathbb{P}_{W'_{i1}} = \mathbb{P}_{V'_{i1}},\dots,\mathbb{P}_{W'_{iS}} = \mathbb{P}_{V'_{iS}}$ and

$$\mathbb{P}\big(|W'_{is} - V'_{is}| > \xi\big) \le 11\Big(\frac{\|V'_{is}+c\|_q}{\xi}\Big)^{\frac{q}{2q+1}} \alpha\big([2\log T]\big)^{\frac{2q}{2q+1}}; \tag{5}$$

- define independent random variables $W''_{i1},\dots,W''_{iS}$ such that $\mathbb{P}_{W''_{i1}} = \mathbb{P}_{V''_{i1}},\dots,\mathbb{P}_{W''_{iS}} = \mathbb{P}_{V''_{iS}}$ and

$$\mathbb{P}\big(|W''_{is} - V''_{is}| > \xi\big) \le 11\Big(\frac{\|V''_{is}+c\|_q}{\xi}\Big)^{\frac{q}{2q+1}} \alpha\big([r-\log T]\big)^{\frac{2q}{2q+1}}. \tag{6}$$

Note that

$$V_{is} + V'_{is} + V''_{is} = \sum_{\substack{t\in \text{ interior}\\ \text{of block } s}} \varepsilon_{it} + \sum_{\substack{t\notin \text{ interior}\\ \text{of block } s}} \varepsilon_{it} = \sum_{t\in \text{ block } s} \varepsilon_{it}.$$

Plugging this into (3) we get the following:

$$\begin{aligned}
\big|\widehat{\phi}_{ij,T}(u,h) - \phi^*_{ij,T}(u,h)\big| &\le 2U_S(u,h) \max_{1\le s\le S} \Big|\sum_{o=1}^{s}\sum_{t\in \text{ block } o} \varepsilon_{it} - \sum_{o=1}^{s}\sigma_i Z_{io}\Big| \le \\
&\le 2U_S(u,h)\max_{1\le s\le S}\Bigg(\Big|\sum_{o=1}^{s} W'_{io} - \sum_{o=1}^{s}\sigma_i Z_{io}\Big| + \Big|\sum_{o=1}^{s} W'_{io} - \sum_{o=1}^{s} V'_{is}\Big| + \Big|\sum_{o=1}^{s}\sum_{\substack{t\notin \text{ interior}\\ \text{of block } o}} \varepsilon_{it}\Big|\Bigg) \le \\
&\le 2U_S(u,h)\max_{1\le s\le S}\Bigg(\Big|\sum_{o=1}^{s} W'_{io} - \sum_{o=1}^{s}\sigma_i Z_{io}\Big| + \Big|\sum_{o=1}^{s} W'_{io} - \sum_{o=1}^{s} V'_{is}\Big| + \\
&\qquad + 2\Big|\sum_{o=1}^{s} W_{io} - \sum_{o=1}^{s} V_{io}\Big| + 2\Big|\sum_{o=1}^{s} W_{io}\Big|\Bigg)
\end{aligned} \tag{7}$$

We need to show:

$$\big|\widehat{\Phi}_{n,T} - \Phi^{\diamond}_{n,T}\big| = o_p\big(\dots\big) \tag{8}$$

or

$$\begin{aligned}
&\max_{1\le i<j\le n}\max_{(u,h)\in\mathcal{G}_T}\big|\widehat{\phi}_{ij,T}(u,h) - \phi^*_{ij,T}(u,h)\big| \le \\
&\quad\le \max_{(u,h)\in\mathcal{G}_T} U_S(u,h)\max_{1\le i<j\le n}\Bigg[\max_{1\le s\le S}\Bigg(\Big|\sum_{o=1}^{s} W'_{io} - \sigma_i\sum_{o=1}^{s} Z_{io}\Big| + \Big|\sum_{o=1}^{s} W'_{io} - \sum_{o=1}^{s} V'_{io}\Big| + \\
&\qquad + 2\Big|\sum_{o=1}^{s} W_{io} - \sum_{o=1}^{s} V_{io}\Big| + 2\Big|\sum_{o=1}^{s} W_{io}\Big|\Bigg)\Bigg] = o_p\big(\dots\big).
\end{aligned}$$

.

First consider $\max_{1\le i<j\le n}\max_{1\le s\le S}\Big|\sum_{o=1}^{s} W'_{is} - \sum_{o=1}^{s} V'_{is}\Big|$. From (4) we know that

$$\mathbb{P}\big(|W'_{is} - V'_{is}| > \xi\big) \le 11\Big(\frac{\|V'_{is} + c\|_q}{\xi}\Big)^{\frac{q}{2q+1}} \alpha\big([2\log T]\big)^{\frac{2q}{2q+1}} \dots$$

Strong approximation

Finally, by arguments very similar to those for Proposition for the anticoncentration bounds, we obtain that

$$\sup_{x \in \mathbb{R}} \mathbb{P}\Big(\big|\Phi^*_{n,T} - x\big| \le \delta_T\Big) = o(1) \tag{9}$$

with $\delta_T = T^{1/q}/\sqrt{Th_{\min}} + \rho_T \sqrt{\log T}$. Combining (1)–(3) with Lemma **??**, we can infer that

$$\sup_{x \in \mathbb{R}} \big|\mathbb{P}\big(\widehat{\Phi}_{n,T} \le x\big) - \mathbb{P}\big(\Phi^*_{n,T} \le x\big)\big| = o(1) \tag{10}$$

$$\sup_{x \in \mathbb{R}} \big|\mathbb{P}\big(\Phi_{n,T} \le x\big) - \mathbb{P}\big(\Phi^*_{n,T} \le x\big)\big| = o(1). \tag{11}$$

From (10) and (11), it immediately follows that

$$\sup_{x \in \mathbb{R}} \big|\mathbb{P}\big(\widehat{\Phi}_{n,T} \le x\big) - \mathbb{P}\big(\Phi_{n,T} \le x\big)\big| = o(1),$$

which in turn implies that $\mathbb{P}(\widehat{\Phi}_{n,T} \le q_{n,T}(\alpha)) = (1-\alpha) + o(1)$. This completes the proof of Theorem 1.1.

References

Wu, W. B. (2005). Nonlinear system theory: another look at dependence. *Proc. Natn. Acad. Sci. USA*, **102** 14150–14154.