Nonparametric comparison of epidemic time trends: the case of COVID-19

Marina Khismatullina[1]
University of Bonn

Michael Vogt[2]
Ulm University

The COVID-19 pandemic is one of the most pressing issues at present. A question which is particularly important for governments and policy makers is the following: Does the virus spread in the same way in different countries? Or are there significant differences in the development of the epidemic? In this paper, we devise new inference methods that allow to detect differences in the development of the COVID-19 epidemic across countries in a statistically rigorous way. In our empirical study, we use the methods to compare the outbreak patterns of the epidemic in a number of European countries.

Key words: simultaneous hypothesis testing; multiscale test; time trend; panel data; COVID-19.
JEL classifications: C12; C23; C54.

1 Introduction

2 Extensions

Now suppose that instead of (**??**), we consider the following nonparametric regression equation:

$$X_{it} = c_i \lambda_i\Big(\frac{t}{T}\Big) + \varepsilon_{it} \quad \text{with} \quad \varepsilon_{it} = \sigma \sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it},$$

where c_i is the country-specific scaling parameter that accounts for the size of the country or population density. We introduce this additional parameter in order to be able to compare countries that differ substantially in terms of the population, i.e. Luxembourg and Russia. In our application, we compare 5 European countries that are similar in many aspects, including size, but for further research we need to be able to account for the differences between the countries.

Hence, we would like to test the hypothesis that the time trends of new COVID-19 cases in different countries are the same up to some scaling parameter. For the identification

[1]Corresponding author. Address: Bonn Graduate School of Economics, University of Bonn, 53113 Bonn, Germany. Email: `marina.k@uni-bonn.de`.
[2]Address: Institute of Statistics, Department of Mathematics and Economics, Ulm University, 89081 Ulm, Germany. Email: `m.vogt@uni-ulm.de`.

purposes, we will need to assume that for each $i \in \mathcal{C}$ we have $\int_0^1 \lambda_i(u)du = 1$. Only then we are able to estimate the scaling parameter c_i. Thus, the testing procedure is as follows.

Step 1

First, we estimate the scaling parameter:

$$\begin{aligned}
\widehat{c}_i &= \frac{1}{T}\sum_{t=1}^{T} X_{it} \\
&= c_i \frac{1}{T}\sum_{t=1}^{T} \lambda_i\Big(\frac{t}{T}\Big) + \sigma \frac{1}{T}\sum_{t=1}^{T} \sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it} \\
&= c_i \frac{1}{T}\sum_{t=1}^{T} \lambda_i\Big(\frac{t}{T}\Big) + o_P(1) \\
&= c_i + o_P(1),
\end{aligned}$$

where in the last inequality we used the normalization $\int_0^1 \lambda_i(u)du = 1$. Hence, $\widehat{c}_i$ is a consistent estimator of c_i.

Step 2

Instead of working with X_{it}, we consider the following variables:

$$\begin{aligned}
X_{it}^* &= \frac{X_{it}}{\frac{1}{T}\sum_{t=1}^{T} X_{it}} \\
&= \frac{c_i}{\widehat{c}_i}\lambda_i\Big(\frac{t}{T}\Big) + \frac{\sigma}{\widehat{c}_i}\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it}.
\end{aligned}$$

A statistic to test the hypothesis $H_0^{(ijk)}$ for a given triple (i,j,k) is then constructed as follows. Instead of $\hat{s}_{ijk,T}$, we work with

$$\hat{s}_{ijk,T}^* = \frac{1}{\sqrt{Th_k}}\sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it}^* - X_{jt}^*).$$

Then

$$\begin{aligned}
\frac{\hat{s}_{ijk,T}^*}{\sqrt{Th_k}} =& \frac{1}{Th_k}\sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)(X_{it}^* - X_{jt}^*) \\
=& \frac{1}{Th_k}\sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\Big(\lambda_i\Big(\frac{t}{T}\Big) - \lambda_j\Big(\frac{t}{T}\Big)\Big) + R_1 + R_2,
\end{aligned}$$

where

$$R_1 = \frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\Big(\Big(\frac{c_i}{\widehat{c}_i}-1\Big)\lambda_i\Big(\frac{t}{T}\Big)-\Big(\frac{c_j}{\widehat{c}_j}-1\Big)\lambda_j\Big(\frac{t}{T}\Big)\Big),$$
$$R_2 = \frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\Big(\frac{\sigma}{\widehat{c}_i}\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it}-\frac{\sigma}{\widehat{c}_j}\sqrt{\lambda_j\Big(\frac{t}{T}\Big)}\eta_{jt}\Big).$$

Since $\widehat{c}_i = c_i + o_P(1)$ and $0 \le \sum_{t=1}^{T}\mathbf{1}\big(\frac{t}{T}\in\mathcal{I}_k\big)\lambda_i\big(\frac{t}{T}\big) \le h_k\lambda_{max}$, we have

$$\begin{aligned}|R_1| &\le \Big|\frac{c_i}{\widehat{c}_i}-1\Big|\frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\lambda_i\Big(\frac{t}{T}\Big)+\Big|\frac{c_j}{\widehat{c}_j}-1\Big|\frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\lambda_j\Big(\frac{t}{T}\Big),\\ &= o_P(1)\cdot\frac{\lambda_{max}}{T}+o_P(1)\cdot\frac{\lambda_{max}}{T} = o_P\Big(\frac{1}{T}\Big). \qquad (2.1)\end{aligned}$$

Furthermore, applying the law of large numbers, we get:

$$\frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it} = o_P(1).$$

Hence, if we uniformly bound the scaling parameters away from 0, i.e. $\exists\, c_{min}$ such that for all $i\in\mathcal{C}$ we have $0 < c_{min} \le c_i$, we can use the fact that $\frac{\sigma}{\widehat{c}_i} = O_P(1)$ to get that

$$\begin{aligned}R_2 &= \frac{\sigma}{\widehat{c}_i}\frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\sqrt{\lambda_i\Big(\frac{t}{T}\Big)}\eta_{it}-\frac{\sigma}{\widehat{c}_j}\frac{1}{Th_k}\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)\sqrt{\lambda_j\Big(\frac{t}{T}\Big)}\eta_{jt}\\ &= o_P(1). \qquad (2.2)\end{aligned}$$

Combining (2.1) and (2.2) together, we get $\hat{s}^*_{ijk,T}/\sqrt{Th_k} = (Th_k)^{-1}\sum_{t=1}^{T}\mathbf{1}(t/T\in\mathcal{I}_k)\{\lambda_i(t/T)-\lambda_j(t/T)\}+o_p(1)$ for any fixed pair of countries (i,j). Hence, the statistic $\hat{s}^*_{ijk,T}/\sqrt{Th_k}$ estimates the average distance between the functions λ_i and λ_j on the interval $\mathcal{I}_k$. The variance of $\hat{s}^*_{ijk,T}$ can not be easily calculated:

$$\begin{aligned}\mathrm{Var}(\hat{s}^*_{ijk,T}) =& \frac{1}{Th_k}\mathrm{Var}\Big(\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)(X^*_{it}-X^*_{jt})\Big)\\ =& \frac{1}{Th_k}\mathrm{Var}\Big(\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)X^*_{it}\Big)+\frac{1}{Th_k}\mathrm{Var}\Big(\sum_{t=1}^{T}\mathbf{1}\Big(\frac{t}{T}\in\mathcal{I}_k\Big)X^*_{jt}\Big)\\ =& \frac{1}{Th_k}\mathrm{Var}\Bigg(\frac{\sum_{t=1}^{T}\mathbf{1}\big(\frac{t}{T}\in\mathcal{I}_k\big)X_{it}}{\frac{1}{T}\sum_{t=1}^{T}X_{it}}\Bigg)+\frac{1}{Th_k}\mathrm{Var}\Bigg(\frac{\sum_{t=1}^{T}\mathbf{1}\big(\frac{t}{T}\in\mathcal{I}_k\big)X_{jt}}{\frac{1}{T}\sum_{t=1}^{T}X_{jt}}\Bigg),\end{aligned}$$

hence, we 'normalize' $\hat{s}^*_{ijk,T}$ intuitively by dividing it by the following value:

$$(\hat{\nu}^*_{ijk,T})^2 = \frac{\hat{\sigma}^2}{Th_k} \sum_{t=1}^{T} \mathbf{1}\Big(\frac{t}{T} \in \mathcal{I}_k\Big)\{X^*_{it} + X^*_{jt}\}.$$

Normalizing the statistic $\hat{s}_{ijk,T}$ by the estimator $\hat{\nu}_{ijk,T}$ yields the expression

$$\hat{\psi}^*_{ijk,T} := \frac{\hat{s}^*_{ijk,T}}{\hat{\nu}^*_{ijk,T}} = \frac{\sum_{t=1}^{T} \mathbf{1}(\frac{t}{T} \in \mathcal{I}_k)(X^*_{it} - X^*_{jt})}{\hat{\sigma}\{\sum_{t=1}^{T} \mathbf{1}(\frac{t}{T} \in \mathcal{I}_k)(X^*_{it} + X^*_{jt})\}^{1/2}},$$

which serves as our test statistic of the hypothesis $H_0^{(ijk)}$. For later reference, we additionally introduce the statistic

$$\hat{\psi}^{*,0}_{ijk,T} = \frac{\sum_{t=1}^{T} \mathbf{1}(\frac{t}{T} \in \mathcal{I}_k)\Big(\big(\frac{c_i}{\hat{c}_i} - \frac{c_j}{\hat{c}_j}\big)\overline{\lambda}_{ij} + \big(\frac{\sigma}{\hat{c}_i} - \frac{\sigma}{\hat{c}_j}\big)\overline{\lambda}_{ij}^{1/2}(\frac{t}{T})(\eta_{it} - \eta_{jt})\Big)}{\hat{\sigma}\{\sum_{t=1}^{T} \mathbf{1}(\frac{t}{T} \in \mathcal{I}_k)(X^*_{it} + X^*_{jt})\}^{1/2}}$$

with $\overline{\lambda}_{ij}(u) = \{\lambda_i(u) + \lambda_j(u)\}/2$, which is identical to $\hat{\psi}_{ijk,T}$ under $H_0^{(ijk)}$.