

Nonparametric comparison of epidemic time trends: the case of COVID-19

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April 2, 2021

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Introduction

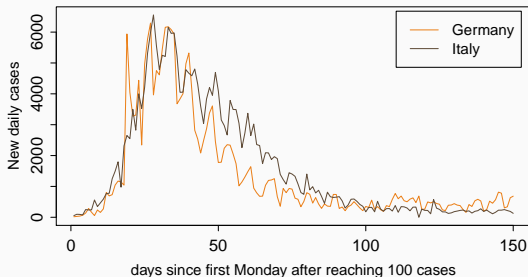
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To develop new inference methods that allow to *identify* and *locate* differences between epidemic time trends.

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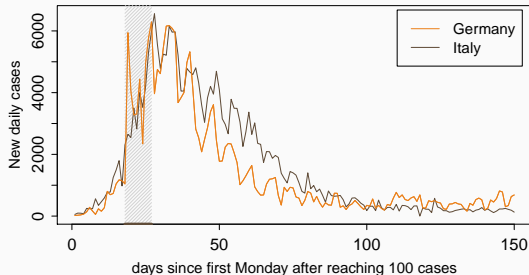
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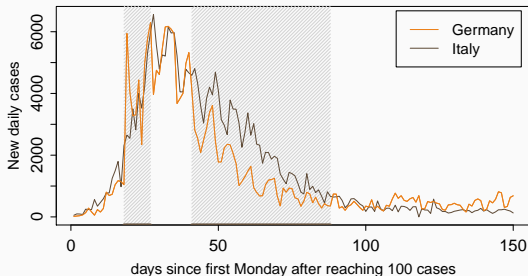
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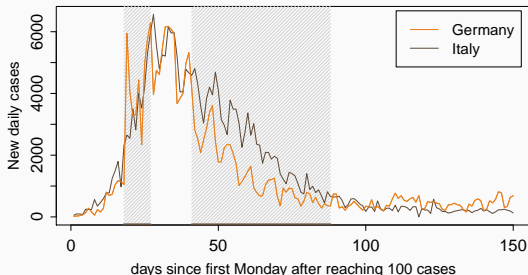
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Research question: Out of many given intervals, how to find those where the trends are significantly different?

Why is it relevant?

Finding systematic differences between trends = basis for further research

⇒ understanding which government policies are more effective.

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Is it limited to COVID-19?

No! Our method = general method for comparing nonparametric trends

⇒ new statistical test for equality of nonparametric trend curves.

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Since $\lambda_i(t/T) = \mathbb{E}[X_{it}] = \text{Var}(X_{it})$, we can rewrite X_{it} as

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In applications the variance can be larger than the mean \Rightarrow quasi-Poisson models.

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- λ_i are unknown trend functions on $[0, 1]$;
- σ is the overdispersion parameter;
- η_{it} are error terms that are independent across i and t and have zero mean and unit variance.

Testing procedure

Testing problem

Let $\mathcal{F} := \{\mathcal{I}_k \subseteq [0, 1] : 1 \leq k \leq K\}$ be a family of rescaled time intervals on $[0, 1]$, and for each triplet (i, j, k) consider the null hypothesis that the functions λ_i and λ_j are equal on an interval \mathcal{I}_k

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$$\text{FWER}(\alpha) = \mathbb{P}\left(\exists(i, j, k) : \text{we wrongly reject } H_0^{(ijk)}\right)$$

Test statistic

For a given interval \mathcal{I}_k and a pair of time series i and j we calculate

$$\hat{s}_{ijk} = \frac{1}{Th_k} \sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(X_{it} - X_{jt}),$$

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Under certain assumptions,

$$\text{Var}(\hat{s}_{ijk}) = \frac{\sigma^2}{T^2 h_k^2} \sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right) \left\{ \lambda_i\left(\frac{t}{T}\right) + \lambda_j\left(\frac{t}{T}\right) \right\}$$

Test statistic, part 2

Under certain assumptions,

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In order to normalize the variance of the statistic \hat{s}_{ijk} , we scale it by an estimator of its variance:

$$\widehat{\text{Var}}(\hat{s}_{ijk}) = \frac{\hat{\sigma}^2}{T^2 h_k^2} \sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right) (X_{it} + X_{jt}),$$

with $\hat{\sigma}^2$ being an appropriate estimator of σ^2 . [Details](#)

Test statistic for the hypothesis $H_0^{(ijk)}$ is defined as

$$\hat{\psi}_{ijk} := \frac{\hat{s}_{ijk}}{\sqrt{\widehat{\text{Var}}(\hat{s}_{ijk})}} = \frac{\sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(X_{it} - X_{jt})}{\hat{\sigma}\left\{\sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(X_{it} + X_{jt})\right\}^{1/2}}$$

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- More modern approach: $c_{ijk}(\alpha)$ depend on the length h_k of the time interval (Dümbgen and Spokoiny (2001)):

$$c_{ijk}(\alpha) = c(\alpha, h_k) := b_k + q(\alpha)/a_k,$$

where a_k and b_k are scale-dependent constants and $q(\alpha)$ is chosen such that we control FWER. [Details](#)

Critical values, part 2

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We want to control FWER. Let $\mathcal{M}_0 := \{(i, j, k) | H_0^{(ijk)} \text{ is true}\}$, then

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Hence, we choose $q(\alpha)$ as the $(1 - \alpha)$ -quantile of the statistic

$$\hat{\Psi}_T = \max_{(i, j, k)} a_k(|\hat{\psi}_{ijk}^0| - b_k),$$

where $\hat{\psi}_{ijk}^0$ is equal to $\hat{\psi}_{ijk}$ under the null.

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Under our assumptions,

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Critical values, part 3

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which can be approximated by a Gaussian version of the test statistic:

$$\phi_{ijk} = \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(Z_{it} - Z_{jt}),$$

where Z_{it} are independent standard normal random variables.

Test procedure

1. Consider the Gaussian test statistic

$$\Phi_T = \max_{(i,j,k)} a_k (|\phi_{ijk}| - b_k),$$

where a_k and b_k are scale-dependent constants and ϕ_{ijk} are weighted averages of the differences of standard normal random variables.

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Test procedure

For the given significance level $\alpha \in (0, 1)$ and for each (i, j, k) , reject $H_0^{(ijk)}$ if $|\hat{\psi}_{ijk}| > c_{\text{Gauss}}(\alpha, h_k)$.

Theoretical properties

$\mathcal{C}1$ The functions λ_i are uniformly Lipschitz continuous:
 $|\lambda_i(u) - \lambda_i(v)| \leq L|u - v|$ for all $u, v \in [0, 1]$.

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- $\mathcal{C6}$ $p := \{\#(i, j, k)\} = O(T^{(\theta/2)(1-b)-(1+\delta)})$ for some small $\delta > 0$.

Proposition

Let \mathcal{M}_0 be the set of triplets (i, j, k) for which $H_0^{(ijk)}$ holds true. Then under C1 – C6, it holds that

$$P\left(\forall (i, j, k) \in \mathcal{M}_0 : |\hat{\psi}_{ijk}| \leq c_{\text{Gauss}}(\alpha, h_k)\right) \geq 1 - \alpha + o(1)$$

Theoretical properties

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Proposition

Consider a sequence of functions $\lambda_i = \lambda_{i,T}$, $\lambda_j = \lambda_{j,T}$ such that

$$\exists \mathcal{I}_k : \lambda_i(w) - \lambda_j(w) \geq c_T \sqrt{\log T / (Th_k)} \quad \forall w \in \mathcal{I}_k, \quad (1)$$

and $c_T \rightarrow \infty$ faster than $\frac{\sqrt{\log T} \sqrt{\log \log T}}{\log \log \log T}$. Let \mathcal{M}_1 be the set of triplets (i, j, k) for which (1) holds true. Then under $\mathcal{C}1 - \mathcal{C}6$, it holds that

$$\mathbb{P}\left(\forall (i, j, k) \in \mathcal{M}_1 : |\hat{\psi}_{ijk}| > c_{\text{Gauss}}(\alpha, h_k)\right) = 1 - o(1)$$

Application

Graphical representation

How to represent the results of the test?

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Plot the results of pairwise comparison $\mathcal{F}_{\text{reject}}(i, j)$:

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An interval $\mathcal{I}_k \in \mathcal{F}_{\text{reject}}(i, j)$ is called **minimal** if there is no other interval $\mathcal{I}_{k'} \in \mathcal{F}_{\text{reject}}(i, j)$ with $\mathcal{I}_{k'} \subset \mathcal{I}_k$.

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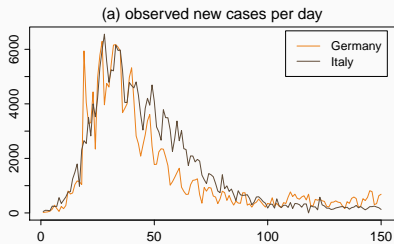
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We can make similar confidence statements about minimal intervals:

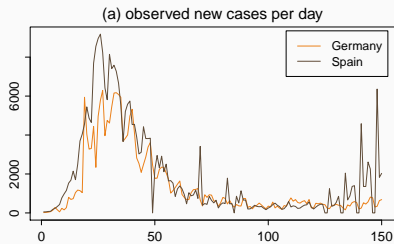
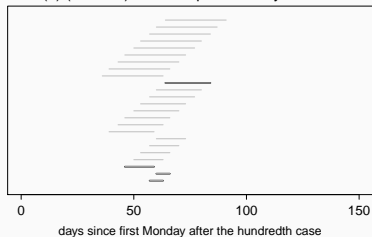
$$P\left(\forall (i, j, k) \in \mathcal{M}_0 : \mathcal{I}_k \notin \mathcal{F}_{\text{reject}}^{\min}(i, j)\right) \geq 1 - \alpha + o(1)$$

- Five countries: Germany, Italy, Spain, France and the UK.
- $T = 150$ days.
- The data is aligned by weekdays: first Monday after reaching 100 cases as $t = 1$.
- Lengths of time intervals 7, 14, 21, 28 days. The intervals start at days 1, 8, 15, ... and 4, 11, 19, ...
- $\alpha = 0.05$.
- 5000 Monte Carlo simulation runs to produce critical values.

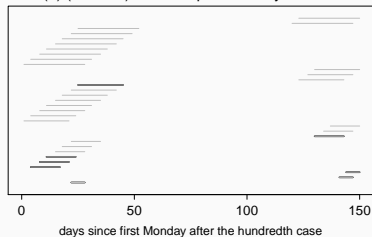
Application results



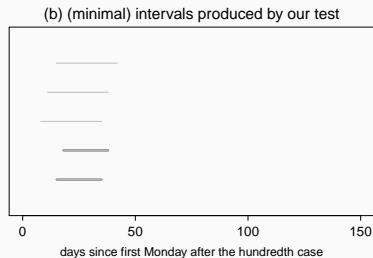
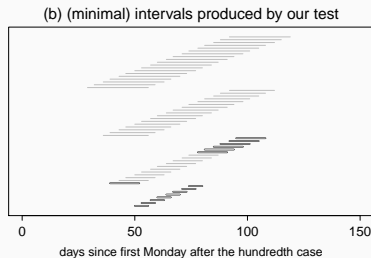
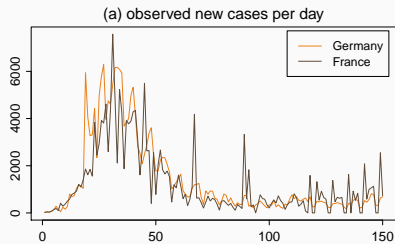
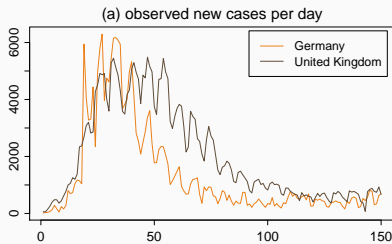
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Application results, part 2



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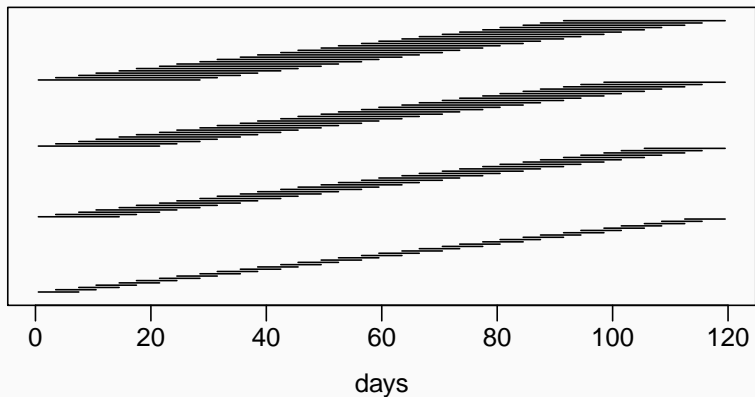
Further possible extensions:

- introduce scaling factor in the trend function, that will allow to adjust for the size of the country (population, density, testing regimes, etc.);
- include dependence in the error terms;
- cluster the countries based on the trends they exhibit.

Thank you!

Family of time intervals

The family of intervals F



Simulation results for the size of the test

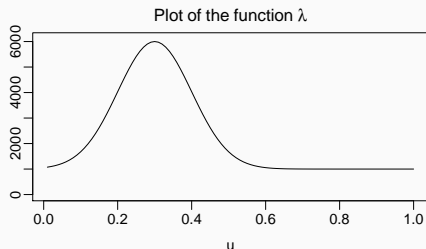


Table 1: Size of the multiscale test

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.011	0.047	0.093	0.010	0.044	0.087	0.008	0.037	0.075
$T = 250$	0.009	0.047	0.091	0.009	0.046	0.087	0.008	0.035	0.069
$T = 500$	0.010	0.044	0.083	0.008	0.048	0.093	0.007	0.035	0.077

Simulation results for the power of the test

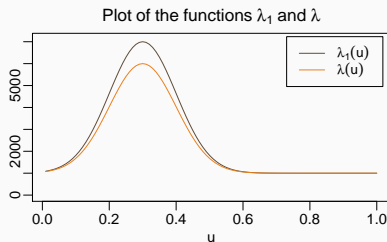


Table 2: Power of the multiscale test for scenario A

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.335	0.518	0.597	0.306	0.474	0.545	0.212	0.352	0.418
$T = 250$	0.615	0.790	0.836	0.580	0.764	0.800	0.470	0.648	0.705
$T = 500$	0.736	0.905	0.917	0.738	0.884	0.890	0.636	0.799	0.830

Simulation results for the power of the test

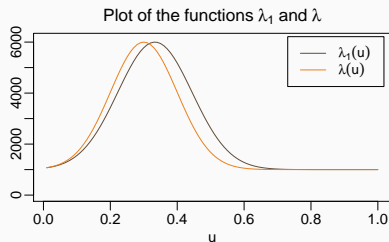
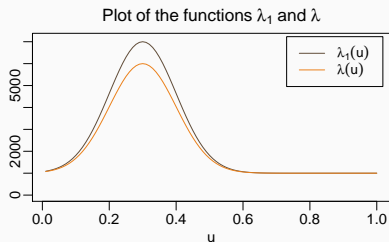


Table 3: Power of the multiscale test for scenario B

	$n = 5$			$n = 10$			$n = 50$		
	significance level α			significance level α			significance level α		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
$T = 100$	0.824	0.910	0.903	0.812	0.893	0.890	0.738	0.847	0.857
$T = 250$	0.991	0.972	0.941	0.991	0.960	0.920	0.991	0.965	0.933
$T = 500$	0.997	0.973	0.949	0.995	0.961	0.923	0.996	0.969	0.932

Estimator of σ^2

We estimate the overdispersion parameter σ^2 by

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_i^2 \text{ and } \hat{\sigma}_i^2 = \frac{\sum_{t=2}^T (X_{it} - X_{it-1})^2}{2 \sum_{t=1}^T X_{it}}$$

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We assume that λ_i is Lipschitz continuous. Then

$$X_{it} - X_{it-1} = \sigma \sqrt{\lambda_i \left(\frac{t}{T} \right)} (\eta_{it} - \eta_{it-1}) + r_{it},$$

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Together with

$$\frac{1}{T} \sum_{t=1}^T X_{it} = \frac{1}{T} \sum_{t=1}^T \lambda_i(t/T) + o_p(1),$$

we get that $\hat{\sigma}_i^2 = \sigma^2 + o_p(1)$ for any i and thus $\hat{\sigma}^2 = \sigma^2 + o_p(1)$.

[Go back](#)

Notation

In order to proceed with the proof, we will need the following notation:

$$\begin{aligned}\hat{\psi}_{ijk,T} &= \frac{\sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(X_{it} - X_{jt})}{\hat{\sigma}\left\{\sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(X_{it} + X_{jt})\right\}^{1/2}} \\ \hat{\psi}_{ijk,T}^0 &= \frac{\sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right) \sigma \bar{\lambda}_{ij}^{-1/2}\left(\frac{t}{T}\right)(\eta_{it} - \eta_{jt})}{\hat{\sigma}\left\{\sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(X_{it} + X_{jt})\right\}^{1/2}} & \hat{\Psi}_T &= \max_{(i,j,k)} a_k(|\hat{\psi}_{ijk,T}^0| - b_k) \\ \psi_{ijk,T}^0 &= \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(\eta_{it} - \eta_{jt}) & \Psi_T &= \max_{(i,j,k)} a_k(|\psi_{ijk,T}^0| - b_k) \\ \phi_{ijk,T} &= \frac{1}{\sqrt{2Th_k}} \sum_{t=1}^T 1\left(\frac{t}{T} \in \mathcal{I}_k\right)(Z_{it} - Z_{jt}) & \Phi_T &= \max_{(i,j,k)} a_k(|\phi_{ijk,T}| - b_k)\end{aligned}$$

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4. It can be shown that $P(\Phi_T \leq q_{\text{Gauss}}(\alpha)) = 1 - \alpha$. From this and (2), it immediately follows that

$$P(\hat{\Psi}_T^0 \leq q_{\text{Gauss}}(\alpha)) = 1 - \alpha + o(1),$$

which in turn implies the desired statement.

Idea behind a_k and b_k

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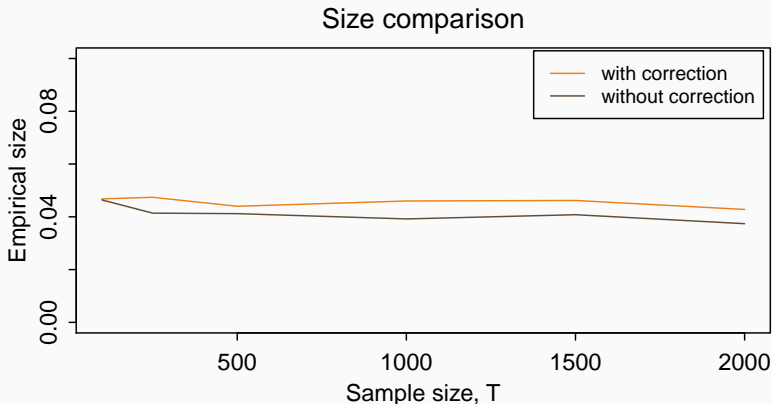
Specifically,

$$c_{ijk}(\alpha) = c(\alpha, h_k) := b_k + q(\alpha)/a_k,$$

where $a_k = \{\log(e/h_k)\}^{1/2} / \log \log(e^e/h_k)$ and $b_k = \sqrt{2 \log(1/h_k)}$ are scale-dependent constants and $q(\alpha)$ is chosen such that we control FWER.

Idea behind a_k and b_k , part 2

This choice of scale-dependent constants helps us balance the significance of hypotheses between the time intervals of different lengths h_k :



[Go back](#)

Idea behind the additive correction

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$\Rightarrow \max_m \dots = \sqrt{2 \log(1/h_l)} + o_P(1) \rightarrow \infty$ as $h \rightarrow 0$ and the stochastic behavior of Φ^{uncor} is dominated by the elements with small bandwidths h_l . [Go back](#)