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ON THE EXISTENCE OF FACTOR SETS BY EXTERNAL EQUIVALENCES IN IST

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Abstract: We study the possibility of defining the factor sets of the real axis by external equivalences through external formulas on using the IST axiomatics of nonstandard analysis. We consider the case of an additive convex equivalence whose equivalence classes are defined by a formula with external universal quantifiers. We show that in this case there exists an external function selecting one representative from each equivalence class if and only if the relation in question coincides up to translation and dilation with the relation of infinite proximity.

Keywords: nonstandard analysis, internal set theory, external formula, external factor set

1. Introduction

In [1] Kanoveĭ considers a series of "external" analogs of theorems of the classical set theory which are of interest for researching within Nelson's axiomatics of nonstandard analysis, internal set theory IST [2]. In this article we study the "external" analog of the construction of factor sets by external equivalences, more precisely, the question of existence of external functions selecting one representative from each equivalence class. This is a particular instance of the "external" analog of the axiom of choice.

In the article we use the following notions from internal set theory [1]. An *external formula* is a formula of IST, which may involve the standardness predicate and consequently fail to be a formula of ZFC. An *external set* is the collection of elements defined by an external formula. Such collections are not sets in general (in the sense of the axiomatics of set theory), and we should exercise certain care while dealing with them. At the same time, they are used in most of the applications of internal set theory. Unlike the so-called "outer" sets, in a constructive approach the external sets are independent of the choice of a model.

The elements of standard sets are said to be *bounded*.

In [3] Gordon defines the predicate of "standardness relative to t" (of being t-standard):

$$x \operatorname{st} t \Leftrightarrow \exists^{\mathrm{st}} \varphi \ (F \operatorname{fin}(\varphi)) \ \ \& \ \ (t \in \operatorname{dom} \varphi) \ \ \& \ \ (x \in \varphi(t))$$

(here $F \operatorname{fin}(\varphi)$ means that φ is a function and all values of φ are finite sets). In the same way the notions are defined of t-infinitesimal, t-unlimited, and t-limited real numbers. We say that x is *t-nearstandard* if there exists a t-standard y t-infinitely close to x [4]. It was proven in [3] that not all points of the interval $I = [0, 1]$ are N-nearstandard for some nonstandard natural N. In general, the author showed in [4, 5] that for an arbitrary nonstandard t of nonmeasurable complexity, where the complexity of t is defined by the formula $\operatorname{compl} t = \min\{\operatorname{card} T : \operatorname{st}(T) \ \& \ t \in T\}$, and for an arbitrary standard Hausdorff topological space X which is not a rare compact space (in particular, for the unit interval I) not all points of X are t-nearstandard.

Thus, there are "too few" t-standard points in I to form the external factor set I/ρ_t, where ρ_t is the external relation of t-infinitely proximity, $\rho_t(x, y) = (x \overset{t}{\approx} y)$. This means that among t-standard real numbers there are "holes," intervals of t-standard (nonzero) length inside which there is no t-standard point. However, might it be possible to fill these "holes" with points of some external set or even to

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define an external set $M_t \subset I$, depending on a parameter $t \in T$, so as to obtain nevertheless a factor set by the relation of t-infinite proximity?

Let us formulate our requirements on $M_t \subset I$:

$$\forall x \in I \, \exists y \in M_t \; x \overset{t}{\approx} y, \quad \forall x_1, x_2 \in M_t \; (x_1 \overset{t}{\approx} x_2) \Rightarrow (x_1 = x_2). \tag{1}$$

2. On the Possibility of Global Definition of a Selecting Function (Defined for All Values of a Parameter)

Theorem 1. *For a standard infinite set T, there is no external formula $\varphi(x,t)$ such that the external set $M_t = {}^E\{x \in I : \varphi(x,t)\}$ would satisfy* (1) *for all values of the parameter $t \in T$.*

Here and throughout the sequel, the word "formula" means " ext-bounded formula" [1], i.e., a formula whose external quantifies ranges over standard sets (such an external quantifier is said to be bounded). As a rule, the concrete applications of IST usually deal with only ext-bounded formula. It is customary to write the superscript E before braces to emphasize that the resultant collection of elements may fail to be an internal set.

This theorem is a particular instance of some more general assertion to be proved below.

Let $\mathscr{E}$ be a standard nonempty set of functions from a standard infinite set T into the set of strictly positive real numbers $\mathbb{R}_+$ and let μ_t be the section of the monad of the filter generated by the family $\Gamma_\varepsilon = \{(t,x) : t \in T, |x| < \varepsilon(t)\}$, $\varepsilon \in \mathscr{E}$, of neighborhoods of the subset $T \times \{0\}$ of the product $T \times \mathbb{R}$:

$$\mu_t = {}^E\{x \in \mathbb{R} : \forall^{\mathrm{st}} \varepsilon \in \mathscr{E} \; |x| < \varepsilon(t)\}. \tag{2}$$

Without loss of generality, we may assume that $\mathscr{E}$ satisfies the following conditions:

$$\begin{gathered} \forall \varepsilon_1, \varepsilon_2 \in \mathscr{E} \quad \min\{\varepsilon_1, \varepsilon_2\} \in \mathscr{E}, \\ \forall \varepsilon_1 \in \mathscr{E} \, \forall \varepsilon_2 \in \mathbb{R}_+^T \quad (\forall t \in T \; \varepsilon_2(t) \geq \varepsilon_1(t)) \Rightarrow (\varepsilon_2 \in \mathscr{E}) \end{gathered} \tag{3}$$

(otherwise we can extend $\mathscr{E}$ to a set that enjoys this property without changing μ_t).

Moreover, we require that μ_t does not cover I for any t; i.e.,

$$\forall t \in T \, \exists^{\mathrm{st}} \varepsilon \in \mathscr{E} \quad \varepsilon(t) \leq 1. \tag{4}$$

Example 1. $\mathscr{E}$ consists only of constants and μ_t is the set of all infinitesimals.

Example 2. $\mathscr{E}$ contains all functions that act from T into $\mathbb{R}_+$ and μ_t is the set of all t-infinitesimals.

Example 3. $\mathscr{E}$ is generated by the power functions $\{t \to t^n\}_{n \in \mathbb{N}}$, $T = (0,1)$. Here μ_t is smaller than in the first example but larger than in the second.

In the last two examples $\mathscr{E}$ possesses one more property: there exists a standard function $\lambda : T \to \mathbb{R}_+$ such that

$$\forall \varepsilon \in \mathscr{E} \; \lambda \cdot \varepsilon \in \mathscr{E}, \quad \exists t_0 \in T \; \lambda(t_0) \approx 0. \tag{5}$$

Theorem 2. *For the standard infinite set T and for $\mathscr{E}$ satisfying conditions* (3)–(5)*, there is no external formula $\varphi(x,t)$ such that the following holds for the external set $M_t = {}^E\{x \in I : \varphi(x,t)\}$ for all values of the parameter $t \in T$:*

$$\forall x \in I \, \exists y \in M_t \; (x - y) \in \mu_t, \quad \forall x_1, x_2 \in M_t \; ((x_1 - x_2) \in \mu_t) \Rightarrow (x_1 = x_2), \tag{6}$$

where μ_t is defined by (2).

Proof. Without loss of generality, we may assume that $\varepsilon \equiv 1$ belongs to $\mathscr{E}$ (otherwise take an arbitrary standard $\varepsilon_0 \in \mathscr{E}$ such that $\varepsilon_0(t_0) \leq 1$ and restrict T to the standard set $T' = \{t \in T : \varepsilon_0(t) \leq 1\}$, $t_0 \in T'$).

Suppose that there exists such a formula $\varphi(x,t)$. Then this formula is equivalent in IST to some Σ_2^{st}-formula [1], i.e., a formula like

$$\exists^{\mathrm{st}} a \in \mathscr{A}\, \forall^{\mathrm{st}} c \in \mathscr{C} \quad \psi(x,a,c,t),$$

where ψ is an internal formula and $\mathscr{A}$ and $\mathscr{C}$ are standard sets. Define the standard set $\mathscr{B} = \mathscr{P}^{\mathrm{fin}}(\mathscr{C})$ of finite subsets of $\mathscr{C}$ and the standard mapping $M : \mathscr{A} \times \mathscr{B} \times T \to \mathscr{P}(I)$:

$$M(a,b,t) = \{x \in I : \forall c \in b \quad \psi(x,a,c,t)\}.$$

Then

$$M_t = {}^{E}\{x \in I : \exists^{\mathrm{st}} a \in \mathscr{A}\, \forall^{\mathrm{st}} b \in \mathscr{B} \quad x \in M(a,b,t)\}; \tag{7}$$

moreover, M possesses the property

$$\forall^{\mathrm{fin}} B \in \mathscr{B}\, \exists b \in B\, \forall a \in \mathscr{A}, t \in T \quad M(a,b,t) = \cap\{M(a,\beta,t) : \beta \in B\} \tag{8}$$

(for example, we can take $\cup B$ as b).

Reformulate (6) in terms of M:

$$\forall t \in T\, \forall x \in I\, \exists y \in I\, \exists^{\mathrm{st}} a \in \mathscr{A}\, \forall^{\mathrm{st}} b \in \mathscr{B}\, \forall^{\mathrm{st}} \varepsilon \in \mathscr{E} \quad y \in M(a,b,t) \;\;\&\;\; |x-y| < \varepsilon(t),$$

$$\forall t \in T\, \forall x_1, x_2 \in I\, \forall^{\mathrm{st}} a_1, a_2 \in \mathscr{A}\, \exists^{\mathrm{st}} b_1, b_2 \in \mathscr{B}\, \exists^{\mathrm{st}} \varepsilon \in \mathscr{E}$$
$$(x_1 \notin M(a_1,b_1,t)) \vee (x_2 \notin M(a_2,b_2,t)) \vee (x_1 = x_2) \vee (|x_1 - x_2| > \varepsilon(t)).$$

Applying Nelson's algorithm [2] to these external formulas, we obtain the equivalent internal formulas

$$\forall b \in \mathscr{B}^{\mathscr{A}}, \varepsilon \in \mathscr{E}^{\mathscr{A}}\, \exists^{\mathrm{fin}} A \subseteq \mathscr{A}\, \forall x \in I, t \in T\, \exists a \in \mathscr{A}\, \exists y \in M(a,b(a),t)\; |x-y| < \varepsilon(a,t),$$

$$\forall a_1, a_2 \in \mathscr{A}\, \exists b_1, b_2 \in \mathscr{B}\, \exists \varepsilon \in \mathscr{E}\, \forall t \in T \forall x_1, x_2 \in I$$
$$(x_1 \notin M(a_1,b_1,t)) \vee (x_2 \notin M(a_2,b_2,t)) \vee (x_1 = x_2) \vee (|x_1 - x_2| > \varepsilon(t)).$$

We can simplify them:

$$\forall b \in \mathscr{B}^{\mathscr{A}}\, \forall \varepsilon \in \mathscr{E}^{\mathscr{A}}\, \exists^{\mathrm{fin}} A \subseteq \mathscr{A}\, \forall t \in T \quad I \subseteq \cup\{U_{\varepsilon(a,t)} M(a,b(a),t) : a \in A\}, \tag{9}$$

$$\forall a_1, a_2 \in \mathscr{A}\, \exists b \in \mathscr{B}, \varepsilon \in \mathscr{E}\, \forall t \in T \quad \bar\rho(M(a_1,b,t), M(a_2,b,t)) > \varepsilon(t). \tag{10}$$

Here $\bar\rho(M_1,M_2) = \inf\{|x_1 - x_2| : x_i \in M_i, x_1 \neq x_2\}$, $U_\varepsilon M$ is the ε-neighborhood of M, and b in (10) is found from $B = \{b_1, b_2\}$ of (8) ($(M_1 \cap M_2) \times (M_1 \cap M_2) \subseteq M_1 \times M_2$ and $\bar\rho$ is a monotone function). In fact, instead of (10) the following weaker assertion would suffice:

$$\forall a \in \mathscr{A}\, \exists b \in \mathscr{B}, \varepsilon \in \mathscr{E}\, \forall t \in T \quad \bar\rho(M(a,b,t)) > \varepsilon(t), \tag{11}$$

where $\bar\rho(M) = \bar\rho(M,M)$.

Let us prove that (9) and (11) are inconsistent conditions. From (11) we find that

$$\exists \bar b \in \mathscr{B}^{\mathscr{A}} \exists \bar\varepsilon \in \mathscr{E}^{\mathscr{A}} \forall a \in \mathscr{A}, t \in T \quad \bar\rho(M(a,\bar b(a),t)) > \bar\varepsilon(a,t).$$

Put $\varepsilon_a(t) = \lambda(t) \cdot \min(1, \bar\varepsilon_a(t))$, and use (9) to define a finite $A \subseteq \mathscr{A}$ from the functions $\bar b_a$ and ε_a. We obtain $I = \cup\{U_{\varepsilon_a(t)} m(a,t) : a \in \mathscr{A}\}$, with $m(a,t) = M(a, \bar b_a, t)$. However, it follows from (11) that $m(a,t)$ is a finite set of points at distance at least $\bar\varepsilon_a(t)$ from one another. Therefore, $\operatorname{card} m(a,t) \le 1 + (\bar\varepsilon_a(t))^{-1}$ and

$$1 = \operatorname{mes} I \le \sum_{a \in A} 2\varepsilon_a(t)(1 + (\bar\varepsilon_a(t))^{-1}) \le 4\lambda(t) \cdot \operatorname{card} A. \tag{12}$$

Since $\lambda(t)$ for $t \in T$ takes arbitrarily small values, (12) cannot hold for all $t \in T$. This contradiction shows that (9) and (11) are inconsistent conditions.

3. Proximity Relations and Auras

By now, we spoke about the construction of a set M_t satisfying certain conditions for all values of the parameter t. We have shown that such a set does not exist. However, this does not exclude the possibility of the existence of M_t for a specific value of t. Moreover, we would like also consider external equivalences with a nonstandard parameter "locally," without being constrained by the whole family of these relations for t ranging over a standard set. Below we study and classify such "locally defined" external equivalences on $\mathbb{R}$, and for some part of them we answer the question of existence of an external selecting function.

Let $\rho(x, y)$ be an external formula (depending possibly on a nonstandard parameter t) which describes an external equivalence on a standard set X, i.e.,

$$\forall x, y, z \in X \ \rho(x, x) \ \& \ (\rho(x, y) \Rightarrow \rho(y, x)) \ \& \ (\rho(x, y) \ \& \ \rho(y, z) \Rightarrow \rho(x, z)).$$

Is there (for a given value of the parameter) an external formula $\varphi(x)$ (depending only on the same parameter as ρ) which selects one element per each equivalence class of X/ρ? Namely, φ should satisfy the following conditions:

$$\begin{gathered}\forall x \in X \, \exists y \in X \ \varphi(y) \ \& \ \rho(x, y), \\ \forall x_1, x_2 \in X \ (\varphi(x_1) \ \& \ \varphi(x_2) \ \& \ \rho(x_1, x_2)) \Rightarrow (x_1 = x_2).\end{gathered} \tag{13}$$

We discuss this question for some class of equivalences on $\mathbb{R}$.

We call an external equivalence ρ (depending possibly on a nonstandard parameter) a *proximity* if (for a given value of the parameter) ρ possesses the properties of additivity:

$$\forall x_1, x_2, y_1, y_2 \in \mathbb{R} \quad (\rho(x_1, y_1) \ \& \ \rho(x_2, y_2)) \Rightarrow \rho(x_1 + x_2, y_1 + y_2),$$

convexity:

$$\forall x, y, z \in \mathbb{R} \quad (\rho(x, z) \ \& \ x \le y \le z) \Rightarrow (\rho(x, y) \ \& \ \rho(y, z)),$$

and nondegeneration:

$$\forall x \in \mathbb{R} \, (\exists y \in \mathbb{R} \quad (y \ne x) \ \& \ \rho(x, y)) \ \& \ (\exists z \in \mathbb{R} \, \neg \rho(x, z)).$$

By additivity, $\rho(x, y) \Leftrightarrow \rho(x - y, 0)$. Consider the external set $\mu = {}^E\{x \in \mathbb{R} : \rho(x, 0)\}$. This set completely describes ρ and enjoys the following properties:

$$\mu + \mu = -\mu = \mu, \quad (x \in \mu, |y| \le x) \Rightarrow y \in \mu, \quad \mu \notin \{\varnothing, \{0\}, \mathbb{R}\}. \tag{14}$$

Conversely, to each set μ with these properties there corresponds some proximity ρ.

In [6, 7] the sets satisfying (14) (i.e., nontrivial convex external additive subgroups of $\mathbb{R}$) were called *auras*. In the same articles, some properties of auras were studied as well as some operations on them were defined. Here we only list those definitions and properties of auras from [6, 7] which are needed for the statement and proof of the existence theorem for a selecting function.

Proposition 1. *Each aura can be defined by a single external quantifier.*

Auras can be classified by the types of their defining formulas: an aura is said to be a Σ- or Π-*aura* if it is defined by a Σ_1^{st}- or Π_1^{st}-formula respectively (i.e., depending on whether the external quantifier in the defining formula of the aura is the existential or universal quantifier). The type of an external formula defining an aura is determined uniquely; i.e., no aura can be a Σ-aura and Π-aura simultaneously.

Note that a Π-aura is a generalization of the notion of π-monad of superinfinitesimals [8] and a Σ-aura is a generalization of the notion of galaxy.

Proposition 2. *Each aura with a bounded parameter t can be written down as*

$$^E\{x \in \mathbb{R} : \forall^{\mathrm{st}}\varepsilon \in \mathscr{E} \quad |x| < \varepsilon(t)\} \tag{15}$$

or

$$^E\{x \in \mathbb{R} : \exists^{\mathrm{st}}\varepsilon \in \mathscr{E} \quad |x| < \varepsilon(t)\} \tag{16}$$

(*depending on its type*), *where $\mathscr{E}$ is some standard set of functions from T into $\mathbb{R}_+$ which satisfies the conditions* (3) *and T is a standard set containing t.*

With respect to multiplication, the family of auras possesses the structure of a semigroup with identity, the aura of limited numbers (denoted by e henceforth). Two auras μ and ν are said to be *similar* if there exists a real $\lambda > 0$ such that $\mu = \lambda\nu$. Obviously, only the auras of the same type may be similar. Define the *derivative* of μ to be the aura $\mu' = {}^E\{c \in \mathbb{R} : c\mu \subseteq \mu\}$.

Proposition 3. *Auras of the same type are similar if and only if their derivatives coincide.*

Examples of auras.

1. The set of infinitesimals and the set of t-infinitesimal reals are Π-auras (we denote the aura of infinitesimals by μ_0).
2. The sets of limited and t-limited real numbers are Σ-auras.
3. The set ${}^E\{x \in \mathbb{R} : \forall^{\mathrm{st}} n \in \mathbb{N} \ \ |x| < t^n\}$, with $0 < t < 1$ and t not infinitely close to 1, is a Π-aura.
4. The set ${}^E\{x \in \mathbb{R} : \forall^{\mathrm{st}}\varepsilon > 0 \ \ |x| < \exp(\varepsilon t)\}$, with $t > 0$ infinitely large, is a Π-aura, too.

4. On the Possibility of Local Definition of a Selecting Function (For Some Values of a Parameter)

Theorem 3. *Let ρ be a proximity* (*depending possibly on a given nonstandard bounded parameter t*) *whose aura μ is a Π-aura. There exists an external formula φ selecting one element per each equivalence class of $\mathbb{R}/\rho$ if and only if μ is similar to the aura $\mu_0 = {}^E\{x \in \mathbb{R} : x \approx 0\}$ of infinitesimals.*

Observe that if there is no selecting external formula for $\mathbb{R}/\rho$ then there is no external selecting formula for $[0, N]/\rho$ for any positive constant N not belonging to μ either.

Proof. Without loss of generality, we may assume that φ has a single nonstandard parameter t (if φ has a nonstandard parameter p then both φ and μ can be considered as depending on a single parameter, the pair $\langle p, t\rangle$).

1. If μ is similar to μ_0, $\mu = \lambda\mu_0$, then we can take a sought formula φ to be

$$\exists^{\mathrm{st}} a \in [0, 1) \exists n \in \mathbb{N} \quad x = \lambda(n + a).$$

So, for $\lambda = 1$, in each equivalence class we choose a representative whose fractional part is standard.

2. Suppose that μ is not similar to μ_0. Without loss of generality, we may assume that $1 \notin \mu$ (otherwise we replace μ with a similar aura).

By analogy to the proof of Theorem 2, we reduce φ to Σ_2^{st}-type:

$$\varphi(x, t) \Leftrightarrow (\exists^{\mathrm{st}} a \in \mathscr{A}\, \forall^{\mathrm{st}} b \in \mathscr{B} \quad x \in M(a, b, t))$$

(the mapping M satisfies (8)), and write down μ in the form (15):

$$\mu = {}^E\{x \in \mathbb{R} : \forall^{\mathrm{st}}\varepsilon \in \mathscr{E} \quad |x| < \varepsilon(t)\}.$$

If μ is not similar to μ_0 and φ satisfies the conditions of the theorem then for the given t the following three conditions are satisfied, the last condition been already encountered in the proof of Theorem 2:

$$\exists^{\mathrm{st}}\lambda \in \mathscr{E}^{\mathscr{E}} \forall^{\mathrm{st}}\varphi \in \mathscr{E}, n \in \mathbb{N}\ f(t) > n\lambda_f(t),$$
$$\exists^{\mathrm{st}} A \in (\mathscr{P}^{\mathrm{fin}}(\mathscr{A}))^{\mathscr{B}^{\mathscr{A}} \times \mathscr{E}^{\mathscr{A}}} \forall^{\mathrm{st}} b \in \mathscr{B}^{\mathscr{A}}, \varepsilon \in \mathscr{E}^{\mathscr{A}}\ \mathbb{R} = \cup\{U_{\varepsilon_a(t)} M(a, b_a, t) : a \in A(b, \varepsilon)\},$$
$$\exists^{\mathrm{st}}\delta \in \mathscr{E}^{\mathscr{A}}, d \in \mathscr{B}^{\mathscr{A}} \forall^{\mathrm{st}} c \in \mathscr{A} \quad \bar\rho(M(c, d_c, t)) > \delta_c(t).$$

Hence, there exists $t \in T$ for which all three conditions are satisfied. Write down the last condition in the form of an equivalent internal formula:

$$\exists A \in (\mathscr{P}^{\mathrm{fin}}(\mathscr{A}))^{\mathscr{B}^{\mathscr{A}} \times \mathscr{E}^{\mathscr{A}}}, \delta \in \mathscr{E}^{\mathscr{A}}, d \in \mathscr{B}^{\mathscr{A}}, \lambda \in \mathscr{E}^{\mathscr{E}} \forall^{\mathrm{fin}} B \subset \mathscr{B}^{\mathscr{A}}, E \subset \mathscr{E}^{\mathscr{A}},$$
$$C \subset \mathscr{A}, F \subset \mathscr{E}, M \subset \mathbb{N}\, \exists t \in T, \forall b \in B, \varepsilon \in E,$$
$$c \in C, f \in F, n \in M \quad t \in T^1(f,n) \cap T^2(b,\varepsilon) \cap T^3(c), \tag{17}$$

where $T^1(f,n) = \{t \in T : f(t) > n\lambda_f(t)\}$, $T^2(b,\varepsilon) = \{t \in T : \mathbb{R} = \cup\{U_{\varepsilon_a(t)} M(a, b_a, t) : a \in A(b,\varepsilon)\}\}$, and $T^3(c) = \{t \in T : \bar{\rho}(M(c, d_c, t)) > \delta_c(t)\}$.

Fix those values A, δ, d, and λ for which (17) holds and let ε_1 be an arbitrary function in $\mathscr{E}$ such that $\varepsilon(t) \le 1$ (such a function exists due to the condition $1 \notin \mu$), $\mu_a = \min(\varepsilon_1, \delta_a) \in \mathscr{E}^{\mathscr{A}}$ (the minimum is taken for each a), $\bar{\varepsilon}_a = \lambda(\mu_a) \in \mathscr{E}^{\mathscr{A}}$, $\overline{A} = A(d, \bar{\varepsilon}) \in \mathscr{P}^{\mathrm{fin}}(\mathscr{A})$, and $\bar{n} = 1 + 4\,\mathrm{card}\,\overline{A}$. There exists t that belongs simultaneously to the three sets

$$\bigcap_{a \in \overline{A}} T^1(\delta_a, \bar{n}), \quad T^2(d, \bar{\varepsilon}), \quad \bigcap_{a \in \overline{A}} T^3(a).$$

For this t we have

$$\forall a \in \overline{A} \quad \bar{\rho}(M(a, d_a, t)) > \delta_a(t) \ge \mu_a(t),$$
$$\forall a \in \overline{A} \quad \bar{\varepsilon}_a(t) < \mu_a(t)/\bar{n},$$
$$I \subseteq \bigcup_{a \in \overline{A}} U_{\bar{\varepsilon}_a(t)} M(a, d_a, t).$$

But then
$$1 = \mathrm{mes}\, I \le \sum_{a \in \overline{A}} 2\bar{\varepsilon}_a(t)\bigl(1 + \mu_a^{-1}(t)\bigr) \le \sum_{a \in \overline{A}} 4\bar{\varepsilon}_a(t)/\mu_a(t) \le 4\,\mathrm{card}\,\overline{A}/\bar{n} < 1.$$

This contradiction shows that if, for some value of t, μ is not similar to μ_0 then, for this value of the parameter, there is no sought function $\varphi(x,t)$ as claimed.

Remark. It is easy to verify that this theorem is a generalization of Theorem 2. For a value of the parameter t_0 satisfying condition (4), the set $\mu = \{x \in \mathbb{R} : \forall^{\mathrm{st}} \varepsilon \in \mathscr{E} \quad |x| < \varepsilon(t_0)\}$ is an aura, since by (4) $\lambda(T_0)^{-1}\mu \le \mu$, while $\lambda(t_0)^{-1}$ is infinitely large. For the same reason, $\lambda(t_0)^{-1}$ belongs to μ' and is not contained in $\mu'_0 = e$; i.e., by Proposition 3 μ is not similar to μ_0. Hence, by Theorem 3 there is no external selecting function for the proximity corresponding to the aura μ and so there is no global external selecting function over the whole set T. Nevertheless, we did not omit the proof of Theorem 2, since it is much more lucid than that of the last theorem.

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