Index theory of unbounded Fredholm operators

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February 22, 2023 / NYC Noncommutative Geometry Seminar

Classical results

Framework

- H — complex separable Hilbert space of infinite dimension
- Linear operator $A\colon H \to H$ is called Fredholm if
  $\mathrm{Im}(A)$ is closed and $\mathrm{Ker}(A)$ and $\mathrm{Coker}(A) = \mathrm{Ker}(A^*)$ are finite-dimensional.
- $\mathcal{B}(H)$ — space of bounded operators on H (with the norm topology)
- $\mathcal{B}_F(H)$ — subspace of bounded Fredholm operators

Index of Fredholm operators

- Single operator: $\mathrm{ind}(A) = \dim \mathrm{Ker}(A) - \dim \mathrm{Coker}(A)$
  determines isomorphism $\mathrm{ind}\colon \pi_0(\mathcal{B}_F) \to \mathbb{Z}$.
- Family case: for X compact, there is natural isomorphism
  $\mathrm{ind}\colon [X, \mathcal{B}_F] \to K^0(X)$, so $\mathcal{B}_F$ is the classifying space for K^0 [Atiyah, Jänich]
- $K^0(X)$ is an Abelian group generated by isomorphism classes $[E]$
  of vector bundles E over X, with relations $[E \oplus F] = [E] + [F]$
- For $\mathcal{A}\colon X \to \mathcal{B}_F$ continuous, $\mathrm{ind}(\mathcal{A}) \approx [\mathrm{Ker}(A)] - [\mathrm{Coker}(\mathcal{A})] \in K^0(X)$
  (roughly speaking)

Self-adjoint Fredholm operators

- $\mathcal{B}_F^{\mathrm{sa}}(H)$ — space of self-adjoint Fredholm operators
- $\mathrm{Coker}(A) \cong \mathrm{Ker}(A^*)$, so $\mathrm{ind}(A) = 0$ for self-adjoint A
  Similarly for families: $\mathrm{ind}(\mathcal{A}) = 0 \in K^0(X)$ for $\mathcal{A}\colon X \to \mathcal{B}_F^{\mathrm{sa}}(H)$
  However, self-adjoint operators have their own index theory, with values in $K^1(X)$ instead of $K^0(X)$.

Theorem [Atiyah and Singer; 1969]

- $\mathcal{B}_F^{\mathrm{sa}} = \mathcal{B}_F^{+} \sqcup \mathcal{B}_F^{-} \sqcup \mathcal{B}_F^{*}$ has three connected components:
  - $\mathcal{B}_F^{+}$ essentially positive operators
  - $\mathcal{B}_F^{-}$ essentially negative operators
  - $\mathcal{B}_F^{*}$ neither essentially positive, nor essentially negative
- $\mathcal{B}_F^{+}$ and $\mathcal{B}_F^{-}$ are contractible
- $\mathcal{B}_F^{*}$ is a classifying space for K^1

$K^1(X) = \lim_{n\to\infty}[X, \mathcal{U}_n] = [X, \mathcal{U}_\infty]$, where $\mathcal{U}_\infty = \lim_{n\to\infty} \mathcal{U}_n$

- $K^0(pt) = \mathbb{Z}$
- $K^1(pt) = 0$, $K^1(S^1) = \mathbb{Z}$

	operator/path	family
Fredholm	index: $\pi_0(\mathcal{B}_F) \to \mathbb{Z}$	index: $[X, \mathcal{B}_F] \to K^0(X)$
Self-adjoint Fredholm	spectral flow: $\pi_1(\mathcal{B}_F^{\mathrm{sa}}) \to \mathbb{Z}$	index: $[X, \mathcal{B}_F^{\mathrm{sa}}] \to K^1(X)$

Spectral flow: path $\mathcal{A}\colon [0,1] \to \mathcal{B}_F^{\mathrm{sa}} \quad \rightsquigarrow \quad \text{sp.flow}(\mathcal{A}) \in \mathbb{Z}$

- The spectral flow counts (with signs) the number of eigenvalues passing through zero from the start of the path to its end

Unbounded operators

- Unbounded operator $A\colon H \to H$ acts from $\mathrm{dom}(A) \subset H$ to H
- A is called closed if its graph is closed
- A is called densely defined if its domain is dense in H
- $\mathcal{R}(H)$ – regular operators ($=$ closed and densely defined)
- $\mathcal{R}^{\mathrm{sa}}(H)$ – regular self-adjoint operators
- $\mathcal{R}_F(H)$ – regular Fredholm operators (the same definition as in bounded case)

Example: elliptic operators on closed manifolds

- E – vector bundle over a closed manifold M
- A – elliptic operator of order d acting on sections of E; it is Fredholm
- If $d > 0$, then A is bounded as an operator $H_s(E) \to H_{s-d}(E)$, but unbounded as an operator $L^2(E) \to L^2(E)$.
- For K^0-index, one can deal with bounded operators $H_s \to H_{s-d}$
- Another option: bounded transform $A \mapsto A(1 + A^*A)^{-1/2}$ provides 0-th order (i.e. bounded) elliptic operator; preserves self-adjointness; behaves well on symbols.

Bounded transform

- Fix a homeomorphism $\chi\colon \overline{\mathbb{R}} = [-\infty, \infty] \to [-1, 1]$, $\chi(A) = A(1 + A^2)^{-1/2}$ (the **bounded transform** map).
- Extension to non-self-adjoint operators: $\chi(A) = A(1 + A^*A)^{-1/2}$.
- A regular $\Longrightarrow \chi(A)$ bounded, of norm $\leqslant 1$
  A self-adjoint $\Longleftrightarrow \chi(A)$ self-adjoint
  A Fredholm $\Longleftrightarrow \chi(A)$ Fredholm
- For elliptic operator A of positive order on a closed manifold, $\chi(A)$ is elliptic of order 0 and depends continuously on A

Riesz topology

Riesz topology on $\mathcal{R}(H)$ is induced by the inclusion $\chi\colon \mathcal{R}(H) \hookrightarrow \mathcal{B}(H)$

Elliptic operators

- Families of elliptic operators on closed manifolds are Riesz continuous
- Families of elliptic boundary value problems on compact manifolds, in general, are **not** Riesz continuous (or are not known to be Riesz continuous), so one needs weaker topology on $\mathcal{R}(H)$

Appropriate topology for boundary value problems is the **graph topology**

Graph topology

- Grassmanian $\mathrm{Gr}(H)$ is the set of all closed subspaces of H with the topology induced by the inclusion $\mathrm{Gr}(H) = \mathcal{P}(H) \subset \mathcal{B}(H)$
- *Graph topology* on $\mathcal{R}(H)$ is induced by the inclusion $\mathcal{R}(H) \hookrightarrow \mathrm{Gr}(H \oplus H)$ taking operator to its graph

Two topologies on $\mathcal{R}(H)$

- Riesz topology is strictly stronger than the graph topology.
- On bounded operators all three topologies coincide: norm = Riesz = graph
- $\mathcal{B}(H)$ is dense and open both in ${}^{r}\mathcal{R}(H)$ and ${}^{g}\mathcal{R}(H)$
- ${}^{r}\mathcal{R}(H)$ / ${}^{g}\mathcal{R}(H)$ – space of regular operators with Riesz / graph topology
- Riesz topology: elliptic operators on closed manifolds
- Graph topology: elliptic boundary value problems on compact manifolds

Index theory for unbounded operators

Subspaces of $\mathcal{R}(H)$

- $\mathcal{R}^{\mathrm{sa}}(H)$ — self-adjoint operators
- $\mathcal{R}_F(H)$ — Fredholm operators
- $\mathcal{R}_K(H)$ — operators with compact resolvent
  (both $(1 + A^*A)^{-1}$ and $(1 + AA^*)^{-1}$ are compact operators)
  - Example: elliptic operators of positive order

Riesz topology

- Natural inclusions ${}^r\mathcal{R}_K \hookrightarrow {}^r\mathcal{R}_F \overset{\varkappa}{\hookrightarrow} \mathcal{B}_F$ and ${}^r\mathcal{R}_K^{\mathrm{sa}} \hookrightarrow {}^r\mathcal{R}_F^{\mathrm{sa}} \overset{\varkappa}{\hookrightarrow} \mathcal{B}_F^{\mathrm{sa}}$
  induce homotopy equivalences
- In particular, ${}^r\mathcal{R}_F^{\mathrm{sa}}$ has three connected components

Families of elliptic operators on closed manifolds are Riesz continuous,
so their index theory is the same as for operators of order 0 (i.e. bounded)

Graph topology

Theorem [Booss-Bavnbek, Lesch, Phillips; 2001]

- ${}^g\mathcal{R}_F^{\mathrm{sa}}(H)$ is path connected
- Spectral flow homomorphism $\pi_1({}^g\mathcal{R}_F^{\mathrm{sa}}) \to \mathbb{Z}$ is surjective

Theorem [Joachim; 2003]

- ${}^g\mathcal{R}_K(H)$ and ${}^g\mathcal{R}_F(H)$ are classifying spaces for K^0
- ${}^g\mathcal{R}_K^{\mathrm{sa}}(H)$ and ${}^g\mathcal{R}_F^{\mathrm{sa}}(H)$ are classifying spaces for K^1

Joachim proved this theorem in a more general situation,
for the Hilbert module $H_C = C \otimes H$ over a unital C^*-algebra C
(the Hilbert space case corresponds to $C = \mathbb{C}$):

- ${}^g\mathcal{R}_K(H_C)$ and ${}^g\mathcal{R}_F(H_C)$ are classifying spaces for $K^0(-; C)$
- ${}^g\mathcal{R}_K^{\mathrm{sa}}(H_C)$ and ${}^g\mathcal{R}_F^{\mathrm{sa}}(H_C)$ are classifying spaces for $K^1(-; C)$

However, his proof is based on Kasparov KK-theory, even in the case $C = \mathbb{C}$

My results

Spaces of unbounded Fredholm operators. I. Homotopy equivalences, arXiv:2110.14359 (2021), 24 pp.

Goals

- Transparent proof of Joachim's results for Hilbert spaces (i.e. for $C = \mathbb{C}$) based on topology and operator theory, without use of KK-theory
- Simple definitions of K^0 and K^1 family index for graph continuous families (convenient for applications; without use of KK-theory)
  works also for non-compact spaces of parameters
- Relate spaces of unbounded operators with graph topology to classical spaces of bounded operators representing K-theory

Relation to classical spaces

- $\mathcal{B}(H) \hookrightarrow {}^r\mathcal{R}(H) \xrightarrow{\mathcal{I}d} {}^g\mathcal{R}(H) \hookrightarrow \mathrm{Gr}(H \oplus H)$
- Zero operator $\mapsto H \oplus 0$
- "Infinite operator" $\mapsto 0 \oplus H$
- Restricted Grassmanian $\mathrm{Gr}_K(H \oplus H) = \{L \in \mathrm{Gr} \mid L \text{ is compact def. of } 0 \oplus H\}$
  $= \{p \in \mathcal{P} \mid p - p_{0 \oplus H} \text{ is compact}\}$
- Fredholm Grassmanian $\mathrm{Gr}_F(H \oplus H) = \{L \in \mathrm{Gr} \mid (L, H \oplus 0) \text{ is Fredholm pair}\}$
  $= \{p \in \mathcal{P} \mid p - p_{H \oplus 0} \text{ is Fredholm}\}$

Theorem 1 [P; 2021]

$$\begin{tikzcd}
 & {}^r\mathcal{R}_K \arrow[r, "\mathcal{I}d"] \arrow[d] & {}^g\mathcal{R}_K \arrow[r, hook] \arrow[d] & \mathrm{Gr}_K \arrow[d] \\
\mathcal{B}_F \arrow[r, hook] & {}^r\mathcal{R}_F \arrow[r, "\mathcal{I}d"] & {}^g\mathcal{R}_F \arrow[r, hook] & \mathrm{Gr}_F
\end{tikzcd}$$

All these maps are homotopy equivalences, so all spaces represent K^0.

Self-adjoint operators

Relation to classical spaces

- Lagrangian Grassmanian instead of Grassmanian: ${}^g\mathcal{R}^{\mathrm{sa}}(H) \hookrightarrow \mathrm{LGr}(H \oplus H)$

$$\begin{tikzcd} {}^g\mathcal{R}^{\mathrm{sa}}(H) \arrow[r, hook] \arrow[dr, "\kappa"'] & \mathrm{LGr}(H \oplus H) \arrow[d, phantom, "\cong"] \\ & \mathcal{U}(H) \end{tikzcd}$$

- Cayley transform $\boldsymbol{\kappa}\colon {}^g\mathcal{R}^{\mathrm{sa}}(H) \hookrightarrow \mathcal{U}(H)$,
  $\boldsymbol{\kappa}(A) = (A - i)(A + i)^{-1} = 1 - 2i(A + i)^{-1}$
  (so graph convergence $\Longleftrightarrow$ convergence in the norm resolvent sense)
- Zero operator $\mapsto -1$, "infinite operator" $\mapsto +1$
- Restricted unitary group $\mathcal{U}_K = \{u \in \mathcal{U} \mid u - 1 \text{ is compact}\} \cong \mathrm{LGr}_K$
- "Fredholm unitary group" $\mathcal{U}_F = \{u \in \mathcal{U} \mid u + 1 \text{ is Fredholm}\} \cong \mathrm{LGr}_F$

Theorem 2 [P; 2021]

$$\begin{tikzcd} & {}^r\mathcal{R}^*_K \arrow[r] \arrow[d, hook] & {}^g\mathcal{R}^{\mathrm{sa}}_K \arrow[r, hook, "\kappa"] \arrow[d, hook] & \mathcal{U}_K \arrow[d, hook] \\ \mathcal{B}^*_F \arrow[r, hook] & {}^r\mathcal{R}^*_F \arrow[r] & {}^g\mathcal{R}^{\mathrm{sa}}_F \arrow[r, hook, "\kappa"] & \mathcal{U}_F \end{tikzcd}$$

All these maps are homotopy equivalences, so all spaces represent K^1.

Essentially positive / negative operators

Question [Booss-Bavnbek, Lesch, and Phillips; 2001]

Are $\mathcal{R}_F^-$ and $\mathcal{R}_F^+$ (two "trivial" parts of $\mathcal{R}_F^{\mathrm{sa}}$) contractible in the graph topology?

The answer is negative:

Theorem 3 [P; 2021]

- Each of the spaces ${}^g\mathcal{R}_F^+$, ${}^g\mathcal{R}_F^-$, and ${}^g\mathcal{R}_F^*$ represents K^1
- Embeddings $\mathcal{R}_F^+ \hookrightarrow \mathcal{R}_F^{\mathrm{sa}}$, $\mathcal{R}_F^- \hookrightarrow \mathcal{R}_F^{\mathrm{sa}}$, and $\mathcal{R}_F^* \hookrightarrow \mathcal{R}_F^{\mathrm{sa}}$ are homotopy equivalences in the graph topology
- The same holds for operators with compact resolvent

The following example demonstrates that

- ${}^g\mathcal{R}_K^+$ is not contractible
- Family of elliptic boundary value problems may fail to be Riesz continuous

Rellich's example (extended from $\mathbb{R}$ to $\mathbb{R}P^1$)

- $A = -d^2/dt^2$ acts on functions $\psi\colon [0,1] \to \mathbb{C}$
- Family of boundary value problems for A parametrized by $\mathbb{R}P^1 \cong S^1$
- For $x = [x_0 : x_1] \in \mathbb{R}P^1$, the domain of $\mathcal{A}_x$ is

  $\left\{\psi \in H^2([0,1];\mathbb{C}) \,|\, \psi(0) = 0 \text{ and } x_0\psi(1) = x_1\psi'(1)\right\}$

Then we have:

- Each $\mathcal{A}_x$ is essentially positive operator with compact resolvent
- $\mathcal{A}$ is a graph continuous loop with non-trivial spectral flow $= 1$
- Spectral graph of this family is a single curve;
  for positive eigenvalues it is an infinite spiral line making an infinite number of rotations in the upward direction over the circle $\mathbb{R}P^1$;
  for negative eigenvalues it goes to $-\infty$ as x goes to $[1 : +0]$

Proof of Theorems 1–3

Our proof handles all the maps on diagrams at once.

The proof is based on the following theorem:

Theorem [tom Dieck; 1971]

A map is a homotopy equivalence if it is locally a homotopy equivalence.

More precisely, let

- $f\colon X \to Y$ continuous map
- (X_λ) and (Y_λ) numerable coverings of X and Y, indexed by $\lambda \in \Lambda$
- $f(X_\lambda) \subset Y_\lambda$ and $f\colon X_\lambda \to Y_\lambda$ is a homotopy equivalence
- Moreover, $f\colon \bigcap_{\lambda\in F} X_\lambda \to \bigcap_{\lambda\in F} Y_\lambda$ is a homotopy equivalence for every finite non-empty $F \subset \Lambda$

Then f itself is a homotopy equivalence

Theorem 4

Let X be one of the spaces of unbounded operators on the diagrams.
Then the subspace $X_0 = \{A \in X \,|\, A \text{ invertible}\}$ is contractible.

Proof of Theorem 2 for the map ${}^r\mathcal{R}^*_K \to {}^g\mathcal{R}^{\mathrm{sa}}_K$

Let $X = {}^r\mathcal{R}^*_K$, $Y = {}^g\mathcal{R}^{\mathrm{sa}}_K$, and $f\colon X \to Y$ the identity map

1. We take open coverings parametrized by $\lambda \in \mathbb{R}$,
   $X_\lambda = \{A \in X \,|\, A - \lambda \text{ invertible}\}$ and $Y_\lambda = \{A \in Y \,|\, A - \lambda \text{ invertible}\}$
2. Each X_λ and Y_λ is contractible, so $X_\lambda \to Y_\lambda$ is a homotopy equivalence
3. For finite intersections, the map is also homotopy equivalence
   (they both are fibered over the same base space with contractible fibers)

Thus tom Dieck's theorem implies that $X \to Y$ is a homotopy equivalence

The proof for all other arrows on diagrams is completely similar

M. Prokhorova. *From graph to Riesz continuity*. arXiv:2202.03337 (2022), 20 pp.

Theorem 5 [P; 2022], homotopy inverse map to ${}^r\mathcal{R}_K \hookrightarrow {}^g\mathcal{R}_K$

There is a map $v \colon \mathcal{R}(H) \to \mathcal{U}(H)$ such that

1. v is both Riesz-to-norm and graph-to-strong continuous
2. $\Phi \colon {}^g\mathcal{R}(H) \to {}^r\mathcal{R}(H)$, $A \mapsto A \cdot v(A)$ is graph-to-Riesz continuous
3. Restriction of Φ to $\mathcal{R}_K$ is homotopy inverse to the identity map ${}^r\mathcal{R}_K \to {}^g\mathcal{R}_K$
4. Restriction of Φ to $\mathcal{R}_F$ is homotopy inverse to the identity map ${}^r\mathcal{R}_F \to {}^g\mathcal{R}_F$

Theorem 6 [P; 2022]

1. X arbitrary topological space
2. $\mathcal{H}$, $\mathcal{H}'$ numerable Hilbert bundles over X
3. $\mathcal{A} = (\mathcal{A}_x)$ **graph** continuous, $\mathcal{A}_x \in \mathcal{R}_K(\mathcal{H}_x, \mathcal{H}'_x)$

Then there are global trivializations of $\mathcal{H}$, $\mathcal{H}'$ taking $\mathcal{A}$ to a **Riesz** continuous map $X \to \mathcal{R}_K(H, H')$.

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Theorem 7: self-adjoint case [P; 2022]

1. X metrizable space of finite covering dimension
2. $\mathcal{H}$ Hilbert bundle over X
3. $\mathcal{A} = (\mathcal{A}_x)$ **graph** continuous, $\mathcal{A}_x \in \mathcal{R}^*_K(\mathcal{H}_x)$
   (self-adjoint operators with compact resolvent,
   neither essentially positive nor essentially negative)

Then there is a global trivializations of $\mathcal{H}$ taking $\mathcal{A}$ to a **Riesz** continuous map $X \to \mathcal{R}^*_K(H)$.

In Rellich example operators are essentially positive, so Thm. 7 is not applicable

Alternative approach to the index of unbounded operators

Recently Ivanov developed a new approach to the index of Fredholm families, which works even under weaker continuity assumptions.
His approach is based on Segal's theory of classifying spaces.

1. N.V. Ivanov. *Topological categories related to Fredholm operators: I. Classifying spaces.* arXiv:2111.14313 (2021), 106 pp.
2. N.V. Ivanov. *Topological categories related to Fredholm operators: II. The analytic index.* arXiv:2111.15081 (2021), 53 pp.