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ON RELATIVE NEARSTANDARDNESS IN IST

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In the article [1], E. Gordon defined the predicate "to be standard relative to t":

$$x \operatorname{st} t \Leftrightarrow \exists^{\mathrm{st}} \varphi : (F\operatorname{fin}(\varphi)) \;\;\&\;\; (t \in \operatorname{dom} \varphi) \;\;\&\;\; (x \in \varphi(t))$$

(here $F\operatorname{fin}(\varphi)$ means that φ is a function whose values are finite sets) and introduced the natural notions of t-infinitesimal, t-illimited number, and t-limited real number. For example, a t-infinitesimal is defined as a number whose modulus is less than every positive t-standard number (this notion is a particular case of the notion of superinfinitesimal or π-monad [2]).

It was demonstrated in [1] that, for some nonstandard natural number N, not all points of the interval $[0,1]$ are N-nearstandard (i.e., there exists $x \in [0,1]$ such that there is no N-standard number N-infinitely close to x). The following question arises: is it possible to choose a nonstandard natural number N so that each point of the interval $[0,1]$ has an N-standard part, i.e., is N-nearstandard. If the answer to this question were positive then we could construct an external function similar to but "more detailed" than the standard part map. Unfortunately, as is shown below, the answer to this question is negative. Moreover, it remains negative in a more general case when we replace the set of natural numbers with an arbitrary set of nonmeasurable cardinality and the interval with an arbitrary Hausdorff space other than a rare compact set [3].

The theorems below are also of interest from the viewpoint of constructing the propositions of bounded internal set theory, BIST, of [4] which are equivalent to the conjecture of existence of measurable cardinals [5] (this conjecture is unprovable in ZFC).

Let (X,τ) be a topological space (τ is the family of open sets). We call t *X-appropriate* if all points of X are t-nearstandard (i.e., for each $x \in X$ there is a t-standard $y \in X$ such that x is contained in every t-standard neighborhood of y). It is clear that such t must be bounded (i.e., t lies in some standard set). We define the *complexity* of a bounded t to be the cardinal

$$\operatorname{compl} t = \min\{\operatorname{card} T : \operatorname{st}(T) \;\;\&\;\; t \in T\}. \tag{1}$$

All standard sets have complexity 1.

Theorem 1. *Suppose that (X,τ) is a standard Hausdorff space other than a rare compact set. If t is nonstandard and X-appropriate then the complexity of t is a measurable cardinal.*

The assumption that X is not a rare compact space is essential in the conditions of the theorem. Take, for instance, $X = \{0\} \cup \{k^{-1} : k \in \mathbb{N}\}$ with the induced topology of $\mathbb{R}$. Then for every natural N (in particular, nonstandard) the point 0 is N-standard and k^{-1} is either N-standard or N-infinitely close to 0.

Unfortunately, existence of t such that all points of the interval [0,1] are t-nearstandard is attained by refusing the hypothesis of existence of t-standard and nonstandard real numbers.

Theorem 2. *Suppose that the topological space (X,τ) is the same as in the conditions of Theorem 1 and $\chi = \operatorname{card} X$ is nonmeasurable. If t is X-appropriate then every t-standard $x \in X$ is standard (in other words, the t-standard part map coincides with the standard part map).*

Theorem 3. *Suppose that X and T are standard infinite sets and* $\operatorname{card} X$ *is nonmeasurable. There is a nonstandard t in T such that all t-standard elements of X are standard if and only if* $\operatorname{card} T$ *is measurable.*

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Suppose that X is a standard infinite set of nonmeasurable cardinality. As Theorem 3 shows, the existence of a nonstandard t such that all t-standard elements of X are standard is equivalent to the conjecture of existence of a measurable cardinal. Putting $X = [0,1]$, we also infer that the existence of a measurable cardinal is equivalent to the existence of a [0,1]-appropriate nonstandard t (if all t-standard elements of [0,1] are standard then the monad of t-infinitesimals coincides with the monad of infinitesimals and t is [0,1]-appropriate).

Proof of Theorem 1. Rewrite the conditions of the theorem in the following equivalent form: if, for a standard T,

$$\exists t \in T \, \forall^{\text{st}}\theta \in T \, \forall x \in X \, \exists^{\text{st}}\varphi \in \varPhi \, \forall^{\text{st}} u \in \mathcal{U} \; (t \neq \theta) \;\; \& \;\; (x \in u(\varphi, t))$$

then $\operatorname{card} T$ is measurable. Here $\varPhi = \mathcal{P}^T$, $\mathcal{P} = \mathcal{P}^{\text{fin}}(X)$ is the set of finite subsets of X, and $\mathcal{U} = \{u \in {}_T{}^{\varPhi\times T} : \forall\varphi \in \varPhi \, \forall t \in T \; \varphi(t) \in u(\varphi, t)\}$ (we deliberately write the statement in the form convenient for further translation). We apply Nelson's algorithm to this formula [6]:

$$\exists\Phi \subseteq (\mathcal{P}^{\text{fin}}(\varPhi))^{T\times\mathcal{U}} \, \forall^{\text{fin}} M \subseteq T, U \subseteq \mathcal{U} \, \exists t \in T$$
$$\forall\theta \in M, u \in U \, \forall x \in X \, \exists\varphi \in \Phi(\theta, u) \; (t \neq \theta) \;\; \& \;\; (x \in u(\varphi, t)).$$

Observe that here we may ignore the dependence of Φ on θ and rewrite the formula in a simpler form:

$$\exists\Phi \subseteq (\mathcal{P}^{\text{fin}}(\varPhi))^{\mathcal{U}} \, \forall^{\text{fin}} U \subseteq \mathcal{U} \{t \in T : X = \cup\{u(\varphi, t) : \varphi \in \Phi(u)\}\} \text{ is infinite.}$$

Denote $Q(u, \Phi) = \{t \in T : X = \cup\{u(\varphi, t) : \varphi \in \Phi\}\}$ and rewrite the last formula as

$$\exists\Phi \subseteq (\mathcal{P}^{\text{fin}}(\varPhi))^{\mathcal{U}} \, \forall^{\text{fin}} U \subseteq \mathcal{U} \bigcap_{u\in U} Q(u, \Phi(u)) \text{ is infinite.} \tag{2}$$

Suppose that χ is an arbitrary infinite cardinal. A filter $\mathcal{F}$ in T is said to be *χ-closed* if every covering $\mathcal{A} \subsetneq \mathcal{P}(T)$ of cardinality χ closed under finite unions intersects $\mathcal{F}$.

It is obvious that the χ-closure property is hereditary (i.e., every extension of a χ-closed filter is χ-closed). Moreover, every χ-closed filter is also χ-multiplicative. We now prove this. Suppose that $\{F_i\}_{i\in\chi}$ is a subset of $\mathcal{F}$ of cardinality χ. Denote $F = \bigcap_{i\in\chi} F_i$ and let $\mathcal{A} = \bigcup_{i\in\chi}\{T - F_i\} \cup \{F\}$ be a covering of T of cardinality $\chi + 1 = \chi$. By the definition of χ-closure, there is an element $\mathcal{F}$ which is the union of finitely many sets in $\mathcal{A}$; i.e., $\exists^{\text{fin}} I \subset \chi : G = \bigcup_{i\in\chi}(T - F_i) \cup F \in \mathcal{F}$. Then $F = \bigcap_{i\in\chi} F_i \cap G \in \mathcal{F}$, as required.

Therefore, if there is a σ-closed free filter $\mathcal{F}$ in T then we can extend $\mathcal{F} \cup \mathcal{P}^{\text{cofin}}(T)$ to a σ-closed ultrafilter in T and thereby prove that $\operatorname{card} T$ is measurable.

We need the following lemma:

Lemma. *There is a system $\nu_n(p)$, $n \in \mathbb{N}$, $p \in \mathcal{P}$, of neighborhoods in X such that p is contained in $\nu_n(p)$ for every n and the set $\{\nu_n(p_i) : i \leq n\}$ of neighborhoods does not cover X for arbitrary $p_1, \dots, p_n \in \mathcal{P}$*

To prove the lemma, we separately consider the following two cases:

1. If X is not compact then suppose that $\mathcal{V}$ is an open covering from which we may extract no finite subcovering. Then we can take $\nu_n(p)$ to be the union (independent of n) of finitely many neighborhoods in $\mathcal{V}$ containing p.

2. If X is compact but not rare then there is a continuous mapping $f : X \xrightarrow{onto} [0,1]$ [7]. Put

$$\nu_n(p) = \{x \in X : 4n\rho(f(x), f(p)) \operatorname{card} p < 1\}$$

(here ρ is the Hausdorff metric). Then

$$[\operatorname{mes} f(\nu_n(p)) \leq 1/2n] \Rightarrow \left[\operatorname{mes} f\Big(\bigcup_{i\leq n} \nu_n(p_i)\Big) \leq \frac{1}{2}\right];$$

i.e., $\bigcup_{i\le n} \nu_n(p_i) \neq X$, as required.

Now, suppose that there is a function $\Phi(u)$ satisfying (2). Then the filter $\mathcal{F}$ on T generated by the family $\mathcal{Q} = \{Q(u, \Phi(u)) : u \in \mathcal{U}\}$ contains only infinite sets; i.e., it is free. Show that $\mathcal{F}$ is σ-closed. Assume that $\mathcal{A} = \{A_i\}$ is a countable covering of T. Put

$$k(t) = \min\{i \in \mathbb{N} : t \in A_i\}, \quad u(\varphi, t) = \nu_{k(t)}(\varphi(t)), \quad n = \operatorname{card} \Phi(u).$$

The family $\{u(\varphi, t) : \varphi \in \Phi(u)\}$ of neighborhoods for such u may cover X only if $k(t) < \operatorname{card} \Phi(u)$. Consequently,

$$\bigcup_{i<n} A_i = \{t \in T : k(t) < \operatorname{card} \Phi(u)\} \supseteq Q(u, \Phi(u)) \text{ i } \bigcup_{i<n} A_i \in \mathcal{F},$$

as required.

Proof of Theorem 2. Applying Nelson's algorithm to the assertion of the theorem, we arrive at the following equivalent statement: prove that $\forall T\, \forall g \in \mathcal{P}^T\, \forall \Phi \in (\mathcal{P}^{\text{fin}}(\Phi))^{\mathcal{U}}$ the filter $\mathcal{F}$ generated by the family $\mathcal{Q} \cup \{T - g^{-1}(p) : p \in \mathcal{P}\}$ coincides with $\mathcal{P}(T)$. Here $\mathcal{P}$, $\mathcal{U}$, and $\mathcal{Q}$ are determined as in Theorem 1.

Fix arbitrary T, g, and Φ. It follows from the proof of Theorem 1 that $\mathcal{F}$ is σ-closed.

Suppose that $\mathcal{F}$ is improper. Extend it to an ultrafilter $\mathcal{F}' \subset \mathcal{P}(T)$. Since $\mathcal{F}'$ is σ-multiplicative and χ is nonmeasurable, $\mathcal{F}'$ is χ-multiplicative [5]. However, $\operatorname{card} \mathcal{P} = \chi$ and $\bigcap\{T - g^{-1}(p) : p \in \mathcal{P}\} = \varnothing$. The so-obtained contradiction shows that $\mathcal{F}$ is proper; i.e., it contains some finite set $M \subset T$. For $p = \cup\{g(t) : t \in M\}$ we have $g^{-1}(p) \supset M$. Therefore, $\varnothing = M \cap (T - g^{-1}(p)) \in \mathcal{F}$ and $\mathcal{F} = \mathcal{P}(T)$, as required.

Proof of Theorem 3. Suppose that $t \in T$ satisfies the following requirement: every t-standard $x \in X$ is standard. This means that, for every function $\varphi : T \to \mathcal{P}$, all elements of the finite set $\varphi(t)$ are standard. Then $\varphi(t)$ itself is standard. Conversely, if $\varphi(t)$ is finite and standard then all elements of $\varphi(t)$ are standard. Thus, we can write down Theorem 3 as follows: Suppose that X and T are standard infinite sets and $\operatorname{card} X$ is nonmeasurable. Then the condition

$$\exists t \in T\, \forall^{\text{st}} \theta \in T\, \forall^{\text{st}} \varphi \in \Phi\, \exists^{\text{st}} p \in \mathcal{P}\ (\varphi(t) = p) \ \&\ (t \neq \theta) \tag{3}$$

is satisfied if and only if $\operatorname{card} T$ is measurable.

Applying Nelson's algorithm to (3), we rewrite (3) in the following equivalent form:

$$\exists q \in \mathcal{P}^{\Phi}\, \forall^{\text{fin}} \Phi \subset \Phi \cap \{Q(\varphi, q(\varphi)) : \varphi \in \Phi\} \text{ is infinite,} \tag{4}$$

where $Q(\varphi, p) = \{t \in T : \varphi(t) = p\}$.

Suppose that there is a function $q(\varphi)$ satisfying (4). Then the filter $\mathcal{F}$ on T generated by the family $\mathcal{Q} = \{Q(\varphi, q(\varphi)) : \varphi \in \Phi\}$ is free. Suppose that $\mathcal{A} = \{A_i\}$ is a countable covering of T. Define the function $k(t)$ as in the proof of Theorem 1 and consider $\varphi(t)$ satisfying the requirement $\operatorname{card} \varphi(t) = k(t)$. Suppose that the number of points in $q(\varphi)$ equals n. Then $Q(\varphi, q(\varphi)) \subset \bigcup_{i=1}^n A_i$; i.e., $\bigcup_{i=1}^n A_i$ is contained in $\mathcal{F}$. Hence, $\mathcal{F}$ is σ-closed and $\operatorname{card} T$ is measurable.

Now, assume that $\operatorname{card} T$ is measurable. Since $\chi = \operatorname{card} \mathcal{P}$ is nonmeasurable, there is a χ-multiplicative ultrafilter $\mathcal{F}$ on T. Fix an arbitrary φ. Since $\cup\{Q(\varphi, p) : p \in \mathcal{P}\} = T$, there exists p such that $Q(\varphi, p)$ is contained in $\mathcal{F}$. By the axiom of choice, there is a function $q(\varphi)$ such that $\forall \varphi \in \Phi\ Q(\varphi, q(\varphi)) \in \mathcal{F}$. Then the intersection of an arbitrary finite collection of sets $Q(\varphi, q(\varphi))$ as well is contained in $\mathcal{F}$; i.e., it is infinite, as required.

Observe that the supposition that $\operatorname{card} X$ is measurable was employed only in the proof of the second part of Theorem 3.

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