Spectral Sections

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Abstract

The paper is devoted to the notion of a spectral section introduced by Melrose and Piazza. In the first part of the paper we generalize results of Melrose and Piazza to arbitrary base spaces, not necessarily compact. The second part contains a number of applications, including cobordism theorems for families of Dirac type operators parametrized by a non-compact base space. In the third part of the paper we investigate whether Riesz continuity is necessary for existence of a spectral section or a generalized spectral section. In particular, we show that if a graph continuous family of regular self-adjoint operators with compact resolvents has a spectral section, then the family is Riesz continuous.

Contents

1 Introduction

In this paper we deal with families of regular (that is, closed and densely defined) linear operators with compact resolvents acting between Hilbert spaces and parametrized by points of an arbitrary topological space X.

Throughout the paper, "Hilbert space" always means a separable complex Hilbert space of infinite dimension; "projection" always means an orthogonal projection, that is, a self-adjoint idempotent; $\mathcal{B}(H, H')$ denotes the space of bounded linear operators from H to H' with the norm topology, $\mathcal{K}(H, H')$ denotes the subspace of $\mathcal{B}(H, H')$ consisting of compact operators, and $\mathcal{P}(H)$ denotes the subspace of $\mathcal{B}(H) = \mathcal{B}(H, H)$ consisting of projections.

Spectral sections. The notion of a spectral section was introduced by Melrose and Piazza in [MP1], in order to give a family version of a global boundary condition of Atiyah-Patody-Singer type. It is convenient to split [MP1, Definition 1] into two parts, as we do below.

Let A be a regular self-adjoint operator with compact resolvents. A projection P is called an r-*spectral section* for A if

$$Au = \lambda u \quad \Longrightarrow \quad \begin{cases} Pu = u & \text{if } \lambda \geqslant r \\ Pu = 0 & \text{if } \lambda \leqslant -r \end{cases} \tag{1.1}$$

In other words,

$$\mathbb{1}_{[r,+\infty)}(A) \leqslant P \leqslant \mathbb{1}_{(-r,+\infty)}(A). \tag{1.2}$$

Here $\mathbb{1}_S$ denotes the characteristic function of the subset $S \subset \mathbb{R}$, and we use the natural partial order on the set of projections. We also say that P is a spectral section for A with a *cut-off parameter* r.

Let now $\mathcal{A} = (\mathcal{A}_x)_{x\in X}$ be a family of regular self-adjoint operators with compact resolvents and $r\colon X \to \mathbb{R}_+$ be a continuous function. A norm continuous family $P = (P_x)_{x\in X}$ of projections is called an r-*spectral section* for $\mathcal{A}$ if, for every $x \in X$, the projection P_x is an r_x-spectral section for the operator $\mathcal{A}_x$. We also say that P is a spectral section for $\mathcal{A}$ with a *cut-off function* r.

Point out that the requirement for the family (P_x) of projections and for the cut-off function r to be continuous is an essential part of the definition of a spectral section.

In (1.1) we replaced strict inequalities used by Melrose and Piazza with non-strict ones, since it simplifies statements of our results. It influences only the cut-off function and does not change the notion of a spectral section.

Generalized spectral sections. The notion of a generalized spectral section was introduced by Dai and Zhang in [DZ] in order to cover both usual spectral sections and the Calderón projection. A projection P is called a generalized spectral section for a self-adjoint operator A if $P - \mathbb{1}_{[0,+\infty)}(A)$ is a compact operator. A norm continuous family $P = (P_x)_{x \in X}$ of projections is called a *generalized spectral section* for a family $\mathcal{A} = (\mathcal{A}_x)$ of self-adjoint operators if P_x is a generalized spectral section for $\mathcal{A}_x$ for every $x \in X$. (We omit a requirement from [DZ, Definition 2.1] for projections P_x to be pseudodifferential, since we work in the general functional-analytic framework in the most part of the paper.)

Compact base spaces. Let $\mathcal{A}$ be a family of first order elliptic self-adjoint differential operators over a closed manifold parametrized by points of a compact space X. As was shown by Melrose and Piazza in [MP1, Proposition 1], for such a family $\mathcal{A}$ the existence of a spectral section is equivalent to the vanishing of the index of $\mathcal{A}$ in $K^1(X)$.

Recall that the *Riesz topology* on the set $\mathcal{R}(H, H')$ of regular operators from H to H' is induced by the *bounded transform* map

$$f\colon \mathcal{R}(H, H') \to \mathcal{B}(H, H'), \quad f(A) = A(1 + A^*A)^{-1/2},$$

from the norm topology on $\mathcal{B}(H, H')$. If a family of elliptic differential operators over a closed manifold has continuously changing coefficients, then the corresponding family of regular operators acting between the Hilbert spaces of square-integrable sections of corresponding vector bundles is Riesz continuous.

The proof of Melrose and Piazza does not use the specifics of differential operators, so [MP1, Proposition 1] can be stated in a more general form:

1.1 Proposition. *Let $\mathcal{A}$ be a Riesz continuous family of self-adjoint regular operators with compact resolvents parametrized by points of a compact space X. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a spectral section.*
2. $\mathrm{ind}(\mathcal{A}) = 0 \in K^1(X)$.

Melrose and Piazza also proved a $\mathbb{Z}_2$-graded analog of this result in [MP2, Proposition 2]. The proofs of these results in [MP1, MP2] depend crucially on the base space being compact.

Arbitrary base spaces. The aim of the first part of this paper is to generalize aforementioned results of Melrose and Piazza to arbitrary base spaces, not necessarily compact. In particular, we prove the following generalization of Proposition 1.1.

Theorem 7.1. *Let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a Riesz continuous family of regular self-adjoint operators with compact resolvents acting on fibers of a numerable Hilbert bundle over a topological space X. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a spectral section.*
2. *$\mathcal{A}$ has a generalized spectral section.*
3. *$\mathcal{A}$ is homotopic to a family of invertible operators.*

If P is a generalized spectral section for $\mathcal{A}$, then for every $\varepsilon > 0$ a spectral section Q for $\mathcal{A}$ can be chosen so that $\|Q - P\|_\infty < \varepsilon$ and Q is homotopic to P as a generalized spectral section.

We also prove a $\mathbb{Z}_2$-graded version of this result, see Theorems 6.6 and 7.2. It deals with $\mathrm{Cl}(1)$ spectral sections for odd self-adjoint operators and generalizes [MP2, Proposition 2] in the same manner as Theorem 7.1 generalizes [MP1, Proposition 1].

Correcting operators of finite rank. Melrose and Piazza showed in [MP1, Lemma 8] that if a self-adjoint family $\mathcal{A}$ over a compact base space has a spectral section P, then there is a correcting family $\mathcal{C}$ of self-adjoint finite rank operators such that the corrected family $\mathcal{A}' = \mathcal{A} + \mathcal{C}$ consists of invertible operators and P is the family of positive spectral projections for $\mathcal{A}'$. They also proved a $\mathbb{Z}_2$-graded analog in [MP2, Lemma 1]. We generalize both these results to arbitrary base spaces in Section 5.

Applications. In Part II we present a variety of applications illustrating how the results of Part I can be used. In particular, in Section 10 we generalize the famous Cobordism Theorem for Dirac operators to arbitrary, not necessarily compact, base spaces. We show in Theorem 10.5 that for an arbitrary family of Dirac type operators over a compact manifold with boundary, the family of symmetrized boundary operators has a spectral section. $\mathbb{Z}/2$-graded case of this result is given by Theorem 11.2. In Section 12 we consider the family of symmetrized tangential operators of a varying Dirac type operator along a varying hypersurface and show that such a family parametrized by pairs (operator, hypersurface) has a spectral section.

Graph continuous families. The Riesz topology on the set of regular operators is well suited for the theory of differential operators on closed manifolds. However, it is not quite adequate for differential operators on manifolds with boundary: it is unknown, except for several special cases, whether families of regular operators defined by boundary value problems are Riesz continuous. In the context of boundary value problems, the better suited topology is the *graph topology*.

The graph topology on $\mathcal{R}(H, H')$ is induced by the inclusion of $\mathcal{R}(H, H')$ to $\mathcal{P}(H \oplus H')$ taking a regular operator to the orthogonal projection onto its graph. A family of elliptic operators and boundary conditions with continuously varying coefficients leads to a graph continuous family of regular operators [P1, Appendix A.5].

As we saw above, a spectral section of a Riesz continuous family always exists locally. There is a topological obstruction for existence of a global spectral section, which in the case of a compact base space takes value in the K^1-group of the base. A *graph continuous* family, however, may admit *no spectral section even locally*. In fact, Riesz continuity is *necessary* for a local existence of a spectral section, as the following result shows.

Theorem 14.1. *Let X be an arbitrary topological space and $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(\mathcal{H})$ be a graph continuous map admitting a spectral section. Then $\mathcal{A}$ is Riesz continuous.*

The situation with *generalized* spectral sections for graph continuous families is more complicated, as we show in Section 15. On the one hand, a generalized spectral section does not necessarily exist even when a base space is an interval and/or a family consists of invertible operators. On the other hand, existence of a generalized spectral section does not imply Riesz continuity.

2 Preliminaries

In this section we recall some basic facts about regular operators (for more detailed exposition see, for example, [Le, BLP, Ka]) and introduce some designations that are used in the rest of the paper.

Regular operators. An unbounded operator A from H to H' is a linear operator defined on a subspace $\mathcal{D}$ of H and taking values in H'; the subspace $\mathcal{D}$ is called the domain of A and is denoted by $\mathrm{dom}(A)$. An unbounded operator A is called closed if its graph is closed in $H \oplus H'$ and densely defined if its domain is dense in H. It is called *regular* if it is closed and densely defined. Let $\mathcal{R}(H, H')$ denote the set of all regular operators from H to H', and let $\mathcal{R}(H) = \mathcal{R}(H, H)$.

The *adjoint operator* of an operator $A \in \mathcal{R}(H, H')$ is an unbounded operator A^* from H' to H with the domain

$$\mathrm{dom}(A^*) = \left\{ u \in H' \mid \text{ there exists } v \in H \text{ such that } \langle Aw, u\rangle = \langle w, v\rangle \text{ for all } w \in H \right\}.$$

For $u \in \mathrm{dom}(A^*)$ such an element v is unique and $A^*u = v$ by definition. The adjoint of a regular operator is itself a regular operator.

An operator $A \in \mathcal{R}(H)$ is called *self-adjoint* if $A^* = A$ (in particular, $\mathrm{dom}(A^*) = \mathrm{dom}(A)$). Let $\mathcal{R}^{sa}(H) \subset \mathcal{R}(H)$ be the subset of self-adjoint regular operators.

Operators with compact resolvents. For a regular operator $A \in \mathcal{R}(H, H')$, the operator $1 + A^*A$ is regular, self-adjoint, and has dense range. Its densely defined inverse $(1 + A^*A)^{-1}$ is bounded and hence can be extended to a bounded operator defined on the whole H. A regular operator $A \in \mathcal{R}(H, H')$ is said to *have compact resolvents* if $(1 + A^*A)^{-1} \in \mathcal{K}(H)$ and $(1 + AA^*)^{-1} \in \mathcal{K}(H')$. We denote by $\mathcal{R}_K(H)$ the subset of $\mathcal{R}(H)$ consisting of regular operators with compact resolvents.

Let $\mathcal{R}^{sa}_K(H) = \mathcal{R}^{sa}(H) \cap \mathcal{R}_K(H)$ be the subset of $\mathcal{R}(H)$ consisting of self-adjoint operators with compact resolvents. Equivalently, a self-adjoint regular operator A is an operator with compact resolvents if $(A + i)^{-1}$ is a compact operator. Such an operator has a discrete real spectrum.

Bounded transform. The *bounded transform* (or the Riesz map) $A \mapsto f(A) = A(1 + A^*A)^{-1/2}$ defines the inclusion of the set $\mathcal{R}(H, H')$ of regular operators to the unit ball in the space $\mathcal{B}(H, H')$ of bounded operators, with the image

$$f(\mathcal{R}(H, H')) = \left\{ a \in \mathcal{B}(H, H') \mid \|a\| \leqslant 1 \text{ and } 1 - a^*a \text{ is injective} \right\}.$$

The inverse map is given by the formula $f^{-1}(a) = a(1 - a^*a)^{-1/2}$.

If A is self-adjoint, then so is $f(A)$. If A has compact resolvents, then $a = f(A)$ is essentially unitary (that is, both $1 - a^*a$ and $1 - aa^*$ are compact operators).

Essentially unitary operators. Let $\mathcal{B}_{eu}(H, H')$ be the subspace of $\mathcal{B}(H, H')$ consisting of essentially unitary operators. The bounded transform takes $\mathcal{R}_K(H, H')$ to

$$f\left(\mathcal{R}_K(H, H')\right) = \left\{ a \in \mathcal{B}_{eu}(H, H') \mid \|a\| \leqslant 1 \text{ and } 1 - a^*a \text{ is injective} \right\}.$$

Let $\mathcal{B}^{sa}_{eu}(H)$ be the the subspace of $\mathcal{B}(H, H')$ consisting of self-adjoint essentially unitary operators.

In Section 3 and the first part of Section 6 we prove our results simultaneously both for regular operators with compact resolvents and for essentially unitary operators.

The notion of a generalized spectral section given in the Introduction works as well for self-adjoint essentially unitary operators. Equivalently, a projection P is a generalized spectral section for $a \in \mathcal{B}^{sa}_{eu}(H)$ if $(2P - 1) - a \in \mathcal{K}(H)$ (since $\mathbb{1}_{[0,+\infty)}(a)$ is a compact deformation of $(a + 1)/2$ for $a \in \mathcal{B}^{sa}_{eu}(H)$).

We call a projection P an r-spectral section for $a \in \mathcal{B}^{sa}_{eu}(H)$ and $r \in (0, 1)$ if $\mathbb{1}_{[r,+\infty)}(a) \leqslant P \leqslant \mathbb{1}_{(-r,+\infty)}(a)$. Let $r\colon X \to (0, 1)$ be a continuous function; we call a norm continuous family of projections $P = (P_x)_{x\in X}$ an r-spectral section (or simply a spectral section) for a family $a = (a_x)$ of self-adjoint essentially unitary operators if P_x is an r_x-spectral section for a_x for every $x \in X$.

A family $P = (P_x)$ of projections is a generalized spectral section, resp. r-spectral section for a family $\mathcal{A} = (\mathcal{A}_x)$ of self-adjoint regular operators with compact resolvents if and only if P is a generalized spectral section, resp. $(f \circ r)$-spectral section for the family $f \circ \mathcal{A}$ of self-adjoint essentially unitary operators.

Two topologies on $\mathcal{R}(H, H')$. There are several natural topologies on the set of regular operators. The two most useful of them are the Riesz topology and the graph topology. They are induced by the inclusions of $\mathcal{R}(H, H')$ to $\mathcal{B}(H, H')$ and to $\mathcal{P}(H \oplus H')$, respectively; see Introduction for details.

We will always specify what topology (Riesz or graph) on $\mathcal{R}(H, H')$ we consider. We will write ${}^{r}\mathcal{R}(H, H')$ or ${}^{g}\mathcal{R}(H, H')$ for the space of regular operators with Riesz or graph topology, respectively. Alternatively, we will write “Riesz continuous” or “graph continuous” for maps from or to $\mathcal{R}(H, H')$ and for families of regular operators.

On the subset $\mathcal{B}(H, H') \subset \mathcal{R}(H, H')$ both Riesz and graph topology coincide with the usual norm topology. Therefore, we always consider $\mathcal{B}(H, H')$ as equipped with the norm topology.

Spectral projections. The spectrum of an operator $A \in \mathcal{R}^{sa}_{K}(H)$ is discrete and real. For real numbers $a < b$, the spectral projection $\mathbb{1}_{[a,b]}(A)$ is defined as $\mathbb{1}_{[f(a),\, f(b)]}(f(A))$; its range is the subspace of H spanned by eigenvectors of A with eigenvalues in the interval $[a, b]$. Similarly, $\mathbb{1}_{[a,+\infty)}(A)$ is defined as $\mathbb{1}_{[f(a),1]}(f(A))$ and $\mathbb{1}_{(-\infty,a]}(A)$ is defined as $\mathbb{1}_{[-1,f(a)]}(f(A))$. The spectral projections for semi-open and open intervals are defined in the same manner.

Let $\mathrm{Res}(A)$ denote the resolvent set of A. For a compact subspace $K \subset \mathbb{R}$, the subset

$$V_K = \{A \in \mathcal{R}^{sa}_{K}(H) \mid K \subset \mathrm{Res}(A)\}$$

is open in both Riesz and graph topology on $\mathcal{R}^{sa}_{K}(H)$.

The map $V_{\{a,b\}} \to \mathcal{P}(H)$ taking A to $\mathbb{1}_{[a,b]}(A)$ is both Riesz-to-norm and graph-to-norm continuous. However, the spectral projection maps $V_a \to \mathcal{P}(H)$ corresponding to unbounded intervals, $A \mapsto \mathbb{1}_{(-\infty,a]}(A)$ and $A \mapsto \mathbb{1}_{[a,+\infty)}(A)$, are only Riesz-to-norm continuous, but not graph-to-norm continuous. This is the major difference between the two topologies in our context.

Part I

Riesz continuous families

Throughout this part, all families of regular operators are supposed to be Riesz continuous.

3 Generalized spectral sections

Homotopy Lifting Property. Recall that a continuous map $\mathcal{G} \to \mathcal{Z}$ is said to have the Homotopy Lifting Property for a space X if for every homotopy $h\colon X \times [0,1] \to \mathcal{Z}$, every lifting $\tilde{h}_0\colon X \times \{0\} \to \mathcal{G}$ of h_0 can be continued to a lifting $\tilde{h}\colon X \times [0,1] \to \mathcal{G}$ of h.

In this section the base space $\mathcal{Z}$ is either ${}^r\mathcal{R}^{sa}_K(H)$ or $\mathcal{B}^{sa}_{eu}(H)$. For the most part of the section, our reasoning works for both these cases simultaneously.

Let $\mathcal{Z}$ be one of these two spaces. Let I denote the range of cut-off parameters for $\mathcal{Z}$, that is, $I = \mathbb{R}_+$ if $\mathcal{Z} = \mathcal{R}^{sa}_K(H)$ and $I = (0,1)$ if $\mathcal{Z} = \mathcal{B}^{sa}_{eu}(H)$.

Let $\mathcal{G}$ denote the subspace of $\mathcal{Z} \times \mathcal{P}(H)$ consisting of pairs (A,P) such that P is a generalized spectral section for A. We consider $\mathcal{G}$ as the total space of a fiber bundle over $\mathcal{Z}$, with the projection $\pi_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{Z}$ taking (A,P) to A.

3.1 Theorem. *The fiber bundle $\pi_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{Z}$ is locally trivial and has the Homotopy Lifting Property for all spaces.*

Proof. The family $(\mathcal{Z}_r)_{r\in I}$, $\mathcal{Z}_r = \{A \in \mathcal{Z} \mid r \in \operatorname{Res}(A)\}$, is an open covering of $\mathcal{Z}$. For every $r \in I$, the map $S^+_r\colon \mathcal{Z}_r \to \mathcal{P}(H)$ defined by the formula $S^+_r(A) = \mathbb{1}_{[r,+\infty)}(A)$ is continuous. The restriction of $\mathcal{G}$ to $\mathcal{Z}_r$ is

$$(3.1) \qquad \mathcal{G}|_{\mathcal{Z}_r} = \left\{(A,P) \mid A \in \mathcal{Z}_r \text{ and } P - S^+_r(A) \in \mathcal{K}(H)\right\}.$$

Our next goal is to trivialize S^+_r locally. To this end, take an open covering $(V_Q)_{Q\in\mathcal{P}(H)}$ of $\mathcal{P}(H)$, where $V_Q = \{P \in \mathcal{P}(H) \mid \|P - Q\| < 1\}$. By [WO, Proposition 5.2.6] for every $Q \in \mathcal{P}(H)$ there is a continuous map $g_Q\colon V_Q \to \mathcal{U}(H)$ such that

$$(3.2) \qquad P = g_Q(P)\, Q\, (g_Q(P))^* \quad \text{for every } P \in V_Q.$$

The inverse images $\mathcal{Z}_{r,Q} = (S^+_r)^{-1}(V_Q) \subset \mathcal{Z}_r$, with r running I and Q running $\mathcal{P}(H)$, form an open covering $(\mathcal{Z}_{r,Q})$ of $\mathcal{Z}$. We claim that the restriction $\mathcal{G}_{r,Q}$ of $\mathcal{G}$ to $\mathcal{Z}_{r,Q}$ is a trivial bundle with the fiber

$$F_Q = \{P \in \mathcal{P}(H) \,|\, P - Q \in \mathcal{K}(H)\}.$$

Indeed, fix an arbitrary pair (r, Q) and consider the map $g = g_Q \circ S_r^+ \colon \mathcal{Z}_{r,Q} \to \mathcal{U}(H)$. It follows from (3.2) that $(g(A))^* S_r^+(A)\, g(A) = Q$ does not depend on $A \in \mathcal{Z}_{r,Q}$. Together with (3.1) this implies that the map

$$\Phi\colon \mathcal{G}_{r,Q} \to \mathcal{Z}_{r,Q} \times F_Q, \quad \Phi(A, P) = (A,\, g(A)^* P\, g(A))$$

is a trivializing bundle isomorphism, which proves the claim.

For both $\mathcal{Z} = {}^r\mathcal{R}^{sa}_K(H)$ and $\mathcal{Z} = \mathcal{B}^{sa}_{eu}(H)$ the topology of $\mathcal{Z}$ is induced by the embedding of $\mathcal{Z}$ to $\mathcal{B}(H)$, so $\mathcal{Z}$ is a metric space. This implies paracompactness of $\mathcal{Z}$ [St, Corollary 1]. Thus $\pi_\mathcal{G}$ is a locally trivial fiber bundle with a paracompact base space. By [Hu, Uniformization Theorem], $\pi_\mathcal{G}$ has the Homotopy Lifting Property for all spaces. This completes the proof of the Proposition. □

Generalized spectral sections. Using Theorem 3.1, we now can prove the following result.

3.2 Theorem. *Let X be an arbitrary topological space and $\mathcal{Z}$ be either ${}^r\mathcal{R}^{sa}_K(H)$ or $\mathcal{B}^{sa}_{eu}(H)$. Let $\mathcal{A}\colon X \to \mathcal{Z}$ be a continuous map. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a generalized spectral section.*
2. *$\mathcal{A}$ is homotopic (as a map from X to $\mathcal{Z}$) to a family of invertible operators.*

Proof. ($2 \Rightarrow 1$) Let $h\colon X \times [0,1] \to \mathcal{Z}$ be a homotopy between $\mathcal{A} = h_1$ and an invertible family h_0. Since h_0 is invertible, it has a spectral section $P_0(x) = \mathbb{1}_{[0,+\infty)}(h_0(x))$. Then the map $\tilde{h}_0\colon X \to \mathcal{G}$ given by the formula $\tilde{h}_0(x) = (h_0(x), P_0(x))$ covers h_0. By Theorem 3.1, $\tilde{h}_0$ can be continued to a map $\tilde{h}\colon X \times [0,1] \to \mathcal{G}$ covering h. Restriction of $\tilde{h}$ to $X \times \{1\}$ gives the map $\tilde{h}_1\colon X \to \mathcal{G}$ covering $\mathcal{A}$. The composition of $\tilde{h}_1$ with the natural projection $\mathcal{G} \to \mathcal{P}(H)$ is a generalized spectral section for A.

($1 \Rightarrow 2$) Let $P\colon X \to \mathcal{P}(H)$ be a generalized spectral section for $\mathcal{A}\colon X \to \mathcal{Z}$. Then $T\colon X \to \mathcal{B}(H)$, $T(x) = 2P(x) - 1$, is a continuous family of symmetries (that is, self-adjoint unitaries).

1. Consider first the case $\mathcal{Z} = \mathcal{B}^{sa}_{eu}(H)$. Then $\mathcal{A}(x) - T(x) \in \mathcal{K}(H)$ for every $x \in X$. Therefore,

$$h\colon X \times [0,1] \to \mathcal{B}^{sa}_{eu}(H), \quad h_t(x) = (1-t)\mathcal{A}(x) + tT(x)$$

is a homotopy from $h_0 = \mathcal{A}$ to $h_1 = T$, with $T(x)$ invertible for every $x \in X$.

2. Let now $\mathcal{Z} = {}^r\mathcal{R}^{sa}_K(H)$. The composition $a = f \circ \mathcal{A}\colon X \to \mathcal{B}^{sa}_{eu}(H)$ is continuous and $a(x) - T(x) \in \mathcal{K}(H)$ for every $x \in X$. One is tempted to apply f^{-1} to the linear homotopy between a and T, as above. But the image of f,

$$f\left(\mathcal{R}^{sa}_K(H)\right) = \{b \in \mathcal{B}^{sa}_{eu}(H) \,|\, \|b\| \leqslant 1 \text{ and } 1 - b^2 \text{ is injective}\}, \tag{3.3}$$

does not contain $T(x)$, so this naive idea does not work. To fix it, we replace T by its compact deformation T' lying in the image of f. Let us fix a strictly positive compact operator $K \in \mathcal{K}(H)$ of norm less than 1. For example, one can identify H with $l^2(\mathbb{N})$ and take the diagonal operator $K = \operatorname{diag}(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$. Then $T'(x) = (1-K)T(x)(1-K)$ is self-adjoint and invertible, $\|T'(x)\| \leqslant 1$, and $T'(x) - T(x) \in \mathcal{K}(H)$. Let $a_t(x) = (1-t)a(x) + tT'(x)$ be the linear homotopy between $a_0 = a$ and $a_1 = T'$. By definition, $a_0(x) = f(\mathcal{A}(x))$ lies in the image of f. For every $t \in (0,1]$, $x \in X$, and $\xi \in H \setminus \{0\}$ we get $\|a_t(x)\xi\| < \|\xi\|$, so $1 - a_t(x)^2$ is injective and thus $a_t(x)$ lies in the image of f. Applying f^{-1} to $a_t(x)$, we obtain a homotopy $h\colon X \times [0,1] \to \mathcal{R}^{sa}_K(H)$, $h_t(x) = f^{-1}(a_t(x))$, connecting $h_0 = \mathcal{A}$ with an invertible family $f^{-1}(T')$. This completes the proof of the theorem. □

4 Spectral sections

Fiber homotopy equivalence. Let $\mathcal{Z}$ be the space $\mathcal{R}^{sa}_K(H)$ equipped with the Riesz topology, and let $\mathcal{S}$ be the subspace of $\mathcal{Z} \times \mathcal{P}(H) \times \mathbb{R}_+$ consisting of triples (A, P, r) such that P is an r-spectral section for A. We consider $\mathcal{S}$ as the total space of a fiber bundle over $\mathcal{Z}$, with the projection $\pi_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{Z}$ taking (A, P, r) to A.

4.1 Theorem. *The bundle map $\iota\colon \mathcal{S} \to \mathcal{G}$ taking (A, P, r) to (A, P) is a fiber homotopy equivalence. Moreover, for every $\varepsilon > 0$, a fiber-homotopy inverse bundle map $\varphi = \varphi_\varepsilon\colon \mathcal{G} \to \mathcal{S}$ can be chosen so that $\|Q - P\| < \varepsilon$ for every $(A, P) \in \mathcal{G}$, $(A, Q) = \iota \circ \varphi(A, P)$.*

As an immediate corollary of this theorem we get the following result.

4.2 Corollary. *The fiber bundle $\pi_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{Z}$ has the Weak Homotopy Lifting Property for all spaces (that is, for any homotopy $h\colon X \times [0,1] \to \mathcal{Z}$ and for any lifting $\tilde{h}_0\colon X \times \{0\} \to \mathcal{S}$ of h_0, there is a lifting $X \times [0,1] \to \mathcal{S}$ of h whose restriction to $X \times \{0\}$ is vertically homotopic to $\tilde{h}_0$).*

Proof of Theorem 4.1. Let us fix $\delta \in (0, \frac{1}{2})$ such that $\delta \leqslant \varepsilon/2$.

Let (A, P) be an arbitrary element of $\mathcal{G}$. For $r \geqslant 0$, define the spectral projections

$$S^-_r(A) = \mathbb{1}_{(-\infty, r]}(A), \quad S^\circ_r(A) = \mathbb{1}_{(-r, r)}(A), \quad \text{and} \quad S^+_r(A) = \mathbb{1}_{[r, +\infty)}(A). \tag{4.1}$$

We approximate P by bounded self-adjoint operators

$$T_r(A, P) = S^+_r(A) + S^\circ_r(A)\, P\, S^\circ_r(A). \tag{4.2}$$

Thus defined map T_r is continuous on the open set $\{(A, P) \in \mathcal{G} \mid \pm r \in \operatorname{Res}(A)\}$. Equality (4.2) can be written equivalently as

$$T_r(A, P) = S^+_0(A) + S^\circ_r(A) \left(P - S^+_0(A)\right) S^\circ_r(A). \tag{4.3}$$

Consider the family

$$\mathcal{U} = (U_r)_{r>0}, \quad U_r = \{(A, P) \in \mathcal{G} \mid \pm r \in \operatorname{Res}(A) \text{ and } \|T_r(A, P) - P\| < \delta\} \tag{4.4}$$

of open subsets of $\mathcal{G}$. We claim that $\mathcal{U}$ is an open covering of $\mathcal{G}$. Indeed, let (A, P) be an arbitrary point of $\mathcal{G}$. Since P is a generalized spectral section for A, the difference $P - S_0^+(A)$ is a compact operator. The net $\{S_r^\circ(A)\}_{r>0}$ is an approximate unit for $\mathcal{K}(H)$. Therefore, the second summand in the right hand side of (4.3) has a limit $P - S_0^+(A)$, and $T_r(A, P) - P \to 0$ as $r \to +\infty$. It follows that (A, P) lies in U_r for r large enough, and thus $\mathcal{U}$ covers $\mathcal{G}$.

The topology of $\mathcal{Z} \times \mathcal{P}(H)$ is induced by its embedding to $\mathcal{B}(H) \times \mathcal{B}(H)$, so $\mathcal{Z} \times \mathcal{P}(H)$, as well as its subspace $\mathcal{G}$, is a metric space. It follows from [St, Corollary 1] that $\mathcal{G}$ is paracompact. Hence there is a partition of unity (u_i) subordinated to $\mathcal{U}$, with $\operatorname{supp}(u_i) \subset U_{r_i}$. (By a partition of unity we always mean *locally finite* partition.) For every $(A, P) \in \mathcal{G}$ we define a bounded self-adjoint operator $T(A, P)$ by the formula

$$T(A, P) = \sum u_i(A, P) \cdot T_{r_i}(A, P). \tag{4.5}$$

The restriction of T_r to U_r is continuous, so all the summands in (4.5) are continuous, and thus T itself is continuous as a map from $\mathcal{G}$ to $\mathcal{B}(H)$.

If $u_i(A, P) \neq 0$, then $(A, P) \in U_{r_i}$ and $\|T_{r_i}(A, P) - P\| < \delta$. Therefore,

$$\|T(A, P) - P\| < \delta \text{ for every } (A, P) \in \mathcal{G}. \tag{4.6}$$

The spectrum of P is $\{0, 1\}$, so the last inequality implies that the spectrum of $T(A, P)$ lies in $\Lambda = [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$. Since $\delta < \frac{1}{2}$, these two intervals are disjoint. The function $\mathbb{1}_{[\frac{1}{2}, +\infty)}$ is continuous (and even smooth) on Λ, so the projection

$$Q(A, P) = \mathbb{1}_{[\frac{1}{2}, +\infty)}(T(A, P)) \tag{4.7}$$

continuously depends on (A, P). Moreover, $\|Q(A, P) - T(A, P)\| \leqslant \delta$, which together with (4.6) provides an estimate $\|Q(A, P) - P\| < \varepsilon$ for every $(A, P) \in \mathcal{G}$.

Let $(A, P) \in \mathcal{G}$ and $r = \max\{r_i \,|\, u_i(A, P) \neq 0\}$. Then $T(A, P) = 0 \oplus T^\circ(A, P) \oplus 1$ and thus $Q(A, P) = 0 \oplus Q^\circ(A, P) \oplus 1$ with respect to the orthogonal decomposition

$$H = H^- \oplus H^\circ \oplus H^+, \quad \text{where } H^\alpha = \operatorname{Im}(S_r^\alpha(A)) \text{ for } \alpha \in \{+, -, \circ\}. \tag{4.8}$$

In other words, $Q(A, P)$ is an r-spectral section for A.

For every point $(A, P) \in \mathcal{G}$ choose its neighbourhood $V_{A,P} \subset \mathcal{G}$ intersecting only a finite number of inverse images $u_i^{-1}(0, 1]$. Let (v_j) be a partition of unity subordinated to the covering $\mathcal{V} = (V_{A,P})$ of $\mathcal{G}$. Let

$$R_j = \max\left\{r_i \,|\, u_i^{-1}(0, 1] \cap v_j^{-1}(0, 1] \neq \emptyset\right\}.$$

We define a continuous map $R \colon \mathcal{G} \to \mathbb{R}_+$ by the formula

$$R(A, P) = \sum v_j(A, P) \cdot R_j. \tag{4.9}$$

Then $R(A, P) \geqslant \max\{r_i \,|\, u_i(A, P) \neq 0\}$ for every $(A, P) \in \mathcal{G}$, so $Q(A, P)$ is an $R(A, P)$-spectral section for A.

Finally, we define the map $\varphi\colon \mathcal{G} \to \mathcal{S}$ by the formula $\varphi(A, P) = (A, Q(A, P), R(A, P))$.

It remains to show that φ is homotopy inverse to ι as a bundle map. To this end, consider the map $h\colon \mathcal{G} \times [0, 1] \to \mathcal{P}(H)$ given by the formula

$$h_t(A, P) = \mathbb{1}_{[\frac{1}{2}, +\infty)}\left(tP + (1 - t)T(A, P)\right). \tag{4.10}$$

Since $\|(tP + (1 - t)T(A, P)) - P\| \leqslant \|T(A, P) - P\| < \delta < 1/2$, the same argument as in the proof of continuity of (4.7) shows that h is continuous. Obviously, $h_0(A, P) = Q(A, P)$ and $h_1(A, P) = P$. Thus the formula $(A, P) \mapsto (A, h_t(A, P))$ defines a bundle homotopy between $\iota \circ \varphi$ and $\mathrm{Id}_{\mathcal{G}}$.

To construct a homotopy between $\varphi \circ \iota$ and $\mathrm{Id}_{\mathcal{S}}$, we use three auxiliary homotopies. We will write Q and R instead of $Q(A, P)$ and $R(A, P)$ for brevity. The first homotopy $h'_t(A, P, r) = (A, Q, R + tr)$ connects $\varphi \circ \iota(A, P, r) = (A, Q, R)$ with $(A, Q, R + r)$. The second one $h''_t(A, P, r) = (A, h_t(A, P), R + r)$ connects $(A, Q, R + r)$ with $(A, P, R + r)$. The third one $h'''_t(A, P, r) = (A, P, (1 - t)R + r)$ connects $(A, P, R + r)$ with (A, P, r). The concatenation of h', h'', and h''' is a desired bundle homotopy between $\varphi \circ \iota$ and $\mathrm{Id}_{\mathcal{S}}$. This completes the proof of the theorem. □

Spectral sections. Using Theorem 4.1, we immediately obtain the following result.

4.3 Theorem. *Let X be an arbitrary topological space and $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(H)$ be a Riesz continuous map. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a spectral section.*
2. *$\mathcal{A}$ has a generalized spectral section.*
3. *$\mathcal{A}$ is homotopic, via a Riesz continuous homotopy $X \times [0, 1] \to \mathcal{R}^{sa}_K(H)$, to a family of invertible operators.*

If P is a generalized spectral section for $\mathcal{A}$ and $\varepsilon > 0$, then a spectral section Q for $\mathcal{A}$ can be chosen so that $\|Q - P\|_\infty < \varepsilon$ and Q is homotopic to P as a generalized spectral section.

Proof. $(1 \Rightarrow 2)$ is trivial.

$(2 \Leftrightarrow 3)$ follows from Theorem 3.2.

$(2 \Rightarrow 1)$ and the last part of the theorem follow from Theorem 4.1. Indeed, if $\varepsilon > 0$ and P is a generalized spectral section for $\mathcal{A}$, then the map $x \mapsto \varphi_\varepsilon(\mathcal{A}_x, P_x) =: (\mathcal{A}_x, Q_x, r_x)$ defines an r-spectral section Q for $\mathcal{A}$. Moreover, $\|Q_x - P_x\| < \varepsilon$ for every $x \in X$. Since $\iota \circ \varphi_\varepsilon$ and $\mathrm{Id}_{\mathcal{G}}$ are vertically homotopic, Q and P are homotopic as generalized spectral sections. This completes the proof of the theorem. □

4.4 Remark. For a self-adjoint family $\mathcal{A}_x$ parametrized by points of a compact space X, Melrose and Piazza showed in [MP1, Proposition 2] that if the set of spectral sections for $\mathcal{A}$ is non-empty, then it contains "arbitrary small" and "arbitrary large" spectral sections, in the following sense: for every given $s \in \mathbb{R}$, there are spectral sections P and Q such that $P_x \leqslant S_x \leqslant Q_x$ for every $x \in X$, where $S_x = \mathbb{1}_{[s,+\infty)}(\mathcal{A}_x)$. This property is no longer true in the general case of a non-compact base space, as the following simple example shows.

4.5 Example. Fix $A \in \mathcal{R}^{sa}_K(H)$ and consider the family $\mathcal{A}_x = A + x$ parametrized by real numbers $x \in X = \mathbb{R}$. The family $\mathcal{A}$ admits a spectral section. Indeed, the constant map taking every $x \in X$ to $\mathbb{1}_{[0,+\infty)}(A)$ is an r-spectral section for $\mathcal{A}$, where $r\colon X \to \mathbb{R}_+$ is an arbitrary function satisfying condition $r(x) > |x|$.

Suppose now that A is unbounded from above. Then, for any given $s \in \mathbb{R}$, $\mathcal{A}$ has no spectral section P dominated by the family $S = (S_x)$ as above. Indeed, suppose that $P\colon X \to \mathcal{P}(H)$ is such a spectral section. Then $P_x \leqslant \mathbb{1}_{[s,+\infty)}(A + x) = \mathbb{1}_{[s-x,+\infty)}(A)$. For $x \leqslant s$, let P'_x be the restriction of P_x to the range H' of $\mathbb{1}_{[0,+\infty)}(A)$. Then (P'_x) is a continuous one-parameter family of projections in H' parametrized by points of the ray $(-\infty, s]$. The kernels of P'_x are finite-dimensional; by continuity, their dimensions should be independent of x. On the other hand, the dimension of $\operatorname{Ker}(P'_x)$ is bounded from below by the rank of $\mathbb{1}_{[0,s-x)}(A)$, which goes to infinity as $x \to -\infty$. This contradiction shows that such a spectral section P does not exist.

Similar argument shows that if A is unbounded from below, then $\mathcal{A}$ has no spectral section dominating the family S.

5 Correcting operators

Melrose and Piazza showed in [MP1, Lemma 8] that if a self-adjoint family $\mathcal{A}$ over a compact base space admits a spectral section P, then $\mathcal{A}$ admits a finite rank correction to an invertible family $\mathcal{A}'$ such that P is the family of positive spectral projections for $\mathcal{A}'$. They also proved a $\mathbb{Z}_2$-graded analog of this result in [MP2, Lemma 1]. Their proofs are based on the existence of "arbitrarily small" and "arbitrarily large" spectral sections in the sense of Remark 4.4. However, for a non-compact base space there may be no such spectral sections, as Example 4.5 shows. In this section we generalize [MP1, Lemma 8] and [MP2, Lemma 1] to arbitrary base spaces using a different method.

Correcting operators. It is convenient to have a special term for operators considered by Melrose and Piazza. Let A be a self-adjoint regular operator with compact resolvents. We say that a self-adjoint bounded operator C is an *r-correcting operator* (or simply a correcting operator) for A if the sum $A + C$ is invertible and the range of C lies in the range of $\mathbb{1}_{(-r,r)}(A)$. We also say that C is r-correcting the operator A.

Obviously, if C is an r-correcting operator for A, then

$$P = \mathbb{1}_{[0,+\infty)}(A + C) \tag{5.1}$$

is an r-spectral section for A. We say that a correcting operator C *agrees* with a spectral section P if (5.1) holds.

These notions are generalized to the family case in a natural way. We say that a norm continuous family $\mathcal{C}\colon X \to \mathcal{B}^{sa}(H)$ is *correcting* a family $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(H)$, or that $\mathcal{C}$ is a *correcting family* for $\mathcal{A}$, if there is a continuous function $r\colon X \to \mathbb{R}_+$ such that $\mathcal{C}_x$

is r_x-correcting the operator $\mathcal{A}_x$ for every $x \in X$. We also call such a family $\mathcal{C}$ an r-*correcting* family for $\mathcal{A}$. We say that a correcting family $\mathcal{C}$ *agrees* with a spectral section P if $P_x = \mathbb{1}_{[0,+\infty)}(\mathcal{A}_x + \mathcal{C}_x)$ for every $x \in X$.

Correcting operators and spectral sections. Let $\mathcal{D}$ be the subspace of ${}^r\mathcal{R}^{sa}_K(H) \times \mathcal{B}^{sa}(H) \times \mathbb{R}_+$ consisting of triples (A, C, r) such that C is an r-correcting operator for A.

Recall that $\mathcal{S}$ denotes the subspace of ${}^r\mathcal{R}^{sa}_K(H) \times \mathcal{P}(H) \times \mathbb{R}_+$ consisting of triples (A, P, r) such that P is an r-spectral section for A. There is a natural map

$$\pi\colon \mathcal{D} \to \mathcal{S}, \quad \pi(A, C, r) = (A, P, r), \text{ where } P = \mathbb{1}_{[0,+\infty)}(A + C). \tag{5.2}$$

The fibers of π considered as subsets of $\mathcal{B}^{sa}(H)$ are convex.

5.1 Proposition. *The map* $\pi\colon \mathcal{D} \to \mathcal{S}$ *is continuous.*

Proof. The map $S\colon \mathcal{D} \to \mathcal{P}(H)$ taking (A, C, r) to $\mathbb{1}_{(-r,r)}(A)$ is continuous on $\mathcal{D}_r = \{(A, C, r) \in \mathcal{D} \mid \pm r \in \operatorname{Res}(A)\}$. The projection $P = \mathbb{1}_{[0,+\infty)}(A + C)$ can be written as an orthogonal sum $\mathbb{1}_{[0,+\infty)}(SAS + C) + \mathbb{1}_{[r,+\infty)}(A)$. The operator $SAS + C$ is bounded and continuously depends on $(A, C, r) \in \mathcal{D}_r$. Therefore, P depends continuously on $(A, C, r) \in \mathcal{D}_r$. Since the open sets $\mathcal{D}_r$ cover $\mathcal{D}$, (5.2) is continuous. $\square$

5.2 Theorem. *The map* $\pi\colon \mathcal{D} \to \mathcal{S}$ *has a section. Such a section* $(A, P, r) \mapsto (A, \gamma(A, P, r), r)$ *can be chosen so that* $\|\gamma(A, P, r)\| < 2r$ *for every* $(A, P, r) \in \mathcal{S}$.

Proof. Let us fix a continuous function $\psi\colon \mathbb{R} \to [0, 1]$ which is equal to 1 on $(-\infty, 0]$ and to 0 on $[1, +\infty)$. Our construction of the map $\gamma\colon \mathcal{S} \to \mathcal{B}^{sa}(H)$ depends on the choice of such a function ψ.

Let $(A, P, r) \in \mathcal{S}$. We take $Q = 1 - P$, $A^+ = PAP$, and $A^- = QAQ$. The operators $r + A^+$ and $r - A^-$ are positive and invertible, and the sum $A^+ + A^-$ is orthogonal. The difference

$$C' = (A^+ + A^-) - A = -PAQ - QAP \tag{5.3}$$

is a self-adjoint operator of norm less than r with the range of C' lying in the range H° of $\mathbb{1}_{(-r,r)}(A)$. Let

$$C^+ = P\,\psi\left(r^{-1}A^+\right)P, \quad C^- = -Q\,\psi\left(-r^{-1}A^-\right)Q, \text{ and } C'' = r\left(C^+ + C^-\right). \tag{5.4}$$

The range of C'' lies in H° and $\|C''\| \leqslant r$, so the range of $C = C' + C''$ lies in H° and $\|C\| < 2r$.

We define γ by the formula $\gamma(A, P, r) = C = C' + C''$, where C' and C'' are defined by formulas 5.3 and 5.4.

The function $\Psi(t) = t + \psi(t)$ is strictly positive on $(-1, +\infty)$. The sum $A + C$ can be written as

$$A + C = (A^+ + A^-) + r\left(C^+ + C^-\right) = r\,P\,\Psi(r^{-1}A^+)P - r\,Q\,\Psi(-r^{-1}A^-)Q,$$

so it is invertible and $\mathbb{1}_{[0,+\infty)}(A+C) = P$.

It remains to show that γ is continuous. For $\lambda > 0$, let

$$\mathcal{S}_\lambda = \{(A, P, r) \in \mathcal{S} \mid \pm\lambda \in \operatorname{Res}(A) \text{ and } r < \lambda\}.$$

The projection $S = \mathbb{1}_{(-\lambda,\lambda)}(A)$ and the cut-off operator $B = SAS = f^{-1}(Sf(A)S) \in \mathcal{B}^{sa}(\mathcal{H})$ continuously depend on $(A, P, r) \in \mathcal{S}_\lambda$. The projections S and P commute, so $C' = SC'S = PBQ + QBP$, and thus the restriction of C' to $\mathcal{S}_\lambda$ is continuous. The restriction of C'' to $\mathcal{S}_\lambda$ can be written in a similar manner:

$$C'' = r\,S\left[P\psi(r^{-1}PBP)P - Q\psi(-r^{-1}QBQ)Q\right]S,$$

which provides continuity of C'' on $\mathcal{S}_\lambda$. Open sets $\mathcal{S}_\lambda$ cover $\mathcal{S}$ when λ runs $\mathbb{R}_+$, so both C' and C'' are continuous on the whole $\mathcal{S}$. Therefore, the sum $\gamma = C' + C''$ is also continuous. This completes the proof of the first two statements of the theorem. □

5.3 Theorem. *Let X be an arbitrary topological space and $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(H)$ be a Riesz continuous map. Suppose that $\mathcal{A}$ admits an r-spectral section P. Then $\mathcal{A}$ admits an r-correcting family $\mathcal{C}\colon X \to \mathcal{B}^{sa}(H)$ that agrees with P and satisfies $\|\mathcal{C}_x\| < 2r_x$.*

Proof. Let $\gamma\colon \mathcal{S} \to \mathcal{B}^{sa}(H)$ be a map satisfying conditions of Theorem 5.2. Then the formula $\mathcal{C}_x = \gamma(\mathcal{A}_x, P_x, r_x)$ determines an r-correcting family for $\mathcal{A}$ satisfying conditions of the theorem. □

5.4 Theorem. *Let P be a spectral section for a Riesz continuous map $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(H)$. Then the space of all correcting families for $\mathcal{A}$ that agree with P is convex and therefore contractible.*

Proof. Let $\mathcal{C}_i$ be an r_i-correcting family for $\mathcal{A}$, $i = 1, 2$. Then both $\mathcal{C}_1$ and $\mathcal{C}_2$ are r-correcting families for $\mathcal{A}$, where $r = \max(r_1, r_2)$. Let $\mathcal{C} = t\mathcal{C}_1 + (1-t)\mathcal{C}_2$, $t \in [0, 1]$. Then the range of $\mathcal{C}(x)$ lies in the range of $\mathbb{1}_{(-r_x,r_x)}(\mathcal{A}(x))$. The sum $\mathcal{A} + \mathcal{C} = t(\mathcal{A} + \mathcal{C}_1) + (1-t)(\mathcal{A} + \mathcal{C}_2)$ commutes with P since both $\mathcal{A} + \mathcal{C}_1$ and $\mathcal{A} + \mathcal{C}_2$ commute with P. The operator $\mathcal{A}(x) + \mathcal{C}(x)$ is strictly positive on the range of $P(x)$ and strictly negative on the kernel of $P(x)$. Therefore, $\mathcal{C}$ is an r-correcting family for $\mathcal{A}$. This completes the proof of the proposition. □

6 $\mathbb{Z}_2$-graded case

Throughout this section $H = H^0 \oplus H^1$ will be a $\mathbb{Z}_2$-graded Hilbert space. Let $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathcal{B}(H)$ be the symmetry defining the grading. We denote by $\mathcal{R}^1_K(H)$ the subset of $\mathcal{R}^{sa}_K(H)$ consisting of odd operators, that is, operators anticommuting with σ. Similarly, we denote by $\mathcal{B}^1(H)$ the subspace of $\mathcal{B}^{sa}(H)$ consisting of odd operators.

Spectral sections for odd operators. The natural inclusion $\mathcal{R}^1_K(H) \hookrightarrow \mathcal{R}^{sa}_K(H)$ admits a spectral section $P\colon \mathcal{R}^1_K(H) \to \mathcal{P}(H)$. Indeed, fix a continuous even function $\psi\colon \mathbb{R} \to \mathbb{R}$ supported on $[-r, r]$ which does not vanish at zero. For every $A \in \mathcal{R}^1_K(H)$, the self-adjoint finite rank operator $C_A = \sigma\,\psi(A)$ is r-correcting the operator A, since

$(A + C_A)^2 = A^2 + \psi(A)^2$ is invertible. Therefore, $P_A = \mathbb{1}_{[0,+\infty)}(A + C_A)$ is an r-spectral section for A. Moreover, the maps $A \mapsto C_A$ and $A \mapsto P_A$ are continuous on $\mathcal{R}^1_K(H)$ and thus determine an r-correcting family and an r-spectral section for the inclusion $\mathcal{R}^1_K(H) \hookrightarrow \mathcal{R}^{sa}_K(H)$.

It follows that, by a trivial reason, *every* family of *odd* self-adjoint operators with compact resolvents has a spectral section. Hence the notion of a spectral section is not very relevant for such operators. Instead, one should consider spectral sections behaving well with respect to the grading. Such a notion of a Cl(1) spectral section was introduced by Melrose and Piazza in [MP2].

Cl(1) **spectral sections.** A norm continuous family of projections $P\colon X \to \mathcal{P}(H)$ is called a Cl(1) spectral section for a family $\mathcal{A}\colon X \to \mathcal{R}^1_K(H)$ if P is a spectral section for $\mathcal{A}$ and satisfies additionally the anticommutation property

$$\sigma P \sigma^{-1} = 1 - P. \tag{6.1}$$

Let $\mathcal{P}^1(H)$ denote the subspace of $\mathcal{P}(H)$ consisting of projections P satisfying (6.1). Equivalently, a projection P lies in $\mathcal{P}^1(H)$ if the symmetry $2P - 1$ anticommutes with σ.

$\mathcal{P}^1(H)$ is naturally homeomorphic to the space of unitary operators $\mathcal{U}(H_0, H_1)$; the corresponding homeomorphism

$$\nu\colon\ \mathcal{P}^1(H) \to \mathcal{U}(H_0, H_1) \tag{6.2}$$

takes $P \in \mathcal{P}^1(H)$ to $\nu \in \mathcal{U}(H_0, H_1)$ such that $2P - 1 = \left(\begin{smallmatrix} 0 & \nu^* \\ \nu & 0 \end{smallmatrix}\right)$.

Generalized Cl(1) **spectral sections.** Again, we consider two cases in parallel:

1. $\mathcal{Z} = {}^r\mathcal{R}^{sa}_K(H)$, $\mathcal{Z}' = {}^r\mathcal{R}_K(H^0, H^1)$, and $I = \mathbb{R}_+$;
2. $\mathcal{Z} = \mathcal{B}^{sa}_{eu}(H)$, $\mathcal{Z}' = \mathcal{B}_{eu}(H^0, H^1)$, and $I = (0, 1)$.

Let $\mathcal{Z}^1$ denote the subspace of $\mathcal{Z}$ consisting of odd operators. The formula

$$A \mapsto \hat{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \tag{6.3}$$

defines a natural homeomorphism $\mathcal{Z}' \to \mathcal{Z}^1$.

Both $\mathcal{Z}'$ and $\mathcal{Z}^1$ are empty if one of H^i is finite-dimensional (recall that H itself is infinite-dimensional). So we will always suppose that both H^0 and H^1 are infinite-dimensional.

We define a *generalized* Cl(1) *spectral section* for $\hat{\mathcal{A}}\colon X \to \mathcal{Z}^1$ as a generalized spectral section P for $\hat{\mathcal{A}}$ satisfying (6.1). Equivalently, a norm continuous map $P\colon X \to \mathcal{P}^1(H)$ is a generalized Cl(1) spectral section for $\hat{\mathcal{A}}$ if a_x is a compact deformation of the unitary $\nu(P_x)$ for every $x \in X$, where $a_x = A_x$ for $\mathcal{Z} = \mathcal{B}^{sa}_{eu}(H)$ and $a_x = f(A_x)$ for $\mathcal{Z} = \mathcal{R}^{sa}_K(H)$.

An element $a \in \mathcal{B}_{eu}(H^0, H^1)$ is a compact deformation of a unitary if and only if the index of a vanishes. For $A \in \mathcal{R}_K(H^0, H^1)$ the indices of A and f(A) coincide. Therefore, the index of $A \in \mathcal{Z}'$ vanishes if and only if the operator $\hat{A}$ given by formula (6.3) has a generalized $\mathrm{Cl}(1)$ spectral section. We denote by $\bar{\mathcal{Z}}'$ the subspace of $\mathcal{Z}'$ consisting of operators with vanishing index and by $\bar{\mathcal{Z}}^1$ the subspace of $\mathcal{Z}^1$ consisting of operators admitting a generalized $\mathrm{Cl}(1)$ spectral section.

Since the index is locally constant on $\mathcal{B}_{eu}(H^0, H^1)$ and thus also on ${}^r\mathcal{R}_K(H^0, H^1)$, $\bar{\mathcal{Z}}'$ is an open and closed subspace of $\mathcal{Z}'$ (in fact, it is a connected component of $\mathcal{Z}'$, but we do not use its connectedness). Homeomorphism (6.3) takes $\bar{\mathcal{Z}}'$ to $\bar{\mathcal{Z}}^1$, so $\bar{\mathcal{Z}}^1$ is an open and closed subspace of $\mathcal{Z}^1$.

6.1 Proposition. *For every $A \in \mathcal{Z}'$, the following conditions are equivalent:*

1. *The index of A vanishes.*
2. *The signature of the restriction of σ to the kernel of $\hat{A}$ vanishes.*
3. *$\hat{A}$ has a $\mathrm{Cl}(1)$ spectral section.*
4. *$\hat{A}$ has a generalized $\mathrm{Cl}(1)$ spectral section.*

Proof. $(3 \Rightarrow 4)$ is trivial.

$(1 \Leftrightarrow 4)$ is explained above.

$(1 \Leftrightarrow 2)$ follows from the equality

$$\begin{aligned}\operatorname{sign}\left(\sigma|_{\operatorname{Ker}\hat{A}}\right) &= \dim\left(H^0 \cap \operatorname{Ker}\hat{A}\right) - \dim\left(H^1 \cap \operatorname{Ker}\hat{A}\right) \\ &= \dim(\operatorname{Ker} A) - \dim(\operatorname{Ker} A^*) = \operatorname{ind}(A).\end{aligned}$$

$(1 \Rightarrow 3)$ If $\dim\operatorname{Ker} A = \dim\operatorname{Ker} A^*$, then there is a unitary $\nu \in \mathcal{U}(\operatorname{Ker} A, \operatorname{Ker} A^*)$. Let $S \in \mathcal{P}(H)$ be the orthogonal projection onto the kernel of $\hat{A}$ and $P_0 = \left(\begin{smallmatrix} 0 & \nu^* \\ \nu & 0 \end{smallmatrix}\right) \in \mathcal{P}(\operatorname{Ker}\hat{A})$. Then $P = P_0 S + \mathbb{1}_{(0,+\infty)}(\hat{A})$ is a $\mathrm{Cl}(1)$ spectral section for $\hat{A}$. □

6.2 Proposition. *For every $A \in \mathcal{Z}^1$ and $r \in I$, the signatures of the restrictions of σ to $\operatorname{Ker}(A)$ and to the range of $\mathbb{1}_{(-r,r)}(A)$ coincide.*

Proof. The range V of $\mathbb{1}_{(-r,r)}(A)$ can be decomposed into the orthogonal sum $V = V^- \oplus \operatorname{Ker}(A) \oplus V^+$ corresponding to the decomposition of the interval $(-r, r) = (-r, 0) \cup \{0\} \cup (0, r)$. Since σ anticommutes with A, σ takes V^- to V^+ and vice versa. Therefore, the signature of the restriction of σ to $V^- \oplus V^+$ vanishes, and so the signatures of $\sigma|_V$ and $\sigma|_{\operatorname{Ker}(A)}$ coincide. □

Homotopy Lifting Property. Let $\mathcal{G}^1$ denote the subspace of $\mathcal{Z}^1 \times \mathcal{P}^1(H)$ consisting of pairs (A, P) such that P is a generalized spectral section for A. Let

$$\mathcal{U}_K(H^0) = \{u \in \mathcal{U}(H^0) \mid u - 1 \in \mathcal{K}(H^0)\}$$

be the group of unitaries which are compact deformation of the identity.

6.3 Theorem. *The natural projection $\mathcal{G}^1 \to \bar{\mathcal{Z}}^1$ is a locally trivial principal $\mathcal{U}_K(H^0)$-bundle. Both $\mathcal{G}^1 \to \bar{\mathcal{Z}}^1$ and $\mathcal{G}^1 \to \mathcal{Z}^1$ have the Homotopy Lifting Property for all spaces.*

Proof. 1. We define the action μ of $\mathcal{U}_K(H^0)$ on the product $\bar{\mathcal{Z}}^1 \times \mathcal{P}^1(H)$ by the formula

$$\mu_u(A, P) = \big(A, \nu^{-1}(\nu(P)u)\big), \quad u \in \mathcal{U}_K(H^0),$$

where $\nu\colon \mathcal{P}^1(H) \to \mathcal{U}(H_0, H_1)$ is homeomorphism (6.2). For every $P, P' \in \mathcal{P}^1(H)$, $P - P' \in \mathcal{K}(H)$ is equivalent to $\nu(P) - \nu(P') \in \mathcal{K}(H^0, H^1)$, which in turn is equivalent to $\nu(P')^{-1}\nu(P) - 1 \in \mathcal{K}(H^0)$. Therefore, $\mathcal{G}^1$ is fixed by the action of μ, and μ acts transitively on the fibers of $\mathcal{G}^1 \to \bar{\mathcal{Z}}^1$. Obviously, this action is free. It follows that $\mathcal{G}^1 \to \bar{\mathcal{Z}}^1$ is a principal $\mathcal{U}_K(H^0)$-bundle.

2. The next step of the proof is local triviality of $\mathcal{G}^1 \to \bar{\mathcal{Z}}^1$. Since this bundle is principal, it is sufficient to show that it allows a local section over a neighbourhood of an arbitrary point $A_0 \in \bar{\mathcal{Z}}^1$. Fix $r > 0$ such that $\pm r \in \operatorname{Res}(A_0)$. Let $S_r^\circ(A)$ and $S_r^+(A)$ be the projections defined by formulae (4.1). They are continuous on the neighbourhood $V_r = \big\{A \in \bar{\mathcal{Z}}^1 \,|\, \pm r \in \operatorname{Res}(A)\big\}$ of A_0. The projections $\frac{1\pm\sigma}{2} S_r^\circ(A)$ are also continuous on V_r, so their ranges are locally trivial vector bundles over V_r. Let $V \subset V_r$ be a neighbourhood of A_0 over which these two vector bundles are trivial; denote their restrictions to V by E^+ and E^-. The ranks of E^+ and E^- are equal to the dimensions of their fibers over A_0; since $A_0 \in \bar{\mathcal{Z}}^1$, these ranks coincide. Choose a unitary bundle isomorphism $\nu\colon E^+ \to E^-$. Let $T = \left(\begin{smallmatrix} 0 & \nu^* \\ \nu & 0 \end{smallmatrix}\right)$ be the corresponding odd symmetry and $P = (T+1)/2 \in \mathcal{P}(E^+ \oplus E^-)$ the bundle projection. Then the formula $A \mapsto (A,\, P(A)S_r^\circ(A) + S_r^+(A))$ defines a section of $\mathcal{G}^1$ over V. This completes the proof of the second step.

3. The same reasoning as in the end of the proof of Theorem 3.1 shows that a locally trivial bundle $\mathcal{G}^1 \to \bar{\mathcal{Z}}^1$ has the Homotopy Lifting Property for all spaces. Since $\bar{\mathcal{Z}}^1$ is closed and open in $\mathcal{Z}^1$, the same is true for $\mathcal{G}^1 \to \mathcal{Z}^1$. This completes the proof of the theorem. □

6.4 Theorem. *Let* X *be an arbitrary topological space and* $\mathcal{Z}$ *be either* ${}^r\mathcal{R}_K^{sa}(H)$ *or* $\mathcal{B}_{eu}^{sa}(H)$. *Let* $\mathcal{A}\colon X \to \mathcal{Z}^1$ *be a continuous map. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a generalized* Cl(1) *spectral section.*
2. *$\mathcal{A}$ is homotopic, as a map from* X *to* $\mathcal{Z}^1$*, to a family of invertible operators.*

Proof. The proof reproduces completely the proof of Theorem 3.2, with (generalized) spectral sections replaced by (generalized) Cl(1) spectral sections and Theorem 6.3 used instead of Theorem 3.1. The only additional care is needed for the choice of a compact operator K: it should commute with σ. Then $T'(x)$ is odd and the homotopy h consists of odd operators. □

Fiber homotopy equivalence. From now on till the end of the section $\mathcal{Z}$ denotes the space $\mathcal{R}_K^1(H)$ equipped with the Riesz topology.

6.5 Theorem. *The bundle map* $\iota\colon \mathcal{S}^1 \to \mathcal{G}^1$ *taking* (A, P, r) *to* (A, P) *is a fiber homotopy equivalence. Moreover, for every* $\varepsilon > 0$*, a fiber-homotopy inverse bundle map* $\varphi = \varphi_\varepsilon\colon \mathcal{G}^1 \to \mathcal{S}^1$ *can be chosen so that* $\|Q - P\| < \varepsilon$ *for every* $(A, P) \in \mathcal{G}^1$ *and* $(A, Q) = \iota \circ \varphi(A, P)$.

Proof. We will show that the bundle map φ from the proof of Theorem 4.1 maps the subspace $\mathcal{G}^1$ of $\mathcal{G}$ to the subspace $\mathcal{S}^1$ of $\mathcal{S}$, and that the restriction of φ to $\mathcal{G}^1$ is homotopy inverse, as a bundle map, to the restriction of ι to $\mathcal{S}^1$.

We use the designations from the proof of Theorem 4.1. It will be convenient to use the following convention: if B is a self-adjoint operator, then we write $\tilde{B}$ as an abbreviation for $2B-1$. We will also use the "sign function"

$$\rho\colon \mathbb{R}\setminus\{0\} \to \{-1,1\}, \quad \rho = \mathbb{1}_{(0,+\infty)} - \mathbb{1}_{(-\infty,0)}\,.$$

Let (A,P) be an arbitrary element of $\mathcal{G}^1$. Equality (4.2) can be written equivalently as

$$\tilde{T}_r(A,P) = (S_r^+(A) - S_r^-(A)) + S_r^\circ(A)\,\tilde{P}\,S_r^\circ(A).$$

Since A and $\tilde{P}$ are odd, $\tilde{T}_r(A,P)$ is also odd. It follows that $\tilde{T}(A,P)$ and $\tilde{Q}(A,P) = \rho\left(\tilde{T}(A,P)\right)$ are odd. Therefore, φ takes $\mathcal{G}^1$ to $\mathcal{S}^1$.

Equality (4.10) can be written equivalently as $\tilde{h}_t(A,P) = \rho\left(t\tilde{P} + (1-t)\tilde{T}(A,P)\right)$. If $\tilde{P}$ and $\tilde{T}$ are odd, then $\tilde{h}_t(A,P)$ is also odd. It follows that h maps $\mathcal{G}^1\times[0,1]$ to $\mathcal{P}^1(H)$. The reasoning in the rest of the proof of Theorem 4.1 shows that the restriction of φ to $\mathcal{G}^1$ is homotopy inverse, as a bundle map, to the restriction of ι to $\mathcal{S}^1$. This completes the proof of the theorem. □

6.6 Theorem. *Let X be an arbitrary topological space and $\mathcal{A}\colon X \to \mathcal{R}^1_K(H)$ be a Riesz continuous map. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a $\mathrm{Cl}(1)$ spectral section.*
2. *$\mathcal{A}$ has a generalized $\mathrm{Cl}(1)$ spectral section.*
3. *$\mathcal{A}$ is homotopic, via a Riesz continuous homotopy $X\times[0,1] \to \mathcal{R}^1_K(H)$, to a family of invertible operators.*

If P is a generalized $\mathrm{Cl}(1)$ spectral section for $\mathcal{A}$ and $\varepsilon > 0$, then a $\mathrm{Cl}(1)$ spectral section Q for $\mathcal{A}$ can be chosen so that $\|Q-P\|_\infty < \varepsilon$ and Q is homotopic to P as a generalized $\mathrm{Cl}(1)$ spectral section.

Proof. ($1 \Rightarrow 2$) is trivial.

($2 \Leftrightarrow 3$) follows from Theorem 6.4.

($2 \Rightarrow 1$) and the last part of the theorem follow from Theorem 6.5, in the same manner as in the proof of Theorem 4.3. □

Correcting operators. Recall that in the previous section we denoted by $\mathcal{D}$ the subspace of ${}^r\mathcal{R}^{sa}_K(H)\times\mathcal{B}^{sa}(H)\times\mathbb{R}_+$ consisting of triples (A,C,r) such that C is an r-correcting operator for A. Let $\mathcal{D}^1$ be the subspace of $\mathcal{D}$ consisting of triples (A,C,r) with A and C *odd* operators. In other words, $\mathcal{D}^1$ is the subspace of ${}^r\mathcal{R}^1_K(H)\times\mathcal{B}^1(H)\times\mathbb{R}_+$ consisting of triples (A,C,r) such that C is an r-correcting operator for A.

If $(A,C,r)\in\mathcal{D}^1$, then $P = \mathbb{1}_{[0,+\infty)}(A+C)$ is a $\mathrm{Cl}(1)$ spectral section for A. Therefore, the restriction of $\pi\colon\mathcal{D}\to\mathcal{S}$ to $\mathcal{D}^1$ defines a natural projection $\pi^1\colon\mathcal{D}^1\to\mathcal{S}^1$.

6.7 Theorem. *The map $\pi^1\colon\mathcal{D}^1\to\mathcal{S}^1$ has a section $(A,P,r)\mapsto(A,\gamma(A,P,r),r)$. It can be chosen so that $\|\gamma(A,P,r)\| < 2r$ for every $(A,P,r)\in\mathcal{S}$.*

Proof. Let $\gamma\colon \mathcal{S} \to \mathcal{B}^{sa}(H)$ be the map constructed in the proof of Theorem 5.2. Then γ takes $\mathcal{S}^1$ to $\mathcal{P}^1(H)$ and thus defines a section satisfying conditions of the theorem. Indeed, if A is odd and P is a $\mathrm{Cl}(1)$ spectral section for A, then the conjugation by σ takes A^+ to $-A^-$ and vice versa. Therefore, both C' and C'' are odd, and thus $C = C' + C''$ is also odd for every $(A, P, r) \in \mathcal{S}^1$. $\square$

6.8 Theorem. *Let X be an arbitrary topological space and $\mathcal{A}\colon X \to \mathcal{R}^1_K(H)$ be a Riesz continuous map. Suppose that $\mathcal{A}$ admits a $\mathrm{Cl}(1)$ spectral section P with a cut-off function r. Then $\mathcal{A}$ admits an odd r-correcting family $\mathcal{C}\colon X \to \mathcal{B}^1(H)$ that agrees with P and satisfies $\|\mathcal{C}_x\| < 2r_x$.*

Proof. Let $\gamma\colon \mathcal{S} \to \mathcal{B}^{sa}(H)$ be a map satisfying conditions of Theorem 6.7. Then the formula $\mathcal{C}_x = \gamma(\mathcal{A}_x, P_x, r_x)$ determines an odd r-correcting family for $\mathcal{A}$ satisfying conditions of the theorem. $\square$

6.9 Theorem. *Let P be a $\mathrm{Cl}(1)$ spectral section for a Riesz continuous map $\mathcal{A}\colon X \to \mathcal{R}^1_K(H)$. Then the space of all odd correcting families for $\mathcal{A}$ that agree with P is convex and therefore contractible.*

Proof. This space is the intersection, inside the vector space $C(X, \mathcal{B}^{sa}(H))$, of the vector subspace $C(X, \mathcal{B}^1(H))$ with the subset of all correcting families for $\mathcal{A}$. The last subset is convex by Theorem 5.4, so their intersection is also convex. $\square$

7 Hilbert bundles

Let $\mathcal{H} \to X$ be a Hilbert bundle over X (that is, a locally trivial fiber bundle over X with the fiber a separable Hilbert space H and the structure group $\mathcal{U}(H)$).

If a base space X is paracompact, then $\mathcal{H}$ is trivial. More generally, every numerable Hilbert bundle is trivial. (Recall that a fiber bundle is called numerable if it admits a local trivialization over a numerable open cover of the base space, that is, an open cover admitting a subordinate partition of unity.) Indeed, for every contractible group G, a principal G-bundle over a point is a universal G-bundle [Do, Theorem 7.5]. In particular, every numerable principal G-bundle E is trivial, and so every fiber bundle associated with E is trivial. By the Kuiper theorem [Ku], the unitary group $\mathcal{U}(H)$ is contractible. Therefore, every numerable Hilbert bundle is trivial.

This allows to reformulate results of the previous sections in terms of families of operators acting between fibers of Hilbert bundles.

A map $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(H)$ is now replaced by a family $\mathcal{A} = (\mathcal{A}_x)_{x\in X}$ such that $\mathcal{A}_x \in \mathcal{R}^{sa}_K(\mathcal{H}_x)$. A (generalized) spectral section for such a family $\mathcal{A}$ is a norm continuous family $P = (P_x)_{x\in X}$ of projections $P_x \in \mathcal{P}(\mathcal{H}_x)$ such that P_x is a (generalized) spectral section for $\mathcal{A}_x$ for every $x \in X$. All the other notions are carried over to the Hilbert bundle case in a similar manner.

7.1 Theorem. *Let $\mathcal{A} = (\mathcal{A}_x)_{x\in X}$ be a Riesz continuous family of regular self-adjoint operators with compact resolvents acting on fibers of a numerable Hilbert bundle $\mathcal{H}$ over a topological space X. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a spectral section.*
2. *$\mathcal{A}$ has a generalized spectral section.*
3. *$\mathcal{A}$ is homotopic to a family of invertible operators.*

If P is a generalized spectral section for $\mathcal{A}$ and $\varepsilon > 0$, then a spectral section Q for $\mathcal{A}$ can be chosen so that $\|Q - P\|_\infty < \varepsilon$ and Q is homotopic to P as a generalized spectral section.

Proof. This follows immediately from Theorem 4.3 and triviality of $\mathcal{H}$. □

A $\mathbb{Z}_2$-graded Hilbert bundle $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ is a locally trivial fiber bundle with the fiber a $\mathbb{Z}_2$-graded Hilbert space $H = H_0 \oplus H_1$ and the structure group $\mathcal{U}(H_0) \oplus \mathcal{U}(H_1)$. Equivalently, a $\mathbb{Z}_2$-graded Hilbert bundle can be defined as a Hilbert bundle $\mathcal{H}$ equipped with a continuous family $\sigma = (\sigma_x)$ of gradings on the fibers $\mathcal{H}_x$ of $\mathcal{H}$. $\mathrm{Cl}(1)$ spectral sections and generalized $\mathrm{Cl}(1)$ spectral sections for families of odd self-adjoint operators acting on sections of $\mathcal{H}$ are defined as in Section 6.

7.2 Theorem. *Let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a Riesz continuous family of regular self-adjoint odd operators with compact resolvents acting on fibers of a $\mathbb{Z}_2$-graded numerable Hilbert bundle $\mathcal{H}$ over a topological space X. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a $\mathrm{Cl}(1)$ spectral section.*
2. *$\mathcal{A}$ has a generalized $\mathrm{Cl}(1)$ spectral section.*
3. *$\mathcal{A}$ is homotopic to a family of invertible operators.*

If P is a generalized $\mathrm{Cl}(1)$ spectral section for $\mathcal{A}$ and $\varepsilon > 0$, then a $\mathrm{Cl}(1)$ spectral section Q for $\mathcal{A}$ can be chosen so that $\|Q - P\|_\infty < \varepsilon$ and Q is homotopic to P as a generalized $\mathrm{Cl}(1)$ spectral section.

Proof. As before, we can suppose that both $\mathcal{H}^0$ and $\mathcal{H}^1$ have infinite rank. Then they both are trivial, and the statement of the theorem follows from Theorem 6.6. □

Correcting operators. Theorems 5.3 and 6.8 can be formulated in terms of Hilbert bundles as follows.

7.3 Theorem. *Let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a Riesz continuous family of regular self-adjoint operators with compact resolvents acting on fibers of a numerable Hilbert bundle $\mathcal{H}$ over a topological space X. Suppose that $\mathcal{A}$ admits an r-spectral section P. Then $\mathcal{A}$ admits an r-correcting family $\mathcal{C} = (\mathcal{C}_x)$ that agrees with P and satisfies $\|\mathcal{C}_x\| < 2r_x$.*

7.4 Theorem. *Let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a Riesz continuous family of regular self-adjoint odd operators with compact resolvents acting on fibers of a $\mathbb{Z}_2$-graded numerable Hilbert bundle $\mathcal{H}$ over a topological space X. Suppose that $\mathcal{A}$ admits a $\mathrm{Cl}(1)$ spectral section P with a cut-off function r. Then $\mathcal{A}$ admits an odd r-correcting family $\mathcal{C} = (\mathcal{C}_x)$ that agrees with P and satisfies $\|\mathcal{C}_x\| < 2r_x$.*

Non-self-adjoint operators. Passing from odd self-adjoint operators to their chiral components, we obtain the following result.

7.5 Theorem. *Let $\mathcal{H}^0$ and $\mathcal{H}^1$ be numerable Hilbert bundles over a topological space X, and let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a Riesz continuous family of regular operators with compact resolvents acting from fibers of $\mathcal{H}^0$ to fibers of $\mathcal{H}^1$. Then the following conditions are equivalent:*

1. *The bounded transform $f \circ \mathcal{A}$ is a compact deformation of a norm continuous family of unitaries.*
2. *$\mathcal{A}$ is Riesz homotopic to a family of invertible operators.*

If this is the case, then there is a norm continuous family $\mathcal{C} = (\mathcal{C}_x)$ of finite rank operators acting from fibers of $\mathcal{H}^0$ to fibers of $\mathcal{H}^1$ such that $\mathcal{A}_x + \mathcal{C}_x$ is invertible for every $x \in X$, the range of $\mathcal{C}_x$ lies in the range of $\mathbb{1}_{[0,\, r_x)}(\mathcal{A}_x\mathcal{A}_x^*)$, and the kernel of $\mathcal{C}_x$ contains the range of $\mathbb{1}_{[r_x, +\infty)}(\mathcal{A}_x^*\mathcal{A}_x)$ for some continuous function $r\colon X \to \mathbb{R}_+$.

Proof. The equivalence of conditions (1) and (2) follows from the part $(2 \Leftrightarrow 3)$ of Theorem 7.2 applied to the family $\hat{\mathcal{A}} = \left(\begin{smallmatrix} 0 & \mathcal{A}^* \\ \mathcal{A} & 0 \end{smallmatrix}\right)$ of regular odd self-adjoint operators with compact resolvents acting on fibers of the $\mathbb{Z}_2$-graded Hilbert bundle $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$.

Suppose now that $\mathcal{A}$ is Riesz homotopic to a family of invertible operators. Then, by Theorem 7.2, $\hat{\mathcal{A}}$ has a $\mathrm{Cl}(1)$ spectral section; let $R\colon X \to \mathbb{R}_+$ be its cut-off function. Applying Theorem 6.8 to $\hat{\mathcal{A}}$, we get a norm continuous family $\hat{\mathcal{C}} = \left(\begin{smallmatrix} 0 & \mathcal{C}^* \\ \mathcal{C} & 0 \end{smallmatrix}\right)$ of odd self-adjoint operators such that $\hat{\mathcal{A}} + \hat{\mathcal{C}}$ is invertible and the range of $\hat{\mathcal{C}}$ lies in the range of $\mathbb{1}_{(-R,\, R)}(\hat{\mathcal{A}})$ (we omit the subscript x for brevity). Since

$$\mathbb{1}_{(-R,\, R)}(\hat{\mathcal{A}}) \;=\; \mathbb{1}_{[0,\, R^2)}(\hat{\mathcal{A}}^2) = \mathbb{1}_{[0,\, R^2)}\left(\mathcal{A}^*\mathcal{A} \oplus \mathcal{A}\mathcal{A}^*\right),$$

the family $\mathcal{C}$ and the function $r = \sqrt{R}$ satisfy conditions of the theorem. □

Part II

Applications

In this part we present a number of applications illustrating how the results of the previous sections can be used.

8 Relatively compact deformations

Let H and H' be Hilbert spaces.

Deformations of a single operator. We give here only two of possible examples. Clearly, one can write a $\mathbb{Z}_2$-analog of Theorem 8.2, using Theorems 6.6 and 6.8 instead of Theorems 4.3 and 5.3; we omit it since it is quite straightforward.

8.1 Theorem. *For a regular operator $B\colon H \to H'$ with compact resolvents, let*

$$\mathcal{Z}_B = \left\{A \in \mathcal{R}_K(H, H') \,|\, f(A) - f(B) \in \mathcal{K}(H, H')\right\}.$$

Then there are Riesz-to-norm continuous maps $\alpha\colon \mathcal{Z}_B \to \mathcal{B}(H, H')$ *and* $r\colon \mathcal{Z}_B \to \mathbb{R}_+$ *such that* $A + \alpha(A)$ *is invertible, the range of* $\alpha(A)$ *lies in the range of* $\mathbb{1}_{[0,r_x)}(AA^*)$*, and the kernel of* $\alpha(A)$ *contains the range of* $\mathbb{1}_{[r_x,+\infty)}(A^*A)$ *for every* $A \in \mathcal{Z}_B$*.*

Proof. The bounded transform $f(B)$ is a compact deformation of some unitary $u \in \mathcal{U}(H, H')$. Therefore, the composition of the bounded transform with the inclusion $\mathcal{Z}_B \hookrightarrow \mathcal{R}_K(H, H')$ is a compact deformation of a norm continuous (even constant!) family of unitaries $\mathcal{Z}_B \ni A \mapsto u$. It remains to apply Theorem 7.5. $\square$

8.2 Theorem. *For a regular self-adjoint operator* $B \in \mathcal{R}_K^{sa}(H)$ *with compact resolvents, let*

$$\mathcal{Z}_B^{sa} = \mathcal{Z}_B \cap \mathcal{R}_K^{sa}(H) = \{A \in \mathcal{R}_K^{sa}(H) \mid f(A) - f(B) \in \mathcal{K}(H)\} \subset \mathcal{R}_K^{sa}(H) \tag{8.1}$$

be the subspace of $\mathcal{R}_K^{sa}(H)$ *equipped with the Riesz topology. Let* $P = \mathbb{1}_{[0,+\infty)}(B)$ *be the positive spectral projection of* B*. Then inclusion* (8.1) *admits both a spectral section and a correcting family. Moreover, for every* $\varepsilon > 0$ *a spectral section* Q *can be chosen so that* $\|Q(A) - P\| < \varepsilon$ *for every* $A \in \mathcal{Z}_B^{sa}$*.*

Proof. The constant map $A \mapsto P$ is a generalized spectral section for inclusion (8.1). It remains to apply Theorems 4.3 and 5.3. $\square$

Essentially self-adjoint operators. Recall that a bounded operator a is called essentially self-adjoint if $a - a^*$ is a compact operator.

8.3 Theorem. *Let* X *be the subspace of* $\mathcal{R}_K(H)$ *consisting of operators* A *whose bounded transform is an essentially self-adjoint operator. Then there are Riesz-to-norm continuous maps* $\alpha\colon X \to \mathcal{B}(H)$ *and* $r\colon X \to \mathbb{R}_+$ *such that* $A + \alpha(A)$ *is invertible, the range of* $\alpha(A)$ *lies in the range of* $\mathbb{1}_{[0,r_x)}(AA^*)$*, and the kernel of* $\alpha(A)$ *contains the range of* $\mathbb{1}_{[r_x,+\infty)}(A^*A)$ *for every* $A \in X$*.*

Proof. Let $A \in X$ and $a = f(A)$. Then $b = (a + a^*)/2$ is a self-adjoint operator of norm $\leqslant 1$ and $u = u_A = b + i\sqrt{1 - b^2}$ is a unitary. Moreover, both b and u are compact deformations of a. The map $X \to \mathcal{U}(H)$ taking A to u_A is Riesz-to-norm continuous. Therefore, the composition of the bounded transform with the embedding $X \hookrightarrow \mathcal{R}_K(H)$ is a compact deformation of a norm continuous family of unitaries. It remains to apply Theorem 7.5. $\square$

Essentially odd operators. Let H be a $\mathbb{Z}_2$-graded Hilbert space, with the grading given by the symmetry σ. A bounded operator $a \in \mathcal{B}(H)$ is called essentially odd if $\sigma a + a\sigma$ is a compact operator.

8.4 Theorem. *Let* X *be the subspace of* ${}^r\mathcal{R}_K^{sa}(H)$ *consisting of operators* A *whose bounded transform is an essentially odd operator. Then the natural embedding* $X \hookrightarrow {}^r\mathcal{R}_K^{sa}(H)$ *admits both a spectral section* $X \to \mathcal{P}(H)$ *and a correcting family* $X \to \mathcal{B}^{sa}(H)$*.*

Proof. Let $A \in X$ and $a = f(A)$. Then $b = (a - \sigma a \sigma)/2$ is an odd self-adjoint operator of norm $\leqslant 1$, $u = b + \sigma\sqrt{1 - b^2}$ is a symmetry, and $P = P_A = (u + 1)/2$ is a projection. Moreover, both b and u are compact deformations of a. The map $X \to \mathcal{P}(H)$ taking A to P_A is Riesz-to-norm continuous. Therefore, this map is a generalized spectral section for the embedding $X \hookrightarrow {}^r\mathcal{R}_K^{sa}(H)$. It remains to apply Theorems 4.3 and 5.3. $\square$

9 Pseudodifferential operators

We show here several examples of applications of our results to pseudodifferential operators over closed manifolds. For simplicity, we restrict ourselves by operators acting on sections of a fixed vector bundle over a fixed manifold. Again, we omit the $\mathbb{Z}_2$-graded case here.

Let M be a closed smooth manifold equipped with a smooth positive measure, and let E, E' be smooth Hermitian bundles over M. Let $\Psi_d(E, E')$ denote the space of pseudodifferential operators of order $d \geqslant 0$ acting from sections of E to sections of E'. We equip it with the topology induced by the inclusion

$$\Psi_d(E, E') \hookrightarrow \mathcal{B}(H^d(E), L^2(E')) \times \mathcal{B}(H^d(E'), L^2(E))$$

taking a pseudodifferential operator A to the pair (A, A^t), where A^t is the operator formally adjoint to A. By [Le, Proposition 2.2], the natural inclusion of the subspace $\Psi_d^{ell}(E, E') \subset \Psi_d(E, E')$ of elliptic operators to $\mathcal{R}(L^2(E), L^2(E'))$ is Riesz continuous.

9.1 Theorem. *Let X be a topological space and $\mathcal{A}\colon X \to \Psi_d^{ell}(E, E')$ be a continuous family of elliptic operators of order $d \geqslant 1$. Suppose that $\mathcal{A}$ is homotopic to a family of invertible operators. Then there is a norm continuous family $\alpha = (\alpha_{x,B})$ of smoothing finite rank operators parametrized by pairs $(x, B) \in X \times \Psi_{d-1}(E, E')$ such that $\mathcal{A}_x + B + \alpha_{x,B}$ is invertible for every $x \in X$ and every $B \in \Psi_{d-1}(E, E')$.*

9.2 Remark. In this theorem, $\mathcal{A}$ is homotopic to a family of invertible operators, in particular, in each of the following cases:

1. X is contractible.
2. X is compact and $\operatorname{ind}(\mathcal{A}) = 0 \in \mathcal{K}^0(X)$.
3. The kernel and cokernel of $\mathcal{A}_x$ have locally constant ranks, and the corresponding vector bundles $\operatorname{Ker}(\mathcal{A})$ and $\operatorname{Coker}(\mathcal{A})$ over X are isomorphic.

Proof. Let $Y = X \times \Psi_{d-1}(E, E')$, $H = L^2(E)$, and $H' = L^2(E')$. The map $\tilde{\mathcal{A}}\colon Y \to \Psi_d^{ell}(E, E')$ taking (x, B) to $\mathcal{A}_x + B$ is continuous. Therefore, the composed map

$$\tilde{\mathcal{A}}\colon Y \to \Psi_d^{ell}(E, E') \hookrightarrow \mathcal{R}_K(H, H')$$

is Riesz continuous. It is Riesz homotopic to the map $Y \to \mathcal{R}_K(H, H')$ taking (x, B) to $\mathcal{A}_x$ via the homotopy $(x, B, t) \mapsto \mathcal{A}_x + tB$. Since $\mathcal{A}$ is homotopic to a family of invertible operators, the same is true for $\tilde{\mathcal{A}}$.

By Theorem 7.5, there is a continuous map $\alpha\colon Y \to \mathcal{B}(H, H')$ such that $\tilde{\mathcal{A}} + \alpha$ is an invertible family, the range of $\alpha_{x,B}$ lies in the range V of $\mathbb{1}_{[0,r)}(\tilde{\mathcal{A}}_{x,B}\tilde{\mathcal{A}}^*_{x,B})$, and the orthogonal complement of the kernel of $\alpha_{x,B}$ lies in the range V' of $\mathbb{1}_{[0,r)}(\tilde{\mathcal{A}}^*_{x,B}\tilde{\mathcal{A}}_{x,B})$ for some $r = r(x, B)$. Since $\tilde{\mathcal{A}}_{x,B}$ is an elliptic operator of positive order, both V and V' are spanned by a finite number of C^∞-sections. Therefore, $\alpha_{x,B}$ is a smoothing operator of finite rank. This completes the proof of the theorem. □

Self-adjoint case. For M and E as above, let $\Psi_d^{sa}(E)$ denote the subspace of $\Psi_d(E)$ consisting of symmetric operators, and let $\Psi_d^{ell,sa}(E) = \Psi_d^{ell}(E) \cap \Psi_d^{sa}(E)$.

9.3 Theorem. *Let* X *be a topological space and* $\mathcal{A}\colon X \to \Psi_d^{\mathrm{ell,sa}}(E, E')$ *be a continuous family of symmetric elliptic operators of order* $d \geqslant 1$. *Suppose that* $\mathcal{A}$ *is homotopic to a family of invertible operators. Then the map*

$$\tilde{\mathcal{A}}\colon Y = X \times \Psi_{d-1}^{\mathrm{sa}}(E) \to \Psi_d^{\mathrm{ell,sa}}(E), \quad (x, B) \mapsto \mathcal{A}_x + B$$

admits both a spectral section $P\colon Y \to \Psi_0^{\mathrm{sa}}(E)$ *and a correcting family with smoothing correcting operators.*

Proof. The proof is completely similar to the proof of Theorem 9.1; one only needs to use Theorems 4.3 and 5.3 instead of Theorem 7.5. □

9.4 Remark. In this theorem, $\mathcal{A}$ is homotopic to a family of invertible operators, in particular, in each of the following cases:

1. X is contractible.
2. X is compact and $\mathrm{ind}(\mathcal{A}) = 0 \in \mathcal{K}^1(X)$.
3. The kernel of $\mathcal{A}_x$ has locally constant rank.

10 Cobordism theorems

Calderón projection. Let M be a smooth compact Riemannian manifold with non-empty boundary ∂M and E, E' be smooth Hermitian vector bundles over M. Denote by E_∂ and E'_∂ the restrictions of E and E' to ∂M.

Let A be a first order elliptic differential operator over M acting from sections of E to sections of E'. The *space of Cauchy data* of A is the closure in $H = L^2(\partial M; E_\partial)$ of the subspace consisting of restrictions to ∂M of all smooth solutions of the equation $Au = 0$. The orthogonal projection $Q = Q(A) \in \mathcal{P}(H)$ onto the Cauchy data space is called the (orthogonal) *Calderón projection* of A; it is a pseudodifferential operator of zeroth order.

At the points of the boundary the operator A can be written as

$$A = -iJ(\partial_z + B), \tag{10.1}$$

where z is the normal coordinate, $J \in \mathrm{Iso}(E_\partial, E'_\partial)$ is the conormal symbol of A, and B is a first order elliptic differential operator over ∂M acting on sections of E_∂. Such an operator B is called the *tangential operator of* A *along the boundary*, or simply the *boundary operator* of A.

Suppose that the principal symbol of B is self-adjoint, that is $B - B^t$ is a bundle endomorphism (here B^t denotes the operator formally adjoint to B). Then the Calderón projection $Q(A)$ has the same principal symbol as the positive spectral projection $\mathbb{1}_{[0,+\infty)}(B + B^t)$. In other words, $Q(A)$ is a generalized spectral section for the symmetrized tangential operator $\tilde{B} = (B + B^t)/2$.

General cobordism theorem. Let $\mathcal{M} \to X$ be a locally trivial fiber bundle over a paracompact Hausdorff space X, with the typical fiber a smooth compact connected manifold M with non-empty boundary. Let $\mathcal{E}$ and $\mathcal{E}'$ be complex vector bundles over $\mathcal{M}$. We suppose that every fiber $\mathcal{M}_x$ is equipped with a Riemannian metric and that $\mathcal{E}_x$ and $\mathcal{E}'_x$ are equipped with structures of smooth Hermitian vector bundles over $\mathcal{M}_x$ for every $x \in X$. We also suppose that these structures and metrics continuously depend on $x \in X$. We consider $\mathcal{E}$ and $\mathcal{E}'$ as two families $(\mathcal{E}_x)$, $(\mathcal{E}'_x)$ of vector bundles over a family $(\mathcal{M}_x)$ of manifolds parametrized by points $x \in X$.

10.1 Theorem. *Let $\mathcal{A} = (\mathcal{A}_x)_{x\in X}$ be a family of first order elliptic differential operators parametrized by points of a paracompact Hausdorff space X, with $\mathcal{A}_x$ acting from sections of $\mathcal{E}_x$ to sections of $\mathcal{E}'_x$. Suppose that the principal symbols $\mathfrak{b}_x$ of the boundary operators $\mathcal{B}_x$ are self-adjoint, as above. Suppose, moreover, that both $\mathfrak{b}_x$ and the Calderón projection $Q(\mathcal{A}_x)$ continuously depend on x. Let $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_x)$ be a continuous family of first order symmetric operators over $\partial\mathcal{M}_x$ having $\mathfrak{b}_x$ as their principal symbols. Then $\tilde{\mathcal{B}}$ admits both a spectral section and a correcting family, with smoothing correcting operators.*

Proof. The operators $\tilde{\mathcal{B}}_x$ act on the fibers $\mathcal{H}_x = L^2(\partial\mathcal{M}_x; \mathcal{E}_x)$ of the Hilbert bundle $\mathcal{H}$ over X. The family $\tilde{\mathcal{B}}$ is Riesz continuous. The Calderón projection $Q_x = Q(\mathcal{A}_x)$ is a generalized spectral section for $\tilde{\mathcal{B}}_x$. Since the map $x \mapsto Q_x$ is continuous, the family (Q_x) of projections is a generalized spectral section for $\tilde{\mathcal{B}}$. It remains to apply Theorems 7.1 and 7.3. Since the range of every correcting operator lies in the span of a finite number of C^∞-sections (namely, eigenvectors of $\tilde{\mathcal{B}}_x$), all correcting operators are smoothing. □

Unique Continuation Properties. There are different criteria for continuity of the family of Calderón projections for different classes of operators. One of such criteria convenient for our purposes is [BLZ, Corollary 7.4] of Booss-Bavnbek, Lesch, and Zhu. It concerns families $\mathcal{A} = (\mathcal{A}_x)$ of first order elliptic operators satisfying, together with their formally adjoints, the Weak inner Unique Continuation Property (weak inner UCP). Recall that a first order operator A is said to have weak inner UCP if the only solution of the equation $Au = 0$ vanishing on the boundary is the trivial solution $u = 0$.

Another useful property is the Weak Unique Continuation Property (weak UCP). An operator A over a connected manifold M is said to have weak UCP if any solution of the equation $Au = 0$ which vanishes on an open subset of M vanishes on the whole M.

Let M be a smooth connected Riemannian manifold, not necessarily compact, and E, E' be smooth Hermitian vector bundles over M. Denote by $\mathcal{W}(E, E')$ the set of first order elliptic differential operators $A\colon C^\infty(M; E) \to C^\infty(M; E')$ whose principal symbol a satisfies the following condition:

$$\text{The fiber endomorphism } ia(\xi)^{-1}a(\eta) \in \operatorname{End}(E_x) \text{ is self-adjoint for every pair of orthogonal cotangent vectors } \xi, \eta \in T^*_x M,\ x \in M. \tag{10.2}$$

10.2 Proposition. *Every $A \in \mathcal{W}(E, E')$ has the Weak Unique Continuation Property.*

Proof. We follow the line of the proof of weak UCP for perturbed Dirac type operators in [BBB, Theorem 1.33], but write it in more detail.

We can suppose without loss of generality that M has no boundary (otherwise replace M by $M \setminus \partial M$). Suppose that a nontrivial solution u of $Au = 0$ vanishes on a non-empty open subset of M. Let V be the union of all open subsets of M where u vanishes, and let $M' = \operatorname{supp}(u) \neq \emptyset$ be the complement of V in M.

1. We claim that there is a point $p \in V$ such that the injectivity radius of p is greater than the distance from p to M', $\operatorname{inj}(p) > \operatorname{dist}(p, M')$. To show this, choose $x \in M' \cap \overline{V}$, and let $r = \operatorname{inj}(x)$. Since the injectivity radius function is lower-semicontinuous, there is a $\delta \in (0, r/4)$ such that the injectivity radius is greater than $r/2$ for all points of the open ball $B_\delta(x) = \{y \in M \,|\, \operatorname{dist}(x, y) < \delta\}$. Since x lies in the boundary of V, the intersection $V \cap B_\delta(x)$ is non-empty; let p be a point in this intersection. Then $\operatorname{dist}(p, M') \leqslant \operatorname{dist}(p, x) < r/4$ and $\operatorname{inj}(p) > r/2$.

2. Let $p \in V$ be such a point that $r = \operatorname{inj}(p) > \operatorname{dist}(p, M') = d$. Then the open ball $B_d(p)$ is contained in V and the larger open ball $B_r(p)$ can be equipped with (geodesical) spherical coordinates. It follows from [BB, Lemmata 5 and 6] that u vanishes on some intermediate ball $B_R(p)$ with $d < R < r$, that is $B_R(p) \subset V$. On the other hand, the radius of $B_R(p)$ is greater than the distance from p to M', so $B_R(p)$ intersects M'. This contradiction shows that A satisfies weak UCP, which completes the proof of the proposition. □

Dirac type operators. Recall that a first order operator A with the principal symbol a is called a Dirac type operator if $a(\xi)^* a(\xi) = \|\xi\|^2 \cdot \mathrm{Id} = a(\xi) a(\xi)^*$ for every $\xi \in T^*M$. Every Dirac type operator acting from sections of E to sections of E' is an element of $\mathcal{W}(E, E')$, but not vice versa.

In the first version of this preprint [P2], we used Proposition 10.2 to prove a cobordism theorem for families of operators of the class $\mathcal{W}(E, E')$, see [P2, Theorem 10.4]. As it happens, that result is not really more general than a cobordism theorem for Dirac type operators. Indeed, composing A with a bundle automorphism of E' does not affect the boundary operator of A. The following proposition shows that every operator $A \in \mathcal{W}(E, E')$ can be obtain from a Dirac type operator by such a composition.

10.3 Proposition. *Let A be a first order operator acting from sections of E to sections of E'. Then the following two conditions are equivalent:*

1. $A \in \mathcal{W}(E, E')$,
2. $A = TD$, *where T is a bundle automorphism of E' and D is a Dirac type operator.*

Proof. Condition (10.2) can be equivalently written as follows:

$$a(\xi) a(\eta)^* + a(\eta) a(\xi)^* = 0 \quad \text{for every } \xi \perp \eta \in T^*_x M,\ x \in M. \tag{10.3}$$

Left hand side of (10.3) is an $\operatorname{End}(E'_x)$-valued symmetric bilinear form on $T^*_x M$,

$$\alpha(\xi, \eta) = a(\xi) a(\eta)^* + a(\eta) a(\xi)^*.$$

$(2 \Leftrightarrow 1)$ If $A = TD$, then $\alpha(\xi,\eta) = 2\langle\xi,\eta\rangle TT^*$ satisfies (10.3), so $A \in \mathcal{W}(E,E')$.

$(1 \Leftrightarrow 2)$ Let $A \in \mathcal{W}(E,E')$. For arbitrary non-zero $\xi,\eta \in T_x^*M$, we write $\xi = \xi' + t\eta$ with ξ' orthogonal to η. This gives

$$\alpha(\xi,\eta) = t\alpha(\eta,\eta) = \langle\xi,\eta\rangle\, S(\eta), \tag{10.4}$$

where

$$S(\eta) = \alpha(\eta,\eta)/\|\eta\|^2 = 2a(\eta)a(\eta)^*/\|\eta\|^2 \in \mathrm{End}(E'_x)$$

is a homogenous function on $S_x^*M \setminus \{0\}$ of degree 0. Since $\alpha(\xi,\eta) = \alpha(\eta,\xi)$, identity (10.4) impies $S(\xi) = S(\eta)$ for every pair of non-orthogonal vectors $\xi,\eta \in S_x^*M$. For every non-zero $\xi,\eta \in T_x^*M$ there is a third vector $\zeta \in S_x^*M$ which is non-orthogonal to both ξ and η, so that $S(\xi) = S(\zeta) = S(\eta)$. Therefore, $S(\eta)$ is independent of $\eta \in S_x^*M$ and depends only on x, $S(\eta) = S_x$. Moreover, S_x is positive for every $x \in M$. Let T_x be the positive square root of $S_x/2$. Then T is a smooth bundle automorphism of E'. The equality $a(\xi)a(\xi)^* = \|\xi\|^2\, T_xT_x^*$ implies $(T_x^{-1}a(\xi))(T_x^{-1}a(\xi))^* = \|\xi\|^2 \cdot \mathrm{Id}$ for every $\xi \in T_x^*M$. Since $T_x^{-1}a(\xi)$ is invertible for $\xi \neq 0$, this implies the second identity $(T_x^{-1}a(\xi))^*(T_x^{-1}a(\xi)) = \|\xi\|^2 \cdot \mathrm{Id}$. Therefore, $T^{-1}A$ is a Dirac type operator. □

10.4 Proposition. *Let M be a smooth connected Riemannian manifold with non-empty boundary and A be a Dirac type operator over M. Then A has the weak inner Unique Continuation Property.*

Proof. The operator A admits an extension across the boundary, that is, A is the restriction to M of some Dirac type operator $\tilde{A}$ over $\tilde{M}$, where $\tilde{M}$ is a smooth Riemannian manifold without boundary containing M as a smooth submanifold of codimension zero. Indeed, let A act from sections of E to sections of E'. The symbol a of A determines the structure of a Clifford module over $T^*M \oplus \mathbb{R}$ on $E \oplus E'$, with the cotangent vector ξ acting as $\hat{a}(\xi) = \begin{pmatrix} 0 & a(\xi)^* \\ a(\xi) & 0 \end{pmatrix}$ and the unit vector in the additional $\mathbb{R}$-direction acting as $\mathrm{Id}_E \oplus (-\mathrm{Id}_{E'})$. Such a structure can be smoothly extended across the boundary of M, to an external collar neighbourhood of ∂M equipped with a metric in a compatible way. Such an extension gives rise to the symbol $\tilde{a}$ of a Dirac type operator extending a and acting from sections of $\tilde{E}$ to sections of $\tilde{E}'$, where $\tilde{E}$ and $\tilde{E}'$ are smooth Hermitian vector bundles over $\tilde{M}$ extending E and E'. Given an extension of the symbol, the whole operator A can be smoothly extended to $\tilde{M}$ using a partition of unity.

Let $u \in L^2(M; E, E')$ be in the kernel of A and let $\tilde{u} \in L^2(\tilde{M}; \tilde{E}, \tilde{E}')$ be the extension of u to $\tilde{M} \setminus M$ by zero. Suppose that u vanishes on the boundary ∂M. Then the Green formula for A implies that $\tilde{u}$ is a weak solution of $\tilde{A}$. Since $\tilde{A}$ is elliptic, $\tilde{u}$ is smooth. By Proposition 10.2, the operator $\tilde{A}$ has weak UCP. Since $\tilde{A}\tilde{u} = 0$ and $\tilde{u}$ vanishes on the open subset $\tilde{M} \setminus M$, we get $\tilde{u} \equiv 0$ and thus $u = \tilde{u}|_M \equiv 0$. Therefore, A has weak inner UCP. □

Cobordism theorem for Dirac type operators. The boundary operator of a Dirac type operator is again a Dirac type operator; moreover, it has a self-adjoint principal symbol. Applying Theorem 10.1 to this situation, we obtain the following generalization of [MP1, Section 2, Corollary].

10.5 Theorem. *Let X be a paracompact Hausdorff space and $\mathcal{M}$, $\mathcal{E}$, $\mathcal{E}'$ be as in Theorem 10.1. Let $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$ be a family of Dirac type operators such that, in every local chart, all the coefficients of the operator $\mathcal{A}_x$, together with their first derivatives in the $\mathcal{M}_x$-direction, continuously depend on $x \in X$. Then the family $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_x)$ of symmetrized boundary operators admits both a spectral section and a correcting family, with smoothing correcting operators.*

Proof. The formal conjugate of a Dirac type operator is again a Dirac type operator. By Proposition 10.4, both $\mathcal{A}_x$ and $\mathcal{A}_x^t$ have weak inner UCP. The family $(\mathcal{A}_x, iJ_x)$ is continuous by $x \in X$ with respect to the strong metric of [BLZ, Definition 7.1]. By [BLZ, Corollary 7.4], the Calderón projections $Q(\mathcal{A}_x)$ continuously depend on x. It remains to apply Theorem 10.1. $\square$

10.6 Remark. While the statement of Theorem 10.5 does not distinguish between even- and odd-dimensional manifolds $\mathcal{M}_x$, the theorem is really interesting only in the even-dimensional case.

If a manifold is odd-dimensional, then its boundary is a closed manifold of even dimension. At the same time, an *arbitrary* family of symmetric Dirac type operators $\mathcal{B} = (\mathcal{B}_x)$ over a closed oriented manifold N of dimension $2k$ (or, more generally, over a family $\mathcal{N} = (\mathcal{N}_x)$ of such manifolds parametrized by X) admits a spectral section and a correcting family, regardless of whether $\mathcal{B}$ is cobordant to zero or not.

Indeed, let B be a symmetric Dirac type operator over N with the symbol b. Let σ be the "normalized orientation", $\sigma_y = i^k\, b(\xi_1) \cdot \ldots \cdot b(\xi_{2k})$, where $(\xi_1, \ldots, \xi_{2k})$ is a positively oriented orthonormal basis of T_y^*N, $y \in N$. Then σ is a bundle symmetry anticommuting with b, so Theorem 8.4 can be applied. In more concrete terms, the operators $\bar{B} = (B - \sigma B \sigma)/2$ and $B' = \bar{B} + \sigma$ are symmetric and have the same symbol as B. Moreover, $\bar{B}$ anticommutes with σ, so $(B')^2 = \bar{B}^2 + 1$ is invertible and thus B' itself is invertible. Therefore, every family $\mathcal{B} = (\mathcal{B}_x)$ of symmetric Dirac type operators over a family $\mathcal{N} = (\mathcal{N}_x)$ of closed even-dimensional manifolds is homotopic to a family $\mathcal{B}' = (\mathcal{B}'_x)$ of invertible operators and thus admits a spectral section and a correcting family by Theorems 7.1 and 7.3.

A relevant cobordism theorem in this case should take into account the grading σ and state the existence of a $Cl(1)$ spectral section for the family $\bar{\mathcal{B}}$ of odd operators. We perform this in the next section, see Theorem 11.2.

11 Cobordism theorems: $\mathbb{Z}_2$-graded case

General cobordism theorem. Let A be a first order *symmetric* operator acting on sections of E, B be its boundary operator, and $J = a(n)$ be the conormal symbol of A. Then J is self-adjoint and the operator $JB + B^tJ$ has zeroth order (that is, $JB + B^tJ$ is a bundle endomorphism). Suppose, in addition, that B has *self-adjoint symbol*. Then both $B - B^t$ and $JB + BJ$ are bundle endomorphisms.

Let σ be a self-adjoint unitary bundle automorphism of E_∂ defined by the formula

$$\sigma = J \cdot |J|^{-1} = J \cdot (J^2)^{-1/2}. \tag{11.1}$$

Then $\sigma B + B\sigma$ is also a bundle endomorphism. Indeed, the symbol b of B satisfies the anticommutation relation

$$J(y)b(\xi) + b(\xi)J(y) = 0 \quad \text{for every } y \in \partial M \text{ and } \xi \in T_y^*\partial M. \tag{11.2}$$

Thus the positive operator $T = J(y)^2$ commutes with $b(\xi)$. Multiplying (11.2) by $T^{-1/2}$, we get $\sigma(y)b(\xi) + b(\xi)\sigma(y) = 0$, which implies $\sigma B + B\sigma \in \mathrm{End}(E)$.

Instead of the symmetrized boundary operator $\tilde{\mathcal{B}} = (B + B^t)/2$, we now consider the supersymmetrized operator

$$\bar{B} = (\tilde{\mathcal{B}} - \sigma\tilde{\mathcal{B}}\sigma)/2 = (B + B^t - \sigma B\sigma - \sigma B^t\sigma)/4, \tag{11.3}$$

which also differs from B by a bundle endomorphism.

In such a way, every first order symmetric elliptic operator A acting on E, whose boundary operator has self-adjoint symbol, determines a grading $\sigma = \sigma_A$ of E_∂ and an odd symmetric elliptic operator $\bar{B} = \bar{B}_A$ acting on E_∂.

By [BLZ, Theorem 6.1.I], the space Λ of Cauchy data of A is a Lagrangian subspace of the symplectic Hilbert space (H, iJ), $H = L^2(\partial M; E_\partial)$. In other words, the orthogonal projection Q of H onto Λ (the Calderón projection) satisfies the anticommutation relation $J(2Q-1) + (2Q-1)J = 0$. Reasoning as above, we obtain $\sigma(2Q-1) + (2Q-1)\sigma = 0$. Therefore, Q is a generalized $\mathrm{Cl}(1)$ spectral section both for $\bar{B}$ and for any other odd symmetric operator with the symbol b.

Applying our previous results to this situation, we obtain a graded version of the general cobordism theorem.

11.1 Theorem. *Let X, $\mathcal{M}$, $\mathcal{E} = \mathcal{E}'$, $\mathcal{A}$, and b be as in Theorem 10.1. Suppose, in addition, that all the operators $\mathcal{A}_x$ are symmetric. Let the grading on $\mathcal{E}_x|_{\partial\mathcal{M}_x}$ be defined by the unitary part of the conormal symbol of $\mathcal{A}_x$, as in (11.1), and let $\bar{\mathcal{B}} = (\bar{\mathcal{B}}_x)$ be a continuous family of odd symmetric operators over $\partial\mathcal{M}_x$ having b_x as the principal symbol. Then $\bar{\mathcal{B}}$ admits both a $\mathrm{Cl}(1)$ spectral section and a correcting family of odd smoothing correcting operators.*

Proof. By the condition of the theorem, the Calderón projection $Q_x = Q(\mathcal{A}_x)$ continuously depend on x. Since Q_x is a generalized $\mathrm{Cl}(1)$ spectral section for $\bar{\mathcal{B}}_x$ for every $x \in X$, the family (Q_x) is a generalized $\mathrm{Cl}(1)$ spectral section for $\bar{\mathcal{B}}$. It remains to apply Theorems 7.2 and 7.4. The same reasoning as in the proof of the Theorem 10.1 shows that all correcting operators are smoothing. □

Dirac type operators. If A is a symmetric Dirac type operator, then the conormal symbol of A is unitary and thus coincides with the grading σ defined by (11.1). Applying Theorem 11.1 to this situation, we immediately obtain the following generalization of [MP2, Corollary 1].

11.2 Theorem. *Let X, $\mathcal{M}$, $\mathcal{E} = \mathcal{E}'$, and a family $\mathcal{A}$ of Dirac type operators be as in Theorem 10.5. Suppose, in addition, that all the operators $\mathcal{A}_x$ are symmetric. Then the family $\bar{\mathcal{B}} = (\bar{\mathcal{B}}_x)$ of supersymmetrized boundary operators admits both a $\mathrm{Cl}(1)$ spectral section and a correcting family with odd smoothing correcting operators, with respect to the grading over $\partial\mathcal{M}_x$ given by the conormal symbol of $\mathcal{A}_x$.*

Proof. As was shown in the proof of Theorem 10.5, the Calderón projection $Q(\mathcal{A}_x)$ continuously depends on x. By the definition of the supersymmetrized boundary operator, $\tilde{\mathcal{B}}_x$ continuously depends on x and its symbol is b_x. It remains to apply Theorem 11.1. □

12 Tangential operators on a moving hypersurface

Let M be a smooth connected Riemannian manifold without boundary, not necessarily compact. Let E and E' be smooth Hermitian vector bundles over M.

Tangential operators. Let Σ be a smooth cooriented hypersurface in M and E_Σ be the restriction of E to Σ. Every first order elliptic differential operator $A\colon C^\infty(M;E) \to C^\infty(M;E')$ determines the tangential operator $B = B_{A,\Sigma}$ of A along Σ acting on sections of E_Σ. It is defined in exactly the same manner as the tangential operator of A along the boundary, see formula (10.1). Let $\tilde{B}_{A,\Sigma} = (B_{A,\Sigma} + B^t_{A,\Sigma})/2$ be the symmetrized tangential operator. Then $(\tilde{B}_{A,\Sigma})$ is a family of symmetric operators parametrized by pairs (A, Σ).

The space of parameters. With a cobordism theorem in mind, we choose a different parametrization of the appropriate part of the family $(\tilde{B}_{A,\Sigma})$.

Let $\mathcal{D}(E, E')$ denote the set of all Dirac type operators acting from sections of E to sections of E'. We equip $\mathcal{D}(E, E')$ with the topology induced by the family of C^1-metrics on coefficients of operators restricted to small (that is, lying in local charts) compact subsets $K \subset M$. If $A \in \mathcal{D}(E, E')$, then $B_{A,\Sigma} \in \mathcal{D}(E_\partial)$ has self-adjoint principal symbol.

Let $\mathcal{S}(M)$ be the set of all smooth compact submanifolds of M of codimension zero, excluding M itself. For every $N \in \mathcal{S}(M)$ its boundary ∂N is a smooth cooriented hypersurface in M, so the symmetrized tangential operator $\tilde{B}_{A,\partial N}$ is well defined.

Let S^*M be the cosphere bundle of M, that is the subbundle of T^*M consisting of unit vectors. A smooth cooriented submanifold $L \subset M$ of codimension one is naturally lifted to S^*M; the corresponding embedding $j_1\colon L \hookrightarrow S^*M$ takes $x \in L$ to $(x, n_x) \in S^*M$, where n_x is the conormal to L at x. Let d_0 be the Hausdorff distance between closed subsets of M and d_1 be the Hausdorff distance between closed subsets of S^*M. We equip $\mathcal{S}(M)$ with the topology induced by the metric

$$d(N, N') = d_0(N, N') + d_1(j_1(\partial N), j_1(\partial N')).$$

This topology does not depend on the Riemannian metric on M, since all $N \in \mathcal{S}(M)$ are compact.

12.1 Theorem. *The family $\tilde{\mathcal{B}} = (\tilde{B}_{A,\partial N})$ of symmetrized tangential operators parametrized by the product $\mathcal{D}(E, E') \times \mathcal{S}(M)$ admits a spectral section and a correcting family, with smoothing correcting operators.*

Proof. We have a continuous family (∂N) of closed manifolds parametrized by $N \in \mathcal{S}(M)$ and a continuous family $(\tilde{B}_{A,\partial N})$ of symmetric elliptic operators over ∂N parametrized by $(A, N) \in X = \mathcal{D}(E, E') \times \mathcal{S}(M)$.

By Proposition 10.4, the restriction of A to N has weak inner UCP. By [BLZ, Corollary 7.4], the Calderón projection $Q(A|_N)$ continuously depends on (A, N), so the family $(Q(A|_N))$ is a generalized spectral section for the family $(\tilde{B}_{A,\partial N})$.

The operator $\tilde{B}_{A,\partial N}$ considered as an unbounded operator acts on the Hilbert space $\mathcal{H}_N = L^2(\partial N; E)$. The corresponding Hilbert bundle over X is the lifting to X of the Hilbert bundle $\mathcal{H}$ over $\mathcal{S}(M)$, where the fiber of $\mathcal{H}$ over $N \in \mathcal{S}(M)$ is $\mathcal{H}_N$. The base $\mathcal{S}(M)$ of $\mathcal{H}$ is a metric space and thus paracompact. Therefore, $\mathcal{H}$ is a numerable Hilbert bundle. Its lifting to X is also a numerable Hilbert bundle.

By Theorems 7.1 and 7.3, the family $\tilde{\mathcal{B}}$ admits a spectral section and a correcting family. The same reasoning as in the proof of Theorem 10.1 shows that all correcting operators are smoothing. This completes the proof of the theorem. □

$\mathbb{Z}_2$-graded case. If A is a symmetric Dirac type operator over M and Σ is a smooth cooriented hypersurface in M, then the conormal symbol of A defines the grading $\sigma_{A,\Sigma}$ on E_Σ. The supersymmetrized tangential operator $\bar{B}_{A,\Sigma}$ given by formula (11.3) is odd with respect to this grading.

Let $\mathcal{D}^{sa}(E)$ denote the subspace of $\mathcal{D}(E)$ consisting of symmetric operators.

12.2 Theorem. *The family $\bar{\mathcal{B}} = (\bar{B}_{A,\partial N})$ of supersymmetrized tangential operators parametrized by the product $\mathcal{D}^{sa}(E) \times \mathcal{S}(M)$ admits both a* Cl(1) *spectral section and a correcting family with odd smoothing correcting operators.*

Proof. As was shown in the previous section, the Calderón projection $Q(A|_N)$ is a generalized Cl(1) spectral section for the operator $\bar{B}_{A,\partial N}$. As was shown in the proof of the previous theorem, $Q(A|_N)$ continuously depends on (A, N), so the family $Q = (Q(A|_N))$ is a generalized Cl(1) spectral section for $\bar{\mathcal{B}}$. The operators $\bar{B}_{A,\Sigma}$ act on the fibers of the lifting of $\mathcal{H}$ to $\mathcal{D}^{sa}(E) \times \mathcal{S}(M)$, where $\mathcal{H}$ is the Hilbert bundle over $\mathcal{S}(M)$ defined in the proof of the previous theorem. Since $\mathcal{H}$ is numerable, the lifted Hilbert bundle is also numerable. By Theorems 7.2 and 7.4, $\bar{\mathcal{B}}$ admits a Cl(1) spectral section and a correcting family with odd correcting operators. The same reasoning as in the proof of Theorem 10.1 shows that all correcting operators are smoothing. □

Part III

Graph continuous families

Throughout this part, all families of regular operators are supposed to be graph continuous.

Recall that the Cayley transform of a regular self-adjoint operator A is the unitary operator defined by the formula $\kappa(A) = (A - i)(A + i)^{-1}$. The Cayley transform $\kappa\colon \mathcal{R}^{sa}(H) \to \mathcal{U}(H)$ is a homeomorphism on the image [BLP, Theorem 1.1]. Therefore, the graph topology on the subspace $\mathcal{R}^{sa}(H)$ of $\mathcal{R}(H)$ can be equivalently described as the topology induced by the inclusion $\kappa\colon \mathcal{R}^{sa}(H) \hookrightarrow \mathcal{U}(H)$. We will use this fact below.

13 Semibounded operators

Positive operators. Let $\mathcal{R}^{+}(\mathcal{H})$ denote the subspace of $\mathcal{R}^{sa}(\mathcal{H})$ consisting of positive operators.

13.1 Proposition. *The restrictions of the graph topology and the Riesz topology to $\mathcal{R}^{+}(\mathcal{H})$ coincide.*

Proof. For $a = f(A)$ the identity $(1 + A^2)^{-1} = 1 - a^2$ implies

$$\kappa(A) = \frac{A - i}{A + i} = \frac{a - i\sqrt{1 - a^2}}{a + i\sqrt{1 - a^2}} = \left(a - i\sqrt{1 - a^2}\right)^2 = \tilde{\kappa}(a),$$

where $\tilde{\kappa}\colon [-1, 1] \to \mathcal{U}(\mathbb{C}) = \{z \in \mathbb{C} \,|\, |z| = 1\}$ is a continuous function given by the formula $\tilde{\kappa}(a) = \left(a - i\sqrt{1 - a^2}\right)^2$. Thus the Cayley transform factors through the bounded transform: $\kappa = \tilde{\kappa} \circ f$. The function $\tilde{\kappa}$ is not invertible. However, its restriction to the interval $[0, 1]$ *is* invertible: it is a homeomorphism from $[0, 1]$ to the bottom half of the unite circle $\Gamma = \left\{e^{it} \,|\, t \in [-\pi, 0]\right\}$. The inverse homeomorphism $\varphi\colon \Gamma \to [0, 1]$ is given by the formula $\varphi(e^{it}) = \cos(t/2)$. Hence the restriction of the bounded transform f to $\mathcal{R}^{+}(H)$ coincides with the composition $\varphi \circ \kappa$ and thus is continuous. It follows that the restriction of the graph topology to $\mathcal{R}^{+}(H)$ coincides with the Riesz topology. □

Semibounded operators. By [CL, Addendum, Theorem 1], the restriction of the graph topology to the subspace of bounded operators coincides with the usual norm topology, and thus with the Riesz topology. The following result generalizes this property to semibounded operators. It is given here for the sake of completeness; we do not use it in the rest of the paper.

Let $\mathcal{R}^{sb}(\mathcal{H})$ denote the subspace of $\mathcal{R}^{sa}(\mathcal{H})$ consisting of semibounded operators, that is, operators bounded from below or above (bounded operators included).

13.2 Theorem. *The restrictions of the graph topology and the Riesz topology to $\mathcal{R}^{sb}(\mathcal{H})$ coincide.*

Proof. For c running $\mathbb{R}$, the subsets

$$V_c = \{A \in \mathcal{R}^{sa}(H) \,|\, \operatorname{Res}(A) \supset (-\infty, c]\} \quad \text{and} \quad V'_c = \{A \in \mathcal{R}^{sa}(H) \,|\, \operatorname{Res}(A) \supset [c, +\infty)\}$$

form an open covering of $\mathcal{R}^{sb}(H)$ in both graph and Riesz topologies. It is sufficient to show that the restriction of the graph topology to each of these subsets coincides with the Riesz topology. Let us prove this for V_c; the proof for V'_c is quite similar.

We work as in the proof of Proposition 13.1, with the same designations. The function $\tilde{\kappa}$ defines a homeomorphism from the interval $[f(c), 1]$ to the arc $\Gamma_c = \{e^{it} \mid t \in [t_c, 0]\}$, where $t_c \in (-2\pi, 0)$, $e^{it_c} = \kappa(c)$. The inverse homeomorphism $\varphi\colon \Gamma_c \to [f(c), 1]$ is given by the same formula as before, $\varphi(e^{it}) = \cos(t/2)$. The restriction of the bounded transform f to V_c coincides with the composition $\varphi \circ \kappa$ and thus is continuous. It follows that the restriction of the graph topology to V_c coincides with the Riesz topology. This completes the proof of the theorem. □

14 Spectral sections

Arbitrary base spaces. It follows from Theorem 4.3 that a Riesz continuous family always has local spectral sections locally, and the only obstruction for existence of a global spectral section is a topological one. In contrast with this, a graph continuous family may have no spectral section even locally. In fact, Riesz continuity is *necessary* for a local existence of a spectral section, as the following result shows.

14.1 Theorem. *Let X be an arbitrary topological space and $\mathcal{A}\colon X \to \mathcal{R}^{sa}_K(\mathcal{H})$ be a graph continuous map having a spectral section. Then $\mathcal{A}$ is Riesz continuous.*

Proof. Let P be a spectral section for $\mathcal{A}$ and $r\colon X \to \mathbb{R}_+$ be the corresponding cut-off function. Let $x_0 \in X$. Choose a positive constant r_0 and a neighborhood V of x_0 such that $\pm r_0 \in \operatorname{Res}(\mathcal{A}_x)$ and $r_0 > r(x)$ for all $x \in V$.

The finite rank projection $S_x = \mathbb{1}_{(-r_0, r_0)}(\mathcal{A}_x)$ continuously depends on $x \in V$ and commutes with P_x. Hence

$$S_x, \quad S_x^+ = (1 - S_x)P_x, \;\text{ and }\; S_x^- = (1 - S_x)(1 - P_x) \tag{14.1}$$

are mutually orthogonal projections continuously depending on $x \in V$. Decreasing V if necessary, one can find a continuous map $g\colon V \to \mathcal{U}(H)$ such that the conjugation by g takes these three projection-valued maps to constant projections S_0, S_0^+, and S_0^-. Indeed, one can first find a neighborhood $V_1 \subset V$ of x_0 and a map $g_1\colon V_1 \to \mathcal{U}(H)$ such that $g_1(x)S_x g_1(x)^* \equiv S_0$ [WO, Proposition 5.2.6]. Next, one can find a neighbourhood $V_2 \subset V_1$ of x_0 and a map $g_2\colon V_2 \to \mathcal{U}(H)$ such that $g_2(x)$ is equal to the identity on the range of S_0 and the conjugation by $g_2(x)$ takes $g_1(x)S_x^+ g_1(x)^*$ to S_0^+. Then $g = g_2 g_1$ is a desired trivialization of projections (14.1) over V_2.

Let H°, H^+, and H^- be the ranges of projections S_0, S_0^+, and S_0^-. Then

$$g_x \mathcal{A}_x g_x^* = \mathcal{A}_x^- \oplus \mathcal{A}_x^\circ \oplus \mathcal{A}_x^+ \tag{14.2}$$

with respect to the orthogonal decomposition $H = H^- \oplus H^\circ \oplus H^+$. The map $g\mathcal{A}g^*\colon V \to \mathcal{R}^{sa}_K(\mathcal{H})$ is graph continuous, so its components $\mathcal{A}^-$, $\mathcal{A}^\circ$, $\mathcal{A}^+$ are also graph continuous. Since $\mathcal{A}^\circ$ acts on the finite-dimensional space H°, it is norm continuous. Both $\mathcal{A}^+$ and $-\mathcal{A}^-$ are bounded from below by a positive constant r_0, so by Proposition 13.1 they are Riesz continuous. Substituting this to (14.2), we see that $g\mathcal{A}g^*$ is Riesz continuous, and thus the restriction of $\mathcal{A}$ to V is also Riesz continuous. Since

$x_0 \in X$ was chosen arbitrarily, $\mathcal{A}$ is Riesz continuous on the whole X. This completes the proof of the theorem. $\square$

Compact base spaces. Taking together Theorem 14.1 and Proposition 1.1, we immediately obtain the following result.

14.2 Theorem. *Let $\mathcal{A} = (\mathcal{A}_x)_{x\in X}$ be a graph continuous family of regular self-adjoint operators with compact resolvents acting on fibers of a Hilbert bundle $\mathcal{H}$ over a compact space X. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has a spectral section.*
2. *$\mathcal{A}$ is Riesz continuous and* $\operatorname{ind}(\mathcal{A}) = 0 \in K^1(X)$.

14.3 Remark. Theorems 14.1 and 14.2 show that the straightforward transfer of [MP1, Proposition 1] and [MP2, Proposition 2] to elliptic operators on manifolds with boundary does not work; one needs to be very careful using spectral sections in this framework. For example, Yu applies [MP1, Proposition 1] to families of Dirac operators with local boundary conditions in [Yu]. However, one needs to ensure first that the corresponding families of unbounded operators are Riesz continuous.

15 Generalized spectral sections

In the previous section we showed that a graph continuous family admitting a spectral section has to be Riesz continuous. Situation with generalized spectral sections, however, is more ambiguous. We give here several illustrating examples.

Graph continuous but Riesz discontinuous family with generalized spectral section. The following example shows that existence of a generalized spectral section does not imply Riesz continuity.

15.1 Example. Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. Consider a family $\mathcal{A} \colon X \to \mathcal{R}^{sa}_K(H)$ of diagonal (in the chosen basis) operators given by the formulae

$$\mathcal{A}_x(e_n) = \begin{cases} n, & n \neq x \\ -n, & n = x \end{cases} \quad \text{for } x \in \mathbb{N}; \quad \mathcal{A}_\infty(e_n) = n.$$

Since $\|\kappa(\mathcal{A}_\infty) - \kappa(\mathcal{A}_x)\| = \|\kappa(x) - \kappa(-x)\| \to 0$ as $x \to \infty$, the family $\mathcal{A}$ is graph continuous. On the other side, $\|f(\mathcal{A}_\infty) - f(A_x)\| = 2f(x) \to 2$ as $x \to \infty$, so $\mathcal{A}$ is Riesz discontinuous at ∞.

The constant function $P \colon X \to \mathcal{P}(H)$ taking every $x \in X$ to the identity is a generalized spectral section for $\mathcal{A}$. Moreover, P_x is even an r_x-spectral section for $\mathcal{A}_x$, where $r \colon X \to \mathbb{R}_+$ is an arbitrary function such that $r_n > n$ for $n \in \mathbb{N}$. However, every such function r is discontinuous at ∞, so $\mathcal{A}$ has no global spectral section. (Otherwise, of course, we would have a contradiction with Theorem 14.1.)

Graph continuous family without generalized spectral sections. A generalized spectral section does not necessarily exist even for a contractible base space (see Example 15.2 below) or for a family of invertible operators (see Example 15.3 below). Note that, in contrast with Example 15.1, we cannot construct such graph continuous families from semibounded operators, in view of Theorem 13.2.

15.2 Example. The space $\mathcal{R}^{sa}_{K}(H)$ equipped with the graph topology is path connected [Jo]. Let $\mathcal{A}\colon [0,1] \to \mathcal{R}^{sa}_{K}(H)$ be a graph continuous path connecting a negative operator $\mathcal{A}_0$ with a positive operator $\mathcal{A}_1$. (Such a path $\mathcal{A}$ can even be chosen consisting of invertible operators, but we do not explore it here for simplicity.) Then $\mathcal{A}$ has no generalized spectral section. Indeed, a generalized spectral section P_0 for $\mathcal{A}_0$ should be compact, while a generalized spectral section P_1 for $\mathcal{A}_1$ should have compact complement $1 - P_1$. Any two such projections P_0 and P_1 lie in the different connected components of the space $\mathcal{P}(H)$, so they cannot be connected by a path $P\colon [0,1] \to \mathcal{P}(H)$.

15.3 Example. Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$ and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H, as in example 15.1. Consider a family $\mathcal{A}\colon X \to \mathcal{R}^{sa}_{K}(H)$ of invertible operators given by the formulae

$$\mathcal{A}_x(e_n) = \begin{cases} -n, & n < x \\ n, & n \geqslant x \end{cases} \text{ for } x \in \mathbb{N}; \quad \mathcal{A}_\infty(e_n) = -n.$$

The same reasoning as in example 15.1 shows that $\mathcal{A}$ is graph continuous. Similar to example 15.2, $\mathcal{A}$ has no generalized spectral section. Indeed, a generalized spectral section P_∞ for $\mathcal{A}_\infty$ should be compact, while a generalized spectral section P_x for $\mathcal{A}_x$, $x \in \mathbb{N}$, should have compact complement $1 - P_x$. Since P_∞ cannot be a limit point of P_x, such a map P cannot be norm continuous.

References

[BB] B. Booss-Bavnbek. Unique continuation property for Dirac operators, revisited. Contemporary Mathematics **258** (2000), 21–32.

[BBB] D. Bleecker and B. Booss-Bavnbek. Spectral invariants of operators of Dirac type on partitioned manifolds. Aspects of boundary problems in analysis and geometry (2004), Birkhäuser, Basel, 1–130.

[BLP] B. Booss-Bavnbek, M. Lesch, and J. Phillips. Unbounded Fredholm operators and spectral flow. Canadian Journal of Mathematics **57**(2) (2005), 225–250; arXiv:math/0108014 [math.FA].

[BLZ] B. Booss-Bavnbek, M. Lesch, and C. Zhu. The Calderón projection: new definition and applications. Journal of Geometry and Physics **59**(7) (2009), 784–826; arXiv:0803.4160 [math.DG].

[CL] H. O. Cordes, J. P. Labrousse. The invariance of the index in the metric space of closed operators. Journal of Mathematics and Mechanics **12**(5) (1963), 693–719.

[DZ] X. Dai and W. Zhang. Splitting of the family index. Communications in Mathematical Physics **182**(2) (1996), 303–317.

[Do] A. Dold. Partitions of unity in the theory of fibrations. Annals of Mathematics **78** (1963), 223–255.

[Hu] W. Hurewicz. On the concept of fiber space. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961.

[Jo] M. Joachim. Unbounded Fredholm operators and K-theory. High-dimensional Manifold Topology, World Sci. Publ., River Edge, NJ (2003), 177–199.

[Ka] T. Kato. Perturbation Theory for Linear Operators. A Series of Comprehensive Studies in Mathematics 132 (1980).

[Ku] N.H. Kuiper. The homotopy type of the unitary group of Hilbert space. Topology, **3**(1) (1965), 19–30.

[Le] M. Lesch. The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators. In: Spectral geometry of manifolds with boundary and decomposition of manifolds (B. Booss-Bavnbek, G. Grubb, and K.P. Wojciechowski, eds.), AMS Contemporary Math Proceedings **366** (2005), 193–224; arXiv:math/0401411 [math.FA].

[MP1] R.B. Melrose and P. Piazza. Families of Dirac operators, boundaries and the b-calculus. J. Differential Geom. **46** (1997), no. 1, 99–180.

[MP2] R.B. Melrose and P. Piazza. An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary. J. Differential Geom, 46 (1997), no.2, 287–334.

[P1] M. Prokhorova. Self-adjoint local boundary problems on compact surfaces. I. Spectral flow. Journal of Geometric Analysis (2019), 46 pp, https://doi.org/10.1007/s12220-019-00313-0; arXiv:1703.06105 [math.AP].

[P2] M. Prokhorova. Spectral Sections. arXiv:2008.04672v1 [math.SP], 34 pp.

[St] A.H. Stone. Paracompactness and product spaces. Bull. Amer. Math. Soc. **54** (1948), 977–982.

[WO] N.E. Wegge-Olsen. K-theory and C*-algebras: a Friendly Approach. Vol. 1050. Oxford: Oxford university press, 1993.

[Yu] Jianqing Yu. Higher spectral flow for Dirac operators with local boundary conditions. Internat. J. Math. 27 (2016), no. 8, 1650068, 19 pp.

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